PERSISTENCE OF STRANGE ATTRACTIONS
WHEN UNFOLDING HOMOCLINIC TANGENCIES

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Abstract. Extending recent results by Benedicks and Carleson on the quadratic family on the plane, we showed in a joint work with L. Mora that any (generic) family of diffeomorphisms on a surface, unfolding a homoclinic tangency, exhibits nonhyperbolic strange attractors or repellers with positive probability in the parameter space. Later, we generalized this result to arbitrary dimension, when there is only one stretching direction and the product of any two eigenvalues has norm less than one. Here we discuss several ideas, questions and conjectures related to these theorems and to the general problem of homoclinic bifurcations. This includes a joint result with L. J. Diaz and J. Rocha on positive density of strange attractors when unfolding certain saddle-node cycles.
§1. Introduction

A central problem in Dynamical Systems concerns the understanding of the changes in the dynamics of a diffeomorphism (or a flow) implied by a homoclinic bifurcation (meaning creation – or destruction – of a transverse homoclinic orbit), namely when this occurs through a homoclinic tangency. This problem has gained a renewed interest in recent times due, to a large extent, to the suggestion by Palis that homoclinic bifurcations are a main mechanism for the nonhyperbolicity of a system, especially in low dimensions: he conjectured that any diffeomorphism on a surface can be approximated either by a hyperbolic diffeomorphism (meaning, with limit set hyperbolic) or by one exhibiting homoclinic tangencies. In this way a global description of nonhyperbolicity, at least in low-dimensional, systems would follow from a good comprehension of homoclinic bifurcations and, specially, of the dynamic types occurring persistently in their unfolding. Here the idea of persistence, which Palis emphasizes, is essentially measure-theoretic and can be precise as follows. A smooth ($C^\infty$) family of diffeomorphisms $\varphi_\mu: M \to M, \mu \in \mathbb{R}$, is said to unfold a homoclinic tangency $q_0$ of a hyperbolic periodic point $p_0$ of $\varphi_0$ if, as $\mu$ changes, the stable and unstable manifolds of $p_0$ (the analytic continuation of $p_0$) move with respect to each other near the tangency so that $q_0$ has a continuation by a transverse homoclinic intersection $q_\mu$, for $\mu > 0$ say. We generally assume the tangency to be quadratic and the unfolding to be generic (nonzero relative velocity of the stable and the unstable manifolds at the tangency). By (measure-theoretic) persistence of some phenomenon on the family $(\varphi_\mu)_\mu$ we just mean that it occurs for a positive Lebesgue measure set of $\mu$-values. We are concerned with phenomena occurring persistently on almost every (in some reasonable probabilistic sense, see comments below) family unfolding a tangency.

A number of important results obtained in the last 2 decades and specially in recent years, in good part motivate this setting of the problem. We summary below some of these
results. For simplicity we restrict here to the 2-dimensional context; extensions to higher dimensions are described later.

Coexistence of infinitely many sinks or sources (Newhouse [Ne], [Ro]). There are intervals $I_j$ in the $\mu$-space, converging to $\mu = 0$, such that for a residual (Baire second category) subset of values of $\mu \in I_j$, $\varphi_\mu$ has infinitely many periodic attractors or repellers (contained in $\Sigma_\mu$, see below).

Contrary to this topological persistence it is generally believed that this phenomenon is not measure-theoretically persistent: conjecturedly it occurs only for a set of parameter values with measure zero.

Relative measure of hyperbolicity. Let the periodic point $p_0$ involved in the tangency be part of a hyperbolic basic set $A_\theta$ of $\varphi_\theta$. Define $\Sigma_\mu = \bigcap_{n \in \mathbb{Z}} \varphi_\mu^n(U \cup V_\mu)$ where $U$ is a fixed small neighborhood of $A_\theta$ and $V_\mu$ is a (const $|\mu|$)-neighborhood of the orbit of tangency (the statements below still hold for slightly larger $V_\mu$, see [PT2, Ch. V]). Let $\mathcal{H}$ be the set of $\mu$-values for which $\Sigma_\mu$ is hyperbolic (and so $\varphi_\mu | \Sigma_\mu$ is topologically stable). Then

- (Palis-Takens [PT1], [PT2])

$$H \text{D}(A_\theta) < 1 \iff \lim_{\epsilon \to 0} \frac{m(\mathcal{H} \cap [-\epsilon, \epsilon])}{2\epsilon} = 1;$$

- (Palis-Yoccoz [PY])

$$H \text{D}(A_\theta) > 1 \iff \lim_{\epsilon \to 0} \frac{m(\mathcal{H} \cap [-\epsilon, \epsilon])}{2\epsilon} < 1;$$

where $H \text{D}(A_\theta)$ is the Hausdorff dimension of $A_\theta$ and $m$ denotes Lebesgue measure.

Strange attractors in the Hénon family (Benedicks-Carlsson [BC]). Let, for $a > 0$ and $b > 0$, $h_{a,b}: \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $h_{a,b}(x, y) = (1 - ax^2 + y, bx)$. For $b > 0$ sufficiently
small there is a positive measure set of $a$-values for which $h_{a,b}$ has strange (nonperiodic, nonhyperbolic) attractors.

The family $h_{a,b}$ may be thought of as a model for the creation of a horseshoe: it is, from the analytical point of view, the simplest family of diffeomorphisms in the plane going through homoclinic tangencies.

§2. Strange Attractors on Surfaces

The announcement of the remarkable result of Benedicks-Carleson led Palis to conjecture that, much more generally, the presence of strange attractors or repellers is a persistent phenomenon on every generic unfolding of a homoclinic tangency of a surface diffeomorphism. This is indeed true:

Theorem A (Mora-V. [MV]). For any generic one-parameter family $(\varphi_\mu)_\mu$ of diffeomorphisms on a surface unfolding a homoclinic tangency, there is $S \subset \mathbb{R}$ such that for all $\varepsilon > 0$ $m(S \cap [\varepsilon, \varepsilon]) > 0$ and for every $\mu \in S \varphi_\mu$ has nonhyperbolic strange attractors or repellers contained in $\Sigma_\mu$.

By an attractor of a transformation $\varphi$ we mean a compact, $\varphi$-invariant set such that

- $\varphi | A$ is transitive, i.e. $\varphi$ has orbits which are dense in $A$;
- the basin $\{x; \lim_{n \to \pm \infty} \text{dist}(\varphi^n(x), A) = 0\}$ of $A$ has nonempty interior.

In many cases the basin of the strange attractors in Theorem A is a full neighbourhood of the attractor and it would be useful to know if the same holds in general. We call an attractor strange if a dense orbit $\{\varphi^n(z_1); n \geq 0\}$ can be found such that

$$\|D\varphi^n(z_1)\| \geq \theta^n \quad \forall n \geq 0 \quad \text{with } \theta > 1.$$

(1)

It is a well known fact ([CE], [No]) that in the 1-dimensional setting an exponential growth of the derivative implies a chaotic behaviour: existence of absolutely continuous
invariant measure with positive Liapounov exponent. For higher dimensions this theory is still rather incomplete and an important task in our setting is the construction of SRB – measures for the strange attractors in the theorem. For the case of the 2-dimensional Hénon family this has been recently announced by Benedicks- Young and it seems likely that the general case of surface diffeomorphisms unfolding a homoclinic tangency can be treated along similar lines.

The generic properties of the family \((\varphi_\mu)_\mu\) assumed for Theorem A are:

- nondegenerate (quadratic) tangency;
- generic unfolding of the tangency;
- \(\det(D\varphi^k_\mu(p_0)) \neq 1\), where \(k\) is the period of \(p_0\).

(The assumption of local linearizability used in [MV] should be removed in a work in preparation.) Clearly, these are \(C^2\) open and dense conditions on the space of one-parameter families passing through a homoclinic tangency, any \(2 \leq k \leq \infty\). Moreover these conditions are satisfied by almost every such family, in the sense that the set they exclude is described by a zero Lebesgue measure subset of a (finite dimensional) euclidean space.

Theorem A has a one-dimensional version for families of endomorphisms on the circle or the interval ([MV]). A (nondegenerate) homoclinic tangency in 1 dimension just means that for some (nondegenerate) critical point \(c_0\) and some hyperbolic (repelling) periodic point \(p_0\) of the endomorphism \(\varphi_0\) we have

- \(\varphi_\ell(c_0) = p_0\) for some \(\ell \geq 1\);
- there is a sequence \((c_n)_n \to p_0\) with \(\varphi_{c_n}(c_n) = c_{n-1}\) \(\forall n \geq 1\).

Then, for any family \((\varphi_\mu)_\mu\) generically unfolding this tangency (i.e. with \(\delta_\mu((\varphi_\mu^k(c_\mu) - p_\mu))_{\mu=0} \neq 0\), denoting by \(c_\mu\) and \(p_\mu\) the analytic continuation of \(c_0\) and \(p_0\), respectively), there is a positive measure set of \(\mu\)-values for which the closure of the critical orbit is a strange attractor for \(\varphi_\mu\).
§3. Higher Dimensions

In the opposite (and, naturally, more difficult) direction, Theorem A also admits an extension to families $(\varphi_\mu)_\mu$ of diffeomorphisms on higher-dimensional manifolds, unfolding a homoclinic tangency. Observe that in general the $\varphi_\mu$, $|\mu|$ small, can be expected to have attractors (periodic or not) near the tangency only if the periodic point $p_0$ involved in the tangency satisfies

(a) $W^s(p_0)$ has dimension 1;

(b) $|\sigma \lambda_i| < 1$ for every $1 \leq i \leq m - 1, m = \dim M$;

where $\sigma, \lambda_1, \lambda_2, \ldots, \lambda_{m-1}$ are the eigenvalues of $D\varphi_{p_0}(p_0)$, $k$ = period of $p_0$, with $|\sigma| > 1 > |\lambda_i|$ for $1 \leq i \leq m - 1$. Under these hypotheses it was proved in [PV] that Newhouse's phenomenon holds in any dimension: for generic families as above $\varphi_\mu$ has infinitely many periodic attractors in $\Sigma_\mu$ (recall definition above) for residual subsets of intervals of values of $\mu$ near $\mu = 0$. Also, extending further the methods in the proof of Theorem A we obtained the following generalization:

Theorem B ([V]). For any generic family $(\varphi_\mu)_\mu$ of diffeomorphisms on an $m$-manifold, $m \geq 2$, unfolding a homoclinic tangency satisfying (a) and (b) above, there is $S \subset \mathbb{R}$ with $m(S \cap [-\varepsilon, \varepsilon]) > 0$ for all $\varepsilon > 0$ and such that for every $\mu \in S, \varphi_\mu | \Sigma_\mu$ has nonhyperbolic strange attractors.

The strange attractors we encounter in the proof of Theorem B are always topologically 1-dimensional: in fact they coincide with the closure of a 1-dimensional unstable manifold of some periodic saddle in $\Sigma_\mu$. Now, diffeomorphisms on an $m$-manifold may exhibit strange attractors of any topological dimension $1 \leq d \leq m - 1$ and it would be interesting to describe "natural" bifurcations yielding such higher-dimensional attractors.

Nonhyperbolicity of the strange attractors in the 2-dimensional setting of Theorem A is a direct consequence of the work of Plykin [PI] on hyperbolic attractors. In the general
In the $m$-dimensional case, we argue as follows. As we said before, the strange attractors $A_{\mu}$ we find can be written as $A_{\mu} = \text{closure}(W^s(P_\mu))$ where $P_\mu$ is some hyperbolic saddle in $\Sigma_\mu$.

As part of proving the strangeness of $A_{\mu}$, we construct in the proof of Theorem B a point $z_1 = z_1(\mu) \in W^s(P_\mu),\, \mu \in S$, satisfying (1) above and also

(2) \[ \|D\phi_\mu^n(z_1)\cdot t\| \to 0 \quad \text{exponentially as } n \to +\infty, \]

where $t$ is any vector tangent to $W^s(P_\mu)$ at $z_1$. This last property immediately implies the nonhyperbolicity of $A_{\mu}$.

§4. Saddle-node Cycles

Theorems A, B lead naturally to the question of when does the set $S$ of values of $\mu$ such that $\phi_\mu$ exhibits strange attractors (or repellers), have positive density at $\mu = 0$, meaning

(3) \[ \lim_{\epsilon \to 0} \frac{m(S \cap [-\epsilon, \epsilon])}{2\epsilon} > 0. \]

The fact that $a = 2$ is a point of density 1 of $\{a: (x \mapsto 1 - ax^2) \text{ has chaotic behaviour}\}$ suggests that this may be the case for homoclinic bifurcations occurring in the Hénon family $(h_{a,b})_a,\, b \neq 0$ fixed and small (more generally in Hénon-like families, see [MV]), at values of $a$ close to 2. Observe that (3) can not be expected to be true in general, as shown by the theorem of Palis-Takens stated above. On the other hand a conjecture of Palis asserts that (3) should indeed hold in the setting of his result with Yoccoz: unfolding of homoclinic tangencies on surfaces, associated to a basic set with Hausdorff dimension greater than 1.

A related situation, where positive density of strange attractors has been proved, is the unfolding of certain saddle-node cycles. A diffeomorphism $\phi_0$ has a saddle-node $k$-cycle, $k \geq 1$, if ([NPT]) there are periodic points $p_1, \ldots, p_k$ of $\phi_0$ such that

* $p_1$ is a saddle-node, $p_1, \ldots, p_k$ are hyperbolic saddles;
• $W^*(p_i)$ intersects $W^*(p_{i+1})$ transversely for every $1 \leq i < k$;

$W^*(p_k)$ intersects $W^*(p_1)$ transversely.

We call the cycle contractive if $\dim W^*(p_1) = 1$ and critical if $W^*(p_2)$ has nontransverse intersections with leaves of the strong stable foliation of $p_1$ (which exists and is unique).

The generic unfolding of a critical contractive saddle-node cycle always involves homoclinic tangencies ([NPT]). By combining Theorem B with the distribution in the $\mu$-space of the parameter values corresponding to these tangencies, we get

**Theorem C** (Diaz-Rocha-V. [DRV]). For generic families of diffeomorphisms $(\varphi_\mu)_\mu$ on an $m$-manifold, $m \geq 2$, unfolding a critical contractive saddle-node cycle, the set $S$ of values of $\mu$ for which $\varphi_\mu$ has nonhyperbolic strange attractors satisfies

$$\lim_{\varepsilon \to 0} \frac{\text{ind} (S \cap [-\varepsilon, \varepsilon])}{2\varepsilon} > 0.$$  

This means that, in a measure-theoretic sense, we get strange attractors for a sizable portion of the parameter interval, when we unfold this kind of saddle-node cycle. We observe that such cycles exist already for diffeomorphisms $\varphi_0$ on the boundary of the set of Morse-Smale diffeomorphisms.

### §5. Vector Fields

We close with some brief comments on recent results concerning the unfolding of homoclinic bifurcations of flows. For simplicity we restrict to the 3-dimensional case. As usually, homoclinic tangencies associated to (regular) periodic orbits may be analyzed through the Poincaré return map, which permits to transport to this setting the results stated above for diffeomorphisms. On the other hand, homoclinic phenomena involving singularities of the vector field exhibit new and important features. Striking examples are the so-called Lorenz-type attractors ([GW]) which are persistent in a very strong sense under a whole open set of perturbations.
Let the singularity involved in the tangency have eigenvalues $\lambda_1 > 0 > \lambda_2 > \lambda_3$. We call the singularity expanding, resp. contracting, if $(\lambda_1 + \lambda_2) > 0$, resp. $(\lambda_1 + \lambda_2) < 0$. The geometric Lorenz flows in [GW] correspond to the expanding case. Lorenz-type flows with a contracting singularity were studied by Rovella [Rv] who constructed a positive measure set of parameter values corresponding to strange attractors, using a Benedicks-Carleson kind of argument. He could also prove that in his situation Axiom A flows occupy an open and dense set of parameter values.

The unfolding of certain singular cycles, i.e. cycles involving a singularity, was studied by Bámón, Labarca, Mañé, Pacifico. Again the type of the singularity determines, in a qualitative way, the behaviour of the unfolding. In the expanding case they obtain an open and dense, full measure set of parameters corresponding to hyperbolicity (Axiom A).
References


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