PISA LECTURES ON LYAPUNOV EXPONENTS

JAIRO BOCHI AND MARCELO VIANA

ABSTRACT. These notes cover the course we taught at the research trimester on Dynamical Systems organized by Stefano Marmi at the Centro di Ricerca Matematica Ennio di Giorgi/Scuola Normale Superiore di Pisa in the Spring of 2002. We are grateful to Stefano for the invitation and all the effort he devoted to the trimester, and to the Scuola Normale for its most charming hospitality.

The purpose of our course was to present certain recent results, mostly from [Boc02, BVa, BGMV, Via], about the Lyapunov exponents of generic dynamical systems and linear cocycles. Since the proofs are long and complex, we aimed at giving fairly complete outlines, which could serve as an introduction to the complete arguments in those papers.

CONTENTS

Date: January 23, 2003.
Partially supported by PRONEX-Dynamical Systems and FAPERJ, Brazil.

1. Lyapunov exponents

Let M be a compact manifold with dimension $d \geq 1$, and $f : M \to M$ be a C^r diffeomorphism, $r \geq 1$. Oseledets theorem [Ose68] says that, relative to any f-invariant probability μ, almost every point admits a splitting of the tangent space

$$T_x M = E_x^1 \oplus \cdots \oplus E_x^k, \quad k = k(x), \tag{1}$$

and real numbers $\lambda_1(f,x) > \cdots > \lambda_k(f,x)$ such that

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \|Df^n(x)v_i\| = \lambda_i(f,x) \quad \text{for every non-zero } v_i \in E_x^i .$$

These objects are uniquely defined and they vary measurably with the point x. Moreover, the Lyapunov exponents $\lambda_i(f,x)$ are constant on orbits, hence they are constant μ-almost everywhere if μ is ergodic.

The results we are going to present address the following two fundamental problems:

(1) *How do Lyapunov exponents vary with the dynamical system ?*
(2) *How often do Lyapunov exponents vanish ?*

We consider f varying in the space $\mathrm{Diff}_\mu^r(M)$ of C^r, $r \geq 1$ diffeomorphisms that preserve a given probability μ, endowed with the corresponding C^r topology. The most interesting case is when μ is Lebesgue measure in the manifold. The second question is to be understood both in topological terms – dense, residual, or even open dense subsets – and in terms of Lebesgue measure inside generic finite-dimensional submanifolds, or parameterized families, of $\mathrm{Diff}_\mu^r(M)$.

1.1. A dichotomy for conservative systems.

First, we are going to see that systems with *zero* Lyapunov exponents are abundant among C^1 volume preserving diffeomorphisms. Let μ be normalized Lebesgue measure on a compact manifold M.

Theorem 1 ([BV02, BVa]). *There is a residual subset $\mathcal{R}$ of $\mathrm{Diff}_\mu^1(M)$ such that, for every $f \in \mathcal{R}$ and μ-almost every point x,*

(a) *either all Lyapunov exponents $\lambda_i(f,x) = 0$ for $1 \leq i \leq d$,*
(b) *or the Oseledets splitting of f is dominated on the orbit of x.*

The second case means there exists $m \geq 1$ such that for any y in the orbit of x

$$\frac{\|Df^m(y)v_i\|}{\|v_i\|} \geq 2\,\frac{\|Df^m(y)v_j\|}{\|v_j\|} \tag{2}$$

for any non-zero $v_i \in E_y^i$, $v_j \in E_y^j$ corresponding to Lyapunov exponents $\lambda_i > \lambda_j$. In other words, the fact that Df^n will eventually expand

E^i_y more than E^j_y can be observed in finite time *uniform over the orbit.* This also implies that the angles between the Oseledets subspaces E^i_y are bounded away from zero along the orbit, in fact *the Oseledets splitting extends to a dominated splitting over the closure of the orbit.*

In many situations, e.g. if the transformation f is ergodic, the conclusion gets a more global form: either (a) all exponents vanish at μ-almost every point or (b) the Oseledets splitting extends to a dominated splitting on the whole ambient manifold. The latter means that $m \geq 1$ as in (2) may be chosen uniform over the whole M.

It is easy to see that a dominated splitting into factors with constant dimensions is necessarily continuous. Now, existence of such a splitting is a very strong property that can often be excluded a priori. In any such case theorem 1 is saying that generic systems must satisfy alternative (a).

A first manifestation of this phenomenon is the 2-dimensional version of theorem 1, proved by Bochi in 2000, partially based on a strategy proposed by Mañé in the early eighties [Mañ96].

Theorem 2 ([Boc02])**.** *For a residual subset $\mathcal{R}$ of C^1 area preserving diffeomorphisms on any surface, either*

(a) *the Lyapunov exponents vanish almost everywhere or*
(b) *the diffeomorphism is uniformly hyperbolic (Anosov) on M.*

Alternative (b) can only occur if M is the torus; so, C^1 *generic area preserving diffeomorphisms on any other surface have zero Lyapunov exponents almost everywhere.*

It is an interesting question whether the theorem can always be formulated in this more global form. Here is a partial positive answer, for symplectic diffeomorphisms on any symplectic manifold (M,ω):

Theorem 3 ([BVa])**.** *There exists a residual set $\mathcal{R} \subset \mathrm{Sympl}^1_\omega(M)$ such that for every $f \in \mathcal{R}$ either the diffeomorphism f is Anosov or Lebesgue almost every point has zero as Lyapunov exponent, with multiplicity ≥ 2.*

1.2. **Linear cocycles.** Let $f : M \to M$ be a continuous transformation on a compact metric space M. A *linear cocycle* over f is a vector bundle automorphism $F : \mathcal{E} \to \mathcal{E}$ covering f, where $\pi : \mathcal{E} \to M$ is a finite-dimensional vector bundle over M. This means that

$$\pi \circ F = f \circ \pi$$

and F acts as a linear isomorphism on every fiber. The quintessential example is the derivative $F = Df$ of a diffeomorphism on a manifold (*dynamical cocycle*).

For simplicity, we focus on the case when the vector bundle is trivial $\mathcal{E} = M \times \mathbb{R}^d$, although this is not strictly necessary for what follows. Then the cocycle has the form

$$F(x, v) = (f(x), A(x)v) \quad \text{for some } A : M \to \mathrm{GL}(d, \mathbb{R}).$$

It is no real restriction to suppose that A takes values in $\mathrm{SL}(d, \mathbb{R})$. Moreover, we assume that A is at least continuous. Observe that $F^n(x, v) = (f^n(x), A^n(x)v)$ for $n \in \mathbb{Z}$, with

$$A^j(x) = A(f^{j-1}(x)) \cdots A(f(x))\, A(x)$$

and

$$A^{-j}(x) = \text{ inverse of } A^j(f^{-j}(x)).$$

The theorem of Oseledets extends to linear cocycles: Given any f-invariant probability μ, then at μ-almost every point x there exists a filtration

$$\{x\} \times \mathbb{R}^d = F_x^0 > F_x^1 > \cdots > F_x^{k-1} > F_x^k = \{0\}$$

and real numbers $\lambda_1(A, x) > \cdots > \lambda_k(A, x)$ such that

$$\lim_{n \to +\infty} \frac{1}{n} \log \|A^n(x) v_i\| = \lambda_i(A, x)$$

for every $v_i \in F_x^{i-1} \setminus F_x^i$. If f is invertible there even exists an invariant splitting

$$\{x\} \times \mathbb{R}^d = E_x^1 \oplus \cdots \oplus E_x^k$$

such that

$$\lim_{n \to \pm\infty} \frac{1}{n} \log \|A^n(x) v_i\| = \lambda_i(A, x)$$

for every $v_i \in E_x^i \setminus \{0\}$. It relates to the filtration by $F_x^j = \oplus_{i>j} E_x^i$.

In either case, the largest Lyapunov exponent $\lambda(A, x) = \lambda_1(A, x)$ describes the exponential rate of growth of the norm

$$\lambda(A, x) = \lim_{n \to +\infty} \frac{1}{n} \log \|A^n(x)\|.$$

If μ is an ergodic probability, the exponents are constant μ-almost everywhere. We represent by $\lambda_j(A, \mu)$ and $\lambda(A, \mu)$ these constants.

Theorem 1 also extends to linear cocycles over any transformation. We state the ergodic invertible case:

Theorem 4 ([Boc02, BV02])**.** *Assume $f : (M, \mu) \to (M, \mu)$ is invertible and ergodic. Let $G \subset \mathrm{SL}(d, \mathbb{R})$ be any subgroup acting transitively on the projective space $\mathbb{RP}^{d-1}$. Then there exists a residual subset $\mathcal{R}$ of maps $A \in C^0(M, G)$ for which either the Lyapunov exponents $\lambda_i(A, \mu)$ are all zero at μ-almost every point, or the Oseledets splitting of A extends to a dominated splitting over the support of μ.*

Remark 1.1. Theorem 4 also carries over to the space $L^\infty(X, \mathrm{SL}(d,\mathbb{R}))$ of measurable bounded cocycles, still with the uniform topology. We also mention that in weaker topologies, cocycles having a dominated splitting may cease to constitute an open set. In fact, for $1 \le p < \infty$, *generic L^p cocycles have all exponents equal*, see Arnold, Cong [AC97] and Arbieto, Bochi [AB].

1.3. **Prevalence of non-zero exponents.** We are now going to see that the conclusions of the previous section change radically if one considers linear cocycles which are better than just continuous: assuming the base dynamics is hyperbolic, the *overwhelming majority of Hölder continuous or differentiable cocycles admit non-zero Lyapunov exponents.*

Let G be any subgroup of $\mathrm{SL}(d,\mathbb{R})$. For $0 < \nu \le \infty$ denote by $C^\nu(M,G)$ the space of C^ν maps from M to G endowed with the C^ν norm. When $\nu \ge 1$ it is implicit that M has a smooth structure. For integer ν the notation is slightly ambiguous: C^ν means either that f is ν times differentiable with continuous ν:th derivative, or that it is $\nu - 1$ times differentiable with Lipschitz continuous derivative. All the statements are meant for both interpretations.

Definition 1.2. Let $f : M \to M$ be a C^1 diffeomorphism with Hölder continuous derivative. An f-invariant probability measure μ is *hyperbolic* if every $\lambda_i(f,x)$ is different from zero at μ-almost every point.

Definition 1.3. A non-atomic hyperbolic probability μ has *local product structure* if it is locally equivalent to the product of the measures μ^u and μ^s obtained by projecting it to the spaces of local stable sets and local unstable sets.

A more formal definition will appear in Section 3. We point out that most interesting invariant measures have local product structure. For instance, Lebesgue measure always has local product structure if it is hyperbolic: this follows from the absolute continuity of Pesin's stable and unstable foliations [Pes76]. The same is true, more generally, for any hyperbolic probability having absolutely continuous conditional measures along unstable manifolds or along stable manifolds. Also, in the uniformly hyperbolic case, every equilibrium state of a Hölder continuous potential [Bow75] has local product structure.

Theorem 5 ([Via]). *Assume $f : (M,\mu) \to (M,\mu)$ is ergodic and hyperbolic with local product structure. Then, for every $\nu > 0$, the set of cocycles A with largest Lyapunov exponent $\lambda(A,x) > 0$ at μ-almost every point contains an open dense subset $\mathcal{A}$ of $C^\nu(M, \mathrm{SL}(d,\mathbb{R}))$. Moreover, its complement has ∞-codimension.*

The last property means that the set of cocycles with vanishing exponents is locally contained inside finite unions of closed submanifolds of $C^\nu(M, \mathrm{SL}(d,\mathbb{R}))$ with arbitrary codimension. Thus, generic parameterized families of cocycles do not intersect this exceptional set at all!

Now suppose $f : M \to M$ is uniformly hyperbolic, for instance, a two-sided shift of finite type, or an Axiom A diffeomorphism restricted to a hyperbolic basic set. Then every invariant measure is hyperbolic. The main novelty is that the set $\mathcal{A}$ may be taken the same for all invariant measures with local product structure.

Theorem 6 ([BGMV, Via]). *Assume $f : M \to M$ is a uniformly hyperbolic homeomorphism. Then, for every $\nu > 0$, the set of cocycles A with largest Lyapunov exponent $\lambda(A, x) > 0$ at μ-almost every point and for every invariant measure with local product structure contains an open dense subset $\mathcal{A}$ of $C^\nu(M, \mathrm{SL}(d,\mathbb{R}))$. Moreover, its complement has ∞-codimension.*

Theorem 6 was first proved in [BGMV], under an additional hypothesis called domination. Under this additional hypothesis [BVb] gets a stronger conclusion: all Lyapunov exponents have multiplicity 1, in other words, the Oseledets subspaces E^i are one-dimensional. We expect this to extend to full generality:

Conjecture. Theorems 5 and 6 should remain true if one replaces $\lambda(A, x) > 0$ by all Lyapunov exponents $\lambda_i(A, x)$ having multiplicity 1.

Theorems 5 and 6 extend to cocycles over non-invertible transformations, respectively, local diffeomorphisms equipped with invariant non-uniformly expanding probabilities (all Lyapunov exponents positive), and uniformly expanding continuous maps, like one-sided shifts of finite type, or smooth expanding maps. Moreover, both theorems remain true if we replace $\mathrm{SL}(d,\mathbb{R})$ by any subgroup G such that

$$G \ni B \mapsto (B\xi_1, \ldots, B\xi_d) \in (\mathbb{RP}^{d-1})^d,$$

is a submersion, for any linearly independent $\{\xi_1, \ldots, \xi_d\} \subset \mathbb{RP}^{d-1}$. In particular, this holds for the symplectic group.

Problem. What are the continuity points of Lyapunov exponents as functions of the cocycle in $C^\nu(M, \mathrm{SL}(d,\mathbb{R}))$, when $\nu > 0$? Analogously, assuming the base system (f, μ) is hyperbolic.

2. Abundance of vanishing exponents

We are going to sketch the proofs of theorems 1 and 3. For complete arguments see [Boc02, BV02].

Let $f \in \mathrm{Diff}^1_\mu(M)$ and Γ be an invariant set. We say that an invariant splitting $T_\Gamma = E \oplus F$ is *m-dominated*, for some $m \in \mathbb{N}$, if for all $x \in \Gamma$

$$\frac{\|Df^m_x|_{F_x}\|}{\mathbf{m}(Df^m_x|_{E_x})} < \frac{1}{2},$$

where $\mathbf{m}(A) = \|A^{-1}\|^{-1}$. We call $E \oplus F$ a *dominated splitting* if it is m-dominated for some m.

2.1. **Volume preserving diffeomorphisms.** Given $f \in \mathrm{Diff}^1_\mu(M)$ and $1 \le p \le d$, we write

$$\Lambda_p(f,x) = \lambda_1(f,x) + \cdots + \lambda_p(f,x) \quad \text{and} \quad \mathrm{LE}_p(f) = \int_M \Lambda_p(f,x)\, d\mu(x).$$

As f preserves volume, $\Lambda_d(f,x) \equiv 0$. It is a well-known fact that the functions $f \in \mathrm{Diff}^1_\mu(M) \mapsto \mathrm{LE}_p(f)$ are upper semi-continuous. Continuity of these functions is much more delicate:

Theorem 7. *Let $f_0 \in \mathrm{Diff}^1_\mu(M)$ be such that the map*

$$\mathrm{Diff}^1_\mu(M) \ni f \mapsto \big(\mathrm{LE}_1(f), \ldots, \mathrm{LE}_{d-1}(f)\big) \in \mathbb{R}^{d-1}$$

is continuous at $f = f_0$. Then for almost every $x \in M$, the Oseledets splitting of f_0 is either dominated or trivial (all $\lambda_p(f,x) = 0$) along the orbit of x.

Since the set of points of continuity of a upper semi-continuous function is always a residual set, we see that theorem 1 is an immediate corollary of theorem 7. Also, theorem 7 remains valid for linear cocycles, and in this setting the necessary condition is also sufficient.

We shall now explain the main steps in the proof of theorem 7.

2.2. **First step: Mixing directions along an orbit segment.** The following notion, introduced in [Boc02], is crucial to the proofs of our theorems. It captures the idea of sequence of linear transformations that can be (almost) realized *on subsets with large relative measure* as tangent maps of diffeomorphisms close to the original one.

Definition 2.1. Given $f \in \mathrm{Diff}^1_\mu(M)$ or $f \in \mathrm{Sympl}^1_\mu(M)$, a neighborhood $\mathcal{U}$ of f in $\mathrm{Diff}^1_\mu(M)$ or $\mathrm{Sympl}^1_\mu(M)$, $0 < \kappa < 1$, and a non-periodic point $x \in M$, we call a sequence of (volume preserving or symplectic) linear maps

$$T_xM \xrightarrow{L_0} T_{fx}M \xrightarrow{L_1} \cdots \xrightarrow{L_{n-1}} T_{f^n x}M$$

an *$(\mathcal{U}, \kappa)$-realizable sequence of length n at x* if the following holds:

For every $\gamma > 0$ there is $r > 0$ such that the iterates $f^j(\overline{B}_r(x))$ are two-by-two disjoint for $0 \le j \le n$, and given any non-empty open set $U \subset B_r(x)$, there are $g \in \mathcal{U}$ and a measurable set $K \subset U$ such that

(i) g equals f outside the disjoint union $\bigsqcup_{j=0}^{n-1} f^j(\overline{U})$;
(ii) $\mu(K) > (1-\kappa)\mu(U)$;
(iii) if $y \in K$ then $\left\| Dg_{g^j y} - L_j \right\| < \gamma$ for every $0 \le j \le n-1$.

To make the definition clear, let us show (informally) that if v, $w \in T_xM$ are two unit vectors with $\measuredangle(v,w)$ sufficiently small then there exists a realizable sequence $\{L_0\}$ of length 1 at x such that $L_0(v) = Df_x(w)$.

Indeed, let $R : T_xM \to T_xM$ be a rotation of angle $\measuredangle(v,w)$ along the plane P generated by v and w, with axis $P^\perp$. We take $L_0 = Df_xR$. In order to show that $\{L_0\}$ is a realizable sequence we must, for any sufficiently small neighborhood U of x, find a perturbation g of f and a subset $K \subset U$ such that conditions (i)-(iii) in definition 2.1 are satisfied. Since this is a local problem, we may suppose, for simplicity, that $M = \mathbb{R}^d = T_xM$. First assume U is a cylinder $B \times B'$, where B and B' are balls centered at x and contained in P and $P^\perp$, respectively. We also assume that $\operatorname{diam} B \ll \operatorname{diam} B' \ll 1$. Define $K \subset U$ as a slightly shrunk cylinder also centered at x, so condition (ii) in definition 2.1 holds. Then there is a volume preserving diffeomorphism h such that h equals the rotation R inside the cylinder K and equals the identity outside U. Moreover, the conditions $\theta \ll 1$ and $\operatorname{diam} B \ll \operatorname{diam} B'$ permit us to take h C^1-close to the identity. Define $g = f \circ h$; then condition (iii) also holds.

This deals with the case where U is a thin cylinder. Now if U is any small neighborhood of x then we only have to cover μ-most of it with disjoint thin cylinders and rotate (as above) each one of them. This "shows" that $\{L_0 = Df_yR\}$ is a realizable sequence.

Our first proposition towards the proof of theorem 7 says that if a splitting $E \oplus F$ is *not* dominated then one can find a realizable sequence that sends one direction from E to F.

Proposition 2.2. *Given $f \in \operatorname{Diff}^1_\mu(M)$, a neighborhood $\mathcal{U} \ni f$ and $0 < \kappa < 1$ be given. Let $m \in \mathbb{N}$ be large. Suppose it is given a non-trivial splitting $T_{\operatorname{orb}(y)}M = E \oplus F$ along the orbit of a non-periodic $y \in M$ satisfying the following "non-dominance" condition:*

$$\frac{\|Df^m_y|_F\|}{\mathbf{m}(Df^m_y|_E)} \ge \frac{1}{2}. \tag{3}$$

Then there exists a $(\mathcal{U},\kappa)$-realizable sequence $\{L_0,\dots,L_{m-1}\}$ at y of length m and there is a non-zero vector $v \in E_y$ such that we have $L_{m-1}\cdots L_0(v) \in F_{f^m y}$.

Let us explain how the sequence is constructed, at least in the simplest case. Assume that $\sphericalangle(E_{f^i y}, F_{f^i y})$ is very small for some $i = 1,\dots,m-1$. We take unit vectors $v_i \in E_{f^i y}$, $w_i \in F_{f^i y}$ such that $\sphericalangle(v_i, w_i)$ is small. As we have explained before, there is a realizable sequence $\{L_i\}$ of length 1 at $f^i x$ such that $L_i(v_i) = w_i$. We define $L_j = Df_{f^j x}$ for $j \neq i$; then $\{L_0,\dots,L_{m-1}\}$ is the desired realizable sequence.

The construction of the sequence is more difficult when $\sphericalangle(E,F)$ is not small, because several rotations may be necessary.

2.3. **Second step: Lowering the norm.** Let us recall some facts from linear algebra. Given a vector space V and a non-negative integer p, let $\wedge^p(V)$ be the p:th exterior power of V. This is a vector space of dimension $\binom{d}{p}$, whose elements are called *p-vectors*. It is generated by the p-vectors of the form $v_1 \wedge \cdots \wedge v_p$ with $v_j \in V$, called the *decomposable p-vectors*. We take the norm $\|\cdot\|$ in $\wedge^p(V)$ such that if $\mathbf{v} = v_1 \wedge \cdots \wedge v_p$ then $\|\mathbf{v}\|$ is the p-dimensional volume of the parallelepiped with edges $v_1, \dots, v_p$. A linear map $L : V \to W$ induces a linear map $\wedge^p(L) : \wedge^p(V) \to \wedge^p(W)$ such that

$$\wedge^p(L)(v_1 \wedge \cdots \wedge v_p) = L(v_1) \wedge \cdots \wedge L(v_p)$$

Let $f \in \mathrm{Diff}^1_\mu(M)$ be fixed from now on. Although it is not necessary, we shall assume for simplicity that f is *aperiodic*, that is, the set of periodic points of f has zero measure.

Given $f \in \mathrm{Diff}^1_\mu(M)$ and $p \in \{1,\dots,d-1\}$, we have, for almost every x,

$$\frac{1}{n}\log\|\wedge^p(Df^n_x)\| \to \Lambda_p(f,x) \quad \text{as } n \to \infty.$$

Suppose the Oseledets splitting along the orbit of a point x is not dominated. Our next task (proposition 2.3) is to construct long realizable sequences $\{\widehat{L}_0,\dots,\widehat{L}_{n-1}\}$ at x such that $\frac{1}{n}\log\|\wedge^p(\widehat{L}_{n-1}\cdots\widehat{L}_0)\|$ is smaller then the expected value $\Lambda_p(f,x)$.

Given p and $m \in \mathbb{N}$, we define $\Gamma_p(f,m)$ as the set of points x such that if $T_{\mathrm{orb}(x)}M = E \oplus F$ is an invariant splitting along the orbit, with $\dim E = p$, then it is not m-dominated. It follows from basic properties of dominated splittings (see section 3.1) that $\Gamma_p(f,m)$ is an open set. Of course, it is also invariant.

Proposition 2.3. *Let $\mathcal{U} \subset \mathrm{Diff}^1_\mu(M)$ be a neighborhood of f, $0 < \kappa < 1$, $\delta > 0$ and $p \in \{1, \dots, d-1\}$. Let $m \in \mathbb{N}$ be large. Then for μ-almost every point $x \in \Gamma_p(f, m)$, there exists an integer $N(x)$ such that for every $n \geq N(x)$ there exists a $(\mathcal{U}, \kappa)$-realizable sequence*

$$\{\widehat{L}_0, \dots, \widehat{L}_{n-1}\} = \{\widehat{L}_0^{(x,n)}, \dots, \widehat{L}_{n-1}^{(x,n)}\}$$

at x of length n such that

$$\frac{1}{n} \log \|\wedge^p(\widehat{L}_{n-1} \cdots \widehat{L}_0)\| \leq \frac{\Lambda_{p-1}(x) + \Lambda_{p+1}(x)}{2} + \delta. \tag{4}$$

Moreover, the function $N : \Gamma_p(f, m) \to \mathbb{N}$ is measurable.

The proof of the proposition may be sketched as follows. Given $x \in \Gamma_p(f, m)$, we may assume $\lambda_p(x) > \lambda_{p+1}(x)$, otherwise we can take the trivial sequence $\widehat{L}_j = Df_{f^j x}$ and there is nothing to prove. Then we can consider the splitting $T_x M = E_x \oplus F_x$, where E_x (resp. F_x) is the sum of the Oseledets spaces corresponding to the exponents $\lambda_1(x)$, $\dots$, $\lambda_p(x)$ (resp. $\lambda_{p+1}(x)$, $\dots$, $\lambda_d(x)$). By assumption, the splitting $E \oplus F$ is not m-dominated along the orbit of x, that is, there exists $\ell \geq 0$ such that

$$y = f^\ell(x) \;\Rightarrow\; \frac{\|Df^m_y|_{F_y}\|}{\mathbf{m}(Df^m_y|_{E_y})} \geq \frac{1}{2}.$$

By Poincaré recurrence, there are infinitely many integers $\ell \geq 0$ such that the above relation is satisfied (for almost every x). Moreover, it can be shown, using Birkhoff's theorem, that for all large enough n, that is, for every $n \geq N(x)$, we can find $\ell \approx n/2$ such that the inequality above holds for ℓ. Here $\ell \approx n/2$ means that $|\frac{\ell}{n} - \frac{1}{2}| < \mathrm{const.}\delta$.

Fix x, $n \geq N(x)$, ℓ as above, $y = f^\ell(x)$ and $z = f^\ell(y)$. Proposition 2.2 gives a $(\mathcal{U}, \kappa)$-realizable sequence $\{L_0, \dots, L_{m-1}\}$, such that there is a non-zero vector $v_0 \in E_y$ for which

$$L_{m-1} \dots L_0(v_0) \in F_z \tag{5}$$

We form the sequence $\{\widehat{L}_0, \dots, \widehat{L}_{n-1}\}$ of length n by concatenating

$$\{Df_{f^i(x)};\ 0 \leq i < \ell\}, \quad \{L_0, \dots, L_{m-1}\}, \quad \{Df_{f^i(x)};\ \ell + m \leq i < m\}.$$

It is not difficult to show that the concatenation is a $(\mathcal{U}, \kappa)$-realizable sequence at x.

We shall give some informal indication why relation (4) is true. Let $\mathbf{v} \in \wedge^p(T_x M)$ be a p-vector with $\|\mathbf{v}\| = 1$, and let

$$\mathbf{v}' = \wedge^p(L_{m-1} \cdots L_0 Df^\ell_x)(\mathbf{v}) \in \wedge^p(T_z M).$$

Since $m \ll n$, and $L_0, \ldots, L_{m-1}$ are bounded, we have

$$\begin{aligned} \frac{1}{n}\log\|\wedge^p(\widehat{L}_{n-1}\cdots\widehat{L}_0)\mathbf{v}\| \lesssim \frac{1}{n}\log\|\wedge^p(Df_z^{n-\ell-m})\mathbf{v}'\| \\ + \frac{1}{n}\log\|\wedge^p(Df_x^\ell)\mathbf{v}\|. \end{aligned} \tag{6}$$

To fix ideas, suppose $\mathbf{v}$ is a decomposable p-vector belonging to the subspace $\wedge^p(E_x)$. Then

$$\frac{1}{\ell}\log\|\wedge^p(Df_x^\ell)\mathbf{v}\| \simeq \Lambda_p(f,x). \tag{7}$$

If we imagine decomposable p-vectors as p-parallelepipeds then, by (5), the parallelepiped $\mathbf{v}'$ contains a direction in F_z. This direction is expanded by the derivative with exponent at most $\lambda_{p+1}(z) = \lambda_{p+1}(x)$. On the other hand, the $(p-1)$-volume of every $(p-1)$-parallelepiped in T_zM grows with exponent at most $\Lambda_{p-1}(x)$. This "shows" that

$$\frac{1}{n-\ell-m}\log\|\wedge^p(Df_z^{n-\ell-m})\mathbf{v}'\| \lesssim \lambda_{p+1}(x) + \Lambda_{p-1}(x). \tag{8}$$

Substituting (7) and (8) in (6), and using that $\ell \approx n-\ell-m \approx n/2$, we obtain

$$\begin{aligned} \frac{1}{n}\log\|\wedge^p(\widehat{L}_{n-1}\cdots\widehat{L}_0)\mathbf{v}\| &\lesssim \frac{\lambda_{p+1}(x)+\Lambda_{p-1}(x)}{2} + \frac{\Lambda_p(x)}{2} \\ &= \frac{\Lambda_{p+1}(x)+\Lambda_{p-1}(x)}{2}. \end{aligned}$$

So the bound from (4) holds at least for p-vectors $\mathbf{v}$ in $\wedge^p(E_x)$. Similar arguments carry over to all $\wedge^p(T_xM)$.

2.4. **Third step: Globalization.** The following proposition renders global the construction of proposition 2.3.

Proposition 2.4. *Let a neighborhood $\mathcal{U} \ni f$, $p \in \{1,\ldots,d-1\}$ and $\delta > 0$ be given. Then there exist $m \in \mathbb{N}$ and a diffeomorphism $g \in \mathcal{U}$ that equals f outside the open set $\Gamma_p(f,m)$ and such that*

$$\int_{\Gamma_p(f,m)} \Lambda_p(g,x)\,d\mu(x) < \delta + \int_{\Gamma_p(f,m)} \frac{\Lambda_{p-1}(f,x)+\Lambda_{p+1}(f,x)}{2}\,d\mu(x). \tag{9}$$

The proof goes as follows. Let $m \in \mathbb{N}$ be large and let $N : \Gamma_p(f,m) \to \mathbb{N}$ be the function given by proposition 2.3 with $\kappa = \delta^2$. For almost every $x \in \Gamma_p(f,m)$ and every $n \geq N(x)$, the proposition provides a realizable sequence $\{\widehat{L}_i\}$ of length n at x satisfying (4). "Realizing" this sequence (see definition 2.1), we obtain a perturbation g of f supported in a small neighborhood of the segment of orbit $\{x,\ldots,f^n(x)\}$, which is a tower $U \sqcup \cdots \sqcup f^n(U)$. Since the set $\Gamma_p(f,m)$ is open and invariant, these towers can always be taken inside it. Each tower $U \sqcup \cdots \sqcup f^n(U) = U \sqcup \cdots \sqcup g^n(U)$ contains a *sub-tower* $K \sqcup \cdots \sqcup f^n(K)$ where the perturbed derivatives are very close to the maps $\widehat{L}_i$. Hence if we choose U small enough then (4) will imply

$$\frac{1}{n} \log \|\wedge^p Dg^n_y\| < \frac{\Lambda_{p-1}(x) + \Lambda_{p+1}(x)}{2} + 2\delta, \qquad \forall\, y \in K. \tag{10}$$

To construct the perturbation g globally, we cover all $\Gamma_p(f,m)$ but a subset of small measure with a (large) finite number of *disjoint* towers as above. Moreover, the towers can be chosen so that they have approximately the same heights (more precisely, all heights are between H and $3H$, where H is a constant). Then we glue all the perturbations (each one supported in a tower) and obtain a C^1 perturbation g of f. Let S be the support of the perturbation, i.e., the disjoint union of the towers. Let $S' \subset S$ be union of the corresponding sub-towers; then $\mu(S \setminus S') < \kappa\mu(S) \le \delta^2$. Moreover, if $y \in S'$ is in the first floor of a sub-tower of height n then (10) holds.

To bound the integral in the left hand side of (9), we want to use the elementary fact (notice $\Gamma_p(f,m)$ is also g-invariant): for all $n \in \mathbb{N}$,

$$\int_{\Gamma_p(f,m)} \Lambda_p(g,x)\, d\mu(x) \le \frac{1}{n} \int_{\Gamma_p(f,m)} \log \|\wedge^p (Dg^n_x)\|\, d\mu(x). \tag{11}$$

Let $n_0 = H/\delta$. Here comes a major step in the proof: To show that *most points (up to a set of measure of order of δ) in $\Gamma_p(f,m)$ are in S' and its positive iterates stay inside S' for at least n_0 iterates.* Intuitively, this is true by the following reason: The set S' is a g-castle[1], whose towers have heights $\approx H$. Therefore a segment of orbit of length $n_0 = \delta^{-1}H$, if it is contained in S', "winds" $\approx \delta^{-1}$ times around S'. Since S' is a castle, there are only δ^{-1} opportunities for the orbit to leave S'. In each opportunity, the probability of leave is of order of δ^2 (the measure of the complementary $\Gamma_p(f,m) \setminus S'$). Therefore the probability of leave S' in n_0 iterates is $\approx \delta^{-1}\delta^2 = \delta$.

Using the fact above, one shows that the right hand side of (11) with $n = n_0$ is bounded by the left hand side of (9), completing the proof of the proposition.

2.5. **Conclusion of the proof.** Let $\Gamma_p(f,m)$ be the set of points where there is no dominated splitting of index p, that is, $\Gamma_p(f,m) = \cap_{m\in\mathbb{N}}\Gamma_p(f,m)$. The following is an easy consequence of proposition 2.4.

[1] That is, a union of disjoint g-towers.

Proposition 2.5. *Given $f \in \mathrm{Diff}^1_\mu(M)$ and $p \in \{1, \dots, d-1\}$, let*

$$J_p(f) = \int_{\Gamma_p(f,\infty)} \frac{\lambda_p(f,x) - \lambda_{p+1}(f,x)}{2} \, d\mu(x).$$

Then for every $\mathcal{U} \ni f$ and $\delta > 0$, there exists a diffeomorphism $g \in \mathcal{U}$ such that

$$\int_M \Lambda_p(g,x)\, d\mu(x) < \int_M \Lambda_p(f,x)\, d\mu(x) - J_p(f) + \delta.$$

Using the proposition we can give the:

Proof of theorem 7. Let $f \in \mathrm{Diff}^1_\mu(M)$ be a point of continuity of all maps $\mathrm{LE}_p(\cdot)$, $p = 1, \dots, d-1$. Then $J_p(f) = 0$ for every p. This means that $\lambda_p(f,x) = \lambda_{p+1}(f,x)$ for almost every x in the set $\Gamma_p(f,\infty)$.

Let $x \in M$ be an Oseledets regular point. If all Lyapunov exponents of f at x vanish, there is nothing to do.

For each p such that $\lambda_p(f,x) > \lambda_{p+1}(f,x)$, we have (if we exclude a zero measure set of x) $x \notin \Gamma_p(f,\infty)$. This means that there is a dominated splitting of index p, $T_{f^n x}M = E_n \oplus F_n$ along the orbit of x. It is not hard to see that E_n is necessarily the sum of the Oseledets spaces of f, at the point $f^n x$, associated to the Lyapunov exponents $\lambda_1(f,x)$, …, $\lambda_p(f,x)$, and F_n is the sum of the spaces associated to the other exponents. This shows that the Oseledets splitting is dominated along the orbit of x. □

2.6. **Symplectic diffeomorphisms.** Now let (M, ω) be a compact symplectic manifold without boundary, of dimension $\dim M = 2q$. subspace of $\mathrm{Diff}^1_\mu(M)$.

The Lyapunov exponents of symplectic diffeomorphisms have a symmetry property: $\lambda_j(f,x) = -\lambda_{2q-j+1}(f,x)$ for all $1 \le j \le q$. In particular, $\lambda_q(x) \ge 0$ and $\mathrm{LE}_q(f)$ is the integral of the sum of all non-negative exponents. Consider the splitting

$$T_xM = E^+_x \oplus E^0_x \oplus E^-_x,$$

where E^+_x, E^0_x, and E^-_x are the sums of all Oseledets spaces associated to positive, zero, and negative Lyapunov exponents, respectively. Then $\dim E^+_x = \dim E^-_x$ and $\dim E^0_x$ is even.

Theorem 8. *Let $f_0 \in \mathrm{Sympl}^1_\mu(M)$ be such that the map*

$$f \in \mathrm{Sympl}^1_\mu(M) \mapsto \mathrm{LE}_q(f) \in \mathbb{R}$$

is continuous at $f = f_0$. Then for μ-almost every $x \in M$, either $\dim E^0_x \ge 2$ or the splitting $T_xM = E^+_x \oplus E^-_x$ is uniformly hyperbolic along the orbit of x.

In the second alternative, what we actually prove is that the splitting is dominated at x. This is enough because, for symplectic diffeomorphisms, dominated splittings into two subspaces of the same dimension are uniformly hyperbolic.

Theorem 3 follows from theorem 8: As in the volume preserving case, the function $f \mapsto \mathrm{LE}_q(f)$ is continuous on a residual subset $\mathcal{R}_1$ of $\mathrm{Sympl}^1_\mu(M)$. Also, there is a residual subset $\mathcal{R}_2 \subset \mathrm{Sympl}^1_\mu(M)$ such that for every $f \in \mathcal{R}_2$ either f is an Anosov diffeomorphism or all its hyperbolic sets have zero measure. The residual set of theorem 3 is $\mathcal{R} = \mathcal{R}_1 \cap \mathcal{R}_2$.

The proof of theorem 8 is similar to that of theorem 7. Actually the only difference is in the first step. In the symplectic analogue of proposition 2.2, we have to suppose that the spaces E and F are Lagrangian [2].

3. Prevalence of exponential behavior

We are going to outline the proof of theorems 5 and 6. In the presentation we focus on the uniform case, with comments about the main additional ingredients in the extension to the general case. Complete proofs can be found in [Via].

Before getting into explaining the arguments, let us make precise what we mean by uniform hyperbolicity and by local product structure.

Uniformly hyperbolic maps. Let $f : M \to M$ be a homeomorphism on a compact metric space M. For $x \in M$ and $\varepsilon > 0$ define

$$W^s_\varepsilon(x) = \{y \in M : \mathrm{dist}(f^n(x), f^n(y)) \le \varepsilon \text{ for all } n \ge 0\}$$

and

$$W^u_\varepsilon(x) = \{y \in M : \mathrm{dist}(f^{-n}(x), f^{-n}(y)) \le \varepsilon \text{ for all } n \ge 0\}$$

We say that f is *uniformly hyperbolic* if there exist $K > 0$, $\tau > 0$, $\bar\varepsilon > 0$, $\bar\delta > 0$ such that

(1) $\mathrm{dist}(f^n(x), f^n(y)) \le Ke^{-\tau n}\,\mathrm{dist}(x,y)$ for all $n \ge 0$ and $y \in W^s_{\bar\varepsilon}(x)$ and $\mathrm{dist}(f^{-n}(x), f^{-n}(y)) \le Ke^{-\tau n}\,\mathrm{dist}(x,y)$ for all $n \ge 0$ and $y \in W^u_{\bar\varepsilon}(x)$.

(2) If $\mathrm{dist}(x,y) < \bar\delta$ then $W^s_{\bar\varepsilon}(x) \cap W^u_{\bar\varepsilon}(y)$ contains exactly one point, denoted $[x,y]$, and this point varies continuously with x and y.

[2] A subspace E of a symplectic vector space (V,ω) is called *Lagrangian* when $\dim E = \frac{1}{2}\dim V$ and $\omega(v_1,v_2) = 0\ \forall\, v_1, v_2 \in E$.

Measures with local product structure. Given any $z \in M$ and small $\delta > 0$, the set of points $W^s_{\bar{\varepsilon}}(x) \cap W^u_{\bar{\varepsilon}}(y)$ with $x \in W^u_\delta(z)$ and $y \in W^s_\delta(z)$ is a neighborhood $\mathcal{B}(z,\delta)$ of z, homeomorphic to $W^u_\delta(z) \times W^s_\delta(z)$. Given a non-atomic f-invariant measure μ, let μ^u be the measure defined on $W^u_\delta(z)$ by

$$\mu^u(E) = \mu\big([E, W^s_\delta(z)]\big).$$

Define μ^s on $W^s_\delta(z)$ analogously. We say that μ has *local product structure* if $\mu \mid \mathcal{B}(z,\delta)$ is equivalent to the product measure defined by

$$(\mu^u \times \mu^s)\big([E,F]\big) = \mu^u(E)\, \mu^s(F),$$

for every z and δ. Equivalently: the unstable holonomy is absolutely continuous with respect to the family of conditional measures of μ along local stable sets in $\mathcal{B}(z,\delta)$. This condition does not change if the roles of stable and unstable are interchanged.

Remark 3.1. This notion extends to the general hyperbolic case as follows. Let μ be a hyperbolic measure for a $C^{1+\varepsilon}$ diffeomorphism f. By Pesin's stable manifold theorem [Pes76], μ-almost every $x \in M$ has a local stable set $W^s_{\mathrm{loc}}(x)$ and a local unstable set $W^u_{\mathrm{loc}}(x)$ which are C^1 embedded disks. Moreover, these disks vary in a measurable fashion with the point. So, for every $\varepsilon > 0$ we may find $M_\varepsilon \subset M$ with $\mu(M_\varepsilon) > 1 - \varepsilon$ such that $W^s_{\mathrm{loc}}(x)$ and $W^u_{\mathrm{loc}}(x)$ vary continuously with $x \in M_\varepsilon$ and, in particular, their sizes are uniformly bounded from zero. Thus for any $x \in M_\varepsilon$ we may construct sets $\mathcal{H}(x,\delta)$ with arbitrarily small diameter δ, such that

(i) $\mathcal{H}(x,\delta)$ contains a neighborhood of x inside M_ε;
(ii) every point of $\mathcal{H}(x,\delta)$ is in the local stable manifold and in the local unstable manifold of some pair of points in M_ε;
(iii) given y, z in $\mathcal{H}(x,\delta)$ the unique point in $W^s(y) \cap W^u(z)$ is also in $\mathcal{H}(x,\delta)$.

Then we say that μ has local product structure if $\mu \mid \mathcal{H}(x,\delta)$ is equivalent to $\mu^u \times \mu^s$, where μ^u (respectively μ^s) is the projection of $\mu \mid \mathcal{H}(x,\delta)$ onto $W^u(x)$ (respectively $W^s(x)$).

The proof of theorem 6 has three main types of ingredients, which are explained in the three sections that follow. We may consider $r \geq 1$, because the Hölder cases $0 < r < 1$ are immediately reduced to the Lipschitz case by a change of the metric. Moreover, it is no restriction to assume that μ is ergodic: any invariant measure measure with local product structure has only finitely (in the general, non-uniform case: countably) many ergodic components, and they also have local product structure. Finally, we consider real-valued cocycles but the arguments apply also (they are even easier at a final step) in the complex case.

3.1. **Dominated behavior and invariant foliations.** Let us consider $f : M \to M$ to be uniformly hyperbolic, with constants K, τ, $\bar{\varepsilon}$, $\bar{\delta}$, and $A : M \to \mathrm{SL}(d, \mathbb{R})$ to be a C^r cocycle, $r \geq 1$.

Definition 3.2. A point $x \in M$ is *dominated* if there are $N \geq 1$ and $\theta < \tau$ such that

$$\prod_{j=0}^{k-1} \|A^N(f^{jN}(x))\| \, \|A^N(f^{jN}(x))^{-1}\| \leq e^{kN\theta} \quad \text{for all } k \geq 1,$$

and analogously with f and A replaced by their inverses. The set of such points is denoted $\mathcal{D}_A(N, \theta)$.

Remark that $\|B\| \, \|B^{-1}\|$ is an upper bound for the expansion exhibited by the projective maps $B_\#$ and $B^1_\#$ induced on the projective space $\mathbb{CP}^{d-1}$ by a linear isomorphism $B : \mathbb{C}^d \to \mathbb{C}^d$ and its inverse B^{-1}.

Proposition 3.3. *Suppose $\lambda(A, \mu) = 0$. Then μ-almost every point is dominated.*

The key to the proof is to express the Lyapunov exponent, which is defined by a sub-additive limit

$$\lambda(A, x) = \lim_{n\to\infty} \frac{1}{n} \log \|A^n(x)\| \, ,$$

in additive terms (a Birkhoff average):

Lemma 3.4. *Suppose $\lambda(A, \mu) = 0$. Given $\delta > 0$ and μ-almost every $x \in M$ there exists $N \geq 1$ such*

$$\frac{1}{k} \sum_{j=0}^{k-1} \frac{1}{N} \log \|A^N(f^{jN}(x))\| < \delta \quad \textit{for all } k \geq 1.$$

Proof. Fix $\varepsilon > 0$ small so that $4\varepsilon \sup \log \|A\| < \delta$. Let $\eta \geq 1$ be large enough so that the set Δ_η of points $x \in M$ such that

$$\frac{1}{\eta} \log \|A^\eta(x)\| < \frac{\delta}{2}$$

has $\mu(\Delta_\eta) > (1 - \varepsilon^2)$. Let $\tau(x)$ be the average sojourn time of the f^η-orbit of x inside Δ_η, and Γ_η be the subset of points for which $\tau(x) \geq 1 - \varepsilon$. By sub-multiplicativity of the norms

$$\frac{1}{k} \sum_{j=0}^{k-1} \frac{1}{l\eta} \log \|A^{l\eta}(f^{jl\eta}(x))\| \leq \frac{1}{kl} \sum_{j=0}^{kl-1} \frac{1}{\eta} \log \|A^\eta(f^{j\eta}(x))\| \tag{12}$$

for any $x \in \Gamma_\eta$ and any $k, l \geq 1$. Fix l large enough so that for any $n \geq l$ at most $(1 - \tau(x) + \varepsilon)n$ of the first iterates n of x under f^η

fall outside Γ_η. Then the right hand side of the previous inequality is bounded by

$$\frac{\delta}{2} + (1-\tau(x)+\varepsilon)\sup\log\|A\| < \frac{\delta}{2} + 2\varepsilon\sup\log\|A\| < \delta.$$

This means that we may take $N = l\eta$. On the other hand,

$$\mu(\Gamma_\eta) + (1-\varepsilon)\mu(M\setminus\Gamma_\eta) \geq \int \tau(x)\,d\mu(x) = \mu(\Delta_\eta) > (1-\varepsilon^2)$$

implies that $\mu(\Gamma_\eta) > (1-\varepsilon)$. Thus, making $\varepsilon \to 0$ we get the conclusion for μ-almost every $x \in M$. □

To deduce Proposition 3.3, choose $4\delta d < \tau$ and observe that $\|A^{-1}\| \leq \|A\|^{d-1}$ because these matrices have determinant 1.

Remark 3.5. Ergodicity is not necessary for Proposition 3.3 nor for Lemma 3.4. On the other hand, in the ergodic case similar arguments show that for every $\delta > 0$ there exists a uniform $N \geq 1$ such that

$$\lim_{k\to\infty}\frac{1}{k}\sum_{j=0}^{k-1}\frac{1}{N}\log\|A^N(f^{jN}(x))\| < \delta \quad \text{for } \mu\text{-almost every } x \in M.$$

Proposition 3.6. *Suppose x is dominated. For every $y,\, z \in W^s_\varepsilon(x)$,*

$$H^s_{y,z} = \lim_{n\to+\infty} A^n(z)^{-1}A^n(y)$$

exists and satisfies $H^s_{y,z} = H^s_{x,z}\circ H^s_{y,x}$ and $H^s_{f(y),f(z)} = A(z)\circ H^s_{y,z}\circ A(y)^{-1}$ and $\|H^s_{y,z} - \mathrm{id}\,\| \leq L\,\mathrm{dist}(y,z)$ for some $L > 0$ that depends only on K,τ,N,θ.

Using domination and exponential contraction of orbits inside $W^s_\varepsilon(x)$, one shows that $A^n(z)^{-1}A^n(y)$ is a Cauchy sequence: for all $n \geq 0$,

$$\|A^{n+1}(z)^{-1}A^{n+1}(y) - A^n(z)^{-1}A^n(y)\| \leq \mathrm{const}\, e^{n(\theta-\tau)}\,\mathrm{dist}(y,z).$$

The other claims in the proposition follow easily. There are analogous results for

$$H^u_{y,z} = \lim_{n\to-\infty} A^n(z)^{-1}A^n(y)$$

when $y,\, z \in W^u_\varepsilon(x)$.

Remark 3.7. We say a point is k-dominated, $k \geq 1$ if the condition in Definition 3.2 holds with $k\theta < \tau$. Proposition 3.3 remains true for k-domination, any $k \geq 1$. Assuming 2-domination we also have

$$\|H^s_{f^j(y),f^j(z)} - \mathrm{id}\,\| \leq \mathrm{const}\, e^{i(2\theta-\tau)}\,\mathrm{dist}(y,z) \leq \mathrm{const}\ \mathrm{dist}(y,z)$$

for all $j \geq 1$, which is useful for the sequel of the arguments.

Now we consider the projective cocycle $f_A : M\times\mathbb{RP}^{d-1} \to M\times\mathbb{RP}^{d-1}$ associated to A, as well as the projectivizations

$$h^s_{y,z} : \mathbb{RP}^{d-1} \to \mathbb{RP}^{d-1} \quad \text{and} \quad h^u_{y,z} : \mathbb{RP}^{d-1} \to \mathbb{RP}^{d-1}$$

of the linear isomorphisms $H^s_{y,z}$ and $H^u_{y,z}$. Given a dominated point $x \in M$, the Lipschitz graphs

$$W^{ss}_\varepsilon(x,\xi) = \{(y, h^s_{x,y}(\xi)) : y \in W^s_\varepsilon(x)\}$$

and

$$W^{uu}_\varepsilon(x,\xi) = \{(z, h^u_{x,z}(\xi)) : z \in W^s_\varepsilon(x)\}$$

are the *strong-stable set* and the *strong-unstable set* of any $(x,\xi) \in \{x\} \times \mathbb{RP}^{d-1}$. Indeed,

Lemma 3.8. *For any $y \in W^s_\varepsilon(x)$ and ξ, $\eta \in \mathbb{RP}^{d-1}$,*

(1) $\displaystyle\limsup_{n\to+\infty} \frac{1}{n} \log \operatorname{dist}\left(f^n_A(x,\xi), f^n_A(y, h^s_{x,y}(\xi))\right) \le -\tau$ *and*

(2) $\displaystyle\limsup_{n\to+\infty} \frac{1}{n} \log \operatorname{dist}\left(f^n_A(x,\xi), f^n_A(y, h^s_{x,y}(\xi))\right) \le -\theta \Leftrightarrow \eta = h^s_{x,y}(\xi)$.

Analogously for $h^u_{x,z}$ with $z \in W^u_\varepsilon(x)$.

We call $h^s_{y,z}$ the *strong-stable holonomies*, and $h^u_{y,z}$ the *strong-unstable holonomies* of the projective cocycle.

Remark 3.9. An easy, yet important, observation is that domination is a robust property: given any N, θ, and $\theta' > \theta$ there exists a C^0 neighborhood $\mathcal{U}$ of A such that

$$\mathcal{D}_A(N,\theta) \subset \mathcal{D}_B(N,\theta') \quad \text{for all } B \in \mathcal{U}.$$

The following proposition summarizes most of this section:

Proposition 3.10. *Suppose $\lambda(A,\mu) = 0$. Then there exists an increasing sequence of compacts M_L whose union has full μ-measure in M, and there exist neighborhoods $\mathcal{U}_L$ of A in $C^r(M, \mathrm{SL}(d,\mathbb{R}))$ such that, given any $x \in M_L$ and any $B \in \mathcal{U}_L$, the strong-stable set and the strong-unstable set of every $(x,\xi) \in \{x\} \times \mathbb{RP}^{d-1}$ for the projective cocycle f_B are L-Lipschitz graphs over $W^s_\varepsilon(x)$ and $W^u_\varepsilon(x)$, respectively.*

Remark 3.11. The definitions and results extend to general hyperbolic systems, considering hyperbolic blocks as in remark 3.1, that is, compact sets where we have

- uniform estimates for the contraction along stable sets and the expansion along unstable sets
- uniform lower bounds for the angles between the Oseledets subspaces and for the sizes of stable sets and unstable sets.

Proposition 3.10 remains true as stated.

3.2. **Invariant measures of projective cocycles.** Consider the projective cocycle

$$f_A : M \times \mathbb{RP}^{d-1} \to M \times \mathbb{RP}^{d-1}$$

associated to $A : M \to \mathrm{SL}(d, \mathbb{R})$. We are going to analyze the f_A-invariant measures m on $M \times \mathbb{RP}^{d-1}$ which project down to μ, that is, such that $(\pi_1)_* m = \mu$.

We consider a family of conditional measures $\{m_x : x \in M\}$ of m along the projective fibers: each m_x is a probability on the fiber $\mathcal{E}_x = \{x\} \times \mathbb{RP}^{d-1}$, depending measurably on x, and

$$m(E) = \int m_x(E \cap \mathcal{E}_x)\, d\mu(x)$$

for every measurable set $E \subset M \times \mathbb{RP}^{d-1}$. Such a disintegration into conditional measures always exists and is essentially unique.

The first result is that if the Lyapunov exponents vanish then the disintegration of any such m is μ-almost everywhere invariant under local strong-stable holonomy and under local strong-unstable holonomy:

Proposition 3.12. *Suppose $\lambda(A, \mu) = 0$. Then for every f_A-invariant probability m and for each $L \geq 1$ there exists a full μ-measure subset E_L of $W^s_{\mathrm{loc}}(M_L)$ such that*

$$m_{y_2} = (h^s_{y_1, y_2})_* m_{y_1}$$

for any y_1, $y_2 \in E_L$ on the same local stable set. Moreover, there is an analogous statement for unstable holonomies.

The proof is based on the following result of Ledrappier:

Proposition 3.13 (Ledrappier [Led84])**.** *Let $(M_*, \mathcal{B}_*, \mu_*)$ be a probability space and $T : M_* \to M_*$ be a measure-preserving transformation. Let $\mathcal{B} \subset \mathcal{B}_*$ be a σ-algebra and $B : M_* \to \mathrm{GL}(d, \mathbb{R})$ be a measurable function such that*

(1) *$T^{-1}(\mathcal{B}) \subset \mathcal{B}$ and $\{T^n(\mathcal{B}) : n \in \mathbb{Z}\}$ generates $\mathcal{B}_*$* μ_*-mod 0
(2) *B is $\mathcal{B}$-measurable* μ_*-mod 0

Suppose $\lambda(B, x) = 0$ at μ_*-almost every $x \in M$. Then for any f_B-invariant measure m on $M \times \mathbb{RP}^{d-1}$ with $(\pi_1)_* m = \mu_*$, the disintegration $\{m_x : x \in M_*\}$ is $\mathcal{B}$-measurable.

If one considers $\mathcal{B} = \sigma$-algebra of measurable subsets of $M_* = M$ consisting of entire local stable manifolds of $T = f$, then the theorem is saying that *if a cocycle is constant along local stable manifolds and the Lyapunov exponents vanish, then the conditional measures of any invariant probability are almost everywhere constant on local stable manifolds.*

Roughly, to deduce Proposition 3.12 we carry out an affine change of coordinates on the fibers to turn A into a cocycle B constant on local stable manifolds. This change of coordinates is defined through local stable holonomies, and it does not affect the Lyapunov exponents, as we are going to explain.

For simplicity, we use some Markov partition $\mathcal{R}$ of f (this is necessary, see Remark 3.14 below concerning the general hyperbolic case). The local stable manifold $W^s_{\mathrm{loc}}(z)$ is the set of points with the same forward itinerary as z, relative to $\mathcal{R}$. Fix an atom $R \in \mathcal{R}$ and let $n(z) \geq 1$ be the first return time of a point $z \in R$ to R. Then define

$$B(\zeta) = \begin{cases} A(f^j(z)) & \text{if } \zeta \in f^j(W^s_{\mathrm{loc}}(z)) \text{ with } 0 \leq j < n(z) \\ A(\zeta) & \text{otherwise.} \end{cases}$$

Observe that in the first case

$$A(f^j(z)) = h^s_{f(\zeta),f^{j+1}(z)} \cdot A(\zeta) \cdot h^s_{f^j(z),\zeta} \, .$$

Hence, the two cocycles are conjugate, by a bounded projective conjugacy. In particular, $\|B^{\pm}1\|$ is bounded, which ensures that Oseledets theorem applies to B, and the two cocycles have the same Lyapunov exponents at almost every point. Under our assumptions that means that

$$\lambda(B,x) = \lambda(A,x) = 0 \quad \mu\text{-almost everywhere.}$$

Consider the σ-algebra $\mathcal{B}$ of measurable sets $E \subset M$ such that, for any $z \in R$,

$$f^j(W_{\mathrm{loc}}(z)) \cap E \neq \emptyset \quad \Rightarrow \quad f^j(W_{\mathrm{loc}}(z)) \subset E.$$

The assumptions of theorem 3.13 are easily checked:

(1) $f^{-1}(\mathcal{B}) \subset \mathcal{B}$ and $\{f^n(\mathcal{B}) : n \in \mathbb{Z}\}$ generates M μ-mod 0.
(2) the cocycle B is $\mathcal{B}$-measurable (constant on each $f^j(W_{\mathrm{loc}}(z))$).

Then, by theorem 3.13, the conditional measures of any f_B-invariant probability measure are constant on local stable manifolds almost everywhere on R. Reversing the conjugacy, this means that the conditional measures of any f_A-invariant probability measure are invariant under local stable holonomies almost everywhere on R.

Remark 3.14. In the general hyperbolic case one proves that for μ-almost every $x \in M_L$ there exists a neighborhood R of x inside M_L which is foliated by a family of stable disks $\{S(z) : z \in H^u\}$ with a Markov type property:

- $S(z) \subset W^s_\varepsilon(z)$ and $S(z)$ contains a disk of uniform size around z inside the stable manifold
- z varies on a subset H^u of the local unstable manifold of x and $R = \cup_{z \in H^u} S(z)$ is a neighborhood of x inside M_L

- $f^j(S(w)) \cap S(z) \neq \emptyset$ implies $f^j(S(w)) \subset S(z)$.

Then we have the statement as before, on a full measure subset of this set R, through similar arguments.

Let us give an informal outline of how the argument will proceed. Suppose the Lyapunov exponents of A vanish. We are going to consider periodic points p_1 and p_2, and a heteroclinic point $q \in W^u(p_1) \cap W^s(p_2)$, all contained in the Markov set R. One may choose the periodic points so that all the eigenvalues of $Df^{k_i}(p_i)$, $k_i = \text{per}(p_i)$ have distinct norms: the cocycles for which such points do not exist form an ∞-codimension subset. Then every Df-invariant probability on the projective fiber of p_i must be a convex combination of the Dirac measures supported on the points in $\mathbb{RP}^{d-1}$ corresponding to the eigenspaces of the derivative.

We want to apply this to the conditional measure m_{p_i} of some f_A-invariant measure m on the projective fiber of p_i. Then we want to argue, using the holonomy invariance proposition 3.12, that

$$h^u_{p_1,q}(E_1) = h^s_{p_2,q}(E_2) \tag{13}$$

for some eigenspaces E_1 of $Df^{k_1}(p_1)$ and E_2 of $Df^{k_2}(p_2)$. See figure 1.

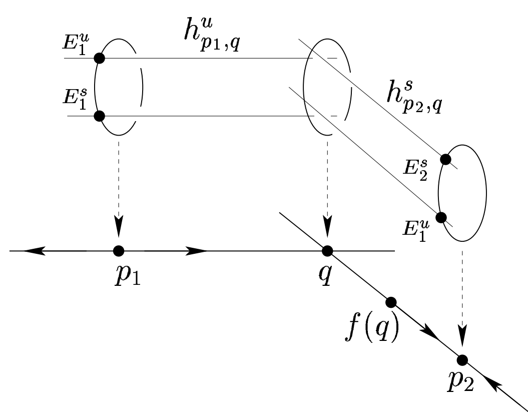

FIGURE 1. Breaking holonomy invariance

Now, we may modify the cocycle near $f(q)$ to change the right hand side of (13) without affecting the left hand side, thus breaking the equality. In this way we shall conclude that the set of cocycles with vanishing exponents is locally contained in a closed hypersurface. Since there are infinitely many periodic points in R, we just have to vary p_1 and p_2 to conclude that this set has ∞-codimension.

One obvious difficulty with this plan is that conditional measures are uniquely defined only almost everywhere, so that it does not really make sense to speak of m_{p_i}. Another, is that proposition 3.12 only gives invariance under holonomy on a full measure subset: a priori, it is not guaranteed that either p_1, p_2, or q are in this set. These difficulties are overcome by the continuity statement that follows. This is the main step where we use the assumption of local product structure.

Proposition 3.15. *Suppose $\lambda(A, \mu) = 0$. Then for every f_A-invariant probability m and for each $L \geq 1$, there exists a disintegration $\{\bar{m}_z\}$ of m such that*

(1) *$z \mapsto \bar{m}_z$ is continuous relative to the weak* topology*
(2) *$\{\bar{m}_z\}$ is invariant under both strong-stable holonomy and strong-unstable holonomy*

everywhere *on the support of $\mu \mid W^u_{\mathrm{loc}}(M_L) \cap W^s_{\mathrm{loc}}(M_L)$.*

Proof. Let $\{m_z\}$ be any disintegration of m. Using proposition 3.12 twice, we find a full μ-measure subset G_L of $W^u_{\mathrm{loc}}(M_L) \cap W^s_{\mathrm{loc}}(M_L)$ such that $\{m_z\}$ is invariant under strong-stable holonomy and under strong-unstable holonomy for any pair of points of G_L (in the same local stable manifold or local unstable manifold, respectively).

Fix some local unstable leaf γ^u with $\mu^u(\gamma^u \setminus G_L) = 0$. Then define $\{m'_z\}$ by forcing h^s-invariance from $\{m_z\}$ restricted to γ^u:

$$m'_w = (h^s_{z,w})_* m_z \quad \text{if } w \in W^s_{\mathrm{loc}}(z),\ z \in \gamma^u.$$

Note that $m'_z = m_z$ almost everywhere, so $\{m'_z\}$ is still a disintegration of m. This disintegration is invariant under local strong-stable holonomy and, consequently, m'_z varies continuously on local stable manifolds.

Next, fix some local stable leaf γ^s with $\mu^s(\gamma^s \setminus G_L) = 0$ and also $\mu(\{z \in \gamma^s : m'_z \neq m_z\}) = 0$. Then define $\{m^u_z\}$ by forcing h^u-invariance from $\{m'_z\}$ restricted to γ^s:

$$m^u_w = (h^u_{z,w})_* m'_z \quad \text{if } w \in W^u_{\mathrm{loc}}(z),\ z \in \gamma^s.$$

Then $\{m^u_z\}$ is a disintegration of m which is invariant under local strong-unstable holonomy and varies continuously with the point z on $W^u_{\mathrm{loc}}(M_L) \cap W^s_{\mathrm{loc}}(M_L)$.

Now construct a disintegration $\{m^s\}$ by a dual procedure, interchanging stable with unstable. Then

- $m^u_z = m^s_z$ at μ-almost every point of $W^u_{\mathrm{loc}}(M_L) \cap W^s_{\mathrm{loc}}(M_L)$.
- m^u_z is h^u-invariant and m^s_z is h^s-invariant.
- $z \mapsto m^u_z$ and $z \mapsto m^s_z$ are continuous on $W^u_{\mathrm{loc}}(M_L) \cap W^s_{\mathrm{loc}}(M_L)$.

This implies that $m_z^u = m_z^s$ on the support of $\mu \mid W^u_{\mathrm{loc}}(M_L) \cap W^s_{\mathrm{loc}}(M_L)$. It suffices to take $\bar{m}_z = m_z^u = m_z^s$ on this set. □

Remark 3.16. By local product structure, the support of $\mu \mid W^u_{\mathrm{loc}}(M_L)$ consists of entire local unstable manifolds, and analogously for the support of $\mu \mid W^s_{\mathrm{loc}}(M_L)$. Moreover,

$$K_L := \operatorname{supp}\big(\mu \,|W^u_{\mathrm{loc}}(M_L) \cap W^s_{\mathrm{loc}}(M_L)\big) = \\ = \operatorname{supp}\big(\mu \mid W^u_{\mathrm{loc}}(M_L)\big) \cap \operatorname{supp}\big(\mu \mid W^s_{\mathrm{loc}}(M_L)\big).$$

In particular, this set K_L has local product structure.

3.3. **Periodic points and obstructions to zero exponents.** Now we implement the strategy outlined in the previous section. Previously, we have constructed compact sets K_L with local product structure and μ-measure going to 1 when $L \to \infty$, such that

(a) points in K_L are dominated, with uniform estimates on the constants N and θ in definition 3.2;
(b) strong holonomies over the stable and unstable manifolds of points in K_L are Lipschitz, with uniform Lipschitz constants;
(c) m admits a disintegration which is continuous and invariant under strong-stable and strong-unstable holonomies over K_L.

Lemma 3.17. *For μ-almost every $z \in K_L \cap \mathcal{D}_A(N,\theta)$ and any $\varepsilon > 0$ there exists a periodic point p of f such that*

(1) *$d(f^j(p), f^j(z)) < \varepsilon$ for all $0 \le j \le \mathrm{per}(p)$*
(2) *p is dominated, with constants $N' = N$ and $\theta' \approx \theta$.*

Proof. By the Poincaré recurrence theorem, given any $\delta > 0$ then the f^N-orbit of almost every $z \in K_L$ returns to K_L in the δ-ball around z: there is $\ell \ge 1$ such that $f^{N\ell}(z) \in K_L \cap B(z,\delta)$. Choosing δ sufficiently small with respect to ε, we may use the shadowing lemma [Bow75] to find a point $p \in M$ such that $f^{N\ell}(p) = p$ and $d(f^j(p), f^j(z)) < \varepsilon$ for all $0 \le j \le N\ell$.

Consider any $\theta' > \theta$. Since z is dominated with constants N and θ, taking $\varepsilon > 0$ to be sufficiently small we ensure (just by continuity) that

$$\prod_{j=0}^{k-1} \|A^N(f^{jN}(p))\| \, \|A^N(f^{jN}(p))^{-1}\| \le e^{kN\theta'} \quad \text{for all } 1 \le k \le \ell.$$

Since p has period $N\ell$, it follows that inequality is true for every $k \ge 1$. So, p is dominated with constants N and θ'. □

Remark 3.18. In the general hyperbolic case use Katok's closing lemma [Kat80] instead: for any hyperbolic block $\mathcal{H}$ and any $\varepsilon > 0$

there exists $\delta > 0$ such that if z and $\ell \geq 1$ are such $d(z, f^{N\ell}(z)) < \delta$ and both points are in $\mathcal{H}$ then there exists some $p \in M$ such that

- $f^{N\ell}(p) = p$ and p is a hyperbolic saddle.
- $d(f^j(p), f^j(z)) < \varepsilon$ for all $0 \leq j \leq N\ell$.

The proof also provides uniform estimates on the eigenvalues of $Df^{N\ell}(p)$ and on the sizes of the Pesin stable and unstable manifolds. In particular, on is able to deduce that p is dominated, with constants $N' = N$ and $\theta' \approx \theta$, and that its stable/unstable manifold intersects the unstable/stable set of K_L.

We construct two such periodic points p_1 and p_2, close enough so that their local stable and unstable manifolds intersect at some point q. An important technical step is to construct a larger compact set

$$\hat{K}_L \supset K_L \cup \{p_1, p_2, q\}$$

with local product structure, and such that properties (a), (b), (c) at the beginning of the section remain true, with slightly worse estimates. Roughly speaking, $\hat{K}_L$ is obtained as the intersection of the saturation

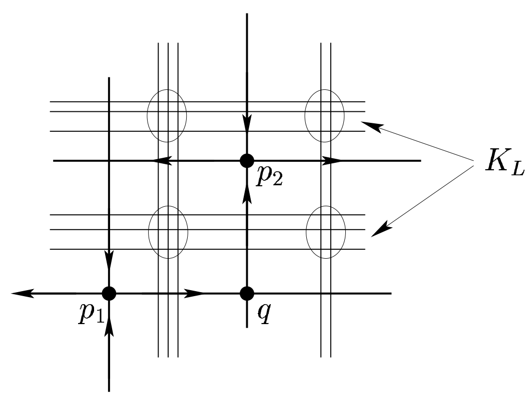

FIGURE 2. Constructing a larger hyperbolic block

towards p_1 and p_2 of the stable and unstable manifolds through K_L. See figure 2. Using that μ has local product structure one ensures that

$$\operatorname{supp}(\mu \mid \hat{K}_L) \supset \{p_1, p_2, q\}. \tag{14}$$

Let $\{\hat{m}_z\}$ be a disintegration of an invariant probability m continuous over $\hat{K}_L$, relative to the weak* topology. Then

Lemma 3.19. *$A^{\kappa_i}(p_i)_* \bar{m}_{p_i} = \bar{m}_{p_i}$ for $i = 1, 2$, where $\kappa_i = \operatorname{per}(p_i)$.*

This is easy to prove. First, since m is invariant $A^{\kappa}(z)_* \bar{m}_z = m_{f(z)}$ for any $\kappa \geq 1$ and almost every z. By continuity, this extends to the support of μ restricted to any positive measure set. Hence, by (14), it holds at our pair of periodic points.

Our previous constructions are robust under small C^0 changes of the cocycle, cf. remark 3.9, so the following does make sense:

Lemma 3.20. *For the majority of nearby cocycles B, the probability $\bar{m}_{p_i}$ is a convex combination of Dirac measures supported on the eigenspaces of $B^{\kappa_i}(p_i)$.*

The proof is easy in the complex case: Almost all complex-valued matrices (the complement of a subset with positive codimension) have all their eigenvalues with different norms. Then the ergodic invariant measures on the projective space are the Dirac measures supported on the corresponding eigenspaces. In the real case the argument is quite more subtle: actually, one may have to replace each periodic point p_i by a new one, with a much higher period but spending most of its iterates close to p_i. This construction is explained in the last section in [BV00]. Of course, one needs to check that these new points satisfy al the relevant estimates, like hyperbolicity or domination, with only slightly worse estimates.

Now we have established the set-up described around figure 1, and the argument may be completed as explained before: If the Lyapunov exponents vanish then eigenspaces at p_1 and at p_2 must be sent to the same point in the projective fiber over q by the strong-unstable holonomy and the strong-stable holonomy, respectively. By considering perturbations of the cocycle near $f(q)$ one sees that this is a codimension 1 phenomenon. Exploring the fact that there are infinitely many distinct periodic points we conclude that vanishing of Lyapunov exponents is an ∞-codimension phenomenon.

References

[AB] A. Arbieto and J. Bochi. L^p-generic cocycles have one-point Lyapunov spectrum. *Stochastics and dynamics.* To appear. Preprint www.preprint.impa.br 2002.

[AC97] L. Arnold and N. D. Cong. On the simplicity of the Lyapunov spectrum of products of random matrices. *Ergod. Th. & Dynam. Sys.*, 17(5):1005–1025, 1997.

[BGMV] C. Bonatti, X. Gomez-Mont, and M. Viana. Généricité d'exposants de Lyapunov non-nuls pour des produits déterministes de matrices. *Annales de l'Inst. Henri Poincaré - Analyse Non-linéaire.* To appear.

[Boc02] J. Bochi. Genericity of zero Lyapunov exponents. *Ergod. Th. & Dynam. Sys.*, 22:1667–1696, 2002.

[Bow75] R. Bowen. *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*, volume 470 of *Lect. Notes in Math.* Springer Verlag, 1975.

[BVa] J. Bochi and M. Viana. The Lyapunov exponents of generic volume preserving and symplectic systems. Preprint www.preprint.impa.br 2002.

[BVb] C. Bonatti and M. Viana. Lyapunov exponents with multiplicity 1 for deterministic products of matrices. Preprint www.preprint.impa.br 2001.

[BV00] C. Bonatti and M. Viana. SRB measures for partially hyperbolic systems whose central direction is mostly contracting. *Israel J. Math.*, 115:157–193, 2000.

[BV02] J. Bochi and M. Viana. Uniform (projective) hyperbolicity or no hyperbolicity: a dichotomy for generic conservative maps. *Annales de l'Inst. Henri Poincaré - Analyse Non-linéaire*, 19:113–123, 2002.

[Kat80] A. Katok. Lyapunov exponents, entropy and periodic points of diffeomorphisms. *Publ. Math. IHES*, 51:137–173, 1980.

[Led84] F. Ledrappier. Quelques propriétés des exposants caractéristiques. *Lect. Notes in Math.*, 1097:305–396, 1984.

[Mañ96] R. Mañé. The Lyapunov exponents of generic area preserving diffeomorphisms. In *International Conference on Dynamical Systems (Montevideo, 1995)*, pages 110–119. Longman, 1996.

[Ose68] V. I. Oseledets. A multiplicative ergodic theorem: Lyapunov characteristic numbers for dynamical systems. *Trans. Moscow Math. Soc.*, 19:197–231, 1968.

[Pes76] Ya. B. Pesin. Families of invariant manifolds corresponding to non-zero characteristic exponents. *Math. USSR. Izv.*, 10:1261–1302, 1976.

[Via] M. Viana. Almost all cocycles over any hyperbolic system have positive Lyapunov exponents. Preprint www.preprint.impa.br 2003.

Jairo Bochi (bochi@impa.br) Marcelo Viana (viana@impa.br)
IMPA, Estrada D. Castorina 110, 22460-320 Rio de Janeiro, Brazil