Lyapunov exponents with multiplicity 1 for deterministic products of matrices

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Abstract

We exhibit an explicit criterium for simplicity of the Lyapunov spectrum of linear cocycles, either locally constant or dominated, over hyperbolic (Axiom A) transformations. This criterium is expressed by a geometric condition on the cocycle's behaviour over periodic points and associated homoclinic orbits. It allows us to prove that for an open dense subset of dominated linear cocycles over a hyperbolic transformation, and for any invariant probability with continuous local product structure (including all equilibrium states of Hölder continuous potentials), all Oseledets subspaces are 1-dimensional. Moreover, the complement of this subset has infinite codimension and, thus, is avoided by any generic family of cocycles described by finitely many parameters.

This improves previous results of Bonatti, Gomez-Mont, Viana where it was shown that some Lyapunov exponent is non-zero, in a similar setting and also for an open dense subset.

To Michael Herman, who preceded us on this path.

Introduction

Lyapunov exponents describe the asymptotic behaviour of products of matrices, positive exponents corresponding to exponential growth, whereas negative exponents correspond to exponential decay of the norm. Numerous situations in Dynamics, and other branches of Mathematics, lead to the problem

*Partially supported by CNRS, Université de Bourgogne, CNPq, Faperj, Pronex-Dynamical Systems, IMPA.

of deciding whether the exponents are different from zero. Important examples come from such areas as smooth dynamics (matrices correspond to the derivative of the system), Schrödinger operators, stochastic processes (products of random matrices), and mathematical models in Economics.

Our own original motivation to address this problem came from two very different directions: the theory of partially hyperbolic diffeomorphisms, where non-zero Lyapunov exponents often permit a very precise description of the dynamics at the ergodic level (Sinai-Ruelle-Bowen measures) [1, 12]; and the study of certain transversely projective foliations, where non-zero exponents imply uniqueness of the harmonic measure on the leaves [10, 11].

Several methods have been devised for proving existence of non-zero exponents: let us mention Furstenberg, Kesten [16, 17], Herman [21], Kotani [25], in various contexts of linear cocycles, and Jakobson [23] and Benedicks, Carleson [5], for smooth transformations. The list is, of course, very far from complete.

The results of Furstenberg [17] about products of *iid* random matrices suggest that non-zero exponents might be typical for linear cocycles, in great generality. However, recent work of Bochi [6] shows that this can not be true without additional assumptions: he proves that generic (Baire second category subset) continuous $\mathrm{SL}(2,\mathbb{R})$-cocycles have zero Lyapunov exponents, or else they are uniformly hyperbolic. Actually, he gets the same conclusion within cocycles given by the derivatives of area-preserving diffeomorphisms, which is much more delicate. Moreover, these results have been extended to arbitrary dimension by Bochi, Viana [7, 8].

Here we require stronger regularity, starting from Hölder continuity, as well as *domination*: the map induced by the cocycle on the projective bundle is partially hyperbolic (this implies hyperbolicity of the base dynamics). The latter condition is motivated by the applications mentioned above, to partially hyperbolic systems and to transversely projective foliations. In this setting, it was proved in [11] that *an explicit condition about the cocycle over some periodic point and some homoclinic orbit associated to it suffices to ensure the existence of at least one non-zero Lyapunov exponent.*

Our main result in the present paper says that *a slightly stronger form of this condition, also satisfied by the vast majority of these cocycles, implies that all Lyapunov exponents are distinct.* By vast majority we mean an open dense subset which has full Lebesgue measure in parameter space within any generic parametrized families of cocycles. In fact, the complement has *infinite codimension:* it is contained in finite unions of closed submanifolds with arbitrary codimension.

Multiplicity 1 for the Lyapunov exponents of *iid* random matrices was proved by Guivarc'h, Raugi [20], thus improving Furstenberg's criterium [17]. See also Gold'sheid, Margulis [19] and LePage [27]. Our proof is an extension of their methods and those of Bonatti, Gomez-Mont, Viana [11].

It is interesting to put our conclusions together with other recent results, by Knill [24], Arnold, Cong [3, 4], and Arbieto, Bochi [2, 6], about Lyapunov exponents of $\mathrm{SL}(d,\mathbb{R})$-cocycles. These are organized in the following table, according to the dimension of the underlying space and the regularity of the cocycle. *Simple* stands for all Lyapunov exponents having multiplicity 1, whereas *zero* means that all the exponents vanish. *Generic* always refers to a Baire second category subset in the corresponding topology, and *full measure* means full Lebesgue measure in parameter space, for an open dense set of parametrized families of cocycles.

Regularity	$d = 2$	$d > 2$
L^p, $1 \le p < \infty$	dense: simple [3] generic: zero [2]	dense: simple [3] generic: zero [2]
L^∞	dense: simple [24] generic: zero or hyperbolic [6]	dense: simple [4] (∗)
C^0	dense: simple ? generic: zero or hyperbolic [6]	dense: simple ? (∗)
C^ν, $\nu > 0$ & domination	open dense & full measure: simple [11]	open dense & full measure: simple [this paper]

(∗) Bochi, Viana [7, 8] prove that, generically, all Lyapunov exponents are zero or else the Oseledets decomposition is dominated.

Recently, [29] extended the main conclusion of [11] to general C^ν cocycles, i.e. without the domination assumption: for an open dense full measure subset there exists some non-zero Lyapunov exponent. Furthermore, this holds even if the base dynamics is just non-uniformly hyperbolic. At this point it is not known whether the multiplicity 1 results in the present paper extend to such generality, although we believe this to be the case.

1 Precise setting and statements

For $d \ge 2$ and T an irreducible $d \times d$ matrix with coefficients in $\{0, 1\}$, let

- $\hat{f} : \hat{\Sigma}_T \to \hat{\Sigma}_T$ be the two-sided subshift of finite type associated to T;
- $\hat{A} : \hat{\Sigma}_T \to \mathrm{SL}(d, \mathbb{C})$ be continuous, and $\hat{f}_{\hat{A}} : \hat{\Sigma}_T \times \mathbb{CP}^{d-1} \to \hat{\Sigma}_T \times \mathbb{CP}^{d-1}$ be the projective cocycle over $\hat{f}$ generated by $\hat{A}$;
- $\hat{\mu}$ be an $\hat{f}$-invariant ergodic probability on $\hat{\Sigma}_T$ with $\operatorname{supp} \hat{\mu} = \hat{\Sigma}_T$ and continuous local product structure.

The last condition means that the restriction of $\hat{\mu}$ to every cylinder $[0; i]$ of $\hat{\Sigma}_T$ satisfies

$$\hat{\mu} \mid [0; i] = \psi\big(\mu^+ \times \mu^-\big) \tag{1}$$

where ψ is continuous and positive, and μ^+ and μ^- are the projections of $\hat{\mu} \mid [0; i]$ to the spaces of one-sided sequences indexed by positive integers and negative integers, respectively. This property holds, in particular, for every equilibrium state of $\hat{f}$ associated to a Hölder continuous potential. See Bowen [13] and Section 2.2 below.

1.1 Stable and unstable holonomies

The *local stable manifold* $W^s_{loc}(\hat{x})$ of a point $\hat{x} = (x_j)_{j\in\mathbb{Z}} \in \hat{\Sigma}_T$ is the set of $\hat{y} = (x_j)_{j\in\mathbb{Z}}$ such that $y_j = x_j$ for all $j \geq 0$. The *local unstable manifold* $W^u_{loc}(\hat{x})$ of $\hat{x} \in \hat{\Sigma}_T$ is the set of $\hat{y} = (x_j)_{j\in\mathbb{Z}}$ such that $y_j = x_j$ for all $j \leq 0$.

We always assume that $\hat{A}$ is either constant on each cylinder $[0; i]$ of $\hat{\Sigma}_T$ or dominated, in the sense of [11]:

Definition 1.1. $\hat{A}$ is *dominated* if there exists a distance d in $\hat{\Sigma}_T$ and constants $\theta < 1$ and $\nu \in (0, 1]$ such that, up to replacing $\hat{A}$ by some power $\hat{A}^N$,

1. $d(\hat{f}(\hat{x}), \hat{f}(\hat{y})) \leq \theta d(\hat{x}, \hat{y})$ and $d(\hat{f}^{-1}(\hat{x}), \hat{f}^{-1}(\hat{z})) \leq \theta d(\hat{x}, \hat{z})$ for every $\hat{y} \in W^s_{loc}(\hat{x})$, $\hat{z} \in W^u_{loc}(\hat{x})$, and $\hat{x} \in \hat{\Sigma}_T$;
2. $\hat{x} \mapsto \hat{A}(\hat{x})$ is ν-Hölder continuous and $\|\hat{A}(\hat{x})\| \, \|\hat{A}(\hat{x})^{-1}\| \theta^\nu < 1$ for every $\hat{x} \in \hat{\Sigma}_T$

(the definition does not depend on the choice of the metric $\|\cdot\|$ on the vector bundle $\hat{\Sigma}_T \times \mathbb{C}^d$, as long as the metric varies ν-Hölder continuously with the base point $\hat{x}$).

Let us explain the geometric meaning of this condition. Given any linear isomorphism $B : \mathbb{C}^d \to \mathbb{C}^d$, the expression $\|B\| \, \|B^{-1}\|$ is a Lipschitz constant for the actions of both B and B^{-1} on the projective space $\mathbb{CP}^{d-1}$. Thus, $\|\hat{A}(\hat{x})\| \, \|\hat{A}(\hat{x})^{-1}\|$ is an upper bound for the expansion, and its inverse is a lower bound for the contraction exhibited by the action of $\hat{A}(\hat{x})$ on the projective fiber $\{\hat{x}\} \times \mathbb{CP}^{d-1}$.

First, consider $\nu = 1$. The inequality in part 2 of the definition becomes

$$\theta < (\|\hat{A}(\hat{x})\| \, \|\hat{A}(\hat{x})^{-1}\|)^{-1} \quad \text{or} \quad \theta^{-1} > \|\hat{A}(\hat{x})\| \, \|\hat{A}(\hat{x})^{-1}\| \, .$$

The first form means that the base map $\hat{f}$ contracts local stable manifolds stronger than the projective cocycle contracts fibers; the second one means that the base map expands local unstable manifolds stronger than the projective cocycle expands fibers. In other words, domination means that $\hat{f}_{\hat{A}}$ is a "partially hyperbolic" transformation, with the fibers as central "leaves" (quotation marks are there just because these notions are usually defined for smooth maps on manifolds).

This interpretation extends immediately to the general case $\nu \in (0, 1]$. It suffices to note that $d(\cdot, \cdot)^\nu$ is also a metric in Σ_T, a function is ν-Hölder with respect to the original metric $d(\cdot, \cdot)$ if and only if it is 1-Hölder with respect to the new one $d(\cdot, \cdot)^\nu$, and θ^ν bounds the expansion and contraction rates of the base map relative to this new metric. This reduces the general case to the previous particular one.

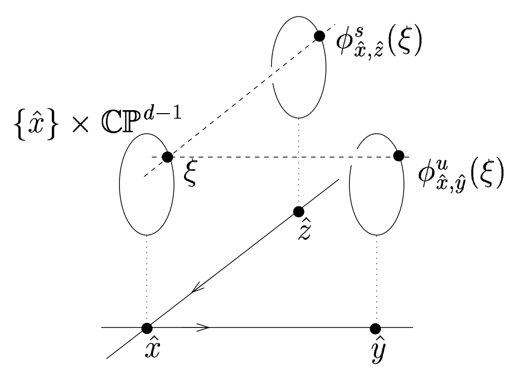

Figure 1: Stable and unstable holonomies

In view of the theory of partially hyperbolic systems [15, 22], one expects such a condition to imply the existence of invariant strong-stable and strong-unstable "foliations" for $\hat{f}_{\hat{A}}$ in $\hat{\Sigma}_T \times \mathbb{CP}^{d-1}$, transverse to the fibers. The next proposition means that this is indeed so, and the corresponding holonomies (projections along the leaves) are projective maps. See also Figure 1.

Proposition 1.2. *If $\hat{A}$ is either dominated or constant on each cylinder* $[0; i]$ *of* $\hat{\Sigma}_T$*, there exists a family* $\phi^u_{\hat{x},\hat{y}}$ *of projective transformations of* $\mathbb{CP}^{d-1}$*, defined for every pair* $\hat{x}, \hat{y} \in \hat{\Sigma}_T$ *in the same local unstable manifold of* $\hat{f}$*, and there exists* $C_1 > 0$ *such that*

1. $\phi^u_{\hat{x},\hat{x}} = \mathrm{id}$ *and* $\phi^u_{\hat{y},\hat{z}} \circ \phi^u_{\hat{x},\hat{y}} = \phi^u_{\hat{x},\hat{z}}$
2. $\hat{A}(\hat{f}^{-1}(\hat{y})) \circ \phi^u_{\hat{f}^{-1}(\hat{x}),\hat{f}^{-1}(\hat{y})} \circ \hat{A}^{-1}(\hat{x}) = \phi^u_{\hat{x},\hat{y}}$
3. $\|\phi^u_{\hat{x},\hat{y}} - \mathrm{id}\| \leq C_1 d(\hat{x}, \hat{y})^\nu$

for all $\hat{x}, \hat{y}, \hat{z} \in \hat{\Sigma}_T$ *in the same local unstable manifold.*

Proof. If $\hat{A}$ is constant on each cylinder $[0; i]$, just define $\phi^u_{\hat{x},\hat{y}} = \mathrm{id}$ for every pair of points in the same local unstable manifold. If $\hat{A}$ is dominated, define $\phi^u_{\hat{x},\hat{y}} = \lim_{n\to\infty} \hat{A}^{-n}(\hat{y})^{-1}\hat{A}^{-n}(\hat{x})$. It is shown in [11, Lemme 1.12] that the limit exists and satisfies the properties in the proposition. □

We shall refer to $\{\phi^u_{\hat{x},\hat{y}}\}$ as the *unstable holonomies* of the cocycle. This is easily extended to pairs $(\hat{x}, \hat{y})$ of points in the same *global* unstable manifold: just define

$$\phi^u_{\hat{x},\hat{y}} = \hat{A}^k(\hat{f}^{-k}(\hat{y})) \circ \phi^u_{\hat{f}^{-k}(\hat{x}),\hat{f}^{-k}(\hat{y})} \circ \hat{A}^{-k}(\hat{x}) \tag{2}$$

for large k and use part 2 of the proposition to see that the definition does not depend on the choice of k. In a dual fashion we construct *stable holonomies* $\{\phi^s_{\hat{x},\hat{z}}\}$, for any points $\hat{x}$ and $\hat{z}$ in the same global stable manifold.

1.2 Typical cocycles

Given a periodic point $\hat{p} \in \hat{\Sigma}_T$ of $\hat{f}$, we say that $\hat{z} \in \Sigma_T$ is a *homoclinic point* associated to $\hat{p}$ if it is in the intersection of the stable manifold and the unstable manifold of $\hat{p}$. Then we define

$$\psi_{\hat{p},\hat{z}} = \phi^s_{\hat{z},\hat{p}} \circ \phi^u_{\hat{p},\hat{z}} \,.$$

This is a projective map from the fiber $\{\hat{p}\} \times \mathbb{CP}^{d-1}$ over $\hat{p}$ back to itself. Up to replacing $\hat{z}$ by some backward iterate, we may suppose that $\hat{z} \in W^u_{loc}(\hat{p})$

and $\hat{f}^l(\hat{z}) \in W^s_{loc}(\hat{p})$ for some $l \geq 1$, which may be taken a multiple of the period of p. Then, by the analogue of (2) for stable holonomies,

$$\psi_{\hat{p},\hat{z}} = \hat{A}^{-l}(p) \circ \phi^s_{\hat{f}^l(z),\hat{p}} \circ \hat{A}^l(\hat{z}) \circ \phi^u_{\hat{p},\hat{z}} \,. \tag{3}$$

Definition 1.3. Suppose $\hat{A} : \hat{\Sigma}_T \to \mathrm{SL}(d, \mathbb{C})$ is either dominated or constant on each cylinder of $\hat{\Sigma}_T$. We say that $\hat{A}$ is 1-*typical* if there exists a periodic point $\hat{p}$ and a homoclinic point $\hat{z}$ associated to $\hat{p}$ such that

1. The eigenvalues of $\hat{A}$ on the orbit of $\hat{p}$ have multiplicity 1 and distinct norms; let $\omega_j \in \mathbb{CP}^{d-1}$ represent the eigenspaces, for $1 \leq j \leq d$.

2. $\{\psi_{\hat{p},\hat{z}}(\omega_i) : i \in I\} \cup \{\omega_j : j \in J\}$ is linearly independent, for all subsets I and J of $\{1, \ldots, d\}$ with $\#I + \#J \leq d$.

For $d = 2$ this second condition means that $\psi_{\hat{p},\hat{z}}(\omega_i) \neq \omega_j$ for $1 \leq i, j \leq 2$. See Figure 2. For $d = 3$ it means that $\psi_{\hat{p},\hat{z}}(\omega_i)$ is outside the plane $\omega_j \oplus \omega_k$ and ω_i is outside the plane $\psi_{\hat{p},\hat{z}}(\omega_j) \oplus \psi_{\hat{p},\hat{z}}(\omega_k)$, for all choices of $1 \leq i, j, k \leq 3$.

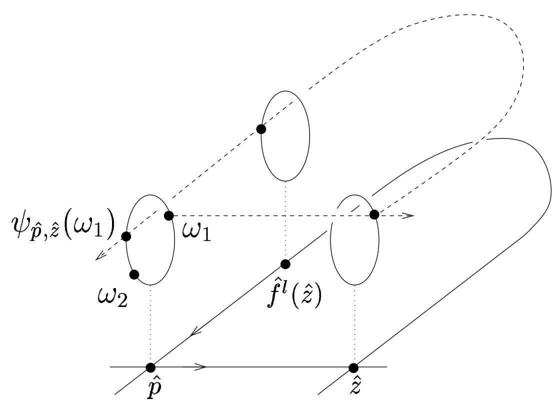

Figure 2: A 1-typical cocycle in dimension $d = 2$

Remark 1.4. In general, condition 2 in Definition 1.3 is equivalent to saying that the algebraic minors of the matrix M of $\psi_{\hat{p},\hat{z}}$ in a basis of eigenvectors of $\hat{A}$ at $\hat{p}$ are all non-vanishing. We say that M is a 1-*typical matrix.* It is also worth stressing that the property of being 1-typical depends only on the map $\hat{A}$, not on the measure $\hat{\mu}$.

Although [11] uses a weaker condition (in any dimension: $\psi_{\hat{p},\hat{z}}(\omega_i) \neq \omega_j$ for every $1 \le i, j \le d$), the same arguments as in [11, Section 2] prove that 1-typical cocycles form an open dense subset of the space of dominated cocycles endowed with a C^ν norm, for any real $\nu > 0$. Moreover, the complement has infinite codimension. We just highlight the main ingredients that make those arguments work.

On the one hand, one may modify $\psi_{\hat{p},\hat{z}}$ by changing $\hat{A}^l(\hat{z})$ without affecting the local holonomies nor the value of $\hat{A}$ on the orbit of $\hat{p}$. In fact, $\hat{A} \mapsto \psi_{\hat{p},\hat{z}}$ is a C^1 submersion restricted to any joint level set of $\phi^s_{\hat{f}^l(\hat{z}),\hat{p}}$, $\phi^u_{\hat{p},\hat{z}}$, and $\hat{A}^j(\hat{p})$, all $j \in \mathbb{Z}$. On the other hand, the conditions in the definition are satisfied by any matrices outside a finite union of closed submanifolds with positive codimension. By varying the fixed point $\hat{p}$ and the homoclinic point $\hat{z}$, one concludes that the exceptional set has infinite codimension.

1.3 Main results

Let $\hat{f}$, $\hat{A}$, $\hat{\mu}$ be as described at the beginning of this section. Firstly, we state

Theorem 1. *If $\hat{A}$ is 1-typical then the largest and the smallest Lyapunov exponents of $\hat{f}_{\hat{A}}$ for $\hat{\mu}$ have multiplicity 1, meaning that the corresponding Oseledets subspaces are 1-dimensional.*

As a consequence of Theorem 1 we obtain a corresponding result for *all* Lyapunov exponents. For the statement we need the following notion:

Definition 1.5. Given $1 \le k \le d$, we say that $\hat{A}$ is *k-typical* if points $\hat{p}$ and $\hat{z}$ as in Definition 1.3 may be chosen such that

1. all the products of k distinct eigenvalues of $\hat{A}$ at $\hat{p}$ have distinct norms;

2. the matrix $M^{\wedge k}$ of the action of M on the k:th external product $\Lambda^k(\mathbb{C}^d)$ is 1-typical, where M is the matrix of $\psi_{\hat{p},\hat{z}}$ introduced in Remark 1.4 (we say that M is a *k-typical matrix*).

We say that $\hat{A}$ is *typical* if it is k-typical for all $1 \le k \le d/2$. In the same way as before, one checks that typical cocycles correspond to an open dense subset of maps $\hat{A}$, whose complement has infinite codimension in parameter space.

Remark 1.6. Roughly speaking, $\hat{A}$ is k-typical if its action $\hat{A}^{\wedge k}$ on the k:th external product is 1-typical. But we need not ask $\hat{A}^{\wedge k}$ to be dominated: existence of stable and unstable holonomies suffices for all our purposes.

Theorem 2. *If $\hat{A}$ is typical then the Lyapunov exponents of $\hat{f}_{\hat{A}}$ with respect to $\hat{\mu}$ have multiplicity* 1. *In fact, if $\hat{A}$ is i-typical for $1 \leq i \leq k$ then the k largest and the k smallest Lyapunov exponents have multiplicity* 1.

Theorems 1 and 2 extend directly to cocycles over hyperbolic basic sets of diffeomorphisms and, in particular, over Anosov diffeomorphisms. The invariant probability should have continuous local product structure, e.g., any equilibrium state associated to a Hölder continuous potential. The proof is by reducing to the shift case via a Markov partition. Such a reduction was described in detail in [11, Section 2] and so we do not repeat it here.

1.4 Outline of the proofs

The first step in the proof of Theorem 1 is to use the existence of stable holonomies to conjugate $\hat{f}_{\hat{A}}$ to a cocycle f_A constant on local stable manifolds or, in other words, defined over the corresponding one-sided shift.

Let $f : \Sigma_T \to \Sigma_T$ be the one-sided subshift associated to T, that is, Σ_T is the space of sequences $(x_n)_{n\geq 0}$ such that $T_{x_n,x_{n+1}} = 1$ for all $n \geq 0$. Let $P : \hat{\Sigma}_T \to \Sigma_T$ be the canonical projection and $\mu = P_*\hat{\mu}$. Then μ is an f-invariant ergodic measure with $\operatorname{supp}\mu = \Sigma_T$. The family of stable holonomies defines a continuous change of linear coordinates on the fibers $\{x\} \times \mathbb{CP}^{d-1}$ that makes $\hat{A}$ constant on each local stable manifold of $\hat{f}$. Compare [11, Corollaire 1.15]. Thus, it is no restriction to suppose that

- $\hat{A} = A \circ P$ for some continuous $A : \Sigma_T \to \mathrm{SL}(d, \mathbb{C})$.

Denote by $f_A : \Sigma_T \times \mathbb{CP}^{d-1} \to \Sigma_T \times \mathbb{CP}^{d-1}$ the projective cocycle over f generated by A.

Let $\hat{m} = \{\hat{m}_{\hat{x}} : \hat{x} \in \hat{\Sigma}_T\}$ be an $\hat{f}_{\hat{A}}$-invariant probability on $\hat{\Sigma}_T \times \mathbb{CP}^{d-1}$ that projects down to $\hat{\mu}$ on $\hat{\Sigma}_T$. We consider the projection $m = \{m_x : x \in \Sigma_T\}$ of $\hat{m}$ to $\Sigma_T \times \mathbb{CP}^{d-1}$: this is an f_A-invariant probability projecting down to μ on Σ_T. A very useful fact (Proposition 3.1 and Remark 3.5) is that $\hat{m}$ may be recovered from m via iteration under A:

$$\lim_{n\to\infty} A^n(\hat{x}_n)_* m_{\hat{x}_n} = \hat{m}_{\hat{x}} \quad \text{for } \hat{\mu}\text{-almost every } \hat{x} \in \hat{\Sigma}_T, \tag{4}$$

where $\hat{x}_n = P(\hat{f}^{-n}(\hat{x}))$. The assumption that A has a simple largest eigenvalue on a periodic point p implies that the iterates $A^{nq}(p)_*\eta$ converge to the

Dirac measure supported on the corresponding eigenspace, for any probability measure η on the fiber of p that neglects the sum of the other eigenspaces. A central part of our strategy is to try and propagate this behaviour to typical points, to conclude that (4) is a Dirac measure for almost every $\hat{x}$. In order to implement this idea, a couple of fundamental issues must be dealt with.

Firstly, we must be able to relate the cocycle's behaviour over the orbit of p (a zero measure subset!) with its behaviour over the typical points of $\hat{\Sigma}_T$. For this purpose, we restrict ourselves to measures $m = P_*(\hat{m})$ such that $\hat{m}$ is invariant also under unstable holonomies: we say that m is (A, ϕ)-invariant. Together with continuous local product structure, this implies (Proposition 4.3) that m admits a disintegration into conditional measures m_x depending *continuously* on the point $x \in \Sigma_T$. Continuity is crucial for showing that the cocycle's behaviour over the periodic orbit is indeed reflected on the typical behaviour.

Secondly, we must make sure that the conditional measures m_x of m on the fibers do neglect the sum of the remaining Oseledets subspaces. In fact, assuming the cocycle is 1-typical, we prove (Proposition 5.1) that μ-almost every conditional measure m_x gives zero weight to any proper subspace of the fiber.

This leads to the proof (Section 6) that the limit in (4) is indeed a Dirac measure at almost every point:

Theorem 3. *If A is 1-typical then for every (A, ϕ)-invariant probability measure m and $\hat{\mu}$-almost every $\hat{x} \in \hat{\Sigma}_T$ the sequence $A^n(\hat{x}_n)_* m_{\hat{x}_n}$ converges to a Dirac measure $\delta_{\xi(\hat{x})}$ in the fiber $\{\hat{x}_0\} \times \mathbb{CP}^{d-1}$ when $n \to \infty$.*

This result alone is not yet sufficient to deduce that the cocycle has a one-dimensional Oseledets subspace corresponding to the largest Lyapunov exponent. For example, the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

has a unique limit Dirac measure δ_ξ in projective space, and yet the eigenvalue has multiplicity 2. To interpret this fact, consider the adjoint matrix A_* acting on the dual space. Then A_* also has a unique limit Dirac measure δ_{ξ_*}, and $\ker \xi_*$ is a candidate to being another eigenspace of A. However, this can not be, because the kernel contains ξ.

So, we apply the previous theory to the adjoint cocycle $\hat{A}_*$ of $\hat{A}$ and, using once more that these cocycles are 1-typical, we check (Lemma 7.3) that the corresponding limit point $\xi_*(\hat{x})$ *does not* vanish at $\xi(\hat{x})$, for almost every point $\hat{x}$. We deduce (Proposition 7.5) that $\xi(\hat{x})$ is indeed the unique direction of strongest expansion for the cocycle at almost every point $\hat{x}$. We conclude that this direction corresponds to a largest Lyapunov exponent (Corollary 8.3), and the $\ker \xi_*(\hat{x})$ is the sum of all the other Oseledets subspaces. This completes the proof of Theorem 1.

In Section 8 we also deduce Theorem 2, simply by applying the previous ideas to the cocycle's external powers $\hat{A}^{\wedge k}$. Finally, all these arguments extend to the case of real-valued cocycles, as explained in Section 9. The main point is to show that, despite the difficulty posed by pairs of complex conjugate eigenvalues, typical cocycles still constitute a dense subset. We prove that, after perturbation, one can always find *some* periodic point near p such that the eigenvalues over the new periodic orbit are all real.

2 Preliminaries

2.1 Continuous local product structure

First we deduce some simple consequences from the assumption (1) that $\hat{\mu}$ has continuous local product structure.

By a slight abuse of language, we call local stable manifold of a point $x \in \Sigma_T$ the set $P^{-1}(x) \subset \hat{\Sigma}_T$. Thus,

$$W^s_{loc}(x) = P^{-1}(x) = W^s_{loc}(\hat{x}) \quad \text{for any } \hat{x} \in P^{-1}(x) \text{ and } x \in \Sigma_T.$$

Given $n \geq 1$ and $\hat{x} \in \hat{\Sigma}_T$, we represent by $\hat{x}_n$ the f^n-pre-image $\hat{x}_n$ of $P(\hat{x})$ defined by $\hat{x}_n = P(\hat{f}^{-n}(\hat{x}))$. Observe that $f(\hat{x}_n) = \hat{x}_{n-1}$ for every $n \geq 1$, and $\hat{x}_0 = P(\hat{x})$.

For x and y in the same cylinder of Σ_T, define the *unstable holonomy map* $h_{x,y} : W^s_{loc}(x) \to W^s_{loc}(y)$ by the condition that $\hat{x} \in W^s_{loc}(x)$ and $\hat{y} = h_{x,y}(\hat{x})$ are in the same local unstable manifold.

Lemma 2.1. *The measure $\hat{\mu}$ has a disintegration into conditional measures $(\hat{\mu}_x)_{x\in\Sigma_T}$ that vary continuously with x in the weak topology. In fact, every*

$$h_{x,y} : (W^s_{loc}(x), \hat{\mu}_x) \to (W^s_{loc}(y), \hat{\mu}_y)$$

is absolutely continuous, with Jacobian $J_{x,y}$ depending continuously on (x, y).

Proof. By assumption, restricted to every $[0;i]$ we may write $\hat{\mu} = \psi(\mu \times \mu^-)$ with ψ continuous and positive. Since $\mu = P_*\hat{\mu}$, we have $\int_{W^s_{loc}(x)} \psi(\hat{x})\, d\mu^- = 1$ over every local stable manifold. Then

$$\hat{\mu}_x = \psi(\hat{x})\mu^- \quad \text{and} \quad J_{x,y}(\hat{x}) = \psi(h_{x,y}(\hat{x}))/\psi(\hat{x})$$

define a disintegration of $\hat{\mu}$ and a Jacobian for $h_{x,y}$ as in the statement. □

The *lift* of an f-invariant probability ν is the probability $\hat{\nu}$ such that

$$\int (\psi \circ P \circ \hat{f}^{-n})\, d\hat{\nu} = \int \psi\, d\nu \tag{5}$$

for every $n \geq 1$ and any measurable function $\psi : \Sigma_T \to \mathbb{R}$. This defines $\hat{\nu}$ uniquely because the $\psi \circ P \circ \hat{f}^{-n}$ generate the space of all measurable functions on $\hat{\Sigma}_T$. Moreover, $\hat{\nu}$ is $\hat{f}$-invariant (the right hand side does not depend on n).

Lemma 2.2. *The measure $\mu = P_*\hat{\mu}$ admits a continuous positive Jacobian $J_\mu f$. Moreover, $\hat{\mu}$ is the lift of μ.*

Proof. Let $y \in \Sigma_T$ and $E \subset \Sigma_T$ be any measurable set containing y and contained in a cylinder $[0;i,j]$. By definition,

$$\mu(f(E)) = \hat{\mu}(P^{-1}f(E)) = \int_{\{x \in f(E)\}} \psi(x,\eta)\, d\mu(x)\, d\mu^-(\eta)$$

and, using also invariance,

$$\mu(E) = \hat{\mu}(P^{-1}(E)) = \hat{\mu}(\hat{f}P^{-1}(E)) = \int_{\{x \in f(E),\ \eta_{-1}=i\}} \psi(x,\eta)\, d\mu(x)\, d\mu^-(\eta)\,.$$

Making $E \to \{y\}$ we get that

$$\frac{\mu(f(E))}{\mu(E)} \to \frac{1}{\int_{\{\eta_{-1}=i\}} \psi(f(y),\eta)\, d\mu^-(\eta)}\,.$$

Define $J_\mu f(y)$ to be the right hand side. Thus $J_\mu f : \Sigma_T \to (0,\infty)$ is a Jacobian for μ, and it is clear that it is continuous. This proves the first part of the lemma. The second part is immediate: the definition $\mu = P_*\hat{\mu}$ gives the relation (5) for $n = 0$, and the fact that $\hat{\mu}$ is $\hat{f}$-invariant extends it to every $n \geq 0$. □

Remark 2.3. Let $n \geq 1$ and A_n be any measurable subset of an $n+1$-cylinder. Write $\hat{\mu} \mid P^{-1}(A_n) = (\mu \mid A_n) \times \hat{\mu}_y$ and $\hat{\mu} \mid \hat{f}^n(P^{-1}(A_n)) = (\mu \mid A) \times \hat{\mu}_x$ (skew-products) where $A = f^n(A_n)$. By invariance, we also have

$$\hat{\mu} \mid \hat{f}^n(P^{-1}(A_n)) = f^n_*(\mu \mid A_n) \times \hat{f}^n_* \hat{\mu}_y = \frac{\mu \mid A}{J_\mu f^n(y)} \times \hat{f}^n_* \hat{\mu}_y \,.$$

Note that $\hat{f}^n(P^{-1}(A_n)) = \cup\{\hat{f}^n(W^s_{loc}(y)) : y \in A_n\}$. Since A_n is arbitrary, uniqueness of the conditional measures gives

$$\hat{\mu}_{f^n(y)} \mid \hat{f}^n(W^s_{loc}(y)) = \frac{1}{J_\mu f^n(y)} \hat{f}^n_* \hat{\mu}_y \tag{6}$$

for μ-almost every $y \in \Sigma_T$ and any $n \geq 1$. Considering the disintegration in Lemma 2.1 and the Jacobian in Lemma 2.2, continuity implies that (6) is true for *every* y.

Notice also that the fact that μ is f-invariant corresponds to the relation

$$\int \Big(\sum_{y:f(y)=x} \frac{1}{J_\mu f(y)} \varphi(y) \Big) d\mu(x) = \int \varphi(y)\, d\mu(y) \tag{7}$$

for every continuous function $\varphi : \Sigma_T \to \mathbb{R}$. Moreover, a probability m in $\Sigma_T \times \mathbb{CP}^{d-1}$ is invariant for f_A if and only if

$$\sum_{y:f(y)=x} \frac{1}{J_\mu f(y)} A(y)_* m_y = m_x \tag{8}$$

for μ-almost every $x \in \Sigma_T$, and any disintegration $(m_x)_{x \in \Sigma_T}$.

2.2 Measures with Hölder continuous Jacobians

In this section we check a partial converse to results in Section 2.1: given any f-invariant probability μ on Σ_T with Hölder continuous Jacobian $J_\mu f > 0$, the lift $\hat{\mu}$ of μ to $\hat{\Sigma}_T$ has continuous local product structure. The assumption also implies that $\hat{\mu}$ is ergodic and supported on the whole $\hat{\Sigma}_T$.

Equilibrium states of $\hat{f}$ are lifts of equilibrium states of the one-sided shift f, which do have Hölder continuous Jacobian if the potential is Hölder continuous. See Bowen [13]. So, this shows that our hypotheses apply to every equilibrium state of $\hat{f}$ associated to a Hölder continuous potential. The proofs of our results are, otherwise, independent of the present section.

Let μ be any f-invariant probability admitting a Jacobian $J_\mu f > 0$ which is γ-Hölder for some $\gamma > 0$. Recall that $\hat{x}_n = P(\hat{f}^{-n}(x))$ for $n \geq 0$.

Lemma 2.4. *Let x and y be in the same cylinder of Σ_T, $\hat{x} \in W^s_{loc}(x)$, and $\hat{y} = h_{x,y}(\hat{x}) \in W^s_{loc}(y)$. Then the limit*

$$J_{x,y}(\hat{x}) = \lim_{n\to\infty} \frac{J_\mu f^n(\hat{x}_n)}{J_\mu f^n(\hat{y}_n)}$$

exists, uniformly in x, y, $\hat{x}$, and it is positive. Moreover, there exists $C_2 > 0$ such that $|J_{x,y}(\hat{x}) - 1| \leq C_2 d(x,y)^\gamma$, for all x, y, and $\hat{x}$.

Proof. The arguments are quite standard. Begin by noting that

$$\log \frac{J_\mu f^n(\hat{x}_n)}{J_\mu f^n(\hat{y}_n)} = \sum_{j=1}^{n} \log J_\mu f(\hat{x}_j) - \log J_\mu f(\hat{y}_j). \tag{9}$$

As $\log J_\mu f$ is γ-Hölder continuous, and $d(\hat{x}_j, \hat{y}_j)$ decreases exponentially fast with j, there exist $C_2' > 0$ and $\tau < 1$ such that $|\log J_\mu f(\hat{x}_j) - \log J_\mu f(\hat{y}_j)| \leq C_2' \tau^j d(x,y)^\gamma$ for all $j \geq 1$. Recall that $x = P(\hat{x}) = \hat{x}_0$, and analogously for y. It follows that the series in (9) converges uniformly and absolutely, and the sum is bounded by $C_2'' d(x,y)^\gamma$, with $C_2'' = C_2' \sum_j \tau^j$. Denoting $\log J_{x,y}(\hat{x})$ the limit, we get that $|\log J_{x,y}(\hat{x})| \leq C_2'' d(x,y)^\gamma$, which implies $|J_{x,y}(\hat{x}) - 1| \leq C_2 d(x,y)^\gamma$ for some $C_2 > 0$ depending only on C_2''. □

Given a subset A of a cylinder $[0; i] \subset \Sigma_T$ and an $(n+1)$-cylinder $\xi_n \subset \hat{\Sigma}_T$ of the form $\xi_n = [-n; a_{-n}, \ldots, a_{-1}, i]$, we shall denote by $A \times \xi_n$ the set of points $(z_j)_{j\in\mathbb{Z}}$ of $\hat{\Sigma}_T$ such that $z_j = a_j$ for $-n \leq j \leq 0$ and $P(z_j) \in A$.

Remark 2.5. Each function $\hat{x} \mapsto J_{n,x,y}(\hat{x}) = J_\mu f^n(\hat{x}_n)/J_\mu f^n(\hat{y}_n)$, $n \geq 1$, is constant on every subset $\{x\} \times \xi_n \subset W^s_{loc}(x)$.

Lemma 2.6. *There exists a disintegration $(\hat{\mu}_x)_x$ of $\hat{\mu}$ relative to the partition $\{W^s_{loc}(x) : x \in \Sigma_T\}$ such that, for any x and y in the same cylinder of Σ_T, the unstable holonomy $h_{x,y} : (W^s_{loc}(x), \hat{\mu}_x) \to (W^s_{loc}(y), \hat{\mu}_y)$ is absolutely continuous, with Jacobian $J_{x,y}$:*

$$\hat{\mu}_y = (h_{x,y})_* \big(J_{x,y} \hat{\mu}_x\big)$$

Proof. For $n \geq 1$ and any $i \in \{1, \ldots, d\}$, let $A \subset [0; i]$ and $\xi_n \subset \hat{\Sigma}_T$ be an $(n+1)$-cylinder of the form $\xi_n = [-n; a_{-n}, \ldots, a_{-1}, i]$. Then $\hat{f}^{-n}(A \times \xi_n)$ may be written as $P^{-1}(A_n)$ for some subset A_n of Σ_T such that f^n maps A_n bijectively onto A. Consequently,

$$\hat{\mu}(A \times \xi_n) = \hat{\mu}(\hat{f}^{-n}(A \times \xi_n)) = \hat{\mu}(P^{-1}(A_n)) = \mu(A_n)$$

and so, by definition of the Jacobian,

$$\frac{\hat{\mu}(A \times \xi_n)}{\mu(A)} = \frac{\mu(A_n)}{\int_{A_n} J_\mu f^n \, d\mu}. \tag{10}$$

Let $(\bar{\mu}_x)_x$ be any disintegration of $\hat{\mu}$ relative to $\{W^s_{loc}(x) : x \in \Sigma_T\}$. For μ-almost any point $x = (x_j)_{j \geq 0}$ in $[0; i] \subset \Sigma_T$ and any cylinder ξ_n as before,

$$\bar{\mu}_x(\xi_n) = \lim_{A \to x} \frac{\hat{\mu}(A \times \xi_n)}{\mu(A)},$$

where the limit is over some basis of neighbourhoods A of x. As $A \to x$, the sets A_n constructed in the previous paragraph converge to the f^n-pre-image $\hat{x}_n$ of x given by $\hat{x}_n = (a_{-n}, \ldots, a_{-1}, i, x_1, \ldots, x_m, \ldots)$. Note that $\hat{x}_n = P(\hat{f}^{-n}(\hat{x}))$ for any choice of $\hat{x} \in \{x\} \times \xi_n$. In view of (10) and the fact that $J_\mu f^n$ is continuous, this gives that

$$\bar{\mu}_x(\xi_n) = \frac{1}{J_\mu f^n(\hat{x}_n)} \tag{11}$$

for any $(n+1)$-cylinder ξ_n and any x in some full μ-measure subset S_n of Σ_T.

We denote by S the intersection of S_n over all $n \geq 1$, which is also a full μ-measure subset. Given any points x and y of S contained in the same cylinder of Σ_T,

$$\bar{\mu}_y(\xi_n) = \frac{J_\mu f^n(\hat{x}_n)}{J_\mu f^n(\hat{y}_n)} \bar{\mu}_x(\xi_n) = J_{n,x,y}(\xi_n)\bar{\mu}_x(\xi_n)$$

for every ξ_n and $n \geq 1$. Recall Remark 2.5. Let $k \geq 1$ and a $(k+1)$-cylinder $\xi_k = [-k; a_{-k}, \ldots, a_{-1}, i]$ be fixed. By definition of the unstable holonomy, $h_{x,y}(\{x\} \times \xi_k) = \{y\} \times \xi_k$. For every $n \geq k$,

$$\bar{\mu}_y(\xi_k) = \sum_{\xi_n} \bar{\mu}_y(\xi_n) = \sum_{\xi_n} J_{n,x,y}(\xi_n)\bar{\mu}_x(\xi_n) = \int_{\xi_k} J_{n,x,y}(\hat{x}) \, d\bar{\mu}_x(\hat{x}),$$

where the sum is over the $(n+1)$-cylinders $\xi_n \subset \xi_k$. Passing to the limit as $n \to \infty$, and using Lemma 2.4, we find that $\bar{\mu}_y(\xi_k) = \int_{\xi_k} J_{x,y}\, d\bar{\mu}_x$. Since the sets $\{x\} \times \xi_k$, with varying ξ_k and $k \geq 1$, generate the σ-algebra of each $W^s_{loc}(x)$, this proves that $h_{x,y}$ is absolutely continuous with respect to $\bar{\mu}_x$, $\bar{\mu}_y$, for every x, y in S.

Since disintegrations are well-defined up to zero measure sets only, it is easy to enforce the absolute continuity relation also outside S. It suffices to fix some $x \in S$ in each cylinder $[0; i]$ and to replace $\bar{\mu}_z$ by $(h_{x,z})_*\big(J_{x,z}\bar{\mu}_x\big)$ for every $z \in [0; i] \setminus S$. □

Given any cylinder $[0; i]$, fix a point x in it. Consider the natural product coordinates on the cylinder obtained identifying (y, η) with $h_{x,y}(\eta)$. In these coordinates, $\hat{\mu}_y = J_{x,y}\hat{\mu}_x$ for every $y \in [0; i]$. So $\hat{\mu} \mid [0; i] = \psi(\mu \times \mu^-)$ with $\mu^- = \hat{\mu}_x$ and $\psi(y, \eta) = J_{x,y}(\eta)$. This shows that $\hat{\mu}$ has local product structure.

3 Convergence of conditional measures

Here we are going to prove

Proposition 3.1. *Let $\hat{m}$ be any $\hat{f}_{\hat{A}}$-invariant probability with $\hat{\pi}_*\hat{m} = \hat{\mu}$, and let $m = (P \times \mathrm{id})_*\hat{m}$. For $\hat{\mu}$-almost every $\hat{x} \in \hat{\Sigma}_T$*

(a) $A^n(\hat{x}_n)_* m_{\hat{x}_n}$ converges in the weak topology as $n \to \infty$ and

(b) for any $k \geq 1$ and any choices of points $y_{n,k}$ with $f^k(y_{n,k}) = \hat{x}_n$

$$\lim_{n\to\infty} A^n(\hat{x}_n)_* m_{\hat{x}_n} = \lim_{n\to\infty} A^{n+k}(y_{n,k})_* m_{y_{n,k}}\,.$$

Let $\mathcal{B}$ be the Borel σ-algebra of Σ_T. Consider the sequence $(\mathcal{B}_n)_n$ of σ-algebras of $\hat{\Sigma}_T$ defined by $\mathcal{B}_0 = P^{-1}(\mathcal{B})$ and $\mathcal{B}_n = \hat{f}(\mathcal{B}_{n-1})$ for $n \geq 1$. That is, each $\mathcal{B}_n$ is the σ-algebra generated by the variables $x_{-n}, \ldots, x_{-1}, x_0, x_1, \ldots$ Equivalently, the elements of $\mathcal{B}_n$ are the measurable sets consisting of entire $\hat{f}^n$-images of local stable manifolds.

Fix a continuous function $\varphi : \mathbb{CP}^{d-1} \to \mathbb{R}$. For $\hat{x} \in \hat{\Sigma}_T$ and $n \geq 0$, define

$$I_n(\hat{x}) = \int \varphi\, d\left(A^n(\hat{x}_n)_* m_{\hat{x}_n}\right) = \int \left(\varphi \circ A^n(\hat{x}_n)\right) dm_{\hat{x}_n}.$$

Then I_n is $\mathcal{B}_n$-measurable: it can be written as $I_n = I_n^0 \circ P \circ \hat{f}^{-n}$, where I_n^0 is the $\mathcal{B}$-measurable function

$$I_n^0(x) = \int (\varphi \circ A^n(x))\, dm_x.$$

Lemma 3.2. *For μ-almost every $x \in \Sigma_T$ and any $k \geq 1$,*

$$I_n^0(x) = \sum_{y: f^k(y)=x} \frac{1}{J_\mu f^k(x)} I_{n+k}^0(y).$$

Proof. The case $k = 1$ is a direct consequence of the invariance relation (8):

$$\begin{aligned} I_n^0(x) &= \int \Big(\varphi \circ A^n(x)\Big) dm_x = \int \Big(\varphi \circ A^n(x)\Big) d\Big(\sum_{f(y)=x} \frac{1}{J_\mu f(y)} A(y)_* m_y \Big) \\ &= \sum_{f(y)=x} \frac{1}{J_\mu f(y)} \int \Big(\varphi \circ A^{n+1}(y)\Big) dm_y = \sum_{f(y)=x} \frac{1}{J_\mu f(y)} I_{n+1}^0(y) \end{aligned}$$

The general case follows analogously, by induction. ☐

The next result says that I_n is the *conditional expectation* of I_{n+1} with respect to the σ-algebra $\mathcal{B}_n$, and so $(I_n, \mathcal{B}_n)_n$ is a martingale.

Lemma 3.3. *For every $n \geq 0$ and $\mathcal{B}_n$-measurable function $\psi : \hat{\Sigma}_T \to \mathbb{R}$,*

$$\int I_{n+1}(\hat{x}) \psi(\hat{x})\, d\hat{\mu}(\hat{x}) = \int I_n(\hat{x}) \psi(\hat{x})\, d\hat{\mu}(\hat{x}).$$

Proof. Let us write $\psi = \psi_n \circ P \circ \hat{f}^{-n}$, for some $\mathcal{B}$-measurable function ψ_n. Since $\hat{\mu}$ is the lift of μ, by Lemma 2.2,

$$\int I_n(\hat{x}) \psi(\hat{x})\, d\hat{\mu}(\hat{x}) = \int I_n^0(x) \psi_n(x)\, d\mu(x).$$

By Lemma 3.2, this is equal to

$$\begin{aligned} &\int \sum_{f(y)=x} \Big(\frac{1}{J_\mu f(y)} I_{n+1}^0(y) \Big) \psi_n(x)\, d\mu(x) \\ &\qquad = \int \sum_{f(y)=x} \Big(\frac{1}{J_\mu f(y)} I_{n+1}^0(y) \psi_n(f(y)) \Big)\, d\mu(x) \end{aligned}$$

Using the invariance relation (7) and the fact that $\hat{\mu}$ is the lift of μ (clearly, $\psi = (\psi_n \circ f) \circ P \circ \hat{f}^{-(n+1)}$), this is equal to

$$\int I^0_{n+1}(y)\psi_n(f(y))\, d\mu(y) = \int I_{n+1}(\hat{y})\psi(\hat{y})\, d\hat{\mu}(\hat{y}),$$

as claimed in the lemma. □

For a point $x \in \Sigma_T$ and $k \geq 0$ we represent by $d\nu_{k,x}$ the probability measure on the f^k-pre-image of x given by

$$d\nu_{k,x} = \sum_{y: f^k(y)=x} \frac{1}{J_\mu f^k(y)} \delta_y.$$

For $n \geq 0$ and $k \geq 1$, define

$$S_{n,k} = \int\int \left(I^0_{n+k}(y) - I_n(\hat{x})\right)^2 d\nu_{k,\hat{x}_n}(y)\, d\hat{\mu}(\hat{x}).$$

Lemma 3.4. *For every $n \geq 0$ and $k \geq 1$,*

$$S_{n,k} = \int I_{n+k}(\hat{x})^2 d\hat{\mu}(\hat{x}) - \int I_n(\hat{x})^2 d\hat{\mu}(\hat{x}).$$

Proof. Using that $\hat{\mu}$ is the lift of μ and the invariance property (7),

$$\begin{aligned}\int\int I^0_{n+k}(y)^2 d\nu_{k,\hat{x}_n}(y)\, d\hat{\mu}(\hat{x}) &= \int\int I^0_{n+k}(y)^2 d\nu_{k,x}(y)\, d\mu(x) \\ &= \int\int I^0_{n+k}(x)^2\, d\mu(x) \\ &= \int\int I_{n+k}(\hat{x})^2\, d\hat{\mu}(\hat{x}).\end{aligned}$$

Similarly, by the invariance of μ

$$\int I_n(\hat{x}) \int I^0_{n+k}(y)\, d\nu_{k,\hat{x}_n}(y)\, d\hat{\mu}(\hat{x}) = \int I_n(\hat{x}) I^0_n(\hat{x}_n)\, d\hat{\mu}(\hat{x}) = \int I_n(\hat{x})^2\, d\hat{\mu}(\hat{x}).$$

The conclusion of the lemma follows. □

Proof of Proposition 3.1. Lemma 3.3 says that the sequence $(I_n, \mathcal{B}_n)_n$ is a martingale. By the martingale convergence theorem (see [14, Chapter 5.4]), I_n converges $\hat{\mu}$-almost everywhere to some $\mathcal{I}_\varphi$. Considering a countable dense subset of the space of continuous functions, we find a full $\hat{\mu}$-measure set of points $\hat{x}$ such that

$$\int \varphi \, d\left(A^n(\hat{x}_n)_* m_{\hat{x}_n}\right) \to \mathcal{I}_\varphi$$

for every continuous function φ. This means that $A^n(\hat{x}_n)_* m_{\hat{x}_n}$ converges weakly to the probability measure $\varphi \mapsto \mathcal{I}_\varphi$. This proves statement (a) in the proposition.

To prove statement (b), let $k \geq 1$ be fixed. Lemma 3.4, together with $\sup_{n,\hat{x}} |I_n(\hat{x})| \leq \sup |\varphi|$, implies that

$$\sum_{n=1}^{l} \int \Big(I_n(\hat{x}) - I^0_{n+k}(y)\Big)^2 \, d\nu_{k,\hat{x}_n}(y) \, d\hat{\mu}(\hat{x}) = \sum_{n=1}^{l} S_{n,k} \leq 2k(\sup|\varphi|)^2$$

for every $l \geq 1$. Consequently,

$$\int \Big(I_n(\hat{x}) - I^0_{n+k}(y)\Big)^2 d\nu_{k,\hat{x}_n}(y) = \sum_{f^k(y)=\hat{x}_n} \frac{1}{J_\mu f^k(y)} \Big(I_n(\hat{x}) - I^0_{n+k}(y)\Big)^2$$

converges to zero when $n \to \infty$, for $\hat{\mu}$-almost every $\hat{x} \in \hat{\Sigma}_T$. Since the Jacobian $J_\mu f^k$ is bounded from zero, it follows that

$$\sum_{f^k(y)=\hat{x}_n} \Big(I_n(\hat{x}) - I^0_{n+k}(y)\Big)^2$$

converges to zero when $n \to \infty$, for $\hat{\mu}$-almost every $\hat{x} \in \hat{\Sigma}_T$. In other words, there exists a full measure subset of points $\hat{x}$ such that

$$\max_{f^k(y)=\hat{x}_n} \left| \int \varphi \, d\left(A^n(\hat{x}_n)_* m_{\hat{x}_n}\right) - \int \varphi \, d\left(A^{n+k}(y)_* m_y\right) \right|$$

converges to zero as $n \to \infty$. The claim (b) in the proposition is an easy consequence, considering a countable dense subset of the space of continuous functions $\varphi : \Sigma_T \to \mathbb{R}$. $\square$

Remark 3.5. For $\hat{\mu}$-almost every $\hat{x}$, the limit of $A^n(x_n)_* m_{\hat{x}_n}$ coincides with the conditional measure $\hat{m}_{\hat{x}}$ of the original measure $\hat{m}$. Indeed, let $\tilde{m}_{\hat{x}}$ denote this limit and let $\tilde{m} = \hat{\mu} \times \{\tilde{m}_x\}$. That is, $\tilde{m}$ is the measure on $\hat{\Sigma}_T \times \mathbb{CP}^{d-1}$ that projects down to $\hat{\mu}$ under $\hat{\pi}_*$ and whose conditional measures along the fibers are the $\tilde{m}_x$. It is easy to see that $\tilde{m}$ is $\hat{f}_{\hat{A}}$-invariant, that is, $\tilde{m}_{\hat{f}(\hat{x})} = \hat{A}(\hat{x})_* \tilde{m}_{\hat{x}}$ at $\hat{\mu}$-almost every point. Moreover, $\tilde{m}$ projects down to m under $P \times \mathrm{id}$. That is because $\tilde{m}$ is the limit of the $\hat{f}_{\hat{A}}$-iterates of $\hat{\mu} \times \{m_{P(\hat{x})} : \hat{x} \in \hat{\Sigma}_T\}$ on $\hat{\Sigma}_T \times \mathbb{CP}^{d-1}$, and all these iterates project down to m. By uniqueness of $\hat{f}_{\hat{A}}$-invariant measures projecting to m, we conclude that $\tilde{m} = \hat{m}$, that is, $\tilde{m}_{\hat{x}} = \hat{m}_{\hat{x}}$ for $\hat{\mu}$-almost every $\hat{x}$.

4 (A, ϕ)-invariant measures

We represent by $\phi_{\hat{x},\hat{y}}$ the expression of the unstable holonomies $\phi^u_{\hat{x},\hat{y}}$ in the new coordinates, introduced in Section 1.4, that render $\hat{A}$ constant on local stable manifolds. Of course, every *local* stable holonomy is the identity in these coordinates. We write $p = P(\hat{p})$ and $z = P(\hat{z})$ and, for simplicity, $\psi_{p,z} = \psi_{\hat{p},\hat{z}}$ and $\phi_{p,z} = \phi_{\hat{p},\hat{z}}$. Then (3) becomes

$$\psi_{p,z} = A^{-l}(p) \circ A^l(z) \circ \phi_{p,z} \,. \tag{12}$$

Let $\pi : \Sigma_T \times \mathbb{CP}^{d-1} \to \Sigma_T$ and $\hat{\pi} : \hat{\Sigma}_T \times \mathbb{CP}^{d-1} \to \hat{\Sigma}_T$ be the canonical projections onto the first factor.

Definition 4.1. A probability measure $\hat{m}$ in $\hat{\Sigma}_T \times \mathbb{CP}^{d-1}$ is *ϕ-invariant* if $\hat{\pi}_* \hat{m} = \hat{\mu}$ and there exists a disintegration $(\hat{m}_{\hat{x}})_{\hat{x}}$ of $\hat{m}$ along the projective fibers $\hat{\pi}^{-1}(x) = \{\hat{x}\} \times \mathbb{CP}^{d-1}$ satisfying

$$(\phi_{\hat{x},\hat{y}})_* \, \hat{m}_{\hat{x}} = \hat{m}_{\hat{y}} \tag{13}$$

for every $\hat{x}$ and $\hat{y}$ in the same local unstable manifold. We say that $\hat{m}$ is *(A, ϕ)-invariant* if, in addition, it is invariant under $\hat{f}_{\hat{A}}$.

We say that a probability measure m in $\Sigma_T \times \mathbb{CP}^{d-1}$ is *ϕ-invariant* if there exists a ϕ-invariant probability measure $\hat{m}$ in $\hat{\Sigma}_T \times \mathbb{CP}^{d-1}$ such that $m = (P \times \mathrm{id})_* \hat{m}$. Note that this implies $\pi_* m = \mu$. We say that m is *(A, ϕ)-invariant* if, in addition, $\hat{m}$ may be taken (A, ϕ)-invariant. In that case m is f_A-invariant. Note also that if $\hat{m}$ is a (A, ϕ)-invariant probability then (13) holds for every $\hat{x}$ and $\hat{y}$ in the same (global) unstable manifold. We are going to prove the following two propositions:

Proposition 4.2. *There exists some (A, ϕ)-invariant measure.*

Proposition 4.3. *Any (A, ϕ)-invariant measure m in $\Sigma_T \times \mathbb{CP}^{d-1}$ admits a disintegration into conditional measures along the projective fibers $(m_x)_{x \in \Sigma_T}$ that vary continuously with x in the weak topology: for every continuous function $g : \mathbb{CP}^{d-1} \to \mathbb{R}$ the function $x \mapsto \int g \, dm_x$ is continuous on Σ_T.*

4.1 Existence of (A, ϕ)-invariant measures

The key step in the proof of Proposition 4.2 is the following compactness property:

Proposition 4.4. *The space $\mathcal{M}_\phi$ of ϕ-invariant probabilities measures $\hat{m}$ in $\hat{\Sigma}_T \times \mathbb{CP}^{d-1}$ is non-empty, convex, and compact relative to the weak topology.*

The idea of the proof is quite simple. Fix a point x_i in each cylinder $[0; i]$, $1 \le i \le d$ of Σ_T and let $\mathcal{M}_i$ be the space of probability measures on $W^s_{loc}(x_i) \times \mathbb{CP}^{d-1}$ which are sent down to $\hat{\mu}_{x_i}$ by the restriction of $\hat{\pi}$. Each $\mathcal{M}_i$ is non-empty, convex, and compact relative to the weak topology. To any $(\lambda^1, \ldots, \lambda^d) \in \mathcal{M}_1 \times \cdots \times \mathcal{M}_d$ we may associate an element $\hat{m}$ of $\mathcal{M}_\phi$ simply by lifting each $\hat{m}_i$ to the cylinder $[0; i]$ along local unstable holonomies. We prove in Lemma 4.6 that this correspondence is a homeomorphism onto $\mathcal{M}_\phi$. Proposition 4.4 is an immediate consequence.

Let us now give the detailed arguments. For each x and y in the same cylinder $[0; i]$ let $\Phi_{x,y} : W^s_{loc}(x) \times \mathbb{CP}^{d-1} \to W^s_{loc}(y) \times \mathbb{CP}^{d-1}$ be the unstable holonomy map defined by $\Phi_{x,y}(\hat{x}, \xi) = (\hat{y}, \eta)$ with $\hat{y} = h_{x,y}(\hat{x})$ and $\eta = \phi_{\hat{x},\hat{y}}(\xi)$. Let $\hat{m}$ be any ϕ-invariant measure and $(\hat{m}_{\hat{x}})_{\hat{x} \in \hat{\Sigma}_T}$ be a disintegration of $\hat{m}$ along the projective fibers as in Definition 4.1. For each $x \in \Sigma_T$ define the skew-product $\hat{m}_x = \hat{\mu}_x \times \hat{m}_{\hat{x}}$, that is, $\hat{m}_x$ is the measure on $W^s_{loc}(x) \times \mathbb{CP}^{d-1}$ whose projection under the $\hat{\pi} \mid W^s_{loc}(x) \times \mathbb{CP}^{d-1}$ coincides with $\hat{\mu}_x$ and which admits $(\hat{m}_{\hat{x}})_{\hat{x} \in W^s_{loc}(x)}$ as a disintegration along the fibers.

Lemma 4.5. *The family $(\hat{m}_x)_{x \in \Sigma_T}$ is a disintegration of $\hat{m}$ along the sets $W^s_{loc}(x) \times \mathbb{CP}^{d-1}$ and it satisfies*

$$\hat{m}_y = J_{x,y} (\Phi_{x,y})_* \hat{m}_x$$

for every x and y in the same cylinder of Σ_T.

Proof. The first claim in the lemma is a simple consequence of the definitions. Let $g : \hat{\Sigma}_T \times \mathbb{CP}^{d-1} \to \mathbb{R}$ be any continuous function. Write the generic point of $\hat{\Sigma}_T \times \mathbb{CP}^{d-1}$ as $(x, \hat{x}, \xi)$ with $x \in \Sigma_T$, $\hat{x} \in W^s_{loc}(x)$, and $\xi \in \{\hat{x}\} \times \mathbb{CP}^{d-1}$. Then

$$\begin{aligned}\int g\, d\hat{m}(x,\hat{x},\xi) &= \int\int g\, d\hat{\mu}(x,\hat{x})\, d\hat{m}_{\hat{x}}(\xi) \\ &= \int\int\int g\, d\mu(x)\, d\hat{\mu}_x(\hat{x})\, d\hat{m}_{\hat{x}}(\xi) = \int\int g\, d\mu(x)\, d\hat{m}_x(\hat{x},\xi)\end{aligned}$$

as claimed. To prove the second claim, let $g : W^s_{loc}(y) \times \mathbb{CP}^{d-1} \to \mathbb{R}$ be any continuous function. Then, making the change of variables $\hat{y} = h_{x,y}(\hat{x})$ and $\eta = \phi_{\hat{x},\hat{y}}(\xi)$,

$$\begin{aligned}\int g\, d\hat{m}_y &= \int d\hat{\mu}_y(\hat{y}) \int g(\hat{y},\eta)\, d\hat{m}_{\hat{y}}(\eta) \\ &= \int J_{x,y}(\hat{x})\, d\hat{\mu}_x(\hat{x}) \int g(h_{x,y}(\hat{x}), \phi_{\hat{x},\hat{y}}(\xi))\, d\hat{m}_{\hat{x}}(\xi) \\ &= \int J_{x,y}(\hat{x})(g \circ \Phi_{x,y})(\hat{x},\xi)\, d\hat{m}_x(\hat{x},\xi),\end{aligned}$$

using $J_{x,y}\,(h_{x,y})_*\hat{\mu}_x = \hat{\mu}_y$ (Lemma 2.1) and the assumption $(\phi_{\hat{x},\hat{y}})_*\hat{m}_{\hat{x}} = \hat{m}_{\hat{y}}$. This completes the proof. □

Given $(\lambda^1, \dots, \lambda^d)$ as before, let $(\lambda^i_{\hat{x}})_{\hat{x} \in W^s_{loc}(x_i)}$ be a disintegration of λ^i along the fibers. Then define $\hat{m} = \Psi(\lambda^1, \dots, \lambda^d)$ to be the probability measure in $\hat{\Sigma}_T \times \mathbb{CP}^{d-1}$ whose projection under $\hat{\pi}$ coincides with $\hat{\mu}$ and which admits as conditional measures along the fibers $\{\hat{x}\} \times \mathbb{CP}^{d-1}$ the probabilities given by

$$\hat{m}_{\hat{x}} = (\phi_{\hat{x}_i,\hat{x}})_*\lambda^i_{\hat{x}_i} \quad \text{for} \quad \hat{x} \in [0; i]$$

where $\hat{x}_i$ represents the point in $W^s_{loc}(x_i) \cap W^u_{loc}(\hat{x})$.

Lemma 4.6. *The map $\Psi : \mathcal{M}_1 \times \cdots \times \mathcal{M}_d \to \mathcal{M}_\phi$ is a homeomorphism.*

Proof. We begin by checking that Ψ is well defined: it is clear that the measure $\hat{m} = \Psi(\lambda^1, \dots, \lambda^d)$ is in $\mathcal{M}_\phi$, but we should explain why it does not depend on the choice of the disintegrations. For this, let $(\tilde{\lambda}^i_{\hat{x}})_{\hat{x} \in W^s_{loc}(x_i)}$ be any other disintegration of λ^i along the fibers. By essential uniqueness, we have $\tilde{\lambda}^i_{\hat{x}_i} = \lambda^i_{\hat{x}_i}$ for $\hat{\mu}_{x_i}$-almost every $\hat{x}_i \in W^s_{loc}(x_i)$. Then, using Lemma 2.1,

$$(\phi_{\hat{x}_i,\hat{x}})_* \tilde{\lambda}^i_{\hat{x}_i} = (\phi_{\hat{x}_i,\hat{x}})_* \lambda^i_{\hat{x}_i}$$

for $\hat{\mu}_x$-almost every $\hat{x} \in W^s_{loc}(x)$ and every x in the cylinder $[0; i]$. This implies that the equality is true for $\hat{\mu}$-almost every $\hat{x} \in [0; i]$, and so the two expressions do define same measure.

Next we check that Ψ is continuous, relative to the weak topologies in the two spaces. Let $\lambda^i(k)$ be sequences of measures converging to some $\lambda^i \in \mathcal{M}_i$ and let $\hat{m}(k) = \Psi(\lambda^1(k), \dots, \lambda^d(k))$. By Lemma 4.5, we have

$$\hat{m}(k)_x = J_{x_i,x}(\Phi_{x_i,x})_* \hat{m}(k)_{x_i} = J_{x_i,x}(\Phi_{x_i,x})_* \lambda^i(k)$$

for every $x \in [0; i]$, $1 \le i \le d$, and $k \ge 1$, and analogously for $\hat{m} = \Psi(\lambda^1, \dots, \lambda^d)$. Since the Jacobian J and the holonomy Φ depend continuously on all the variables, this implies that $\hat{m}(k)_x$ converges to $\hat{m}_x$ in the weak topology, uniformly on $x \in \Sigma_T$. Consequently, $\hat{m}(k)$ converges to $\hat{m}$.

Finally, we check Ψ is bijective. Let $\Psi(\lambda^1, \dots, \lambda^d) = \hat{m} = \Psi(\eta^1, \dots, \eta^d)$. Then, by Lemma 4.5,

$$J_{x_i,x}(\Phi_{x_i,x})_* \lambda^i = \hat{m}_x = J_{x_i,x}(\Phi_{x_i,x})_* \eta^i$$

for all x in every cylinder $[0; i]$. Fixing any such x we obtain that

$$\lambda^i = (1/J_{x_i,x})(\Phi_{x,x_i})_* \hat{m}_x = \eta^i\,,$$

which proves injectivity. To prove surjectivity, let $\hat{m} \in \mathcal{M}_\phi$ and $(\hat{m}_x)_{x \in \Sigma_T}$ be the disintegration given by Lemma 4.5. Take $\lambda^i = \hat{m}_{x_i}$ for each $1 \le i \le d$. Recall that $\hat{\pi}_* \hat{m}_x = \hat{\mu}_x$ for every x, by the definition of these measures. In particular, $\hat{\pi}_* \lambda^i = \hat{\mu}_{x_i}$ and so $\lambda^i \in \mathcal{M}_i$. Let $\tilde{m} = \Psi(\lambda^1, \dots, \lambda^d) \in \mathcal{M}_\phi$. By Lemma 4.5, the conditional measures $\tilde{m}_x$ and $\hat{m}_x$ at any point $x \in [0; i]$ are determined by the corresponding value at x_i, which is λ^i in both cases. It follows that $\tilde{m}_x = \hat{m}_x$ for every x, and so $\tilde{m} = \hat{m}$. □

Proof of Proposition 4.4. The claim that $\mathcal{M}_\phi$ is non-empty and compact is an immediate consequence of Lemma 4.6. Moreover, given any $\alpha, \beta > 0$ with $\alpha + \beta = 1$,

$$\Psi(\alpha\lambda^1 + \beta\theta^1, \dots, \alpha\lambda^d + \beta\theta^d) = \alpha\Psi(\lambda^1, \dots, \lambda^d) + \beta\Psi(\theta^1, \dots, \theta^d)$$

and convexity also follows immediately. □

Proof of Proposition 4.2. Let $\bar{m}$ be any ϕ-invariant probability measure in $\hat{\Sigma}_T \times \mathbb{CP}^{d-1}$. By compactness, the sequence

$$\hat{m}_n = \frac{1}{n}\sum_{j=0}^{n-1}(\hat{f}_{\hat{A}}^j)_*\bar{m}.$$

has weak accumulation points $\hat{m} \in \mathcal{M}_\phi$. Since $\hat{f}_{\hat{A}}$ is continuous, any such accumulation point is $\hat{f}_{\hat{A}}$-invariant and, hence, (A, ϕ)-invariant. □

4.2 Continuity of conditional measures

In order to prove Proposition 4.3 we need the following simple fact:

Lemma 4.7. *Let $\hat{m}$ be an $\hat{f}$-invariant measure on $\hat{\Sigma}_T \times \mathbb{CP}^{d-1}$ with $\hat{\pi}_*\hat{m} = \hat{\mu}$, and let $m = (P \times \mathrm{id})_*\hat{m}$. If $(\hat{m}_{\hat{x}})_{\hat{x}}$ is a disintegration of $\hat{m}$ relative to the partition $\{\hat{\pi}^{-1}(\hat{x}) : \hat{x} \in \hat{\Sigma}_T\}$ and $(\hat{\mu}_x)_x$ is a disintegration of $\hat{\mu}$ relative to $\{W^s_{loc}(x) : x \in \Sigma_T\}$ then*

$$m_x = \int \hat{m}_{\hat{x}}\, d\hat{\mu}_x(\hat{x})$$

defines a disintegration of m relative to $\{\pi^{-1}(x) : x \in \Sigma_T\}$.

Proof. For any $\varphi : \Sigma_T \times \mathbb{CP}^{d-1} \to \mathbb{R}$ and $\hat{\varphi} = \varphi \circ (P \times \mathrm{id})$,

$$\begin{aligned}
\int\int \varphi\, dm_x\, d\mu(x) &= \int\int\Big(\int \varphi(x, v)\, d\hat{m}_{\hat{x}}(v) d\hat{\mu}_x(\hat{x})\Big) d\mu(x) \\
&= \int\int\Big(\int \hat{\varphi}(x, v)\, d\hat{m}_{\hat{x}}(v)\Big) d\hat{\mu}_x(\hat{x})\, d\mu(x) \\
&= \int\Big(\int \hat{\varphi}(x, v)\, d\hat{m}_{\hat{x}}(v)\Big)\, d\hat{\mu}(\hat{x}) = \int \hat{\varphi} d\hat{m} = \int \varphi dm
\end{aligned}$$

and this proves that $(m_x)_x$ is a disintegration of m. □

Proof of Proposition 4.3. Let $(\hat{m}_{\hat{x}})_{\hat{x}}$ be a disintegration of $\hat{m}$ as in Definition 4.1, $(\hat{\mu}_x)$ be a disintegration of $\hat{\mu}$ as in Lemma 2.1, and $(m_x)_x$ be the disintegration of m given by Lemma 4.7: $\int g\, dm_y = \int\int g\, d\hat{m}_{\hat{y}}\, d\hat{\mu}_y(\hat{y})$ for every continuous $g : \mathbb{CP}^{d-1} \to \mathbb{R}$ and every $y \in \Sigma_T$. Let x be in the same cylinder as y. Changing variables $\hat{y} = h_{x,y}(\hat{x})$,

$$\int g\, dm_y = \int\int \Big(g \circ \phi_{\hat{x},\hat{y}}\Big) d\hat{m}_{\hat{x}}\, J_{x,y}(\hat{x})\, d\hat{\mu}_x(\hat{x}),$$

by Lemma 2.1 and the definition of (A,ϕ)-invariant measure. Therefore,

$$|\int g\,dm_y - \int g\,dm_x| \le \int\int |(g\circ\phi_{\hat{x},\hat{y}})J_{x,y}(\hat{x}) - g|\,d\hat{m}_{\hat{x}}\,d\hat{\mu}_x(\hat{x}).$$

Lemma 2.1 and Proposition 1.2 imply that $\|J_{x,y}-1\|_0$ and $\|\phi_{\hat{x},\hat{y}}-\mathrm{id}\|_0$ are close to zero if $d(x,y)$ is close to zero. Thus, given any $\varepsilon>0$ there exists $\delta>0$ such that $d(x,y)<\delta$ implies $\|(g\circ\phi_{\hat{x},\hat{y}})J_{x,y}(\hat{x})-g\|_0<\varepsilon$, and so $|\int g\,dm_y - \int g\,dm_x|<\varepsilon$. □

Remark 4.8. When m is (A,ϕ)-invariant, we may choose a continuous disintegration and then (8) is valid for *every* $x\in\Sigma_T$.

5 Invariant measures of 1-typical cocycles

Proposition 5.1. *Suppose A is 1-typical. Let m be any (A,ϕ)-invariant measure and $(m_x)_x$ be a continuous disintegration of m, as in Proposition 4.3. Then $m_x(V)=0$ for every $x\in\Sigma_T$ and any projective subspace V of $\pi^{-1}(x)$.*

Proof. The proof is by contradiction. Suppose there were ℓ, some $x\in\Sigma_T$, and some ℓ-dimensional projective subspace of $\pi^{-1}(x)$ with positive m_x-measure. Fix such an ℓ minimum. Let γ_0 be the supremum of all $\gamma\in(0,1]$ such that $m_x(V)\ge\gamma$ for some $x\in\Sigma_T$ and some ℓ-dimensional projective subspace V. The supremum is attained, because the conditional measures m_x vary continuously with x, and the space of ℓ-dimensional projective subspaces is compact. More than that,

Lemma 5.2. *For every $x\in\Sigma_T$ there exists some ℓ-dimensional projective subspace V such that $m_x(V)=\gamma_0$. Besides, $m_x(V)=\gamma_0$ if and only if $m_y(A(y)^{-1}V)=\gamma_0$ for every $y\in f^{-1}(x)$.*

Proof. Let Γ_0 be the set of points $x\in\Sigma_T$ such that $m_x(V)=\gamma_0$ for some ℓ-dimensional projective subspace V. Using continuity of the conditional measures (Proposition 4.3) and compactness of the set of ℓ-dimensional subspaces, we conclude that Γ_0 is closed in Σ_T. Moreover, by Remark 4.8,

$$m_x(V) = \sum_{f(y)=x} \frac{1}{J_\mu f(y)}\, m_y(A(y)^{-1}V)$$

for all $x \in \Sigma_T$. Since $\sum_{f(y)=x} 1/J_\mu f(y) = 1$ at every point, and γ_0 is the maximum measure of any ℓ-dimensional subspace, we get that $m_x(V) = \gamma_0$ if and only if $m_y(A(y)^{-1}V) = \gamma_0$ for every $y \in f^{-1}(x)$, as stated. It follows that Γ_0 is f-invariant: $x \in \Gamma_0$ if and only if $f^{-1}(x) \subset \Gamma_0$. Since the subshift $f : \Sigma_T \to \Sigma_T$ is transitive, full backward orbits are dense in Σ_T. Thus, any non-empty closed invariant subset coincides with Σ_T. Hence, $\Gamma_0 = \Sigma_T$. □

Let $(\hat{\mu}_x)_x$ be a disintegration of $\hat{\mu}$ as in Lemma 2.1, and $(\hat{m}_{\hat{x}})_{\hat{x}}$ be a disintegration of $\hat{m}$ as in Definition 4.1. Since $\hat{m}$ is $\hat{f}_{\hat{A}}$-invariant,

$$\hat{m}_{\hat{f}^n(\hat{y})} = A^n(y)_* \hat{m}_{\hat{y}} \tag{14}$$

for all $n \geq 1$ and $\hat{\mu}$-almost all $\hat{y} \in \Sigma_T$. Actually, (14) is true for $\hat{\mu}_y$-almost every $\hat{y} \in W^s_{loc}(y)$ and *every* $y \in \Sigma_T$. Indeed, given any $y \in \Sigma_T$ we may approximate it by some $z \in \Sigma_T$ such that (14) holds $\hat{\mu}_z$-almost everywhere in $W^s_{loc}(z)$. Using that the holonomy $h_{z,y}$ is absolutely continuous (Lemma 2.1), the conditional measures of $\hat{m}$ are preserved by the family of maps $\{\phi_{\hat{y},\hat{z}}\}$ (Definition 4.1), and the latter commute with the cocycle (Proposition 1.2), we conclude that (14) holds for $\hat{\mu}_y$-almost every point in $W^s_{loc}(y)$.

Lemma 5.3. *Given $x \in \Sigma_T$ and any ℓ-dimensional projective subspace V, we have $\hat{m}_{\hat{x}}(V) \leq \gamma_0$ for $\hat{\mu}_x$-almost every $\hat{x} \in W^s_{loc}(x)$. Thence, $m_x(V) = \gamma_0$ if and only if $\hat{m}_{\hat{x}}(V) = \gamma_0$ for $\hat{\mu}_x$-almost every $\hat{x} \in W^s_{loc}(x)$.*

Proof. Suppose there was V, $x \in \Sigma_T$, $\gamma_1 > \gamma_0$, and a positive $\hat{\mu}$-measure subset X of $W^s_{loc}(x)$ such that $\hat{m}_{\hat{x}}(V) \geq \gamma_1$ for every $\hat{x} \in X$. For each $n \geq 1$, let us consider the partition

$$\big\{\hat{f}^n(W^s_{loc}(y)) : y \in f^{-n}(x)\big\}.$$

of the local stable manifold of x into $\hat{f}^n$-images of local stable manifolds. The diameters of these partitions go to zero as $n \to \infty$. Thus, by regularity of the measure $\hat{\mu}_x$, given any $\varepsilon > 0$ we may find $n \geq 1$ and $y \in f^{-n}(x)$ such that

$$\hat{\mu}_x\big(X \cap \hat{f}^n(W^s_{loc}(y))\big) \geq (1-\varepsilon)\hat{\mu}_x\big(\hat{f}^n(W^s_{loc}(y))\big).$$

Fix $\varepsilon > 0$ small enough that $(1-\varepsilon)\gamma_1 > \gamma_0$. Using (14), and excluding a zero $\hat{\mu}_x$-measure subset of X if necessary,

$$\hat{m}_{\hat{y}}(A^n(y)^{-1}V) = \hat{m}_{\hat{f}^n(\hat{y})}(V) \geq \gamma_1$$

for every $\hat{y} \in \hat{f}^{-n}(X) \cap W^s_{loc}(y)$. From (6) we obtain

$$\hat{\mu}_y\big(\hat{f}^{-n}(X) \cap W^s_{loc}(y)\big) = J_\mu f^n(y)\, \hat{\mu}_x\big(X \cap \hat{f}^n(W^s_{loc}(y))\big)$$
$$\geq (1-\varepsilon)\, J_\mu f^n(y)\, \hat{\mu}_x\big(\hat{f}^n(W^s_{loc}(y))\big) = (1-\varepsilon).$$

It follows that

$$m_y(A^n(y)^{-1}V) = \int \hat{m}_{\hat{y}}(A^n(y)^{-1}V)\, d\hat{\mu}_y(\hat{y}) \geq (1-\varepsilon)\gamma_1 > \gamma_0\,,$$

which contradicts the definition of γ_0. This contradiction proves the first part of the lemma.

The second one is a consequence of the first, and the fact that $m_x(V)$ is the $\hat{\mu}_x$-average of all $\hat{m}_{\hat{x}}(V)$. □

Fix p, z, and $l \geq 1$ as in the Definition 1.3 of typical cocycle. Recall that $\phi_{p,z} = \phi_{\hat{p},\hat{z}}$ and $\psi_{p,z} = A^l(z)\phi_{p,z}$, where $\hat{p}$ is the periodic point of $\hat{f}$ projecting to p, and $\hat{z} = \hat{z}(p)$ is the point of the local unstable manifold of $\hat{p}$ projecting to z. Let $q \geq 1$ be the period of p.

By Lemma 5.2, we may find an ℓ-dimensional projective subspace V such that $m_p(V) = \gamma_0$. Define $V^n = A^{-nq}(p)V$ for each $n \geq 0$. Using Lemma 5.2 once more we obtain $m_p(V^n) = \gamma_0$ for all $n \geq 0$. Moreover, since the eigenvalues of $A^q(p)$ have distinct norms, V^n converges to an invariant subspace (sum of eigenspaces) $\tilde{V}$ of $A^q(p)$, with dimension ℓ. It follows that $m_p(\tilde{V}) = \gamma_0$. This means that we may suppose right from the start that V is an invariant subspace of $A^q(p)$. We do so in all that follows.

Define $W = A^l(z)^{-1}V$. From Lemmas 5.2 and 5.3 we get $m_z(W) = \gamma_0$ and $\hat{m}_{\hat{z}}(W) = \gamma_0$ for $\hat{\mu}_z$-almost every $\hat{z} \in W^s_{loc}(z)$. For each $\hat{w} \in W^s_{loc}(p)$, denote $\hat{z} = h_{p,z}(\hat{w})$ and $V_{\hat{w}} = \phi_{\hat{z},\hat{w}}W$. It is clear that each $V_{\hat{w}}$ is an ℓ-dimensional projective subspace and depends continuously on the point $\hat{w}$. Moreover, by Definition 4.1, $\hat{m}_{\hat{w}}(V_{\hat{w}}) = \hat{m}_{\hat{z}}(W) = \gamma_0$ for $\hat{\mu}_p$-almost every $\hat{w}$. For each $j \geq 0$, let $V^j_{\hat{w}} = A^{-jq}(p)V_{\hat{f}^{jq}(\hat{w})}$. Using (14) we deduce that $\hat{m}_{\hat{w}}(V^j_{\hat{w}}) = \gamma_0$ for every $j \geq 0$ and $\hat{\mu}_p$-almost every $\hat{w}$.

In the next lemma we denote $U_n = \hat{f}^{nq}\big(W^s_{loc}(p)\big)$. The lemma says that, for a sizable fraction of points $\hat{w}$ close to $\hat{p}$ in $W^s_{loc}(p)$, two spaces $V^j_{\hat{w}}$ and $V^k_{\hat{w}}$ either coincide or their intersection has small measure.

Lemma 5.4. *Given $0 \leq j < k$, $\varepsilon > 0$, $\delta > 0$, there exists $n_0 \geq 1$ such that for all $n \geq n_0$*

$$\hat{\mu}_p\big(\{\hat{w} \in U_n : \dim(V^j_{\hat{w}} \cap V^k_{\hat{w}}) < \ell \ \text{ and } \ \hat{m}_{\hat{w}}(V^j_{\hat{w}} \cap V^k_{\hat{w}}) \geq \varepsilon\}\big) \leq \delta \hat{\mu}_p(U_n).$$

Proof. Suppose there were $0 \le j < k$, $\varepsilon > 0$, $\delta > 0$ such that

$$\hat{\mu}_p\big(\{\hat{w} \in U_n : \dim(V^j_{\hat{w}} \cap V^k_{\hat{w}}) < \ell \;\text{ and }\; \hat{m}_{\hat{w}}(V^j_{\hat{w}} \cap V^k_{\hat{w}}) \ge \varepsilon\}\big) > \delta \hat{\mu}_p\big(U_n\big)$$

for values of n arbitrarily large. Let $j, k, \varepsilon, \delta$ be fixed. Taking pre-images under the nq:th iterate, and keeping (6) in mind, we get

$$\hat{\mu}_p\big(\{\hat{w} \in W^s_{loc}(p) : \dim V^{j,k,n}_{\hat{w}} < \ell \;\text{ and }\; \hat{m}_{\hat{w}}(V^{j,k,n}_{\hat{w}}) \ge \varepsilon\}\big) > \delta$$

for arbitrarily large n, where $V^{j,k,n}_{\hat{w}} = A^{-nq}(p)(V^j_{\hat{w}} \cap V^k_{\hat{w}})$. Recall that we took V an invariant subspace of $A^q(p)$. Then, by part 2 of the Definition 1.3,

$(*)$ $V_{\hat{p}} = \psi^{-1}_{p,z} V$ does not contain any eigenspace of $A^q(p)$.

By continuity, the same is true for any $V_{\hat{w}}$ with $\hat{w}$ close enough to $\hat{p}$. It follows that, as n goes to infinity, $V^{j,k,n}_{\hat{w}}$ converges (uniformly on a neighbourhood of $\hat{p}$) to an invariant space $\tilde{V}$ of $A^q(p)$, corresponding to the smallest eigenvalues. By construction, $\dim \tilde{V} < \ell$ and $\hat{m}_{\hat{w}}(\tilde{V}) \ge \varepsilon$ for a subset of $\hat{w} \in W^s_{loc}(p)$ with $\hat{\mu}_p$-measure larger than δ. Restricting to a positive $\hat{\mu}_p$-measure subset of $\hat{w}$ for which the dimension of $\tilde{V}$ is constant, we have that $\tilde{V}$ itself is independent of $\hat{w}$ on that subset. It follows that $m_p(\tilde{V}) > 0$, which contradicts the choice of ℓ. This contradiction proves Lemma 5.4. □

Now we may conclude the proof of Proposition 5.1. Let N be an integer satisfying $N > 2/\gamma_0$. Fix $\varepsilon = \gamma_0/N$ and $\delta > 0$ arbitrary. Let n_0 be the largest of all integers provided by Lemma 5.4, over all $0 \le j < k < N$. Then, for any $n \ge n_0$ there exists a subset E_n of U_n with

$$\hat{\mu}_p(E_n) \ge (1-\delta)\hat{\mu}_p(U_n) > 0$$

such that, for any $\hat{w} \in E_n$ and $0 \le j < k < N$, either (a) $\dim(V^j_{\hat{w}} \cap V^k_{\hat{w}}) = \ell$ or (b) $\hat{m}_{\hat{w}}(V^j_{\hat{w}} \cap V^k_{\hat{w}}) < \varepsilon$. Moreover, given any $\hat{w}$, alternative (b) can not be true for *all* choices of j, k. Indeed, that would imply

$$\begin{aligned} \hat{m}_{\hat{w}}\big(\hat{\pi}^{-1}(\hat{w})\big) &\ge \hat{m}_{\hat{w}}\big(\cup_{i=0}^{N-1} V^i_{\hat{w}}\big) \ge \sum_{0<i<N} \hat{m}_{\hat{w}}\big(V^i_{\hat{w}}\big) - \sum_{0 \le j<k<N} \hat{m}_{\hat{w}}\big(V^j_{\hat{w}} \cap V^k_{\hat{w}}\big) \\ &> N\gamma_0 - (N^2/2)\varepsilon > N\gamma_0/2 > 1, \end{aligned}$$

which is a contradiction. Thus, for every $\hat{w} \in E_n$ there exist $0 \le j < k < N$ such that $\dim(V^j_{\hat{w}} \cap V^k_{\hat{w}}) = \ell$, that is, $V^j_{\hat{w}} = V^k_{\hat{w}}$. Since we are dealing with a finite number of pairs (j,k), we may fix $0 \le j < k$ such that $V_{\hat{w}}^j = V_{\hat{w}}^k$ for a positive $\hat{\mu}_p$-measure (hence non-empty) subset of every U_n. By continuity, making $\hat{w} \to \hat{p}$, we obtain

$$V_{\hat{p}}^j = V_{\hat{p}}^k \quad \text{that is,} \quad A^{(k-j)q}(p) V_{\hat{p}} = V_{\hat{p}} \, .$$

This implies that $V_{\hat{p}}$ is an invariant subset (sum of eigenspaces) of $A^q(p)$, which contradicts $(*)$ above. This contradiction proves that there is no projective subspace with positive m_x-measure, for any point $x \in \Sigma_T$, as claimed in Proposition 5.1. □

6 Convergence to a Dirac measure

In this section we prove Theorem 3. We begin by recalling the following useful notion, that was introduced by Furstenberg [18].

6.1 Quasi-projective transformations

Let $v \mapsto [v]$ be the natural projection from $\mathbb{C}^d \setminus \{0\}$ to $\mathbb{CP}^{d-1}$. A map $P : \mathbb{CP}^{d-1} \to \mathbb{CP}^{d-1}$ is *projective* if there is $\tilde{P} \in \mathrm{GL}(d, \mathbb{C})$ that induces P through $P([v]) = [\tilde{P}(v)]$. We are going to embed the space of projective maps of $\mathbb{CP}^{d-1}$ into a larger space of *quasi-projective* transformations. The quasi-projective transformation Q induced by a non-zero, possibly non-invertible, linear map $\tilde{Q} : \mathbb{C}^d \to \mathbb{C}$ is given by $Q([v]) = [\tilde{Q}(v_1)]$ where v_1 is any vector such that $v - v_1$ is in $\ker \tilde{Q}$. Observe that Q is defined and continuous on the complement of the projective subspace $\ker Q = \{[v] : v \in \ker \tilde{Q}\}$.

The space of quasi-projective transformations inherits a topology from the space of linear maps, through the natural projection $\tilde{Q} \mapsto Q$. Clearly, every quasi-projective transformation Q is induced by some linear map $\tilde{Q}$ such that $\|\tilde{Q}\| = 1$. It follows that the space of quasi-projective transformations is compact for this topology. In particular, every sequence $(P_n)_n$ of projective transformations has a subsequence converging to some quasi-projective transformation Q.

We also point out the following consequence of the definitions, whose proof we leave for the reader: If $(P_n)_n$ is a sequence of projective transformations converging to a quasi-projective transformation Q then $(P_n)_n$ converges uniformly to Q outside every neighbourhood of $\ker Q$.

Lemma 6.1. *If $(P_n)_n$ is a sequence of projective transformations converging to a quasi-projective transformation Q, and $(\nu_n)_n$ is a sequence of probability measures in $\mathbb{CP}^{d-1}$ weakly converging to some probability ν_0 with $\nu_0(\ker Q) = 0$, then $(P_n)_*\nu_n$ converges weakly to $Q_*\nu_0$.*

Proof. Let $(K_m)_m$ denote a basis of neighbourhoods of $\ker Q$ such that we have $\nu_0(\partial K_m) = 0$ for all m. Given any continuous function $\varphi : \mathbb{CP}^{d-1} \to \mathbb{R}$, and given $\varepsilon > 0$, fix $m \geq 1$ large enough so that $\nu_0(K_m) \leq \varepsilon$. Then fix $n_0 \geq m$ so that $\nu_n(K_m) \leq \nu_0(K_m) + \varepsilon \leq 2\varepsilon$,

$$\Big|\int_{K_m^c} (\varphi \circ Q)\, d\nu_n - \int_{K_m^c} (\varphi \circ Q)\, d\nu_0\Big| \leq \varepsilon \quad \text{and} \quad \sup_{K_m^c} \Big|\varphi \circ P_n - \varphi \circ Q\Big| \leq \varepsilon$$

for all $n \geq n_0$. Then, $\big|\int (\varphi \circ P_n)\, d\nu_n - \int (\varphi \circ Q)\, d\nu_0\big|$ is bounded by

$$\Big|\int_{K_m^c} (\varphi \circ P_n - \varphi \circ Q)\, d\nu_n\Big| + \Big|\int_{K_m^c} (\varphi \circ Q)\, d\nu_n - \int_{K_m^c} (\varphi \circ Q)\, d\nu_0\Big| +$$
$$+ \Big|\int_{K_m} (\varphi \circ P_n)\, d\nu_n\Big| + \Big|\int_{K_m} (\varphi \circ Q)\, d\nu_0\Big| \leq 2\varepsilon + 3\varepsilon \sup|\varphi|$$

for all $n \geq n_0$. This proves the lemma. ☐

6.2 Proof of Theorem 3

Having proved Proposition 3.1, at this point we only have to show that for $\hat{\mu}$-almost every $\hat{x} \in \hat{\Sigma}_T$ there exists a subsequence $(n_j)_j$ and a point $\xi(\hat{x}) \in \{\hat{x}_0\} \times \mathbb{CP}^{d-1}$ such that

$$A^{n_j}(\hat{x}_{n_j})_* m_{\hat{x}_{n_j}} \to \delta_{\xi(\hat{x})} \quad \text{when } j \to \infty. \tag{15}$$

We begin by settling a special case, contained in the next lemma.

Let p be a periodic point of f, of period $q \geq 1$, and z be a point in the same cylinder $[0; i]$ of Σ_T that contains p. Let $\hat{p}$ be the unique periodic point in $W^s_{loc}(p)$ and $\hat{z}$ be the point of intersection between $W^u_{loc}(\hat{p})$ and $W^s_{loc}(z)$. As before, denote $\hat{z}_n = P(\hat{f}^{-n}(\hat{z}))$ for $n \geq 0$. Note that $\hat{z}_{qk}$ is in the cylinder $[0; i]$ for all $k \geq 0$. We assume that $A^q(p)$ has a simple eigenvalue with largest norm; recall Definition 1.3. Take $\omega_1 \in \pi^{-1}(p)$ to be the eigenspace corresponding to that largest eigenvalue, and let $\xi(\hat{z}) = \phi_{p,z}(\omega_1)$.

Lemma 6.2. *The sequence $A^{qk}(\hat{z}_{qk})_* m_{\hat{z}_{qk}}$ converges to $\delta_{\xi(\hat{z})}$ when $k \to \infty$.*

Proof. Recall that $\phi_{p,z} = \phi_{\hat{p},\hat{z}} = \lim_{n\to\infty} \hat{A}^{-qn}(\hat{z})^{-1}\hat{A}^{-qn}(\hat{p})$. Using the relations $A^{qk}(p)^{-1} = \hat{A}^{-qk}(\hat{p})$ and $A^{qk}(\hat{z}_{qk})^{-1} = \hat{A}^{-qk}(\hat{z})$, we find that

$$A^{qk}(\hat{z}_{qk})_* m_{\hat{z}_{qk}} = \left(\hat{A}^{-qk}(\hat{z})^{-1}\hat{A}^{-qk}(\hat{p})\right)_* A^{qk}(p)_* m_{\hat{z}_{qk}}.$$

On the one hand, $\hat{A}^{-qk}(\hat{z})^{-1}\hat{A}^{-qk}(\hat{p})$ converges to $\phi_{p,z}$ when $k \to \infty$. On the other hand, $A^{qk}(p)_* m_{\hat{z}_{qk}}$ converges to δ_{ω_1} when $k \to \infty$. That is because the eigenvalue corresponding to the eigenspace ω_1 is strictly larger in norm than all the others. Note also that $m_{\hat{z}_{qk}}$ converges to m_p, by Proposition 4.3, and m_p gives zero weight to the sum $\oplus_{j>1}\omega_j$ of all the other eigenspaces, by Proposition 5.1. It follows that $A^{qk}(\hat{z}_{qk})_* m_{\hat{z}_{qk}}$ converges to $(\phi_{p,z})_*\delta_{\omega_1} = \delta_{\xi(\hat{z})}$ when $k \to \infty$, as stated in the lemma. ☐

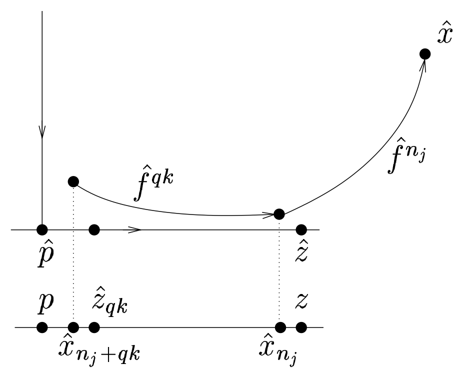

Figure 3: Proof of Theorem 3: case $\xi(\hat{z})$ not in $\ker Q$

Now we are going to propagate this behaviour to almost every point. For $\hat{\mu}$-almost every $\hat{x}$ there exists a sequence $(n_j)_j$ such that $\hat{f}^{-n_j}(\hat{x})$ converges to $\hat{z}$. That is because $\hat{\mu}$ is ergodic, and so almost every orbit is dense in $\operatorname{supp}\hat{\mu} = \hat{\Sigma}_T$. See Figure 3. Since the space of quasi-projective transformations is compact, up to replacing $(n_j)_j$ by a subsequence we may suppose that $A^{n_j}(\hat{x}_{n_j})$ converges to a quasi-projective map Q from $\pi^{-1}(z)$ to $\pi^{-1}(\hat{x}_0)$. Let $k \geq 1$ be fixed. By Proposition 3.1(a),

$$\lim_{j\to\infty} A^{n_j}(\hat{x}_{n_j})_* m_{\hat{x}_{n_j}} = \lim_{j\to\infty} A^{n_j}(\hat{x}_{n_j})_* A^{qk}(\hat{x}_{n_j+qk})_* m_{\hat{x}_{n_j+qk}}. \qquad (16)$$

By construction, $\hat{x}_{n_j+qk}$ converges to $\hat{z}_{qk}$ when $j \to \infty$. Thus, $A^{qk}(\hat{x}_{n_j+qk})$ converges to $A^{qk}(\hat{z}_{qk})$ and, using Proposition 4.3, $m_{\hat{x}_{n_j+qk}}$ converges to $m_{\hat{z}_{qk}}$

when $j \to \infty$. So,

$$A^{qk}(\hat{x}_{n_j+qk})_* m_{\hat{x}_{n_j+qk}} \to A^{qk}(\hat{z}_{qk})_* m_{\hat{z}_{qk}} \quad \text{when } j \to \infty.$$

Now, $A^{n_j}(\hat{x}_{n_j})$ converges to Q. By Proposition 5.1, the subspace $\ker Q$ has zero measure for $A^{qk}(\hat{z}_{qk})_* m_{\hat{z}_{qk}}$. Therefore, we may apply Lemma 6.1 to conclude that

$$\lim_{j\to\infty} A^{n_j}(\hat{x}_{n_j})_* A^{qk}(\hat{x}_{n_j+qk})_* m_{\hat{x}_{n_j+qk}} = Q_* A^{qk}(\hat{z}_{qk})_* m_{\hat{z}_{qk}}, \tag{17}$$

for every $k \geq 1$. This shows that $A^{n_j}(\hat{x}_{n_j})_* m_{\hat{x}_{n_j}}$ converges to $Q_* A^{qk}(\hat{z}_{qk})_* m_{\hat{z}_{qk}}$ when $j \to \infty$, for every $k \geq 1$ (in particular, the latter expression does not depend on k).

Suppose, for the time being, that $\xi(\hat{z})$ is in the domain $\mathbb{CP}^{d-1} \setminus \ker Q$ of the quasi-projective map Q. Denote $\xi(\hat{x}) = Q(\xi(\hat{z}))$. Lemma 6.2 implies that

$$Q_* A^k(\hat{z}_{qk})_* m_{\hat{z}_{qk}} \to Q_* \delta_{\xi(\hat{z})} = \delta_{\xi(\hat{x})} \quad \text{when } k \to \infty. \tag{18}$$

Putting (16), (17), (18) together, we find that $A^{n_j}(x_{n_j})_* m_{x_{n_j}}$ converges to the Dirac measure $\delta_{\xi(\hat{x})}$ when $j \to \infty$. This proves (15) and Theorem 3 in this case.

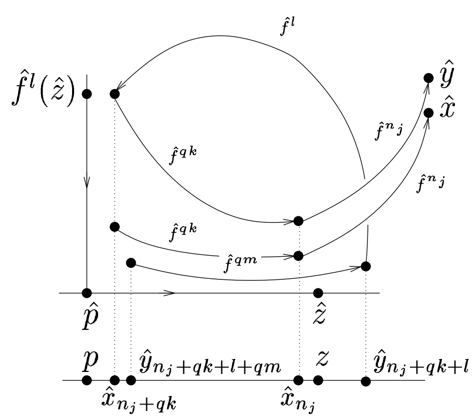

Figure 4: Proof of Theorem 3: avoiding $\ker Q$

Next, we show that one can always reduce the proof to the previous case. It is no restriction to assume that $\hat{z}$ has been taken a homoclinic point associated to $\hat{p}$, and they satisfy the conditions in Definition 1.3. Fix $l \geq 1$ such that $p = f^l(z)$, as before. For each $j \gg k$, let $\hat{y} = \hat{y}(j,k)$ be defined by

$$\hat{f}^{-n_j-qk}(\hat{y}) \in W^s_{loc}(\hat{f}^{-n_j-qk}(\hat{x})) \cap W^u_{loc}(\hat{f}^l(\hat{z})).$$

See Figure 4. The fact that $\hat{y}$ depends on j and k will not be important in the sequel, because the projections to the one-sided shift do not:

$$\hat{f}^{-i}(\hat{y}) \in W^s_{loc}(\hat{f}^{-i}(\hat{x})) \quad \text{and so} \quad \hat{y}_i = \hat{x}_i\,, \quad \text{for all } 0 \leq i \leq n_j + qk. \tag{19}$$

Denote $\hat{w} = \hat{w}(k)$ the point such that $\hat{f}^l(\hat{w}) \in W^s_{loc}(\hat{f}^{-qk}(\hat{z})) \cap W^u_{loc}(\hat{f}^l(\hat{z}))$. As $j \to \infty$ we have $\hat{f}^{-n_j-qk}(\hat{z}) \to \hat{f}^l(\hat{w})$ and so

$$\hat{f}^{-n_j-qk-l-qm}(\hat{y}) \to \hat{f}^{-qm}(\hat{w}) \quad \text{for each fixed } m \text{ (and } k). \tag{20}$$

Let k and m be fixed. By construction, $f^{l+qm}(\hat{y}_{n_j+qk+l+qm}) = \hat{x}_{n_j+qk}$ for every j. Hence, using (16) and Proposition 3.1(b),

$$\begin{aligned}\lim_{j\to\infty} A^{n_j}(\hat{x}_{n_j})_* m_{\hat{x}_{n_j}} &= \lim_{j\to\infty} A^{n_j+qk}(\hat{x}_{n_j+qk})_* m_{\hat{x}_{n_j+qk}} \\ &= \lim_{j\to\infty} A^{n_j+qk+l+qm}(\hat{y}_{n_j+qk})_* m_{\hat{y}_{n_j+qk+l+qm}}\,.\end{aligned}$$

We are going to prove that the third limit is indeed a Dirac measure. Recall that $\hat{y}_{n_j+qk+l} \to w$ as $j \to \infty$, by (20). Thus,

$$A^{n_j+qk+l}(\hat{y}_{n_j+qk+l}) = A^{n_j}(\hat{x}_{n_j}) A^{qk+l}(\hat{y}_{n_j+qk+l})$$

converges to $\tilde{Q} = Q \circ A^{qk+l}(w)$ in the space of quasi-projective transformations, as $j \to \infty$. The key observation is

Lemma 6.3. *Assuming k is large enough, $\xi(\hat{w})$ is not contained in* $\ker \tilde{Q}$.

Assuming this fact for a while, we can complete the proof of the theorem, using the same argument as in the previous particular case, with n_j and z replaced by $m_j = n_j + qk + l$ and $w = P(\hat{w})$, respectively and qm in the role of qk. Indeed, using that $A^{qm}(\hat{y}_{n_j+qk+l+qm})_* m_{\hat{y}_{n_j+qk+l+qm}}$ converges to $A^{qm}(\hat{w}_{qm})_* m_{\hat{w}_{qm}}$ as $j \to \infty$, we find that

$$\lim_{j\to\infty} A^{n_j+qk+l+qm}(\hat{y}_{n_j+qk})_* m_{\hat{y}_{n_j+qk+l+qm}} = \tilde{Q}_* A^{qm}(\hat{w}_{qm})_* m_{\hat{w}_{qm}}$$

for every m, which is the analogue of (17). By Lemma 6.2, the sequence $A^{qm}(\hat{w}_{qm})_* m_{\hat{w}_{qm}}$ converges to $\delta_{\xi(\hat{w})}$ when $m \to \infty$. Hence, using Lemmas 6.1 and 6.3 and arguing as in (18),

$$\lim_{j\to\infty} A^{n_j+qk+l+qm}(\hat{y}_{n_j+qk})_* m_{\hat{y}_{n_j+qk+l+qm}} = \delta_{\xi(\hat{x})}$$

where $\xi(\hat{x}) = \tilde{Q}_* \delta_{\xi(\hat{w})}$. This means that we are left to give the

Proof of Lemma 6.3. Recall that $\hat{w} = \hat{w}(k)$ converges to $\hat{z}$ as $k \to \infty$ and $\hat{f}^l(\hat{w})$ is in the local stable manifold of $\hat{f}^{-qk}(\hat{z})$. On the one hand,

$$\ker \tilde{Q} = A^{kq+l}(w)^{-1}(\ker Q) = A^l(w)^{-1} A^{qk}(\hat{z}_{qk})^{-1}(\ker Q).$$

Notice that

$$A^{qk}(\hat{z}_{qk})^{-1}(\ker Q) = \phi_{\hat{z}_{qk},p} A^{qk}(p)^{-1} \phi_{z,p}(\ker Q).$$

Since the eigenvalues of $A^q(p)$ have distinct norms, $A^{qk}(p)^{-1}\phi_{z,p}(\ker Q)$ converges when $k \to \infty$, and the limit coincides with some sum of eigenspaces of $A^q(p)$:

$$\lim_{k\to\infty} A^{qk}(p)^{-1}\phi_{z,p}(\ker Q) = \oplus_{j\in J}\omega_j$$

for some subset $J \subset \{1,\dots,d\}$ with $\#J < d$ (the cardinal of J coincides with the dimension of $\ker Q$ as a vector subspace). Moreover, $A^l(w)$ converges to $A^l(z)$ and $\phi_{\hat{z}_{qk},p}$ converges to the identity as $k \to \infty$, since $\hat{z}_{qk}$ converges to p. This proves that

$$\ker \tilde{Q} \to A^l(z)^{-1}\big(\oplus_{j\in J}\,\omega_j\big) \quad \text{as } k \to \infty. \tag{21}$$

On the other hand, using (12),

$$\xi(\hat{w}) = A^l(w)^{-1}A^l(p)\psi_{p,w}\omega_1 \to A^l(z)^{-1}A^l(p)\psi_{p,z}\omega_1 \quad \text{as } k \to \infty. \tag{22}$$

Now we claim that

$$A^l(z)^{-1}A^l(p)\psi_{p,z}\omega_1 \notin A^l(z)^{-1}\big(\oplus_{j\in J}\,\omega_j\big). \tag{23}$$

Indeed, otherwise we would have

$$A^l(p)\psi_{p,z}\omega_1 \in \oplus_{j\in J}\omega_j \quad \Leftrightarrow \quad \psi_{p,z}\omega_1 \in \oplus_{j\in J}\omega_j$$

and this would contradict the second condition in Definition 1.3. Relations (21), (22), (23) imply that $\xi(\hat{w})$ is not in $\ker \tilde{Q}$ if k is large enough, as claimed. □

Now the proof of Theorem 3 is complete.

Remark 6.4. Comparing Remark 3.5 and Theorem 3 we get $\hat{m} = \hat{\mu} \times \{\delta_{\xi(\hat{x})}\}$. In Section 8 we shall see that ξ is the direction of the Oseledets subspace corresponding to the largest Lyapunov exponent of $\hat{\mu}$ and, consequently, is uniquely defined almost everywhere. This proves that the (A, ϕ)-invariant measure is unique, if the cocycle is 1-typical.

Remark 6.5. From the expression $\xi(\hat{x}) = \tilde{Q}(\phi_{p,w}\omega_1)$ we easily deduce that the direction $\xi(\,\cdot\,)$ is invariant under the family of unstable holonomies $\phi_{\hat{x},\hat{x}'}$. Indeed, for every point $\hat{x}'$ in the local unstable manifold of $\hat{x}$, we may choose the same sequence $(n_j)_j$ as for $\hat{x}$. Then $Q' = \lim A^{n_j}(\hat{x}'_{n_j})$ coincides with $\phi_{\hat{x},\hat{x}'} \circ Q$, because both $(\hat{x}_{n_j})_j$ and $(\hat{x}'_{n_j})_j$ converge to the point z.

7 Direction of strongest expansion

We shall see that $\hat{x} \mapsto \xi(\hat{x})$ defines the direction of the Oseledets subspace of the cocycle corresponding to the largest Lyapunov exponent. To prove Theorem 1, we need to exhibit the hyperplane corresponding to the other Lyapunov exponents. It is useful to introduce the adjoint cocycle of $\hat{A}$, which describes the action of $\hat{A}$ on linear forms and, thus, hyperplanes of $\mathbb{C}^d$.

7.1 The adjoint cocycle

For $\hat{x} \in \Sigma_T$ let $\hat{A}_*(\hat{x}) : (\mathbb{C}^d)^* \to (\mathbb{C}^d)^*$ be the adjoint operator of $\hat{A}(\hat{f}^{-1}(\hat{x}))$, defined by

$$\big(\hat{A}_*(\hat{x})u\big)v = u\big(\hat{A}(\hat{f}^{-1}(\hat{x}))v\big) \quad \text{for each } u \in (\mathbb{C}^d)^* \text{ and } v \in \mathbb{C}^d. \tag{24}$$

For notational simplicity we fix some Riemannian metric $\cdot$ on $\mathbb{C}^d$ and identify the dual space $(\mathbb{C}^d)^*$ with $\mathbb{C}^d$ through that metric. Then we may consider $\hat{A}_* : \hat{\Sigma}_T \to \mathrm{SL}(d, \mathbb{C})$, and (24) becomes

$$\hat{A}_*(\hat{x})u \cdot v = u \cdot \hat{A}(\hat{f}^{-1}(\hat{x}))v \quad \text{for each } u \in \mathbb{C}^d \text{ and } v \in \mathbb{C}^d.$$

The *adjoint cocycle* of $\hat{A}$ is the projective cocycle generated by $\hat{A}_*$ over the transformation $\hat{f}^{-1} : \hat{\Sigma}_T \to \hat{\Sigma}_T$:

$$\hat{f}^*_{\hat{A}} = \big(\hat{f}^{-1}\big)_{\hat{A}_*} : \hat{\Sigma}_T \times \mathbb{CP}^{d-1} \to \hat{\Sigma}_T \times \mathbb{CP}^{d-1}$$

Remark 7.1. The choice of the Riemannian metric is unimportant: different metrics yield adjoint operators that are conjugate.

Lemma 7.2. *1. $\hat{A}_*$ is dominated if and only if $\hat{A}$ is dominated.*

2. The stable/unstable holonomies of $\hat{f}^*_{\hat{A}}$ are adjoint to the unstable/stable holonomies of $\hat{f}_{\hat{A}}$: $\phi^{*,u}_{\hat{x},\hat{z}} = \left(\phi^s_{\hat{x},\hat{z}}\right)_*$ *and* $\phi^{*,s}_{\hat{x},\hat{y}} = \left(\phi^u_{\hat{x},\hat{y}}\right)_*$.

3. $\hat{A}$ is 1-typical if and only if $\hat{A}_*$ is 1-typical.

Proof. Fix a basis of $\mathbb{C}^d$ orthogonal for the Riemannian metric chosen above. Relative to this basis, the matrix of $\hat{A}_*$ is the transposed of the matrix of $\hat{A}$. Hence, $\hat{A}$ is ν-Hölder if and only if $\hat{A}_*$ is. Moreover, the definition of the adjoint implies $\|\hat{A}\| = \|\hat{A}_*\|$ and the same for their inverses. This gives the second property in Definition 1.1, and so it proves the first claim in the lemma.

The second claim is a consequence of the expression of the holonomies in the proof of Proposition 1.2:

$$\phi^{*,u}_{\hat{x},\hat{z}} = \lim_{n\to\infty} \hat{A}_*^{-n}(\hat{z})^{-1}\,\hat{A}^{-n}(\hat{x}) = \lim_{n\to\infty} \left(\hat{A}^n(\hat{x})^{-1}\,\hat{A}^n(\hat{z})\right)_* = \left(\phi^s_{\hat{x},\hat{z}}\right)_*.$$

Since the eigenvalues of $\hat{A}$ and $\hat{A}_*$ are the same, the first condition in Definition 1.3 coincides for $\hat{A}$ and for $\hat{A}_*$. As for the second condition, it is easy to see that it is not affected if we replace the cocycle by another conjugate to it. Indeed, one gets the same transformation $\psi_{\hat{p},\hat{z}}$. So, in view of Remark 7.1, we may suppose that the Riemannian metric is such that eigenvectors of $\hat{A}$ at p form an orthogonal basis of $\mathbb{C}^d$. By the previous statement, the matrices M and M_* of

$$\psi_{\hat{p},\hat{z}} = \phi^s_{\hat{z},\hat{p}} \circ \phi^u_{\hat{p},\hat{z}} \quad \text{and} \quad \psi^*_{\hat{p},\hat{z}} = \phi^{*,s}_{\hat{z},\hat{p}} \circ \phi^{*,u}_{\hat{p},\hat{z}}$$

in this basis are transposed. In particular, M and M_* have the same algebraic minors, and so the third claim in the lemma follows from Remark 1.4. □

This lemma means that the previous theory applies equally to the adjoint cocycle $\hat{f}^*_{\hat{A}}$. In this way we find a map $\xi_* : \hat{x} \mapsto \mathbb{CP}^{d-1}$ invariant under $\hat{f}^*_{\hat{A}}$ and, by Remark 6.5 under the corresponding unstable holonomies.

The next important result means that $\xi(\hat{x})$ is outside the kernel of the linear form represented by $\xi_*(\hat{x})$:

Lemma 7.3. *$\xi_*(\hat{x})$ is not orthogonal to $\xi(\hat{x})$, for $\hat{\mu}$-almost every $\hat{x}$.*

Proof. As in Section 4, let us consider projective coordinates on the fibers such that $\hat{A}$ is constant on local stable manifolds of $\hat{f}$. Since our Riemannian metric is constant, in these coordinates $\hat{A}_*^{-1}$ is also constant on local stable manifolds of $\hat{f}$, which are the local unstable manifolds of $\hat{f}^{-1}$. Consequently, the unstable holonomies $\phi_{\hat{x},\hat{z}}^{*,u} = \mathrm{id}$ for any points $\hat{x}$ and $\hat{z}$ in the same local stable manifold of $\hat{f}$. Remark 6.5 gives that ξ_* is invariant under unstable holonomies of $\hat{A}_*$. In view of what we just said, this means that its expression in these coordinates is constant on every local stable manifold of $\hat{f}$. In other words, $\xi_*(\hat{x})$ depends only on $P(\hat{x})$.

Now fix any $x \in \Sigma_T$. By the preceding comments, the hyperplane H_x orthogonal to $\xi_*(\hat{x})$ is the same for all $\hat{x} \in W^s_{loc}(x)$. Using Remark 6.4,

$$m_x(H_x) = \int \delta_{\xi(\hat{x})}(H_x)\, d\hat{\mu}_x(\hat{x}) = \hat{\mu}_x\big(\{\hat{x} \in W^s_{loc}(x) : \xi(\hat{x}) \in H_x\}\big).$$

From Proposition 5.1, we have that $m_x(H_x) = 0$. It follows that

$$\hat{\mu}\big(\{\hat{x} \in \hat{\Sigma}_T : \xi(\hat{x}) \in H_x\}\big) = \int d\mu(x)\hat{\mu}_x\big(\{\hat{x} \in W^s_{loc}(x) : \xi(\hat{x}) \in H_x\}\big) = 0,$$

as claimed. □

Given $(\hat{x}, \eta) \in \hat{\Sigma}_T \times \mathbb{CP}^{d-1}$, let $\xi(\hat{x})$ and $\xi_*(\hat{x})$ be as constructed before. Observe that ξ and ξ_* are invariant under the corresponding cocycles:

$$\hat{A}(\hat{x})\xi(\hat{x}) = \xi(\hat{f}(\hat{x})) \quad \text{and} \quad \hat{A}_*(\hat{x})\xi_*(\hat{x}) = \xi_*(\hat{f}^{-1}(\hat{x}))$$

$\hat{\mu}$-almost everywhere. The first relation also gives that the orthogonal complement $\xi^\perp$ is invariant under the adjoint $\hat{A}_*$. Let

$$\xi(\hat{x})^\perp = V^2_{\hat{x}} \oplus \cdots \oplus V^k_{\hat{x}}$$

be the Oseledets decomposition, corresponding to the measure $\hat{\mu}$, of the restriction of $\hat{A}_*$ to the subbundle $\xi^\perp$ in decreasing order of the Lyapunov exponents: $\lambda_2 > \cdots > \lambda_k$. We also let λ_1 be the Lyapunov exponent of $\hat{A}_*$ along the direction of ξ_*. Since $\hat{\mu}$ is ergodic, the dimensions of the subspaces are constant almost everywhere, and so are the Lyapunov exponents.

Definition 7.4. Given a linear map $L : \mathbb{C}^d \to \mathbb{C}^d$ and a subspace V of $\mathbb{C}^d$, we denote by $\det(L, V)$ the *determinant of L along V,* defined as the quotient of the volumes of the parallelograms determined by $\{Lv_1, \dots, Lv_s\}$ and $\{v_1, \dots, v_s\}$, respectively for any basis $v_1, \dots, v_s$ of V.

Denote $V_{\hat{x}} = \mathbb{C}\xi_*(\hat{x}) \oplus V^2_{\hat{x}}$ and let $s = 1 + \dim V^2_{\hat{x}}$ be the dimension of $V_{\hat{x}}$. For $n \geq 1$, define

$$\Delta^n(\hat{x}) = \frac{\det(\hat{A}^n_*(\hat{x}), V_{\hat{x}})}{\|\hat{A}^n_*(\hat{x})\xi_*(\hat{x})\|^s}. \tag{25}$$

Denoting $\Delta(\cdot) = \Delta^1(\cdot)$, we get that

$$\Delta^n(\hat{x}) = \Delta(\hat{x})\, \Delta(\hat{f}^{-1}(\hat{x})) \cdots \Delta(\hat{f}^{-n+1}(\hat{x})) \tag{26}$$

for all $n \geq 1$, because both ξ_* and V are $\hat{A}_*$-invariant.

Proposition 7.5. *For $\hat{\mu}$-almost every $\hat{x} \in \Sigma_T$ and every $\eta \in \mathbb{CP}^{d-1}$,*

$$\lim_{n\to\infty} \Delta^n(\hat{x}) = 0.$$

For the proof we need a few elementary facts from linear algebra:

7.2 Eccentricity of linear maps

Let $L : \mathbb{C}^d \to \mathbb{C}^d$ be a linear map. If there is a unique direction that is most expanding for L, we define the *eccentricity* of L to be

$$E(L) = \frac{\|L(\xi_a)\|}{\|L \mid \xi_a^\perp\|}$$

where ξ_a is a norm 1 maximally expanding vector, and $\xi_a^\perp$ is the hyperplane orthogonal to it. If the most expanding subspace has dimension larger than 1, we just set $E(L) = 1$.

Let $C_\alpha(\eta)$ denote the cone of width α around a vector η. We have $L(C_\alpha(\xi_a)) \subset C_\beta(L(\xi_a))$ with $\tan\beta = \tan\alpha / E(L)$. Moreover, these are the most contracted cones: given any α, β, and any vectors ξ, η,

$$L(C_\alpha(\eta)) \subset C_\beta(\xi) \quad \Rightarrow \quad \tan\beta \geq \tan\alpha / E(L). \tag{27}$$

Lemma 7.6. *Let $\mathcal{N}$ be any weakly compact family of probability measures on $\mathbb{CP}^{d-1}$ such that every $\nu \in \mathcal{N}$ gives zero weight to all projective hyperplanes. Let $L_n : \mathbb{C}^d \to \mathbb{C}^d$ be a sequence of linear maps such that $(L_n)_*\nu_n$ converges to a Dirac measure δ_ξ, for some sequence ν_n in $\mathcal{N}$.*

Then $E(L_n) \to \infty$ as $n \to \infty$, and the image $L_n\xi_{a,n}$ of the most expanding direction $\xi_{a,n}$ of L_n converges to ξ.

Proof. By the compactness of $\mathcal{M}$ and the space of projective hyperplanes, there exists $\delta > 0$ such that

$$\nu(H_\delta) < \frac{1}{2}$$

for every $\nu \in \mathcal{N}$ and any cone H_δ of width δ around a hyperplane H. Let $\xi \in \mathbb{CP}^{d-1}$ and ν_n be as in the statement. For every $\varepsilon > 0$

$$\nu_n(L_n^{-1}(C_\varepsilon(\xi))) = (L_n)_*\nu_n(C_\varepsilon(\xi)) \to 1 \quad \text{as } n \to \infty.$$

Hence, in particular, the ν_n-measure of the cone $L_n^{-1}(C_\varepsilon(\xi))$ is bigger than $1/2$ if $n > n(\varepsilon)$. It follows that $L_n^{-1}(C_\varepsilon(\xi))$ is not contained in H_δ, for any hyperplane H. Another useful remark is that, given $\delta > 0$, there exists $\alpha > 0$ such that any convex cone that is not contained in H_δ for any hyperplane H, contains the cone $C_\alpha(\eta)$ of width α around some vector η.

As an application, we get that every cone $L_n^{-1}(C_\varepsilon(\xi))$, $n > n(\varepsilon)$, contains some $C_\alpha(\eta_{n,\varepsilon})$. Then $L_n(C_\alpha(\xi)) \subset C_\varepsilon(\xi)$ and, in view of (27), this implies $E(L_n) \geq \tan\alpha/\tan\varepsilon$. When $n \to \infty$, we may take $\varepsilon \to 0$, and then $E(L_n)$ goes to ∞. The second statement in the lemma is a consequence. Given any fixed width $\varepsilon > 0$ we can find $\alpha_n \to \infty$ such that

$$L_n(C_{\alpha_n}(\xi_{a,n})) \subset C_\varepsilon(L_n(\xi_{a,n})).$$

This implies that the $(L_n)_*\nu_n$-mass of the ε-neighbourhood of $L_n\xi_{a,n}$ converges to 1 as $n \to \infty$. Since the $(L_n)_*\nu_n$ converge to δ_ξ, by assumption, this implies that $L_n\xi_{a,n}$ converges to ξ. □

7.3 Proof of Proposition 7.5

Recall that we identify $(\mathbb{C}^d)^*$ with $\mathbb{C}^d$ through some fixed Riemannian metric. Expansion and contraction are with respect to that metric.

Proof. For each $n \geq 1$ let $\xi_n(\hat{x})$ be the most expanding direction for the map $\hat{A}_*^n(\hat{x})$ or, equivalently, the image of the direction most expanded by

$\hat{A}^n(\hat{f}^{-n}(\hat{x})) = A^n(\hat{x}_n)$. Observe that this is well-defined: using Lemma 7.6 with $\nu_n = m_{\hat{x}_n}$, $L_n = \hat{A}^n(\hat{f}^{-n}(\hat{x}))$, $\xi = \xi(\hat{x})$ we find that the eccentricity

$$E_n = E(\hat{A}^n_*(\hat{x})) = E(\hat{A}^n(\hat{f}^{-n}(\hat{x})))$$

goes to infinity as $n \to \infty$. Moreover, $\xi_n(\hat{x})$ tends to $\xi(\hat{x})$ as $n \to \infty$. So, according to Lemma 7.3, the direction $\xi_*(\hat{x})$ is uniformly far from the orthogonal hyperplane $\xi_n(\hat{x})^\perp$ for all large n.

Let us consider any norm 1 vector $\eta \in V_{\hat{x}}$. We split $\eta = \eta_n + \zeta_n$, with $\eta_n \in \xi_n(\hat{x})^\perp$ and ζ_n collinear to $\xi_*(\hat{x})$. The remark we just made ensures that

$$\|\eta_n\| \le C \quad \text{and} \quad \|\hat{A}^n_*(\hat{x})\xi_*(\hat{x})\| \ge \frac{1}{C}\|\hat{A}^n_*(\hat{x})\xi_n(\hat{x})\|$$

for some constant $C > 0$ independent of n. It follows that

$$\|\hat{A}^n_*(\hat{x})\eta_n\| \le \frac{\|\hat{A}^n_*(\hat{x})\xi_n(\hat{x})\|}{E_n}\|\eta_n\| \le \frac{C^2}{E_n}\|\hat{A}^n_*(\hat{x})\xi_*(\hat{x})\|.$$

Now we consider any basis $\{\xi_*(\hat{x}), \eta^2, \dots, \eta^s\}$ of $V_{\hat{x}}$, and decompose each $\eta^i = \eta^i_n + \zeta^i_n$ in the same way as before. Observe that the parallelogram determined by the $\hat{A}^n_*(\hat{x})$-images of the basis vectors does not change if we replace $\hat{A}^n_*(\hat{x})\eta^i$ by $\hat{A}^n_*(\hat{x})\eta^i_n$. Moreover, the volume of a parallelogram is always bounded by the product of the norms of the corresponding edges. So, the previous inequality gives

$$\det(\hat{A}^n_*(\hat{x}), V_{\hat{x}}) \le \frac{C^{2(s-1)}}{E_n^{s-1}}\|\hat{A}^n_*(\hat{x})\xi_*(\hat{x})\|^s,$$

and so $\Delta(\hat{x}) \le C^{2(s-1)}E_n^{1-s}$. The proposition follows, using Lemma 7.6. $\square$

8 Proof of the main results

Now we are ready for the proof of Theorem 1. To turn Proposition 7.5 into an exponential estimate, we use the following general statement from ergodic theory; see for instance [26, Corollary 6.10].

Lemma 8.1. *Let $T : X \to X$ be a measurable transformation preserving a probability measure ν in X, and $\varphi : X \to \mathbb{R}$ be a ν-integrable function such that $\lim_{n\to\infty}\sum_{j=0}^{n-1}(\varphi \circ T^j) = -\infty$ at ν-almost every point. Then $\int \varphi \, d\nu < 0$.*

If ν is ergodic for T, it follows that $\lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1}(\varphi \circ T^j) = \int \varphi \, d\nu < 0$ at ν-almost every point.

Proposition 8.2. *For $\hat{\mu}$-almost every point*

$$\lim_{n\to\infty} \frac{1}{n} \log \Delta^n(\hat{x}) < 0.$$

Proof. By (26),

$$\frac{1}{n} \log \Delta^n(\hat{x}) = \frac{1}{n} \sum_{j=0}^{n-1} \log \Delta(\hat{f}^j(\hat{x})).$$

So, the claim follows from Proposition 7.5 and Lemma 8.1. □

As a direct consequence we get that λ_1 is the largest Lyapunov exponent of $\hat{A}_*$, and it has multiplicity 1:

Corollary 8.3. *We have $\lambda_1 > \lambda_2$.*

Proof. On the one hand, $\lim_{n\to\infty} \frac{1}{n} \log \|\hat{A}_*^n(\hat{x})\xi_*(\hat{x})\| = \lambda_1$. On the other hand, by Oseledets theorem [28],

$$\lim_{n\to\infty} \frac{1}{n} \log \det(\hat{A}_*^n(\hat{x}), V_{\hat{x}}) = \lambda_1 + (s-1)\lambda_2 \,.$$

It follows that

$$\lambda_2 - \lambda_1 = \frac{1}{s-1} \lim_{n\to\infty} \frac{1}{n} \log \Delta^n(\hat{x}) < 0,$$

where we used (25) and Proposition 8.2. □

Replacing $\hat{A}$ by its inverse ($\hat{A}^{-1}$ is 1-typical if and only if $\hat{A}$ is), we get that the smallest Lyapunov exponent of $\hat{f}_{\hat{A}}$ also has multiplicity 1. The proof of Theorem 1 is complete.

Now the deduction of Theorem 2 is quite standard; see [20] or [19] where this kind of argument was used before. The Lyapunov exponents of the action $\hat{A}^{\wedge k}$ of $\hat{A}$ on the k:th external product $\Lambda^k(\mathbb{C}^d)$ are just the sums

$$\lambda_{i_1} + \cdots + \lambda_{i_k}$$

of k distinct exponents of $\hat{A}$. The assumption ensures that the previous theory can be applied to $\hat{A}^i$ for every $1 \le i \le k$. A simple induction argument gives that the k largest exponents have multiplicity 1. A similar argument follows for the k smallest ones, dealing with the inverse instead. The case $d = [k/2]$ implies that all the exponents have multiplicity 1, completing the proof of the theorem.

9 Extension to real-valued cocycles

The conclusions of Theorems 1 and 2 remain valid for $\mathrm{SL}(d,\mathbb{R})$-cocycles. The only point that needs an extra comment is the proof that typical cocycles form an open and dense subset of all cocycles: *a priori* we can not ask that all eigenvalues have distinct norms, due to the possibility of pairs of complex conjugate eigenvalues. The way to bypass this is to show that, after perturbation, we can always find a periodic point q such that the eigenvalues of the cocycle at q are all real.

As before, $\hat{f} : \hat{\Sigma}_T \to \hat{\Sigma}_T$ is the two-sided subshift of finite type associated to an irreducible $d \times d$ matrix T with coefficients in $\{0,1\}$. We always consider the C^ν-norm in the space of C^ν maps $\hat{A} : \hat{\Sigma}_T \to SL(d,\mathbb{R})$. Here $0 \le \nu \le 1$, the case $\nu = 1$ corresponding to Lipschitz cocycles. However, the arguments that follow apply just the same if one replaces $\hat{\Sigma}_T$ by a hyperbolic basic set of a diffeomorphism g (endowed with a Markov partition), and in that setting we may consider any $\nu \in [0, +\infty]$.

Proposition 9.1. *For every C^ν-neighbourhood $\mathcal{V}$ of $\hat{A}$ there exists $\hat{B} \in \mathcal{V}$ and a periodic orbit $q \in \hat{\Sigma}_T$ such that all the eigenvalues of $\hat{B}^{\mathrm{per}(q)}(q)$ are real and have distinct norms.*

The proof of the proposition occupies the remainder of this section. For simplicity of the presentation, we suppose that $\hat{f}$ has some fixed point $\hat{p}$. The general case follows along the same lines.

Up to an initial perturbation of the cocycle, we may assume from the start that *all the eigenvalues of $\hat{A}(\hat{p})$ have multiplicity* 1 *and distinct norms, except for the existence of $c \ge 0$ pairs of complex conjugate eigenvalues.* This is our formulation of the first condition in Definition 1.3 in the present setting. In the sequel we suppose $c \ge 1$, since for $c = 0$ there is nothing to prove. Let

$$E^1_{\hat{p}} \oplus \cdots \oplus E^k_{\hat{p}}$$

be the splitting of $\mathbb{R}^d$ into eigenspaces of $\hat{A}(\hat{p})$, ordered according to increasing norm of the eigenvalues. The dimension of each $E^i_{\hat{p}}$ is either 1 or 2, corresponding to real and complex eigenvalues, respectively.

Let $[0;i]$ be the cylinder of $\hat{\Sigma}_T$ that contains $\hat{p}$. Fix some homoclinic point $\hat{z} = (\cdots, i, \cdots, i, i_1, \cdots, i_{l-1}, i, \cdots, i, \cdots) \in [0;i]$ associated to $\hat{p}$. As before, let

$$\psi_{\hat{p},\hat{z}} = \phi^s_{\hat{z},\hat{p}} \circ \phi^u_{\hat{p},\hat{z}} \, .$$

Corresponding to the second condition in Definition 1.3 we also assume that $\psi_{\hat{p},\hat{z}}(E^I)$ *is linearly independent of* E^J *for any sums* $E^I = \oplus_{i\in I} E^i_{\hat{p}}$ *and* $E^J = \oplus_{j\in J} E^j_{\hat{p}}$ *of eigenspaces such that* $\dim E^I + \dim E^J \leq d$. This may be obtained by an additional perturbation of the cocycle.

For every n let x_n be the periodic point of period $l+n$ defined by the itinerary $(i, i_1, \cdots, i_{l-1}, i, \cdots, i)$, where the symbol i appears $n+1$ times. Then let K_n be the closure of the union of the orbits of x_m over all $n \geq m$. Moreover, let K_∞ the closure of the orbit of $\hat{z}$, that is, the union of the orbit itself with the point $\hat{p}$. Note that each K_n is just the union of the orbits of the x_m, $m \geq n$, with K_∞.

Given an $\hat{f}$-invariant set $K \subset \hat{\Sigma}_T$, an $\hat{A}$-invariant decomposition

$$K \times \mathbb{R}^d = V^1 \oplus \cdots \oplus V^k$$

is called *dominated* if the dimensions of each subspace $V^i_{\hat{x}}$ are independent of the point $\hat{x} \in K$, and there exist constants $C > 0$ and $\lambda < 1$ such that

$$\frac{\|A^n(\hat{x})u\|}{\|A^n(\hat{x})v\|} \leq C\lambda^n \frac{\|u\|}{\|v\|}$$

for every $n \geq 1$, $\hat{x} \in K$, $1 \leq i < k$, and any non-zero vectors $u \in V^1 \oplus \cdots \oplus V^i$ and $v \in V^{i+1} \oplus \cdots \oplus V^k$.

A dominated decomposition is necessarily continuous, that is, each $V^i_{\hat{x}}$ depends continuously on the point $\hat{x} \in K$ (assuming the cocycle is continuous). Existence of dominated decomposition is equivalent to existence of invariant cone fields over K. In particular, this is a robust property: if $\hat{A}$ admits a dominated decomposition over an invariant set K then so does any cocycle $\hat{B}$ in a C^0 neighbourhood of $\hat{A}$, the subspaces in the decomposition of $\hat{B}$ having the same dimensions and being uniformly close to the corresponding ones in the decomposition of $\hat{A}$. More information on the notion of dominated decomposition may be found e.g. in [9].

Lemma 9.2. *For every large enough n, the cocycle $\hat{A}$ admits a dominated decomposition $E^1 \oplus \cdots \oplus E^k$ over the invariant set K_n coinciding with the decomposition into eigenspaces at the point $\hat{p}$.*

Proof. First we construct a dominated decomposition over K_∞. Then we explain why this implies the conclusion for every large n.

For each $1 \le i \le k$ let $F^i_{\hat{p}} = E^1_{\hat{p}} \oplus \cdots \oplus E^i_{\hat{p}}$ and $G^i_{\hat{p}} = E^i_{\hat{p}} \oplus \cdots \oplus E^s_{\hat{p}}$. Then take $E^i_{\hat{z}} = F^i_{\hat{z}} \cap G^i_{\hat{z}}$, where

$$F^i_{\hat{z}} = \phi^s_{\hat{p},\hat{z}} F^i_{\hat{p}} \quad \text{and} \quad G^i_{\hat{z}} = \phi^u_{\hat{p},\hat{z}} G^i_{\hat{p}}\,.$$

Our second assumption above implies that $F^i_{\hat{z}} \oplus G^{i+1}_{\hat{z}} = \mathbb{R}^d = F^{i-1}_{\hat{z}} \oplus G^i_{\hat{z}}$, and so $\dim E^i_{\hat{z}} = \dim E^i_{\hat{p}}$. Then extend E^i to the whole orbit of $\hat{z}$ by iteration under $\hat{A}$. To prove that these E^i define a dominated decomposition over K_∞, we are going to show that every E^i is continuous at $\hat{p}$.

For this, it suffices to show that

$$F^i_{\hat{f}^n(\hat{z})} \to F^i_{\hat{p}} \quad \text{and} \quad G^i_{\hat{f}^n(\hat{z})} \to G^i_{\hat{p}}, \quad \text{when } n \to \pm\infty.$$

We do this for $n \to +\infty$, the arguments for $n \to -\infty$ being analogous. Consider continuous coordinates as in Section 4, given by the stable holonomies ϕ^s, rendering the cocycle $\hat{A}$ constant along local stable manifolds. In these coordinates $\phi^s = \mathrm{id}$ and so

$$F^j_{\hat{f}^n(\hat{z})} = F^j_{\hat{p}} \quad \text{for every } n \ge l \text{ and every } j.$$

Taking $j = i$ we, immediately, get the claim for F^i. Moreover, the fact that $G^{j+1}_{\hat{f}^l(\hat{z})}$ is transverse to $F^j_{\hat{p}} = F^j_{\hat{f}^l(\hat{z})}$ implies that $G^{j+1}_{\hat{f}^n(\hat{z})}$ converges to $G^j_{\hat{p}}$. Choosing $j = i - 1$ we get the claim for G^i.

Clearly, every point of K_∞ spends all but a finite number of iterates near $\hat{p}$. This, together with continuity of the E^i and the assumption that the norms of the eigenvalues of $\hat{A}(\hat{p})$ are all distinct, implies that the decomposition we have exhibited is dominated. This finishes the construction over K_∞.

It follows that a dominated decomposition exists over every K_n with large enough n. That is because K_n is contained in a small neighbourhood of K_∞, and dominated decompositions always extend to any invariant set in some neighbourhood. This last fact is easily checked as follows. One considers forward and backward invariant cone fields around the subspaces G^i and F^i, respectively and extends these cone fields continuously to some neighbourhood V of K_∞. Up to reducing V, these extensions are still invariant under the dynamics, just by continuity, and they yield a dominated decomposition over any invariant set contained in V. $\square$

By robustness, every cocycle $\hat{B}$ in a C^0 neighbourhood $\mathcal{U}$ of $\hat{A}$ has a dominated decomposition $E^1_{\hat{B}} \oplus \cdots \oplus E^k_{\hat{B}}$ over K_n, for every large n (independent of $\hat{B}$), depending continuously on $\hat{B}$, and with $\dim E^i_{\hat{B}} = \dim E^i$ for all i. Let ℓ be smallest such the dimension of E^ℓ is equal to 2.

The case when $\hat{A}^{l+n}(x_n)$ reverses the orientation of $E^\ell_{x_n}$ is easy, as we shall see right after the statement of the next lemma. For the time being, we suppose that $\hat{A}^{l+n}(x_n)$ preserves the orientation of $E^\ell_{x_n}$. Thence, the same is true for every nearby cocycle $\hat{B}$. Then we denote $\rho(n, \hat{B})$ the rotation number associated to $\hat{B}^{l+n}(x_n)$. Moreover, given a continuous arc $\mathcal{B} = \{\hat{B}_t\}$ of cocycles close to A, we denote $\delta(n, \mathcal{B})$ the oscillation of $\rho(n, \hat{B}_t)$ over the whole parametrization interval.

The main step in the proof of Proposition 9.1 is the following

Lemma 9.3. *There exists a continuous arc $\mathcal{A} = \{A_t : t \in [0,1]\}$ of C^ν cocycles in $\mathcal{U}$ with $\hat{A}_0 = \hat{A}$ and such that for every $t > 0$ there exists $n_t \geq 1$ so that $\delta\big(n, \{\hat{A}_s : s \in [0,t]\}\big) > 1$, for every $n \geq n_t$.*

Let us explain how Proposition 9.1 follows from Lemma 9.3, after which we shall prove the lemma.

Firstly, for every t and every large n, the matrix $\hat{A}_t^{k+n}(x_n)$ has at most c pairs of complex eigenvalues. Secondly, in the orientation preserving case we may use Lemma 9.3 to conclude that there exists t arbitrarily close to zero and $n \geq 1$ for which the rotation number $\rho(n, \hat{A}_t)$ is integer. This means that $\hat{A}_t^{k+n}(x_n)$ has some real eigenvector along the subspace $E^\ell_{\hat{A}_t}$. Observe that in the orientation reversing case this conclusion comes for free. So, in general, by an arbitrarily small perturbation close to x_n and preserving $E^\ell_{\hat{A}_t}$, we can obtain a cocycle $\hat{A}'$ for which there are two real and distinct eigenvalues along that subspace. Thus, $(\hat{A}')^{l+n}(x_n)$ has at most $c - 1$ pairs of complex eigenvalues.

Repeating this procedure, with $p' = x_n$ and $\hat{A}'$ in the place of p and $\hat{A}$, respectively in not more than c steps we find a periodic point q and a continuous cocycle $\hat{B}$, arbitrarily close to the initial p and $\hat{A}$, such that all the eigenvalues of $\hat{B}$ over the orbit of q are real. This concludes the proof of Proposition 9.1.

Finally, we prove Lemma 9.3: The arguments are quite simple, and so we focus on presenting the main ideas in a transparent way, rather than giving complete details.

Proof. We begin by fixing, once and for all, a basis of $\mathbb{R}^d$ coherent with the decomposition $E^1_{\hat{p}} \oplus \cdots \oplus E^k_{\hat{p}}$: every vector in the basis is in some $E^i_{\hat{p}}$, and the matrix of $\hat{A}(\hat{p})$ restricted to each 2-dimensional eigenspace $E^i_{\hat{p}}$ is a rotation (of angle ρ_i), relative to this basis. We always consider the (constant) system of coordinates on the fibers $\{\hat{x}\} \times \mathbb{R}^d$ defined by this basis. Given any θ, we define R_θ to be the linear map given by the rotation of angle θ along $E^\ell_{\hat{p}}$, and by the identity along all the other $E^i_{\hat{p}}$. We choose

$$\hat{A}_t(\hat{x}) = R_{t\varepsilon} \cdot A(\hat{x}), \quad \text{for } t \in [0,1],$$

where $\varepsilon > 0$ is fixed small enough so that all these cocycles be in $\mathcal{U}$. Reducing $\varepsilon > 0$ if necessary, we may find $r > 0$ small enough so that every $E^i_{\hat{A}_t}$ is a graph over $E^i_{\hat{p}}$ restricted to the r-neighbourhood of p. We identify $E^\ell_{\hat{A}_t}$ with $E^\ell_{\hat{p}}$ on that neighbourhood, via projection parallel to the other $E^i_{\hat{p}}$. Moreover, we write

$$\hat{A}_t^{l+n}(x_n) \mid E^\ell_{\hat{A}_t} = \alpha_{t,n,n} \cdots \alpha_{t,n,1} \cdot \beta_{t,n}$$

where the $\alpha_{t,n,j}$ correspond to iterates inside the r-neighbourhood of p, and $\beta_{t,n}$ encompasses the iterates outside that neighbourhood Since there are finitely many of the latter, $\beta_{t,n}$ converges uniformly to some β_t, as $n \to \infty$. Thus, in order to obtain the conclusion of the lemma, it suffices to show that the variation of the rotation number of the matrix $\alpha_{s,n,n} \cdots \alpha_{s,n,1}$ over every interval $[0,t]$ goes to infinity when $n \to \infty$. For this we observe that, by the definition of $\hat{A}_t$, the $\alpha_{t,n,i}$ are uniformly close to the rotation of angle $t\varepsilon + \rho_\ell$, if the radius r is chosen small enough. Since the $\alpha_{t,n,i}$ preserve the orientation, all their contributions to the rotation number roughly add up, yielding the claim. □

Acknowledgments: We are most grateful to Philippe Thieullen for his careful reading of the paper and useful comments, as well as for pointing out a gap in the original proof of Proposition 4.2.

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