PERSISTENCE OF STRANGE ATTRACTORS WHEN UNFOLDING HOMOCLINIC TANGENCIES

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Abstract. Extending recent results by Benedicks and Carleson on the quadratic family on the plane, we showed in a joint work with L. Mora that any (generic) family of diffeomorphisms on a surface, *unfolding a homoclinic tangency*, exhibits *nonhyperbolic strange attractors or repellers with positive probability in the parameter space*. Later, we generalized this result to arbitrary dimension, when there is only one stretching direction and the product of any two eingenvalues has uorm less than one. Here we discuss several ideas, questions and conjectures related to these theorems and to the general problem of homoclinic bifurcations. This includes a joint result with L. J. Diaz and J. Rocha on *positive density of strange attractors* when unfolding certain saddle-node cycles.

§1. Introduction

A central problem in Dynamical Systems concerns the understanding of the changes in the dynamics of a diffeomorphism (or a flow) implied by a homoclinic bifurcation (meaning creation – or destruction – of a transverse homoclinic orbit), namely when this occurs through a homoclinic tangency. This problem has gained a renewed interest in recent times due, to a large extent, to the suggestion by Palis that homoclinic bifurcations are a main mechanism for the nonhyperbolicity of a system, specially in low dimensions: he conjectured that any diffeomorphism on a surface can be approximated either by a hyperbolic diffeomorphism (meaning, with limit set hyperbolic) or by one exhibiting homoclinic tangencies. In this way a global description of nonhyperbolicity, at least in low-dimensional, systems would follow from a good comprehension of homoclinic bifurcations and, specially, of the dynamic types occuring persistently in their unfolding. Here the idea of persistence, which Palis emphasizes, is essentially measure-theoretic and can be precised as follows. A smooth (C^∞) family of diffeomorphisms $\varphi_\mu : M \to M$, $\mu \in \mathbb{R}$, is said to unfold a homoclinic tangency q_0 of a hyperbolic periodic point p_0 of φ_0 if, as μ changes, the stable and unstable manifolds of p_μ (the analytic continuation of p_0) move with respect to each other near the tangency so that q_0 has a continuation by a transverse homoclinic intersection q_μ, for $\mu > 0$ say. We generally assume the tangency to be quadratic and the unfolding to be generic (nonzero relative velocity of the stable and the unstable manifolds at the tangency). By (measure-theoretic) persistence of some phenomenon on the family $(\varphi_\mu)_\mu$ we just mean that it occurs for a positive Lebesgue measure set of μ–values. We are concerned with phenomena occurring persistently on almost every (in some reasonable probabilistic sense, see comments below) family unfolding a tangency.

A number of important results obtained in the last 2 decades and specially in recent years, in good part motivate this setting of the problem. We summary below some of these results. For simplicity we restrict here to the 2– dimensional context; extensions to higher dimensions are described later.

Coexistence of infinitely many sinks or sources (Newhouse [Ne], [Ro]). There are intervals I_j in the μ–space, converging to $\mu = 0$, such that for a residual (Baire second cathegory) subset of values of $\mu \in I_j$, φ_μ has infinitely many periodic attractors or repellers (contained in Σ_μ, see below).

Contrary to this *topological* persistence it is generally believed that this phenomenon is not measure-theoretically persistent: conjecturedly it occurs only for a set of parameter values with measure zero.

Relative measure of hyperbolicity. Let the periodic point p_0 involved in the tangency be part of a hyperbolic basic set Λ_0 of φ_0. Define $\Sigma_\mu = \bigcap_{n\in\mathbb{Z}} \varphi_\mu^n(U \cup V_\mu)$ where U is a fixed small neighbourhood of Λ_0 and V_μ is a (const $|\mu|$)–neighbourhood of the orbit of tangency (the statements below still hold for slightly larger V_μ, see [PT2, Ch. V]). Let $\mathcal{H}$ be the set of μ–values for which Σ_μ is hyperbolic (and so $\varphi_\mu|\Sigma_\mu$ is topologically stable). Then

- (Palis-Takens [PT1], [PT2])

$$HD(\Lambda_0) < 1 \Longrightarrow \lim_{\varepsilon\to 0} \frac{m(\mathcal{H}\cap[-\varepsilon,\varepsilon])}{2\varepsilon} = 1;$$

- (Palis-Yoccoz [PY])

$$HD(\Lambda_0) > 1 \Longrightarrow \liminf_{\varepsilon\to 0} \frac{m(\mathcal{H}\cap[-\varepsilon,\varepsilon])}{2\varepsilon} < 1;$$

where $HD(\Lambda_0)$ is the Hausdorff dimension of Λ_0 and m denotes Lebesgue measure.

Strange attractors in the Hénon family (Benedicks-Carleson [BC]). Let, for $a > 0$ and $b > 0$, $h_{a,b}\colon \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $h_{a,b}(x,y) = (1 - ax^2 + y, bx)$. For $b > 0$ sufficiently small there is a positive measure set of a−values for which $h_{a,b}$ has strange (nonperiodic, nonhyperbolic) attractors.

The family $h_{a,b}$ may be thought of as a model for the creation of a horseshoe: it is, from the analytical point of view, the simplest family of diffeomorphisms in the plane going through homoclinic tangencies.

§2. Strange Attractors on Surfaces

The announcement of the remarkable result of Benedicks-Carleson led Palis to conjecture that, much more generally, the presence of strange attractors or repellers is a persistent phenomenon on every generic unfolding of a homoclinic tangency of a surface diffeomorphism. This is indeed true:

Theorem A (Mora-V. [MV]). *For any generic one-parameter family $(\varphi_\mu)_\mu$ of diffeomorphisms on a surface unfolding a homoclinic tangency, there is $S \subset \mathbb{R}$ such that for all $\varepsilon > 0$ $m(S \cap [-\varepsilon, \varepsilon]) > 0$ and for every $\mu \in S$ φ_μ has nonhyperbolic strange attractors or repellers contained in Σ_μ.*

By an *attractor* of a transformation φ we mean a compact, φ−invariant set such that

- $\varphi|\Lambda$ is transitive, i.e. φ has orbits which are dense in Λ;
- the basin $\{z: \lim_{n\to+\infty} \text{dist}(\varphi^n(z), \Lambda) = 0\}$ of Λ has nonempty interior.

In many cases the basin of the strange attractors in Theorem A is a full neighbourhood of the attractor and it would be useful to know if the same holds in general. We call an attractor *strange* if a dense orbit $\{\varphi^n(z_1): n \geq 0\}$ can be found such that

$$(1) \qquad \|D\varphi^n(z_1)\| \geq \theta^n \quad \forall n \geq 0 \quad \text{with } \theta > 1.$$

It is a well known fact ([CE], [No]) that in the 1-dimensional setting an exponential growth of the derivative implies a chaotic behaviour: existence of absolutely continuous invariant measure with positive Liapounov exponent. For higher dimensions this theory is still rather incomplete and an important task in our setting is the construction of SRB – measures for the strange attractors in the theorem. For the case of the 2-dimensional Hénon family this has been recently announced by Benedicks– Young and it seems likely that the general case of surface diffeomorphisms unfolding a homoclinic tangency can be treated along similar lines.

The generic properties of the family $(\varphi_\mu)_\mu$ assumed for Theorem A are:

- nondegenerate (quadratic) tangency;
- generic unfolding of the tangency;
- $\left|\det(D\varphi_0^k(p_0))\right| \neq 1$, where k is the period of p_0.

(The assumption of local linearizability used in [MV] should be removed in a work in preparation.) Clearly, these are C^k ***open and dense*** conditions on the space of one-parameter families passing through a homoclinic tangency, any $2 \leq k \leq \infty$. Moreover these conditions are satisfied by ***almost every*** such family, in the sense that the set they exclude is described by a zero Lebesgue measure subset of a (finite dimensional) euclidean space.

Theorem A has a one-dimensional version for families of endomorphisms on the circle or the interval ([MV]). *A (nondegenerate) homoclinic tangency in 1 dimension* just means that for some (nondegenerate) critical point c_0 and some hyperbolic (repelling) periodic point p_0 of the endomorphism φ_0 we have

- $\varphi_0^\ell(c_0) = p_0$ for some $\ell \geq 1$;
- there is a sequence $(c_n)_n \to p_0$ with $\varphi_0(c_n) = c_{n-1}$ $\forall\, n \geq 1$.

Then, for any family $(\varphi_\mu)_\mu$ generically unfolding this tangency (i.e. with $\partial_\mu(\varphi_\mu^\ell(c_\mu) - p_\mu)_{\mu=0} \neq 0$, denoting by c_μ and p_μ the analytic continuation of c_0 and p_0, respectively), there is a positive measure set of μ–values for which the closure of the critical orbit is a strange attractor for φ_μ.

§3. Higher Dimensions

In the opposite (and, naturally, more difficult) direction, Theorem A also admits an extension to families $(\varphi_\mu)_\mu$ of diffeomorphisms on higher- dimensional manifolds, unfolding a homoclinic tangency. Observe that in general the φ_μ, $|\mu|$ small, can be expected to have attractors (periodic or not) near the tangency only if the periodic point p_0 involved in the tangency satisfies

(a) $W^u(p_0)$ has dimension 1;

(b) $|\sigma\lambda_i| < 1$ for every $1 \leq i \leq m-1$, $m = \dim M$;

where $\sigma, \lambda_1, \lambda_2, \ldots, \lambda_{m-1}$ are the eigenvalues of $D\varphi_0^k(p_0)$, k =period of p_0, with $|\sigma| > 1 > |\lambda_i|$ for $1 \leq i \leq m-1$. Under these hypotheses it was proved in [PV] that Newhouse's phenomenon holds in any dimension: for generic families as above φ_μ has infinitely many periodic attractors in Σ_μ (recall definition above) for residual subsets of intervals of values of μ near $\mu = 0$. Also, extending further the methods in the proof of Theorem A we obtained the following generalization:

Theorem B ([V]). *For any generic family $(\varphi_\mu)_\mu$ of diffeomorphisms on an $m-$manifold, $m \geq 2$, unfolding a homoclinic tangency satisfying (a) and (b) above, there is $S \subset \mathbb{R}$ with $m(S \cap [-\epsilon, \epsilon]) > 0$ for all $\epsilon > 0$ and such that for every $\mu \in S$ $\varphi_\mu \mid \Sigma_\mu$ has nonhyperbolic strange* **attractors.**

The strange attractors we encounter in the proof of Theorem B are always topologically 1-dimensional: in fact they coincide with the closure of a 1- dimensional unstable manifold of some periodic saddle in Σ_μ. Now, diffeomorphisms on an $m-$manifold may exhibit strange attractors of any topological dimension $1 \leq d \leq m-1$ and it would be interesting to describe "natural" bifurcations yielding such higher-dimensional attractors.

Nonhyperbolicity of the strange attractors in the 2-dimensional setting of Theorem A is a direct consequence of the work of Plykin [Pl] on hyperbolic attractors. In the general

m-dimensional case, we argue as follows. As we said before, the strange attractors Λ_μ we find can be written as Λ_μ =closure $(W^u(P_\mu))$ where P_μ is some hyperbolic saddle in Σ_μ. As part of proving the strangeness of Λ_μ, we construct in the proof of Theorem B a point $z_1 = z_1(\mu) \in W^u(P_\mu)$, $\mu \in S$, satisfying (1) above and also

$$\|D\varphi_\mu^n(z_1) \cdot t\| \to 0 \quad \text{(exponentially) as } n \to +\infty, \tag{2}$$

where t is any vector tangent to $W^u(P_\mu)$ at z_1. This last property immediately implies the nonhyperbolicity of Λ_μ.

§4. Saddle-node Cycles

Theorems A, B lead naturally to the question of when does the set S of values of μ such that φ_μ exhibits strange attractors (or repellers), have positive density at $\mu = 0$, meaning

$$\lim_{\varepsilon \to 0} \frac{m(S \cap [-\varepsilon, \varepsilon])}{2\varepsilon} > 0. \tag{3}$$

The fact that $a = 2$ is a point of density 1 of $\{a: (x \mapsto 1 - ax^2) \text{ has chaotic behaviour}\}$ suggests that this may be the case for homoclinic bifurcations occuring in the Hénon family $(h_{a,b})_a$, $b \neq 0$ fixed and small (more generally in Hénon-like families, see [MV]), at values of a close to 2. Observe that (3) can not be expected to be true in general, as shown by the theorem of Palis-Takens stated above. On the other hand a conjecture of Palis asserts that (3) should indeed hold in the setting of his result with Yoccoz: unfolding of homoclinic tangencies on surfaces, associated to a basic set with Hausdorff dimension greater than 1.

A related situation, where positive density of strange attractors has been proved, is the unfolding of certain saddle-node cycles. A diffeomorphism φ_0 has a *saddle-node k-cyle,* $k \geq 1$, if ([NPT]) there are periodic points $p_1, \ldots, p_k$ of φ_0 such that

- p_1 is a saddle-node; $p_2, \ldots, p_k$ are hyperbolic saddles;

- $W^u(p_i)$ intersects $W^s(p_{i+1})$ transversely for every $1 \leq i < k$;
  $W^u(p_k)$ intersects $W^s(p_1)$ transversely.

We call the cycle *contractive* if $\dim W^u(p_1) = 1$ and *critical* if $W^u(p_k)$ has nontransverse intersections with leaves of the strong stable foliation of p_1 (which exists and is unique). The generic unfolding of a critical contractive saddle-node cycle always involves homoclinic tangencies ([NPT]). By combining Theorem B with the distribution in the μ-space of the parameter values corresponding to these tangencies, we get

Theorem C (Diaz-Rocha-V. [DRV]). *For generic families of diffeomorphims $(\varphi_\mu)_\mu$ on an m-manifold, $m \geq 2$, unfolding a critical contractive saddle-node cycle, the set S of values of μ for which φ_μ has nonhyperbolic strange attractors satisfies*

$$\liminf_{\varepsilon \to 0} \frac{m(S \cap [-\varepsilon, \varepsilon])}{2\varepsilon} > 0.$$

This means that, in a measure-theoretic sense, we get strange attractors for a sizable portion of the parameter interval, when we unfold this kind of saddle- node cycle. We observe that such cycles exist already for diffeomorphisms φ_0 on the boundary of the set of Morse-Smale diffeomorphisms.

§5. Vector Fields

We close with some brief comments on recent results concerning the unfolding of homoclinic bifurcations of flows. For simplicity we restrict to the 3- dimensional case. As usually, homoclinic tangencies associated to (regular) periodic orbits may be analysed through the Poincaré return map, which permits to transport to this setting the results stated above for diffeomorphisms. On the other hand, homoclinic phenomena involving singularities of the vector field exhibit new and important features. Striking examples are the so-caled Lorenz-type attractors ([GW]) which are persistent in a very strong sense: under a whole open set of perturbations.

Let the singularity involved in the tangency have eigenvalues $\lambda_1 > 0 > \lambda_2 > \lambda_3$. We call the singularity *expanding*, resp. *contracting*, if $(\lambda_1 + \lambda_2) > 0$, resp. $(\lambda_1 + \lambda_2) < 0$. The geometric Lorenz flows in [GW] correspond to the expanding case. Lorenz-type flows with a contracting singularity were studied by Rovella [Rv] who constructed a positive measure set of parameter values corresponding to strange attractors, using a Benedicks-Carleson kind of argument. He could also prove that in his situation Axiom A flows occupy an open and dense set of parameter values.

The unfolding of certain *singular cycles*, i.e. cycles involving a singularity, was studied by Bamón, Labarca, Mañé, Pacifico. Again the type of the singularity determines, in a qualitative way, the behaviour of the unfolding. In the expanding case they obtain an open and dense, full measure set of parameters corresponding to hyperbolicity (Axiom A).

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