Physical measures

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Asymptotic behavior

Let us consider smooth transformations $f : M \to M$ on some (compact) manifold M.

Analogous considerations apply for continuous-time systems.

Problem

How to describe the behavior of the orbit $f^n(z)$ as time $n \to \infty$, for "most" initial states $z \in M$?

Often, this behavior is very complex and depends sensitively on the state z.

Physical measures

Definition

An invariant probability measure μ on M is called physical (or empirical) if the set $B(\mu)$ of initial states z such that

$$\frac{1}{n}\sum_{j=0}^{n-1}\varphi(f^j(z)) \to \int \varphi \, d\mu$$

for all continuous functions $\varphi : M \to \mathbb{R}$, has positive volume.

This means that, for any $z \in B(\mu)$ and $V \subset M$ with $\mu(\partial V) = 0$,

$$\mu(V) = \text{average time the orbit of } z \in B(\mu) \text{ spends in } V.$$

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Conservative systems

If $f : M \to M$ preserves a (normalized) volume measure *vol* on M then, by the ergodic theorem,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(z))$$

exists for all continuous functions $\varphi : M \to \mathbb{R}$ and almost all z.

If f is ergodic, then $B(vol)$ has full measure. More generally, if *vol* decomposes into finite or countably many ergodic components, these components are the physical measures.

Existence of physical measures

Problem

Do typical (conservative or dissipative) systems admit finite or countably many physical measures such that the unions of their basins has full volume ?

Uniform hyperbolicity

The limit set $L(f)$ is the closure of the set of all forward and backward accumulation points of orbits.

The system $f : M \to M$ is uniformly hyperbolic if the tangent space admits an equivariant splitting

$$T_z M = E^s_z \oplus E^u_z \quad \text{over all } z \in L(f)$$

such that vectors in E^s_z are contracted by Df and vectors in E^u_z are expanded by Df, at uniform rates.

Example

Any $A \in \mathrm{SL}(\mathbb{Z}, d)$ with no eigenvalues in the unit circle induces a uniformly hyperbolic system f on $M = \mathbb{T}^d$, with $L(f) = M$: here E^s_z corresponds to the eigenvalues of A of norm < 1 and E^u_z corresponds to the eigenvalues of norm > 1.

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Existence and finiteness

Theorem [Sinai, Ruelle, Bowen]

If f is uniformly hyperbolic then it admits finitely many physical measures and the union of their basins has full volume.

Ingredients to the proof

- The general hyperbolic case can be reduced to the purely expanding one, where $E^u_z = T_z M$ and $E^s_z = 0$.
- In the expanding case, one considers the transfer operator

$$(\mathcal{L}\varphi)(y) = \Sigma_{x:f(x)=y} \frac{\varphi(x)}{|\det Df(x)|}$$

acting on a convenient function space.
- $\mathcal{L}$ admits 1 as a simple eigenvalue, and the rest of the spectrum is contained in a disk of radius < 1.
- Let ψ be an eigenfunction, normalized by $\int \psi \, dvol = 1$. Then ψvol is an invariant probability measure with finitely many ergodic components, and these are the physical measures.

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- The general hyperbolic case can be reduced to the purely expanding one, where $E_z^u = T_z M$ and $E_z^s = 0$.
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Beyond hyperbolicity

In many other cases, one can obtain uniform hyperbolicity through replacing the original system f by a "variable time iterate" g:

$$g(x) = f^{n(x)}(x) \quad \text{with } n(x) \geq 1.$$

When the function $n(\cdot)$ is not too big (integrability condition), one can use this feature to reduce the study of such "weakly hyperbolic" systems to the uniformly hyperbolic case.

Collet-Eckmann maps

Example

Suppose a logistic map $f : [-2, 2] \to [-2, 2]$, $f(x) = c - x^2$ satisfies a Collet-Eckmann condition

$$|Df^n(c)| \geq \sigma^n \quad \text{for all large } n \text{ and some } \sigma > 1.$$

Then $[-2, 2]$ partitions into subdomains I_k such that

$$|Df^k(x)| \geq \sigma^{k/3} \text{ for } x \in I_k \quad \text{and} \quad \sum_k k\, length(I_k) < \infty.$$

Using a similar construction, Jakobson proved that logistic maps have a unique physical measure for a subset of parameter values with positive measure.

One dimensional transformations

This approach has been sharpened considerably, by several people, eventually leading to

Theorem [Avila-de Melo-Lyubich, Avila-Moreira]

For typical parametrized families of unicritical maps of the interval, and for almost every parameter value, there is a physical measure whose basin contains almost every point.

This strategy has been extended to many other situations: Hénon maps (Benedicks-Young), billiards (Young, Chernov); and it has been axiomatized, by Young, to yield many more ergodic properties.

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Partially hyperbolic systems

A diffeomorphism $f : M \to M$ is partially hyperbolic if the tangent space splits, equivariantly, as

$$T_z M = E^s_z \oplus E^c_z \oplus E^u_z$$

such that $Df \mid E^s_z$ is uniformly contracting, $Df \mid E^u_z$ is uniformly expanding, and $Df \mid E^c_z$ is "in between" the two.

Example

Examples on the torus $M = \mathbb{T}^d$ may be obtained from matrices $A \in \mathrm{SL}(\mathbb{Z}, d)$ whose spectrum intersects the unit circle.

The contracting and expanding bundles E^s and E^u are always uniquely integrable. A central foliation is an integrable foliation to E^c, when it exists.

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Construction of physical measures

Physical measures for partially hyperbolic systems have been constructed as limits of iterates $f_*^j(vol)$ of the volume measure on the manifold: in some cases, one considers

$$\lim_{k\to\infty} \frac{1}{n_k} \sum_{j=0}^{n_k-1} f_*^j(vol)$$

for convenient subsequences n_k; more generally, one takes restrictions $f_*^j(vol \mid H_j)$ to approppriate subsets H_j.

Construction of physical measures

Theorem [Alves-Bonatti-Viana]

Assume E^c is either mostly contracting or mostly expanding. Then f admits a finite number of physical measures whose basins cover almost every point in M.

mostly contracting $\leftrightarrow \lim_{n\to\infty} \frac{1}{n} \log \|Df_z^n(v^c)\| < 0$

mostly expanding $\rightarrow \lim_{n\to\infty} \frac{1}{n} log\|Df_z^n(v^c)\| \geq c > 0$

for every $v^c \in E_z^c$ and most initial states $z \in M$.

Non-uniform hyperbolicity suffices

Conjecture 1

Assume the tangent space splits at *vol*-almost every point, equivariantly, as

$$T_z M = S_z \oplus U_z$$

where $Df \mid S_z$ is mostly contracting and $Df \mid U_z$ is mostly expanding. Then f admits some physical measure.

True for many interval maps, by a classical result of Keller.

Substantial recent progress in any dimension, by Pinheiro.

One-dimensional center suffices

Conjecture 2

Assume $f : M \to M$ is partially hyperbolic with $\dim E^c = 1$. Then, generically, f admits finitely many physical measures, and the union of their basins has full volume.

True for surface maps, by an important recent result of Tsujii.

True on any manifold (?), assuming the leaves of the central foliation are circles (Viana-Yang).

Conservative systems

Physical measures do not always exist!

Example (Herman)

Conservative diffeomorphisms may exhibit, in a robust fashion, positive volume sets formed by invariant tori of codimension 1.

Such sets do not intersect the basin of any physical measure.

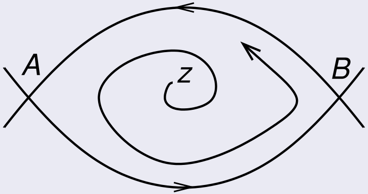
Each invariant torus carries one ergodic component of the volume measure: uncountably many ergodic components.

Nevertheless, time averages do converge for all those points, and such examples actually admit a very simple description.

Oscillating behavior

Example (Bowen)

For the following two-dimensional flow, time averages fail to exist on a whole open set of initial states z:



Random walks

Example (Bonatti)

The symmetric random walk on the one-dimensional lattice exhibits a similar oscillation of time averages.

In this case, convergence may be recuperated by considering higher order Cesaro averages: let $a_n^0 = \varphi(z_n)$ and

$$a_n^k = \frac{1}{n} \sum_{j=0}^{n-1} a_j^{k-1} \quad \text{for } k \geq 1.$$

Although a_n^1 oscilates, a_n^2 converges almost surely.