Prevalence of Hénon-like attractors in the unfolding of saddle-node cycles

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Abstract

We discuss here recent joint work with L.J. Díaz and J. Rocha concerning bifurcations of diffeomorphisms through the unfolding of *cycles with a saddle-node*. For a large class of such bifurcations we have shown that the presence of *Hénon-like attractors* is a *prevalent phenomenon*: they occur for a *set of parameter values with positive density* at the bifurcation value. Further developments of this work indicate the presence, in some relevant situations, of a unique *global strange attractor* and this is also announced here.

Introduction

In a recent but already celebrated paper [BC], Benedicks and Carleson have shown that the quadratic (Hénon) family of diffeomorphisms on the plane exhibits strange attractors for a *positive Lebesgue measure set of parameter values*. Based on their work, it was proved in [MV] that this *(measure-theoretic) persistence* of strange attractors (or repellers) is a quite common phenomenon: it holds, much in general, for every (generic) one-parameter family of surface diffeomorphisms unfolding a homoclinic tangency. Moreover [V], it also holds in arbitrary dimension, for a large class of families with a homoclinic bifurcation.

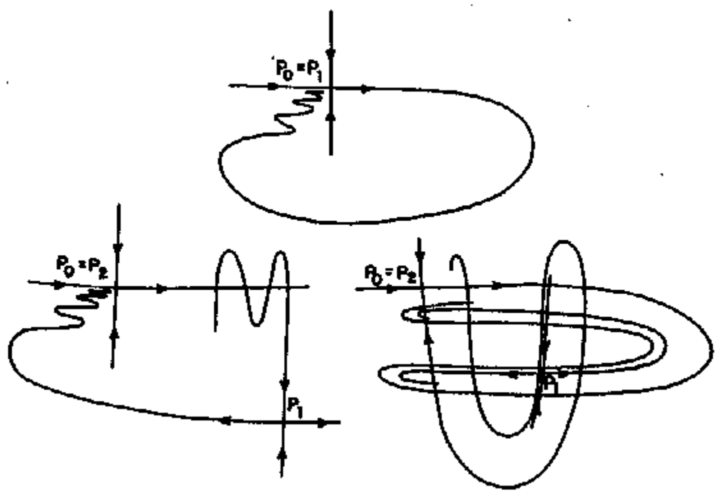
Even more recently, a considerably stronger statement on the frequency of strange attractors was obtained in [DRV1] for a related type of dynamical bifurcations: via the formation of a critical contractive saddle-node cycle (see below for the definitions). These relate closely to homoclinic bifurcations in that ([NPT]) the unfolding of such a cycle yields the appearance of homoclinic tangencies and conversely (L. Mora), the unfolding of a tangency gives rise to the formation of saddle-node cycles. As a consequence, parametrized families of diffeomorphisms passing through one with a critical contractive saddle-node cycle exhibit all the dynamical complexity associated to the creation/destruction of homoclinic orbits, in particular persistent strange attractors. The novelty in [DRV1] is the proof that for such families the set of parameters for which strange attractors are present has *positive density at the parameter value corresponding to the cycle*. In other words, *chaotic dynamics* is observed for a *definite positive fraction* of parameters close to the bifurcation value. We say that the presence of strange attractors is a *(measure-theoretically) prevalent phenomenon* in such families.

The precise statement of this result is to be given below. Presently we want to make a few comments on the nature of the strange attractors encountered in this setting. The construction in [DRV1], based on the results of [MV], [V], leads to attractors which are related to the dynamics near homoclinic tangencies created in the unfolding of the cycle and which are, therefore, of a somewhat local type. While this is probably unavoidable, due to the generality of our statement here, one may expect to get sharper results in relevant particular cases where more information is available.

A particularly interesting situation is that of 1-cycles, i.e. cycles involving one single periodic point, the saddle-node itself. It is fairly easy to see that such a cycle admits a positively invariant neighbourhood R, homeomorphic to the solid torus. In this case it is natural to ask whether a homotopically nontrivial strange attractor contained in R exists (in a persistent - or even prevalent - way). On-going work ([DRV2]) suggests that, at least in a large (open) set of cases, this is indeed true and even more: *the maximal invariant set in R* (i.e. the set of points which never leave R under negative iterations) *is a strange attractor*, for a set of parameter values with positive density. This means that for a large set of parameter values one is able to give a "complete" description of the assymptotic dynamics related to the unfolding of the cycle: all the points close to the cycle converge to a (unique, transitive) *global strange attractor*. Note that R can be defined to depend only on φ_0 and so the strange attractor has an à priori defined global basin of attraction.

We close this introduction by observing that these results may be viewed in the scenario of the program recently proposed by Palis (see [P], [PT2]) for a global description of complicated, nonhyperbolic dynamics:

- to find a sufficiently large set of well-defined bifurcating systems (diffeomorphisms, flows), ideally dense in the interior of the nonhyperbolic ones;
- to describe the persistent and/or prevalent phenomena occurring in generic parametrized families passing through those bifurcating systems.



Definitions and statement of results

By a *saddle-node k-cycle*, $k \geq 1$, of a diffeomorphism φ we mean a finite set $p_0, p_1, \ldots, p_{k-1}$, $p_k = p_0$ of periodic points of φ such that

- $p_0 = p_k$ is a saddle-node; $p_1, \ldots, p_{k-1}$ are hyperbolic saddles.
- $W^u(\mathcal{O}(p_{i-1}))$ has transverse intersections with $W^s(\mathcal{O}(p_i))$ for every $1 \leq i \leq k$ (where $\mathcal{O}(p_i) =$ orbit of p_i).

We call the cycle *contractive* if dim $W^u(p_0) = 1$ and *critical* if $W^u(\mathcal{O}(p_{k-1}))$ has non-transverse intersections with leaves of the strong stable foliation of $\mathcal{O}(p_0)$ ([NPT]). In the case of 1-cycles we also require in what follows that $W^u(\mathcal{O}(p_0))$ be totally contained in the interior of $W^s(\mathcal{O}(p_0))$. The figure illustrates some examples of 1- and 2-cycles.

By a *strange attractor* of φ we mean a compact φ-invariant set A such that

- $W^s(A) = \{z: \text{dist}(\varphi^n(z), A) \to 0 \text{ as } n \to +\infty\}$ is a neighbourhood of A.
- there exists $z_1 \in A$ such that $\{\varphi^n(z_1): n \geq 0\}$ is dense in A and moreover $\|d\varphi^n(z_1)\| \geq \sigma^n$ for some $\sigma > 1$ and every $n \geq 0$.

The strange attractors we encounter here always coincide with the closure of the unstable manifold of some periodic saddle point and they are never (uniformly) hyperbolic. We call them *Hénon-like strange attractors*

Theorem A ([DRV1]). *Let $(\varphi_\mu)_{\mu \in \mathbb{R}}$ be a generic smooth 1-parameter family of diffeomorphisms such that φ_0 has a critical contractive saddle-node cycle. Let SA be the set of values of μ for shich φ_μ has Hénon-like strange attractors. Then*

$$\liminf_{\varepsilon \to 0} \frac{m(SA \cap [-\varepsilon, \varepsilon])}{2\varepsilon} > 0,$$

where m denotes the Lebesgue measure.

Now we turn to the statement on global strange attractors. Let a diffeomorphism φ_0 have a critical contractive saddle-node 1-cycle. It is no restriction to suppose that the saddle-node p_0 is a fixed point and we do so from now on. Then there exists a compact neighbourhood R of the cycle (more precisely: of the closure of $W^u(p_0)$) homeomorphic to the solid torus $S^1 \times B^{m-1}$, m = dimension of the ambient manifold, such that $\varphi_0(R) \subset$ interior (R_0). For $(\varphi_\mu)_\mu$ a smooth 1-parameter family of diffeomorphisms passing through φ_0 and μ small, we denote $A_\mu = \bigcap_{n\geq 0} \varphi_\mu^n(R)$. We also let GSA be the set of values of μ for which A_μ is a Hénon-like attractor.

Theorem B ([DRV2]). *There exists an open set of smooth families $(\varphi_\mu)_\mu$ as above, for which*

$$\liminf_{\varepsilon\to 0} \frac{m(GSA \cap [-\varepsilon,\varepsilon])}{2\varepsilon} > 0.$$

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