ON THE CONTINUITY OF HAUSDORFF DIMENSION AND LIMIT CAPACITY FOR HORSESHOES

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1. Introduction

We begin by recalling some definitions and known facts in hyperbolic dynamics (see [3] for details).

Let M be a compact manifold without boundary, $f \in \mathrm{Diff}^1(M)$ and $\Lambda \subset M$ a basic set for f, i.e., a compact, invariant, hyperbolic, transitive set, with $\Lambda = \bigcap_{n\in\mathbb{Z}} f^n(U)$ for some neighbourhood U of Λ. Then there is a neighbourhood $\mathcal{N}$ of f in $\mathrm{Diff}^1(M)$, such that, for $g \in \mathcal{N}$, $\Lambda(g) = \bigcap_{n\in\mathbb{Z}} g^n(U)$ is a basic set for g and there is $h_g\colon \Lambda \to \Lambda_g$, a homeomorphism conjugating $g|_{\Lambda(g)}$ and $f|_\Lambda$, with h_g C^0-close to $h_f = id$ if g is C^1-close to f.

Let, for $g \in \mathcal{N}$, $T_{\Lambda(g)}M = E^s(g) \oplus E^u(g)$ be the hyperbolic splitting of $\Lambda(g)$. The local unstable set of $z \in \Lambda(g)$ is $W^u_\beta(g,z) = \{w \in M\colon d(g^n(w),g^n(z)) \le \beta \text{ for all } n \le 0\}$, where $\beta > 0$ is small. This is an embedded C^1-disk with $T_z W^u_\beta(g,z) = E^u(g,z)$. Moreover $(W^u_\beta(g,z))_{z\in\Lambda(g)}$ is continuous on g in the following sense: there is $(\theta_{g,x})_{x\in\Lambda}$ where $\theta_{g,x}\colon W^u_\beta(f,x) \to W^u_\beta(g,h_g(x))$ is a C^1 diffeomorphism with $\theta_{g,x}(x) = h_g(x)$, such that if g is C^1 close to f then, for all $x \in \Lambda$, $\theta_{g,x}$ is uniformily C^1 close to the inclusion of $W^u_\beta(f,x)$ in M. The (global) unstable set of $z\in\Lambda(g)$ is $W^u(g,z) = \{w \in M\colon d(g^n(w),g^n(z)) \to 0 \text{ when } n \to -\infty\} = \bigcup_{n\ge 0} g^n(W^u_\beta(g,g^{-n}(z)))$. We also consider local and global stable sets defined by $W^s_\beta(g,z) = W^u_\beta(g^{-1},z)$, $W^s(g,z) = W^u(g^{-1},z)$.

In the following we will always assume $\dim E^u = 1$. Consider, for $z \in \Lambda(g)$, the set $W^u(g,z) \cap \Lambda(g)$. Its limit capacity is a measure of how "fat" $\Lambda(g)$ is in the unstable direction and has been used in [7] to estimate the ammount of bifurcations through a homoclinic Ω-explosion involving Λ (see also [8] for the heteroclinic case). Recall that the limit capacity of a compact metric space X is defined by $d(X) = \limsup_{\epsilon\to 0} \log n(X,\epsilon)/\log \epsilon^{-1}$, where $n(X,\epsilon)$ is the minimum number of ϵ-balls that cover X.

A related notion is the one of Hausdorff dimension of X defined by $HD(X) = \sup\{d \geq 0:\ m_d(X) = \infty\} = \inf\{d \geq 0:\ m_d(X) = 0\}$, where $m_d(X) = \lim_{\epsilon\to 0} \inf \sum_{U\in\mathcal{U}} (\operatorname{diam} U)^d$, the inf being taken over all finite covers $\mathcal{U}$ of X by sets of diameter not greater than $\epsilon > 0$.

Clearly both the limit capacity and Hausdorff dimension are invariant by Lipschtz homeomorphisms.

With the above notations and assumptions we state:

<u>THEOREM A</u> Suppose $\dim E^u = 1$. Then $HD(W^u_\beta(g,h_g(x)) \cap \Lambda(g))$ is a continuous function of g and its value is independent of $x \in \Lambda$. The same is true for $d(W^u_\beta(g,h_g(x)) \cap \Lambda(g))$. Moreover in our context these fractional dimensions are equal.

In [5], McCluskey - Manning proved this result for Hausdorff dimension using techniques of the thermodynamic formalism. Here we give a direct and perhaps more geometric proof based on the following theorem.

<u>THEOREM B</u> Suppose $\dim E^u = 1$. Then there is $C > 0$ and, for any $\gamma \in (0,1)$, there is a neighbourhood $\mathcal{N}_\gamma$ of f in $\mathrm{Diff}^1(M)$ such that, for $g \in \mathcal{N}_\gamma$ and $x \in \Lambda$, $h_g\big|_{W^u(f,x)\cap\Lambda}$ and

$\left(h_g\Big|_{W^u_\beta(f,x)\cap\Lambda}\right)^{-1}$ are (C,γ)-Hölder continuous.

Recall that a morphism of metric spaces $\varphi: X \to Y$ is said to be (C,γ)-Hölder continuous if there is $\delta > 0$ such that $d(\varphi(x),\varphi(y)) \leq C(d(x,y))^\gamma$ for all $x,y \in X$ with $d(x,y) \leq \delta$.

We also note that the same techniques yield a proof that the Hausdorff dimension of the Julia set of a hyperbolic rational map on the Riemann sphere is a continuous function of the map. Notice however that Ruelle ([10]) has proven that this dependence is actually analytical.

Let $x \in \Lambda$. Then, $(W^u_\beta(f,x) \cap \Lambda) \times (W^s_\beta(f,x) \cap \Lambda) \longrightarrow \Lambda$, $(y,z) \mapsto W^s_\beta(f,y) \cap W^u_\beta(f,z)$, is a homeomorphism onto a neighbourhood V_x of x and so it induces continuous projections $\pi^s: V_x \to (W^u_\beta(f,x)\cap\Lambda)$, $\pi^u: V_x \to (W^s_\beta(f,x) \cap \Lambda)$. In our context we can say somewhat more.

<u>THEOREM C</u> Suppose $\dim E^u = 1$. Let $x \in \Lambda$ and $\bar{x} \in W^s_\beta(f,x)$ close to x. Let $\pi^s: (W^u_\beta(f,x) \cap \Lambda) \longrightarrow (W^u_\beta(f,\bar{x}) \cap \Lambda)$, $\pi^s(y) = W^s_\beta(f,y) \cap W^u_\beta(f,\bar{x})$. Then, for any $\gamma \in (0,1)$, π^s and $(\pi^s)^{-1}$ are (C_γ,γ)-Hölder continuous for some $C_\gamma > 0$.

In other words, Theorem C means that the (partial) stable foliation $(W^s_\beta(f,x))_{x\in\Lambda}$ is Hölder continuous with exponential Hölder constant arbitrarily close to one. This is the main step in the proof of the following result on the dimension of the whole basic sets, in the two-dimensional case.

<u>THEOREM D</u> Let $\dim M = 2$, $\dim E^u = \dim E^s = 1$. Then

$$HD(\Lambda(g)) = HD(W^u_\beta(g,z) \cap \Lambda(g)) + HD(W^s_\beta(g,z) \cap \Lambda(g))$$

$$d(\Lambda(g)) = d(W^u_\beta(g,z) \cap \Lambda(g)) + d(W^s_\beta(g,z) \cap \Lambda(g))$$

and so $HD(\Lambda(g)) = d(\Lambda(g))$ is a continuous function of $g \in \mathrm{Diff}^1(M)$.

Finally we would like to make some comments about on the Hausdorff dimension of basic sets. It was proved by Bowen and Ruelle ([2]) that if f is C^2 then all its basic sets have measure zero. On the other hand, Bowen has given in [1] an example of a C^1-diffeomorphism on the sphere exhibiting a horseshoe with positive measure. Theorem D allows us to prove that Bowen's example is nongeneric. Actually, generically the Hausdorff dimension of the horseshoes is smaller than two unless the horseshoe is all of M.

Corollary Let $\dim M = 2$, $\dim E^u = \dim E^s = 1$ and $\mathfrak{h}$ be a small neighbourhood of f in $\mathrm{Diff}^1(M)$. Then, for g is an open and dense subset of $\mathfrak{h}$ in the C^1-topology, we have $HD(\Lambda(g)) < 2$ and so $\Lambda(g)$ has Lebesgue measure zero.

After the preparation of this paper, Mañé [4] showed that the Hausdorff dimension of a horseshoe in M^2 is a C^{r-1} function of $f \in \mathrm{Diff}^r(M^2)$, $r > 1$.

We now proceed to prove Theorems A-D and the Corollary in the next sections.

2. Proof of Theorem B. For $y,z \in W^u_\beta(f,x)$, $x \in \Lambda$, we denote by $d_u(y,z)$ the length of the segment $[y,z] \subset W^u_\beta(f,x)$, i.e., d_u is the induced distance on $W^u_\beta(f,x)$. We also denote, for $y \in W^u_\beta(f,x)$, $Df^u(y) = Df\big|_{T_y W^u_\beta(f,x)}$ as the derivative of f in the unstable direction.

Let $\lambda = \inf\{|Df^u(\xi)|: \xi \in W^u_\beta(f,x),\ x \in \Lambda\} > 1$. For $\gamma \in (0,1)$, take $\epsilon > 0$ such that $(\lambda - 2\epsilon) \geq \lambda^\gamma$, and $\delta > 0$ such that the following is true for all $x \in \Lambda$

(1) $d_u(y,z) \le 4\delta \Longrightarrow |Df^u(y) - Df^u(z)| \le \epsilon$, for $y,z \in W^u_\beta(f,x)$.

Take $\mathcal{U}_\gamma$ a small neighbourhood of f such that for all $g \in \mathcal{U}_\gamma$ and $x \in \Lambda$, we have

(2) $d_u(\theta^{-1}_{g,x}\, h_g(y), y) \le \delta/2$, for $y \in W^u_\beta(f,x) \cap \Lambda$

and

(3) $|D(\theta^{-1}_{g,f(x)} \circ g \circ \theta_{g,x})(y) - Df^u(y)| \le \epsilon$, for $y \in W^u_\beta(f,x)$.

Let now $x \in \Lambda$ and $y,z \in W^u_\beta(f,x) \cap \Lambda$ be such $d_u(y,z) \le \delta$. Since $W^u_\beta(f,x)$ is 1-dimensional, we have

$$d_u(f(y),f(z)) = d_u(y,z)\cdot|Df^u(\xi_0)|, \text{ for some } \xi_0 \in [y,z] \subset W^u_\beta(f,x)$$

and, more generally

$$d_u(f^n(y),f^n(z)) = d_u(y,z)\cdot \prod_0^{n-1} |Df^u(\xi_j)|, \text{ with } \xi_j \in [f^j(y),f^j(z)]$$

Take $N \ge 0$ such that

(4) $$d_u(f^N(y),f^N(z)) \le \delta \le d_u(f^{N+1}(y),f^{N+1}(z))$$

Then, for $0 \le n \le N$, $d_u(f^n(y),f^n(z)) \le \delta$ and so, by (2), $d_u(\theta_n^{-1}\, h_g(f^n(y)),\ \theta_n^{-1}\, h_g(f^n(z))) \le 2\delta$, where $\theta_n = \theta_{g,f^n(x)}$.
On the other hand,

$$d_u(\theta_n^{-1}\, h_g(f^n(y)),\ \theta_n^{-1}\, h_g(f^n(z))) =$$

$$= d_u(\theta^{-1}_{g,x}\, h_g(y),\ \theta^{-1}_{g,x}\, h_g(z))\cdot \prod_0^{n-1} |D(\theta^{-1}_{j+1} \circ g \circ \theta_j)(\eta_j)|$$

with $\eta_j \in [\theta_j^{-1} h_g(f^j(y)), \theta_j^{-1} h_g(f^j(z))]$, $0 \le j \le n$.

In particular, by (4) we get

$$(5) \quad d_u(\theta_{g,x}^{-1} h_g(y), \theta_{g,x}^{-1} h_g(z)) \cdot \prod_0^{N-1} |D(\theta_{j+1}^{-1} \circ g \circ \theta_j)(\eta_j)| \le 2\delta \le$$

$$\le 2\delta^\gamma \le 2(d_u(y,z))^\gamma \cdot \prod_0^N (|Df^u(\xi_j)|)^\gamma .$$

Now $d_u(\xi_j, \eta_j) \le 4\delta$ and so, by (3) and (1),

$$|D(\theta_{j+1} \circ g \circ \theta_j)(\eta_j)| \ge |Df^u(\eta_j)| - \epsilon \ge |Df^u(\xi_j)| - 2\epsilon$$

and so, since $|Df^u(\xi_j)| \ge \lambda$,

$$|D(\theta_{j+1} \circ g \circ \theta_j)(\eta_j)| \ge (|Df^u(\xi_j)|)^\gamma .$$

Then, it follows from (5) that

$$d_u(\theta_{g,x}^{-1} h_g(y), \theta_{g,x}^{-1} h_g(z)) \le 2 \cdot (|Df^u(\xi_N)|)^\gamma \cdot (d_u(y,z))^\gamma$$

and so

$$d_u(h_g(y), h_g(z)) \le C \cdot (d_u(y,z))^\gamma$$

where $C = 2 \cdot \sup\{|D\theta_{g,x}(\xi)| : g \in \mathfrak{N}, x \in \Lambda, \xi \in W_\beta^u(f,x)\} \cdot$

$$\cdot \sup\{|Df^u(\xi)| : \xi \in W^u(f,x), x \in \Lambda\}.$$

The Hölder continuity of $(h_g|_{W_\beta^u(f,x)} \cap \Lambda)^{-1}$ is proved in the same way. The proof of the theorem is complete.

3. Proof of Theorem A

It is easy to check that if there is an onto, (C,γ)-Hölder continuous map $X \to Y$, where X,Y are metric spaces, then $m_d(Y) \leq C^d \cdot m_{d\gamma}(X)$ for all $d \geq 0$ and $n(Y,C\epsilon^\gamma) \leq n(X,\epsilon)$ for $\epsilon > 0$ small, so $HD(Y) \leq \gamma^{-1} HD(X)$ and $d(Y) \leq \gamma^{-1} d(X)$.

Since $h_g\colon W^u_\beta(f,x) \cap \Lambda \longrightarrow W^u_\beta(g,h_g(x)) \cap \Lambda(g)$, it follows from Theorem B that for $g \in \mathcal{U}_\gamma$

$$HD(W^u_\beta(g,h_g(x)) \cap \Lambda(g)) \in [\gamma,\gamma^{-1}]\ HD(W^u_\beta(f,x) \cap \Lambda)$$

and

$$d(W^u_\beta(g,h_g(x)) \cap \Lambda(g)) \in [\gamma,\gamma^{-1}]\ d(W^u_\beta(f,x) \cap \Lambda).$$

This obviously implies the claimed countinuity of the Hausdorff dimension and the limit capacity.

As to the second part of the theorem, let us first suppose that f is C^2. In this case the family of (codimension one) stable manifolds $(W^s(f,x))_{x\in\Lambda}$ can be extended to an invariant C^1 foliation $\mathcal{F}^s$, defined in a neighbourhood of Λ (see comments in [7]). Then if $x,y \in \Lambda$ are close, the holonomy of $\mathcal{F}^s$ defines a C^1-diffeomorphism from $W^u(f,x)$ to $W^u(f,y)$, sending $(W^u(f,x) \cap \Lambda)$ to $(W^u(f,y) \cap \Lambda)$.

Let first $x_0 \in \Lambda$ be transitive. We claim that the value of $HD(W^u_\beta(f,f^j(x_0)) \cap \Lambda)$ is independent of $j \geq 0$ and $\beta > 0$ small. In fact we only have to prove this last assertion, since f^j is Lipschitz and $W^u_\beta(f,f^j(x_0)) \cap \Lambda = f^j(W^u_\alpha(f,x_0) \cap \Lambda)$ for some small $\alpha > 0$. Take $N \geq 1$ such that $f^N(x_0)$ is close to x_0. By the remarks above we have

$$HD(W^u_\beta(f,x_o) \cap \Lambda) \leq HD(W^u_{\beta_o}(f^N(x_o)) \cap \Lambda) \leq HD(W^u_{\alpha_o}(f,x_o) \cap \Lambda)$$

for some $\beta_o > 0$, $\alpha_o > 0$. Moreover, by taking N arbitrarily large, we can suppose that $\beta_o > 0$, is close to $\beta > 0$ and $\alpha_o > 0$ is arbitrarily small. The claim now follows easily.

Take now any $x \in \Lambda$ and choose $j \geq 0$ such that $f^j(x_o)$ is close to x. Again by the differentiability of $\mathcal{F}^s$ we have

$$HD(W^u_{\beta_1}(f,f^j(x_o)) \cap \Lambda) \leq HD(W^u_\beta(f,x) \cap \Lambda) \leq HD(W^u_{\beta_2}(f,f^j(x_o)) \cap \Lambda)$$

for some $\beta_1 > 0$, $\beta_2 > 0$ close to $\beta > 0$. By the claim above $HD(W^u_\beta(f,x) \cap \Lambda) = HD(W^u_\beta(f,x_o) \cap \Lambda)$. This ends the proof in the C^2 case. The proof of the general case is now quite easy: just use the first part of the theorem and the fact that any $f \in \mathrm{Diff}^1(M)$ can be aproximated by C^2-diffeomorphisms. The proof for the limit capacity is similar.

Finally, the Hausdorff dimension and the limit capacity of a basic set Λ take the same value if f is C^2 (see Takens [11]). Again, by the first part of the theorem the same is true even if f is just C^1. The proof of Theorem A is complete.

4. Proof of Theorems C-D and the Corollary.

Proof of Theorem C. Take $g \in \mathcal{U}_\gamma \cap \mathrm{Diff}^2(M)$. As observed in Section 3, the stable foliation $(W^u_\beta(g,z))_{z\in\Lambda(g)}$ is differentiable, in the sense that it can be extended to an invariant C^1 foliation in a neighbourhood of $\Lambda(g)$. It follows that its holonomy maps, in particular $\pi^s_g: (W^u_\beta(g,h_g(x)) \cap \Lambda(g)) \to (W^u_\beta(g,h_g(\bar{x})) \cap \Lambda(g))$, $\pi^s_g(z) = W^s_\beta(g,z) \cap W^u_\beta(g,h_g(\bar{x}))$, are Lipschitz.

On the other hand, for $y \in W^u(f,x)$, $h_g(\pi^s_g(y)) =$
$= h_g(W^s(f,y) \cap W^u(f,\bar{x})) = W^s_\beta(g,h_g(y)) \cap W^u_\beta(g,h_g(\bar{x})) = \pi^s_g(h_g(y))$,
since h_g preserves local invariant sets. Then $\pi^s = h_g^{-1} \circ \pi^s_g \circ h_g$
and the result follows from Theorem B and the remarks above.

<u>Proof of Theorem D</u>. From the definitions we easily get that, for any metric space A, $HD(A) \leq d(A)$. Consider the product space $A_x = (W^u_\beta(f,x) \cap \Lambda) \times (W^s_\beta(f,x) \cap \Lambda)$. It is quite easy to check that $d(A_x) \leq d(W^u_\beta(f,x) \cap \Lambda) + d(W^s_\beta(f,x) \cap \Lambda)$. On the other hand, by Marstrand [6], $HD(A_x) \geq HD(W^u_\beta(f,x) \cap \Lambda) + HD(W^s_\beta(f,x) \cap \Lambda)$. Then, by Theorem A, $HD(A_x) = HD(W^u_\beta(f,x) \cap \Lambda) + HD(W^s_\beta(f,x) \cap \Lambda) =$
$= d(W^u_\beta(f,x) \cap A) + d(W^s_\beta(f,x) \cap \Lambda) = d(A_x)$.

Let now $\Phi: A_x \to \Lambda$ be given by $\Phi(y,z) = W^s_\beta(f,y) \cap W^u_\beta(f,y)$. Φ is a homeomorphism onto a neighbourhood V_x of x in Λ. We claim that Φ and Φ^{-1} are (C_γ,γ)-Hölder continuous for any $\gamma \in (0,1)$. From this it follows that $HD(V_x) \in [\gamma,\gamma^{-1}]HD(A_x)$, $d(V_x) \in [\gamma,\gamma^{-1}]d(A_x)$ for any $\gamma \in (0,1)$ and so $HD(V_x) =$
$= HD(A_x)$, $d(V_x) = d(A_x)$. Now, $(V_x)_{x\in\Lambda}$ is an open cover of Λ, and so $HD(\Lambda) = HD(V_x) = HD(A_x)$, $d(\Lambda) = d(V_x) = d(A_x)$, these values being independent of $x \in \Lambda$.

We are now left to prove the claim above. Let $w_1 = \Phi(y_1,z_1)$, $w_2 = \Phi(y_2,z_2)$ and $w = W^u_\beta(f,w_1) \cap W^s_\beta(f,w_2) = \Phi(y_2,z_1)$. Clearly, by Theorem C,

$$d(w_1,w_2) \leq d_u(w_1,w) + d_s(w,w_2) \leq C_\gamma(d_u(y_1,y_2))^\gamma + C_\gamma(d_s(z_1,z_2))^\gamma \leq$$

$$\leq 2C_y(\max\{d_u(y_1,y_2),\ d_s(z_1,z_2)\})^\gamma$$

and this proves the Hölder continuity of Φ.

Now, from the continuous dependence of $W^u_\beta(f,z)$ and $W^s_\beta(f,z)$ on $z \in \Lambda$, we get in a fairly easy way that

$$d(w_1,w_2) > k\cdot\max[d_u(w_1,w),d_s(w,w_2)]$$

for some $k > 0$. Then, again by Theorem C,

$$\max[d_u(y_1,y_2),d_s(z_1,z_2)] \leq \max[C_\gamma(d_u(w_1,w))^\gamma,C_\gamma(d_s(w,w_2))^\gamma] \leq$$
$$\leq C_\gamma(\max[d_u(w_1,w),d_s(w,w_2)])^\gamma \leq C_\gamma\ k^{-\gamma}(d(w_1,w_2))^\gamma.$$

This ends the proof of the theorem.

5. Proof of the Corollary.

Openess is clear from the continuity statement in Theorem D. Density is a trivial consequence of the fact that $HD(\Lambda(g)) < 2$ whenever g is C^2. This follows from the results of Bowen and Ruelle in [2] together with McCluskey and Manning [5]. For the sake of being complete we give here a different (and elementary) proof of this last fact not using the thermodynamic formalism.

Let g be C^2 and let $p \in \Lambda(g)$ be a periodic point for g. Take an interval J in $W^u(p,g)$ containing p and let $K = W^u(p,g) \cap J \cap \Lambda(g)$. One can prove (see [9]) that there are $I_1,\ldots,I_k$ disjoint compact subintervals of J and $\psi\colon I = (I_1 \cup\ldots\cup I_k) \to J$ such that

1) $K \subset I$ and the boundary of each I_j is contained in K
2) $\psi(K) = K$ and $K = \bigcap_{n\geq 0} \psi^{-n}(I)$
3) For each j, $\psi(I_j)$ is the convex hull of some subset of $\{I_1,\ldots,I_k\}$

4) ψ is C^1 on (a neighborhood of) each I_j

5) ψ is expanding, i.e., there is $\lambda > 1$ such that $|\psi'(x)| \geq \lambda$ for all $x \in I$.

Moreover, ψ has the bounded distortion property: there is $C > 0$ such that $|(\psi^n)'(x)|/|(\psi^n)'(y)| \leq C$ whenever $x,y \in I$ are such that $\psi^i(x)$ and $\psi^i(y)$ belong to the same I_{j_i} for all $0 \leq i \leq n-1$. See remark at the end of this section.

Let $\mathcal{U}_n$, $n \geq 1$, be the covering of K formed by the connected components of $\psi^{-n}(I)$. We are going to construct $d \in (0,1)$ such that $(\sum_{U \in \mathcal{U}_n} (\operatorname{diam} U)^d)_n$ is bounded (and in fact non-increasing). Since $\sup\{\operatorname{diam} U: U \in \mathcal{U}_n\}$ goes to zero when n goes to infinity this will imply $HD(W^u_{loc}(g,p) \cap \Lambda(g)) = HD(K) \leq d < 1$, which proves the Corollary.

Let $U \in \mathcal{U}_{n-1}$: then $\psi^{(n-1)}(U) = I_j$ for some $1 \leq j \leq k$. By property (3) above $\psi^n(U) = \psi(I_j)$ can be written in the form $\psi^n(U) = I_r \cup G_{r+1} \cup I_{r+1} \cup \ldots \cup G_{r+s} \cup I_{r+s}$, where the union is assumed disjoint and in increasing order ($\sup I_{i-1} = \inf G_i < \sup G_i = \inf I_i$). Then, $U = I^n_r \cup G^n_{r+1} \cup I^n_{r+1} \cup \ldots \cup G^n_{r+s} \cup I^n_{r+s}$, with $I^n_i \in \mathcal{U}_n$, G^n_i disjoint from K, $\psi^n(I^n_i) = I_i$, $\psi^n(G^n_i) = G_i$.

Now for $r \leq p \leq r+s$ and $r+1 \leq q \leq r+s$ we have $(\operatorname{diam} I_p)/(\operatorname{diam} G_q) = [(\operatorname{diam} I^n_p)/(\operatorname{diam} G^n_q)] \cdot [(\psi^n)'(\xi)/(\psi^n)'(\eta)]$ for some $\xi \in I^n_p$, $\eta \in G^n_q$. Note that for $0 \leq i \leq n-1$, $\psi^i(\xi)$ and $\psi^i(\eta)$ belong to the same element of $\{I_1,\ldots,I_k\}$ because $\xi,\eta \in U$, a connected component of $\psi^{-(n-1)}(I)$. Hence, by the bounded distortion property, $(\operatorname{diam} I_p)/(\operatorname{diam} G_q) \geq$ $\geq [(\operatorname{diam} I^n_p)/(\operatorname{diam} G^n_q)] \cdot C^{-1}$ and so $(\operatorname{diam} I^n_p)/(\operatorname{diam} G^n_q) \leq \theta$

for some θ independent of p, q, n and U.

We now need the following fact.

Lemma: Given $\theta > 0$ and $S \geq 1$ there is $\beta_o = \beta_o(\theta,S) \in (0,1)$ such that for any $x_o, y_1, x_1, \ldots, y_s, x_s > 0$ with $x_o+y_1+x_1+\ldots+y_s+x_s = 1$ and $x_i/y_i \leq \theta$, $x_{i-1}/y_i \leq \theta$ for all $1 \leq i \leq S$, we have $\sum_o^s x_i^\beta \leq 1$ for $\beta_o \leq \beta \leq 1$.

Taking $x_i = (\text{diam } I^n_{r+i}/\text{diam } U)$, $y_i = (\text{diam } G^n_{r+i}/\text{diam } U)$ in the lemma we get $\sum_o^s (\text{diam } I^n_{r+i})^\beta \leq (\text{diam } U)^\beta$ for all $\beta_o(\theta,S) \leq \beta \leq 1$. Now taking $d = \max\{\beta_o(\theta,S) : 1 \leq S \leq k\}$ we have, by the argument above, $\sum_{\substack{V \in \mathcal{U}_n \\ V \subset U}} (\text{diam } V)^d \leq (\text{diam } U)^d$ for every $U \in \mathcal{U}_{n-1}$ and so $\sum_{V \in \mathcal{U}_n} (\text{diam } V)^d \leq \sum_{U \in \mathcal{U}_{n-1}} (\text{diam } U)^d$. This ends the proof of the Corollary.

We now want to make a final remark about this proof. The assumption that g is C^2 is essential to assure that ψ is C^1. In fact the construction of ψ involves holonomy maps of the stable foliation $\mathcal{F}^s$ and, as noted earlier, these maps are not necessarily smooth, unless the diffeomorphism generating $\mathcal{F}^s$ is C^2. If we assume that g is C^3 the proof becomes somewhat simpler because then ψ can be taken to be $C^{1+\gamma}$ (i.e. ψ' exists and is Hölder continuous). In such a case the bounded distortion property is easier to prove: it just follows from the fact that $|\psi'| \geq \lambda > 1$ and it is Hölder continuous. In the general case, when ψ is induced by a C^2 diffeomorphism, the bounded distortion property is still true but the proof is more subtle and it is due to Newhouse (see [9]).

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