ABSOLUTE CONTINUITY OF FOLIATIONS

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1. Introduction

In what follows, $U$ is an open neighborhood in a compact Riemannian manifold $M$, and $F$ is a local foliation of $U$. By this we mean there exists a homeomorphism $\varphi : U \to \mathbb{R}^n \times \mathbb{R}^m$ sending the $F$-plaque $F_p$ through $p \in U$ to the hyperplane $\mathbb{R}^n \times \{\varphi(p)_{\mathbb{R}^m}\}$. We assume that the leaves of $F$ are uniformly smooth, at least $C^1$.

A transverse local foliation to $F$ is a local foliation $T$ of $U$ whose leaves are transverse to the leaves of $F$ and of complementary dimension. We assume that $F$ has a smooth transverse local foliation (which means that $U$ has the structure of a product neighborhood).

We denote by $m$ the Riemannian volume on $U$. For $\Sigma$ a smooth submanifold of $U$, we denote by $\lambda_\Sigma$ the induced Riemannian measure on $\Sigma$. The terms “absolutely continuous,” “leafwise absolutely continuous,” and so on are defined here for local foliations; the corresponding definitions for global foliations can be given in terms of foliation charts.

Denote by $\mu$ the projection of $m$ to the quotient space $U/F$ (which is homeomorphic to $\mathbb{R}^m$). Denote by $\{m_p : p \in U\}$ the Rohlin decomposition of $m$ along $F$-leaves.

2. Leafwise absolute continuity

Definition 2.1. $F$ is absolutely continuous if, for any pair of smooth transversals $\tau_1, \tau_2$ in $U$, the $F$-holonomy map $h_F$ between $\tau_1$ and $\tau_2$ is absolutely continuous with respect to $\lambda_{\tau_1}$ and $\lambda_{\tau_2}$.

Definition 2.2. $F$ is leafwise absolutely continuous I, if, for any zero-set $Z \subset U$ for $m$, and for $m$-almost every $p \in U$, $\lambda_{F_p}(Z) = 0$.

$F$ is leafwise absolutely continuous II, if for any measurable set $Z \subset U$, if $\lambda_{F_p}(Z) = 0$ for $m$-almost every $p \in U$, then $Z$ is a zero-set for $m$.

$F$ is leafwise absolutely continuous III if $F$ is both leafwise absolutely continuous I and leafwise absolutely continuous II.

We can rephrase these definitions in terms of disintegrations.
Lemma 2.3. \( \mathcal{F} \) is leafwise absolutely continuous I if and only if, for \( m \)-almost every \( p \in U \), the measure \( \lambda_{\mathcal{F}_p} \) is absolutely continuous with respect to the disintegration \( m_p \).

\( \mathcal{F} \) is leafwise absolutely continuous II if and only if, for \( m \)-almost every \( p \in U \), the disintegration \( m_p \) is absolutely continuous with respect to \( \lambda_{\mathcal{F}_p} \).

\( \mathcal{F} \) is leafwise absolutely continuous III if and only if, for \( m \)-almost every \( p \in U \), the disintegration \( m_p \) is equivalent to \( \lambda_{\mathcal{F}_p} \).

Proof. Immediate consequence of properties of disintegration.

Lemma 2.4. \( \mathcal{F} \) is leafwise absolutely continuous III if there exists a transverse local foliation \( \mathcal{T} \) to \( \mathcal{F} \) such that \( \mathcal{T} \) is absolutely continuous, and such that the \( \mathcal{F} \)-holonomy between almost every pair of \( \mathcal{T} \)-leaves is absolutely continuous.

Proof. Rectify.

Corollary 2.5. If \( \mathcal{F} \) is absolutely continuous, then \( \mathcal{F} \) is leafwise absolutely continuous III.

The converse to the corollary does not hold. But there is a converse to Lemma 2.4.

Lemma 2.6. If \( \mathcal{F} \) is leafwise absolutely continuous III, then for every absolutely continuous transverse local foliation \( \mathcal{T} \) to \( \mathcal{F} \), the local \( \mathcal{F} \)-holonomy map \( h_{\mathcal{F}} \) between \( m \)-almost every pair of \( \mathcal{T} \)-leaves is absolutely continuous.

Proof. The plaques of \( \mathcal{F} \) are diffeomorphic to \( \mathbb{R}^n \). Fixing a leaf \( L_0 \) of \( \mathcal{F} \) and identifying it with \( \mathbb{R}^n \), we obtain an action of the group \( G = \mathbb{R}^n \) on \( L_0 \), which is the pullback of the action of \( G \) on itself by translations. This action of \( G \) on \( L_0 \) is transitive (for every \( x, x' \in L_0 \), there exists a \( g \in G \) such that \( gx = x' \)), free (if \( gx = x \), for some \( x \), then \( g = id \)), absolutely continuous (\( g(X) \) is a zero-set for \( \lambda_{L_0} \) if and only if \( X \) is), and even smooth.

Let \( \mathcal{T} \) be a transverse local foliation to \( \mathcal{F} \). Assume that \( \mathcal{T} \) is absolutely continuous. Then each element of \( G \) lifts via \( \mathcal{F} \)-holonomy to a homeomorphism \( h_g \) of the ambient space \( U \). The restriction of \( h_g \) to \( \mathcal{T}_x \) is the \( \mathcal{F} \)-holonomy to \( \mathcal{T}_{gx} \). Note that \( h_{g_1g_2} = h_{g_1} \circ h_{g_2} \). Absolute continuity of the transverse local foliation \( \mathcal{T} \) implies that \( h_g \) preserves the Lebesgue measure class on all \( \mathcal{F} \)-leaves. Moreover, \( h_g \) is absolutely continuous, since \( h_g \) preserves the transverse measure \( \mu \).

Lemma 2.7. For every \( g \in G \), there is a set \( X_g \subset L_0 \), of full measure, such that, for every \( x \in X_g \), the restriction of \( h_g \) to \( \mathcal{T}_x \) is absolutely continuous.
Proof. Otherwise there exists a positive $\lambda_{L_0}$-measure set $B \subset L_0$ and, for each $x \in B$, a zero set $Z_x \subset T_x$ for $\lambda_{T_x}$ whose image under $h_g$ has positive $\lambda_{T_{gx}}$-measure in $T_{gx}$. From this we (carefully...) obtain a set $Z \subset U$ of $m$-measure 0 whose image under $h_g$ has positive $m$-measure (this uses absolute continuity of the transverse local foliation $T$). Leafwise absolute continuity of $\mathcal{F}$ implies that $Z$ meets $m$-almost every $\mathcal{F}$-leaf in a zero set for $\lambda_\mathcal{F}$. Since $h_g$ preserves the Lebesgue measure class on $\mathcal{F}$-leaves, the set $h_g(Z)$ meets $m$-almost every $\mathcal{F}$-leaf in a zero set for $\lambda_\mathcal{F}$. Leafwise absolute continuity of $\mathcal{F}$ implies that $h_g(Z)$ is a zero-set for $m$. This contradicts the fact that $h_g(Z)$ has positive measure.

Looking at the space $L_0 \times L_0$, we have a foliation. The leaf through the point $(x_0, g(x_0))$ is $G_{g} = \{(x, g(x)) : x \in L_0\}$. This foliation is smooth (diffeomorphic to a foliation by affine spaces). By the lemma, each leaf $G_{g}$ contains a full measure set $\{(x, g(x)) : x \in X_g\}$ with the property that for every $(x, x')$ in this set, the holonomy from $T_x$ to $T_{x'}$ is absolutely continuous. But Fubini then implies that for almost every $(x, x') \in L_0 \times L_0$, the holonomy from $T_x$ to $T_{x'}$ is absolutely continuous.

Corollary 2.8. The following are equivalent
1. $\mathcal{F}$ is leafwise absolutely continuous III;
2. there exists an absolutely continuous, transverse local foliation $T$, such that the local $\mathcal{F}$-holonomy map between $m$-almost every pair of $T$-leaves is absolutely continuous; and
3. for every absolutely continuous, transverse local foliation $T$, the local $\mathcal{F}$-holonomy map between $m$-almost every pair of $T$-leaves is absolutely continuous.

2.1. A lemma on saturates.

Lemma 2.9. Assume $\mathcal{F}$ is leafwise absolutely continuous. Given any zero $m$-measure set $Z$, the $\mathcal{F}$-saturate $Z_x$ through almost any leaf $\Sigma_x$ of a chosen transverse foliation $(Z_x = \text{union of the } \mathcal{F}\text{-leaves through } Z \cap \Sigma_x)$ has zero $m$-measure.

Proof. Let $(x, y)$ be smooth local coordinates such that the leaves are graphs over the $x$-axis. The quotient space may be identified with the vertical cross-section $\Sigma_a$ at any fixed $x = a$. By Rokhlin, $m$ is a skew-product $m_F \times \mu$, where $m_F$ are the conditional measures and $\mu$ is a measure on $\Sigma_a$. The assumption means
$$m = \rho_1(\cdot) \lambda_F \times \mu \quad \text{with} \quad 0 < \rho_1(\cdot) < \infty.$$
Rectify the foliation through $\phi : (x, y) \mapsto (x, h_{a,x}(y))$. The pull-back of $m$ under $\phi$ is a measure

$$n = n_y \times \mu = \rho_2(\cdot)dy \times \mu$$

where $n_y$ are the conditional measures on horizontals, $0 < \rho_2() < \infty$ (because $\phi$ is smooth on horizontals). Let $Y = \phi^{-1}(Z)$. Then

$$m(Z) = 0 \iff n(Y) = 0$$

$$\iff \mu(\Sigma_x \cap Y) = 0 \text{ for } dx\text{-almost every } x$$

$$\iff n(Y_x) = 0 \text{ for } dx\text{-almost every } x$$

$$\iff m(Z_x) = 0 \text{ for } dx\text{-almost every } x$$

where $Y_x$ is the horizontal saturate of $\Sigma_x \cap Y$ and $Z_x$ is the $F$-saturate of $\Sigma_x \cap Z$. \hfill \Box

3. Absolute continuity with bounded Jacobians

A local foliation $F$ is transversely absolutely continuous with bounded Jacobians if for every angle $\alpha \in (0, \pi/2]$, there exists $C \geq 1$ such that, for any two smooth transversals $\tau_1, \tau_2$ to $F$ in $U$ of angle at least $\alpha$ with the leaves of $F$, and any $\lambda_{\tau_1}$-measurable set $A$ contained in $\tau_1$:

$$C^{-1}\lambda_{\tau_1}(A) \leq \lambda_{\tau_2}(h_F(A)) \leq C\lambda_{\tau_1}(A). \quad (1)$$

A foliation $F$ with smooth leaves is absolutely continuous with bounded Jacobians if, for every $\alpha \in (0, \pi/2]$, there exists $C \geq 1$ such that, for any smooth transversal $\tau$ to $F$ in $U$ of angle at least $\alpha$ with $F$, and any measurable set $A$ contained in $U$, we have the inequality:

$$C^{-1}m(A) \leq \int_{\tau} \lambda_F(A \cap F_{\text{loc}}(x)) d\lambda_{\tau}(x) \leq Cm(A). \quad (2)$$

In [10] it is asserted that absolute continuity with bounded Jacobians does not imply transverse absolute continuity, with bounded Jacobians. This assertion is wrong. In fact the two notions are equivalent.

**Lemma 3.1.** $F$ is transversely absolutely continuous with bounded Jacobians if and only if $F$ is absolutely continuous with bounded Jacobians.

**Proof.** Suppose $F$ is transversely absolutely continuous with bounded Jacobians. Let $\tau$ be a transversal in $U$. Fix a smooth, transverse local foliation $T$ in $U$ containing $\tau$ in a leaf. Rectify, and apply the standard argument to get formula (2). Thus $F$ is absolutely continuous with bounded Jacobians.
Suppose $F$ is absolutely continuous with bounded Jacobians. Then (by compactness of $M$) inside $U$, the plaques of $F$ all have comparable induced Riemannian $\lambda_F$-measure. For any transversal $\tau$ in $U$, and any saturated set $A$ in $U$, formula (2) gives that the $m$-volume of $A$ is uniformly comparable to the $\lambda_\tau$-volume of $A \cap \tau$. So if we take two transversals $\tau_1$ and $\tau_2$ in $U$ and a set $X \subset \tau_1$, the $\lambda_{\tau_1}$-measure of $X$ is comparable to the $\lambda_{\tau_2}$-measure of the holonomy image of $X$. Hence $F$ is transversely absolutely continuous with bounded Jacobians.

The next lemma shows that it suffices to check bounded Jacobians along a fixed, sufficiently nice, transverse local foliation.

**Lemma 3.2.** Suppose there exists a transverse local foliation $T$ that is absolutely continuous with bounded Jacobians and such that the $F$-holonomy between $T$-leaves is absolutely continuous, with a uniform bound on the Jacobians. Then $F$ is absolutely continuous with bounded Jacobians.

**Proof.** Let $\tau$ be a smooth transversal to $F$. At each point $p \in \tau$, there is a local projection $\pi_p$ along $F$-leaves from $\tau$ to $T_p$ (a local homeo). This projection is bi-Lipschitz (in fact, differentiable, with nonsingular inverse) at $p$ (though it might not have any nice properties away from $p$). The Lipschitz norms of $\pi_p, \pi_p^{-1}$ are bounded uniformly in $p$ by a constant $L \geq 1$ that depends only on the angles between $\tau$, the leaves of $T$, and the leaves of $F$, plus the $C^1$ data from $\tau$, the leaves of $T$ and the leaves of $F$.

Now suppose $Y$ is a zero-set in $\tau$. Cover $Y$ with balls in $\tau$ whose total $\lambda_\tau$-volume is small. For each ball $B = B_r(p, r)$ in this cover, centered at $p \in \tau$ of radius $r > 0$, consider the projection $\pi_p(B)$ into $T_p$. Since $\pi_p$ is bi-Lipschitz at $p$, the set $\pi_p(B)$ is contained in a ball of radius $Lr$ and contains a ball of radius $L^{-1}r$; hence the volumes $\lambda_r(B)$ and $\lambda_{T_p}(\pi_p(B))$ are comparable.

Since $F$ has bounded Jacobians between $T$-leaves, the image of the set $\pi_p(B)$ under $F$-holonomy between $T_p$ and any other $T$-leaf also has small $\lambda_T$-volume, comparable to the $\lambda_\tau$-volume of $B$. Thus if we saturate the cover of $Y$ by $F$ we get a cover of the $F$-saturate $Z$ of $Y$ of small $m$-volume (comparable to the volume of the original cover of $Y$ in $\tau$). This implies that the saturate of any 0-set $\lambda_\tau$ is a 0 set in $M$; moreover, we’ve shown:

**Lemma 3.3.** There exists a constant $C \geq 1$ such that, for every $F$-saturated set $A$,

$$m(A) \leq C\lambda_r(A \cap \tau),$$
and for every $p \in M$,
\[
\lambda_{T_p}(A \cap T_p) \leq C \lambda_{\tau}(A \cap \tau).
\]

Now suppose that $A$ is an arbitrary $\mathcal{F}$-saturated set. Suppose that $B = B_r(p, r)$ is a ball in $\tau$ such that $A \cap \tau$ has $\lambda_{\tau}$-density in $B$ at least $1 - \delta$. Then $\pi_p(B)$ contains a ball $B'$ in $T_p$ of radius $L^{-1}r$ in which $A \cap T_p$ has $\lambda_{T_p}$-density at least $1 - C\delta c'L^d$, where $d$ is the codimension of $\mathcal{F}$, and $c'$ is some universal constant (the point is that the $\lambda$-measure of a ball of radius $L^{-1}r$ in any leaf of $T$ can be uniformly compared to the $\lambda_{\tau}$-measure of a ball of radius $r$ in $\tau$). Saturating $B'$ by $\mathcal{F}$, we obtain a set $X \subset M$ in which $A$ has $m$-density at least $1 - C^2\delta c'L^d$ and whose $m$-measure is comparable to $\lambda_{\tau}(B)$. So we have shown:

**Lemma 3.4.** There exists a constant $\gamma > 0$ such that, for every $\varepsilon > 0$, there exists $\delta > 0$ satisfying the following. Let $B \subset \tau$ be any round ball in $\tau$, and let $A$ be any $\mathcal{F}$-saturated set. Suppose that $\lambda_{\tau}(A \cap \tau : B) > 1 - \delta$.

Then there exists an $\mathcal{F}$-saturated set $X$ with the following properties:
1. $X \cap \tau \subset B$,
2. $m(X) \geq \gamma \lambda_{\tau}(B)$, and
3. $m(A : X) > 1 - \varepsilon$.

This allows us to prove:

**Lemma 3.5.** There exists a constant $D \geq 1$ with the following property. If $A$ is any $\mathcal{F}$-saturated set, then
\[
D^{-1}\lambda_{\tau}(A \cap \tau) \leq m(A) \leq D\lambda_{\tau}(A \cap \tau).
\]

The constant $D$ depends uniformly on the transversal $\tau$.

Lemma 3.5 implies that $\mathcal{F}$ is absolutely continuous with bounded Jacobians.

**Proof of Lemma 3.5.** Lemma 3.3 gives the upper bound, so let’s find $D$ giving the lower bound. Let $\gamma$ be given by Lemma 3.4, and let $D = 1/(9\gamma)$. Let $\varepsilon = .1$, and let $\delta$ be given by Lemma 3.4. Let $A$ be given, and assume $\lambda_{\tau}(A \cap \tau) > 0$. Fix a (mod 0) Vitali cover of $A \cap \tau$ by disjoint balls in $\tau$ so that $A$ has density at least $1 - \delta$ in each ball. Then Lemma 3.4 gives a disjoint collection of sets in $M$ of total $m$-measure at least $\gamma \lambda_{\tau}(A \cap \tau)$. The $m$ density of $A$ in the union of these sets is at least .9. Thus the $m$-measure of $A$ is at least $.9 \gamma m(A \cap \tau)$, which is what we wanted to show.

This completes the proof of Lemma 3.2.
Corollary 3.6. The following are equivalent
1. $\mathcal{F}$ is absolutely continuous with bounded Jacobians;
2. there exists a transverse local foliation $\mathcal{T}$ such that $\mathcal{T}$ is absolutely continuous with bounded Jacobians, and such that the local $\mathcal{F}$-holonomy map between every pair of $\mathcal{T}$-leaves is absolutely continuous with bounded Jacobians; and
3. for any transverse local foliation $\mathcal{T}$, if $\mathcal{T}$ is absolutely continuous with bounded Jacobians, then the local $\mathcal{F}$-holonomy map between every pair of $\mathcal{T}$-leaves is absolutely continuous.

4. Smoothness

In this section $r$ denotes a real number greater than 1, not equal to an integer.

Definition 4.1. A local foliation $\mathcal{F}$ is $C^r$ if there exists a $C^r$ embedding $\psi: U \rightarrow \mathbb{R}^n \times \mathbb{R}^m$ sending each plaque $\mathcal{F}_p$ into the hyperplane $\mathbb{R}^n \times \{\psi(p)\}$.

The main technical result we use in this section is:

Theorem 4.2 (Journé [12]). Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be transverse foliations of a manifold $M$ whose leaves are uniformly $C^r$. Let $\psi: M \rightarrow \mathbb{R}$ be any continuous function such that the restriction of $\psi$ to the leaves of $\mathcal{F}_1$ is uniformly $C^r$ and the restriction of $\psi$ to the leaves of $\mathcal{F}_2$ is uniformly $C^r$. Then $\psi$ is $C^r$, with estimates on the $C^r$ norm that are uniform in the estimates along the leaves of $\mathcal{F}_1$ and $\mathcal{F}_2$.

This has the following corollary:

Corollary 4.3 (see [17]). A local foliation with uniformly $C^r$ leaves and uniformly $C^r$ holonomies (with respect to a fixed $C^r$ transverse local foliation) is a $C^r$ local foliation.

Proof. Let $\mathcal{F}$ be a local foliation with uniformly $C^r$ leaves, and let $\mathcal{T}$ be a $C^r$ transverse local foliation to $\mathcal{F}$. By a $C^r$ change of coordinates, we may assume that $\mathcal{T}$ is the foliation by vertical coordinate planes in $\mathbb{R}^n$. Now, the standard rectification of $\mathcal{F}$ in $\mathbb{R}^n$ (via holonomy between $\mathcal{T}$-leaves) sends $\mathcal{F}$-leaves to horizontal vertical planes. The assumption that the leaves of $\mathcal{F}$ are uniformly $C^r$ implies that the rectification is $C^r$ along leaves of $\mathcal{F}$. The assumption that the holonomy maps between $\mathcal{T}$-leaves are uniformly $C^r$ implies that the rectification is uniformly $C^r$ along vertical planes. Journé’s Theorem implies that the rectification is $C^r$, so that $\mathcal{F}$ is a $C^r$ foliation.

Corollary 4.4. Let $\mathcal{F}$ be a local foliation with uniformly $C^r$ leaves. The following are equivalent:
1. $F$ is $C^r$;
2. there exists a $C^r$ transverse local foliation $T$ such that the local $F$-holonomy map between every pair of $T$-leaves is uniformly $C^r$; and
3. for any $C^r$ transverse local foliation $T$, the local $F$-holonomy map between every pair of $T$-leaves are uniformly $C^r$.

Proof. The nontrivial part of the proof ($2 \implies 1$) is Corollary 4.3. 

5. Intersection properties of local foliations

**Lemma 5.1.** Let $F$ be a local foliation, and let $N$ be a smooth submanifold of $U$, diffeomorphic to a disk, transverse to the leaves of $F$, but not necessarily of complementary dimension. Let $F_N$ be the local foliation of $N$ given by intersecting leaves of $F$ with $N$.

If $F$ is an absolutely continuous foliation of $M$, then $F_N$ is an absolutely continuous foliation of $N$ (with respect to $\lambda_N$)

If $F$ is absolutely continuous with bounded Jacobians, then $F_N$ is absolutely continuous, with bounded Jacobians (with respect to $\lambda_N$).

If $F$ is $C^r$, and $N$ is $C^r$, then $F_N$ is $C^r$.

Proof. A transversal in $N$ to $F_N$ is also a transversal in $M$ to $F$. $F$-holonomy and $F_N$ holonomy coincide on transversals contained in $N$.

If $F$ is absolutely continuous, then $F$-holonomy is absolutely continuous, and therefore so is $F_N$-holonomy. This implies that $F_N$ is absolutely continuous.

If $F$ is absolutely continuous with bounded Jacobians, then $F$-holonomy is absolutely continuous with bounded Jacobians, and therefore so is $F_N$-holonomy. This implies that $F_N$ is absolutely continuous with bounded Jacobians.

If $F$ is $C^r$, and $N$ is $C^r$, then the leaves of $F_N$ are uniformly $C^r$, and $F_N$-holonomy is $C^r$. Corollary 4.3 implies that $F_N$ is $C^r$.

**Lemma 5.2.** Let $G_1$ and $G_2$ be local foliations whose leaves intersect transversely in a local foliation $F$. Let $G_1$ and $G_2$ be local foliations (not necessarily of complementary dimension).

If $G_1$ and $G_2$ are both absolutely continuous, then so is $F$.

If $G_1$ and $G_2$ are both absolutely continuous with bounded Jacobians, then so is $F$.

If $G_1$ and $G_2$ are both $C^r$, then so is $F$.

Proof. Suppose that $G_1$ and $G_2$ are both absolutely continuous. Let’s prove absolute continuity of $F$. 

Let $\tau$ and $\tau'$ be smooth transversals to $\mathcal{F}$, and let $A$ be a zero-set in $\tau$. By Lemma 5.1, the induced foliations $\mathcal{G}_{1,\tau}$ of $\tau$ and $\mathcal{G}_{1,\tau'}$ of $\tau'$ are both absolutely continuous. Under $\mathcal{F}$-holonomy between $\tau$ and $\tau'$, the leaves of $\mathcal{G}_{1,\tau}$ are taken to the leaves of $\mathcal{G}_{1,\tau'}$.

Since $\mathcal{G}_{1,\tau}$ is absolutely continuous, almost every leaf of $\mathcal{G}_{1,\tau}$ meets $A$ in a zero-set. Now, each of the leaves of $\mathcal{G}_{1,\tau}$ is also a transversal to $\mathcal{G}_{2}$. On these $\mathcal{G}_{1,\tau}$-leaves, $\mathcal{F}$ holonomy and $\mathcal{G}_{2}$ holonomy coincide. Since $\mathcal{G}_{2}$ holonomy is absolutely continuous, for almost every $\mathcal{G}_{1,\tau}$-leaf, the image of $A$ in this leaf under $\mathcal{G}_{2}$ holonomy is a zero-set in the corresponding leaf of $\mathcal{G}_{1,\tau'}$. But since $\mathcal{G}_{1,\tau'}$ is absolutely continuous, Fubini implies that the image of $A$ under $\mathcal{F}$ holonomy is a 0-set in $\tau'$. Hence $\mathcal{F}$ is absolutely continuous.

The proofs of the other parts are similar. \hfill \Box

6. Technical lemmas for proving regularity

**Lemma 6.1.** Let $\mathcal{G}_{1}$ and $\mathcal{G}_{2}$ be local foliations whose leaves intersect transversely in a local foliation $\mathcal{F}$. Suppose there exist local foliations $\mathcal{F}_{1}$ and $\mathcal{F}_{2}$ with the following properties

1. both $\mathcal{F}_{1}$ and $\mathcal{F}_{2}$ are absolutely continuous with bounded Jacobians,
2. $\mathcal{F}_{1}$ is transverse to $\mathcal{G}_{2}$, and $\mathcal{F}_{2}$ is transverse to $\mathcal{G}_{1}$,
3. $\mathcal{F}_{1}$ subfoliates the leaves of $\mathcal{G}_{1}$, and the restriction of $\mathcal{F}_{1}$ to $\mathcal{G}_{1}$-leaves is absolutely continuous, with bounded Jacobians; $\mathcal{F}_{2}$ subfoliates the leaves of $\mathcal{G}_{2}$, and the restriction of $\mathcal{F}_{2}$ to $\mathcal{G}_{2}$-leaves is absolutely continuous, with bounded Jacobians;
4. $\mathcal{F}$ holonomy between $\mathcal{F}_{1}$ leaves is absolutely continuous with bounded Jacobians, and $\mathcal{F}$ holonomy between $\mathcal{F}_{2}$ leaves is absolutely continuous with bounded Jacobians.

Then $\mathcal{G}_{1}$, $\mathcal{G}_{2}$ and $\mathcal{F}$ are all absolutely continuous with bounded Jacobians, as are the restrictions of $\mathcal{F}$ to the leaves of $\mathcal{G}_{1}$ and $\mathcal{G}_{2}$.

**Proof.** We first show that the restriction of $\mathcal{F}$ to any leaf of $\mathcal{G}_{1}$ is absolutely continuous, with bounded Jacobians.

To show this, we use the fact that the $\mathcal{F}$-holonomy maps between $\mathcal{F}_{1}$ leaves are absolutely continuous, with bounded Jacobians. Since the restriction of $\mathcal{F}_{1}$ to $\mathcal{G}_{1}$ is absolutely continuous with bounded Jacobians, Corollary 3.6 now implies that the restriction of $\mathcal{F}$ to $\mathcal{G}_{1}$-leaves is absolutely continuous, with bounded Jacobians. Similarly, the restriction of $\mathcal{F}$ to $\mathcal{G}_{2}$ leaves is absolutely continuous, with bounded Jacobians.

To show that $\mathcal{G}_{1}$ is absolutely continuous, with bounded Jacobians, it suffices (by Corollary 3.6) to prove that $\mathcal{G}_{1}$-holonomy between $\mathcal{F}_{1}$-leaves is absolutely continuous, with bounded Jacobians.
Let \( \tau \) and \( \tilde{\tau} \) be transversals to \( G_1 \), both lying inside of \( F_2 \) leaves, and let \( A \) be a measurable set in \( \tau \). Saturate \( A \) by \( F \) to get a set \( A^c \) inside of the \( G_2 \) leaf containing \( \tau \). The \( \lambda_{G_2} \) measure of \( A^c \) is comparable to the \( \lambda_{F_2} \) measure of \( A \) (since the restriction of \( F \) to \( G_2 \)-leaves is absolutely continuous, with bounded Jacobians).

Apply \( F_1 \) holonomy to \( A^c \) to get a \( F \)-saturated set \( \tilde{A}^c \) inside of the \( G_2 \) leaf containing the second transversal \( \tilde{\tau} \). Since \( F_1 \) is absolutely continuous, with bounded Jacobians, the \( \lambda_{G_2} \) measure of \( \tilde{A}^c \) is comparable to the \( \lambda_{G_2} \) measure of \( A^c \).

Project \( \tilde{A}^c \) along \( F \) leaves to get a set \( \tilde{A} \) in \( \tilde{\tau} \); its \( \lambda_{F_2} \) measure is comparable to the \( \lambda_{G_2} \) measure of \( \tilde{A}^c \), and therefore comparable to the \( \lambda_{F_2} \) measure of \( A \). Furthermore, the set \( \tilde{A} \) is the image of \( A \) under \( G_1 \) holonomy. Hence \( G_1 \) holonomy between \( F_1 \) transversals is absolutely continuous, with bounded Jacobians. Corollary 3.6 implies that \( G_1 \) is absolutely continuous, with bounded Jacobians. A similar argument shows that \( G_2 \) is absolutely continuous, with bounded Jacobians.

This proves that \( G_1 \) and \( G_2 \) are absolutely continuous with bounded Jacobians. Absolute continuity of \( F \) with bounded Jacobians now follows from Lemma 5.2.

Lemma 6.2. Let \( G_1 \) and \( G_2 \) be local foliations whose leaves intersect transversely in a local foliation \( F \). Suppose there exist local foliations \( F_1 \) and \( F_2 \) with the following properties

1. \( F_1 \) is transverse to \( G_2 \), and \( F_2 \) is transverse to \( G_1 \),
2. \( F_1 \) \( C^r \) subfoliates the leaves of \( G_1 \), and \( F_2 \) \( C^r \) subfoliates the leaves of \( G_2 \),
3. \( F \) holonomy between \( F_1 \) leaves is uniformly \( C^r \), and \( F \) holonomy between \( F_2 \) leaves is uniformly \( C^r \).

Then \( F \) is a \( C^r \) foliation, as are the restrictions of \( F \) to \( G_1 \) and \( G_2 \).

Proof. This is very similar to the proof of Lemma 6.1. Since the leaves of \( F \) are uniformly \( C^r \), to prove the proposition, by Corollary 4.3, it suffices to show that the \( F \) holonomy maps are uniformly \( C^r \). To this end, fix a \( C^\infty \) local foliation \( T \) transverse to \( F \). Fix one leaf \( T_p \) and for \( q \in F_p \), consider the associated family of \( F \) holonomy maps \( \psi_{p,q} : T_p \to T_q \). We will use Theorem 4.2 to prove that \( \psi_{p,q} \) is \( C^r \), uniformly in \( q \).

To do this, we first show that the restriction of \( F \) to the leaves of \( G_1 \) is uniformly \( C^r \), and the restriction of \( F \) to the leaves of \( G_2 \) is uniformly \( C^r \). To see this, observe that by assumption \( F_1 \) is (uniformly) a \( C^r \) subfoliation of \( G_1 \), and the \( F \)-holonomy maps between \( F_1 \) leaves are uniformly \( C^r \). The leaves \( F \) are uniformly \( C^r \), since the leaves of \( G_1 \)
and $G_2$ are. Corollary 4.3 then implies that the restriction of $F$ to the leaves of $G_1$ is uniformly $C^r$. Similarly, the restriction of $F$ to the leaves of $G_2$ is uniformly $C^r$.

Intersecting the leaves of $T$ with the leaves of $G_1$, we obtain a foliation $T_1$ with uniformly $C^r$ leaves that subfoliates both $T$ and $G_1$. Restricting our attention to the leaves of $G_1$, since $F$ is a $C^r$ subfoliation of $G_1$, we obtain that the $F$-holonomy maps between $T_1$ transversals are uniformly $C^r$. Similarly, intersecting the leaves of $T$ with the leaves of $G_2$, we obtain foliation $T_2$ with uniformly $C^r$ leaves that subfoliates both $T$ and $G_2$; the $F$-holonomy maps between $T_2$ transversals are uniformly $C^r$.

The foliations $T_1$ and $T_2$ transversely subfoliate the leaves of $T$ and have uniformly $C^r$ leaves. For a fixed $q \in M$, we have just shown that the holonomy map $\psi_{p,q}$ defined above is uniformly $C^r$ along $T_1$-leaves and uniformly $C^r$ along $T_2$-leaves. Now Theorem 4.2 implies that $\psi_{p,q}$ is $C^r$, uniformly in $q$, completing the proof of Lemma 6.2.

\[ \square \]

7. Miscellania

Given a submanifold $\Sigma$ of $M$, let $\lambda_\Sigma$ be the measure induced by the Riemann metric on $\Sigma$.

**Definition 7.1.** A foliation $F$ on $M$ is **(strongly) absolutely continuous** if the holonomy map $h_{\Sigma,\Sigma'}$ between any pair of smooth cross-sections $\Sigma$ and $\Sigma'$ is absolutely continuous: $(h_{\Sigma,\Sigma'})_*\lambda_\Sigma$ is absolutely continuous with respect to $\lambda_{\Sigma'}$.

Reversing the roles of the two-cross sections, one immediately deduces that $(h_{\Sigma,\Sigma'})_*\lambda_\Sigma$ is actually equivalent to $\lambda_{\Sigma'}$. Here we need weaker notions of absolute continuity:

**Definition 7.2.** A foliation $F$ on $M$ is **leafwise absolutely continuous** if for every zero $m$-measure set $Y \subset M$ and $m$-almost every $z \in M$, the leaf $F$ through $z$ meets $Y$ in a zero $\lambda_F$-measure set.

In other words, the Riemann measure $\lambda_F$ on almost every leaf $F$ is absolutely continuous with respect to the conditional measure $m_F$ of $m$ along the leaf.

**Remark 7.3.** If $F$ is invariant under a diffeomorphism $g$ and $g$ is ergodic, these two measures must actually be equivalent. Indeed, suppose some set $Y$ meets almost every leaf $F$ on a zero $\lambda_F$-measure set. Since the restriction of $g$ to leaves preserves the class of zero measure sets (because $g$ is smooth), we may suppose $Y$ is invariant. If $Y$ has
full measure then its complement is a zero $m$-measure that intersects leaves $F$ in full $\lambda_F$-measure subsets, a contradiction. Thus, $Y$ has zero $m$-measure, and this proves our claim.

Definition 7.4. A foliation $\mathcal{F}$ on $M$ is transversely absolutely continuous if for any cross-section $\Sigma$ and any zero $\lambda_{\Sigma}$-measure set $Z \subset \Sigma$, the union of the leaves through the points of $Z$ is a zero $m$-measure set.

Remark 7.5. Absolute continuity implies transverse and leafwise absolute continuity. This can be seen as follows: Fixing a smooth foliation transverse to $\mathcal{F}$, and using the fact that the holonomies are absolutely continuous, one define a local change of coordinates 

$$ (x, y) \mapsto (x, h(0, x)(y)) $$

dying

that rectifies the leaves of $F$ and transforms $m$ to a measure of the form $J(x, y)dx dy$ with $J > 0$. Leafwise and transverse absolute continuity are clear in these coordinates.

The previous argument only needs the property in Definition 7.1 to hold for some choice of a transverse foliation, and then only for a full measure subset of its leaves. Is the following definition any good?

Definition 7.6. A foliation $\mathcal{F}$ on $M$ is weakly absolutely continuous if near any point there exists a transverse foliation and a full measure subset of leaves such that the corresponding holonomies send zero measure sets to zero measure sets, with respect to the Riemann measures on the leaves.

Problem 7.7. Does leafwise (or transversely) absolutely continuous imply weakly absolutely continuous? What about leafwise + transversely absolutely continuous?

Remark 7.8. Leafwise absolutely continuous alone does not imply absolutely continuous: you can destroy absolute continuity of holonomy at a single transversal while keeping leafwise absolute continuity. This is an exercise in [8].

References


