# Multidimensional nonhyperbolic attractors

Marcelo Viana *

**Abstract**

We construct smooth transformations and diffeomorphisms exhibiting nonuniformly hyperbolic attractors with multidimensional sensitiveness on initial conditions: typical orbits in the basin of attraction have several expanding directions. These systems also illustrate a new robust mechanism of sensitive dynamics: despite the nonuniform character of the expansion, the attractor persists in a full neighbourhood of the initial map.

## 1 Introduction

Let $\varphi: M \longrightarrow M$ be a smooth map on a manifold $M$, admitting some compact invariant region $U$, that is, $\varphi(U) \subset \operatorname{int}(U)$. Here we say that $\varphi$ has *expanding behaviour* on $U$ if typical points $x \in U$ have tangent vectors $v$ whose iterates grow exponentially fast: $\log \|D\varphi^n(x)v\|^{1/n}$ has positive limit (or lim inf) as $n \to +\infty$. In general, we call *Lyapunov exponents* of $\varphi$ at $x$ to all values of this limit, for all nonzero tangent vectors $v$.

Clearly, Lyapunov exponents measure the asymptotic exponential rate at which infinitesimally nearby points approach or move away from each other as time increases to $+\infty$. Hence, presence of positive exponents indicates sensitive dependence of trajectories starting near $x$ with respect to the corresponding initial point ("chaotic" dynamics).

A classical example are the uniformly hyperbolic (or Axiom A, see [Sm]) diffeomorphisms with nonperiodic attractors. In this case the number of positive Lyapunov exponents is constant on the basin of attraction $U$, and the dynamics of the attractor is (structurally) stable. In particular, any nearby map also has a nonperiodic attractor, close to the initial one and with the same number of positive exponents.

The mathematical study of *nonuniform* expanding behaviour is much more incomplete, in fact it has been mostly restricted to systems with a unique positive

*Partially supported by a J. S. Guggenheim Foundation Fellowship.

Lyapunov exponent. A first important result was due to [Ja], who showed that many quadratic maps of the interval admit an absolutely continuous invariant probability measure $\mu$. Then such maps have positive Lyapunov exponent at all $\mu$-generic points (a positive Lebesgue measure set). Other proofs of this result were given e.g. by [CE], [BC1].

In higher dimensions, [BC2] showed that many Hénon diffeomorphisms of the plane have strange attractors containing dense orbits on which the diffeomorphism has a positive Lyapunov exponent. See [BY], [MV], [Vi] for further developments. In all these cases, expanding behaviour exhibits a rather subtle form of persistence: "many" means positive Lebesgue measure in parameter space. On the other hand, [Yo] constructed open sets of nonuniform hyperbolicity in a space of linear cocycles.

The purpose of this paper is to introduce the study of certain dynamical systems exhibiting nonuniform *multidimensional* expansion. We construct smooth maps, both invertible and noninvertible, having nonuniformly hyperbolic attractors with a high dimensional character: *the map has several expanding directions at Lebesgue almost every point $x$ in the basin.* More precisely, one may write $T_xM = E^+ \oplus E^-$ with

$$\liminf_{n\to+\infty} \log \left\| D\varphi^n(x)v^+ \right\| \geq c > 0 > -c \geq \limsup_{n\to+\infty} \log \left\| D\varphi^n(x)v^- \right\|$$

for all $v^\pm \in E^\pm \setminus \{0\}$ ($c$ independent of $x$ or $v^\pm$) and the number $\dim E^+$ of expanding directions (or of positive Lyapunov exponents) is larger than 1.

The basic strategy is to couple nonuniform models, namely quadratic or Hénon maps, with convenient "fast" systems such as expanding maps or solenoid diffeomorphisms. The expanding behaviour observed in these multidimensional examples originates from a different mechanism, of a statistical type, which makes them much more robust than their low-dimensional counterparts: *the expanding attractor persists in a whole $C^3$-neighbourhood of the initial map.*

Although the main ingredients are quite general, we illustrate this strategy through some concrete situations, in order to keep our presentation as transparent as possible. Further extension of these methods is briefly discussed at the end of this Introduction.

## 1.1 Statement of results

First we consider the, simpler, noninvertible case. Let $\varphi_\alpha: S^1 \times \mathbb{R} \longrightarrow S^1 \times \mathbb{R}$ be a $C^3$ map given by $\varphi_\alpha(\theta, x) = (\hat{g}(\theta), a(\theta) - x^2)$, where $\hat{g}: S^1 \longrightarrow S^1$ is an expanding map of the circle $S^1 = \mathbb{R}/\mathbb{Z}$, and $a(\theta) = a_0 + \alpha\phi(\theta)$. Here $\phi(\theta)$ is some Morse function and $a_0 \in (1,2)$ is fixed such that $x = 0$ is a preperiodic point for the map $h(x) = a_0 - x^2$. For the sake of definiteness, we take $\phi(\theta) = \sin 2\pi\theta$ and we also suppose $\hat{g}$ to be linear, $\hat{g}(\theta) = d\theta \bmod 1$ for some $d \geq 2$. It is easy to check that, since $a_0 < 2$, there exists a compact interval $I_0 \subset (-2, 2)$ such that $\varphi_\alpha(S^1 \times I_0) \subset \operatorname{int}(S^1 \times I_0)$ for any small $\alpha$.

**Theorem A** *Assume $d$ to be large enough, $d \geq 16$ say. Then for every sufficiently small $\alpha > 0$ the map $\varphi_\alpha$ has two positive Lyapunov exponents at Lebesgue almost every point $(\theta, x) \in S^1 \times I_0$. Moreover, the same holds for every map $\varphi$ sufficiently close to $\varphi_\alpha$ in $C^3(S^1 \times \mathbb{R})$.*

Here $C^3(S^1 \times \mathbb{R})$ denotes the space of all $C^3$ maps from $S^1 \times \mathbb{R}$ to itself: in the second part of the theorem $\varphi$ is *not* assumed to have a skew-product form. In our construction, the cylinder $S^1 \times \mathbb{R}$ may be replaced by other surfaces, e.g., the torus $S^1 \times S^1$. Furthermore, examples of the same kind having any preassigned number of positive Lyapunov exponents may be obtained replacing the factor map $g$ by other hyperbolic transformations, like expanding maps on the $m$-torus, $m \geq 2$. All our arguments extend directly to these situations, cf. Section 2.5.

Now we describe a corresponding construction for diffeomorphisms. We take $\varphi_{\alpha,b}: T_3 \times \mathbb{R}^2 \longrightarrow T_3 \times \mathbb{R}^2$ given by $\varphi_{\alpha,b}(\Theta, X) = (\hat{g}(\Theta), \hat{f}_{\alpha,b}(\Theta, X))$, where

- $T_3 = S^1 \times B^2$ is the solid 3-torus, $\Theta = (\theta, T)$, and $\hat{g}: T_3 \longrightarrow T_3$ is a solenoid embedding $\hat{g}(\theta, T) = (d\theta \bmod 1, G(\theta, T))$, see [Sm];
- $\hat{f}_{\alpha,b}(\Theta, X) = (a(\theta) - x^2 + by, -bx)$, with $X = (x, y)$ and $a(\theta)$ defined in the same way as before.

Clearly, $\varphi_{\alpha,b}$ is a diffeomorphism (onto its image) for every nonzero value of $b$. In the same way as before, $a_0 < 2$ ensures the existence of some compact interval $I_0 \subset (-2, 2)$ such that $\varphi_{\alpha,b}(T_3 \times I_0^2) \subset \operatorname{int}(T_3 \times I_0^2)$ for every small $\alpha$ and $b$.

**Theorem B** *Assume $d$ to be large enough. Then there exists an open set of (small positive) values of $(\alpha, b)$ for which $\varphi_{\alpha,b}$ has two positive Lyapunov exponents at Lebesgue almost every point $(\Theta, X) \in T_3 \times I_0^2$. Moreover, the same holds for every $\varphi$ close enough to $\varphi_{\alpha,b}$ in $C^3(T_3 \times \mathbb{R}^2)$.*

As in the previous case, all the arguments can be easily extended to yield similar examples in compact manifolds without boundary and or with any given number of positive Lyapunov exponents. For the proof of Theorem B we take $b$ to be larger than $100/\sqrt{d}$, cf. Section 3. It is an interesting problem to decide whether such a lower bound is indeed necessary, or just a requirement of our approach.

## 1.2 General comments

Let us briefly comment on main ideas in the proofs of the theorems we have stated, as well as on relations between these and other results in the literature.

Classical examples of [Sh] and [Ma] show that certain topological features of the dynamics, such as transitivity, may be persistent under perturbations even in the absence of uniform hyperbolicity. Their systems are obtained by deformation of Anosov diffeomorphisms and retain many uniform features of the initial map (continuous invariant cone fields, invariant foliations), obstruction to Axiom A coming from the existence of saddle points with different stable indices. Recently, [BD] have given a new, more general construction of such partially hyperbolic, persistently transitive systems.

The nonhyperbolicity of the maps in Theorems A and B results from a different mechanism, which they inherit from the nonuniform models (quadratic maps, Hénon diffeomorphisms) involved in their construction. Indeed, a main feature here is coexistence of hyperbolic dynamics (uniform expansion, resp. invariant stable and unstable cone fields) in large portions of phase space, together with highly nonhyperbolic behaviour (infinite contraction, resp. breakdown of the cone fields due to interchange of expanding and contracting directions) in certain *folding* (or *critical*) *regions* of phase space. The presence of these critical regions is a major drawback for expanding behaviour, so let us sketch how it is dealt with in the proof of our theorems.

In the context of real quadratic maps, expanding behaviour relies on a delicate control of the recurrence of the critical orbit, more precisely, on appropriate lower bounds for the distance between the critical point and its $n$-th iterate, $n \geq 1$. This translates into a sequence of conditions on the parameter, which must then be proven to hold for a positive measure set of parameter values, [Ja]. A similar approach, in a more sophisticated form, is also central in the study of Hénon maps with small jacobian. Actually, a main guideline in [BC2] is to try and view such maps as a kind of perturbation of 1-dimensional maps, and the accuracy of this point of view itself depends on bounding the recurrence of the "critical set".

It is not difficult to see that the nature of the critical regions renders such a strategy of recurrence control hopeless for the present multidimensional systems. For instance, in the setting of Theorem A, the critical region $\{\det D\varphi = 0\}$ is a codimension-1 submanifold and, hence, it is likely to intersect its iterates. Such intersections can not be destroyed by small parameter variations, regardless of the number of parameters involved, which means that *critical points can not be prevented from hitting back the critical region in finite time*, with the corresponding accumulation of nonhyperbolic effects.

Instead, our arguments are based on a statistical (large deviations type) analysis of these returns to the vicinity of the critical region. Roughly, we prove that for most trajectories the total nonhyperbolicity associated with returns is not strong enough to annihilate the hyperbolicity acquired at iterates far from the critical region. Also, these arguments must have an essentially high-dimensional character in the case of Theorem B: necessity to allow for arbitrarily close returns implies that we must deal with tangent vectors of unbounded slope right from early iterates, and so the situation is far from being "nearly 1-dimensional".

This analysis relies on the fact that the driving maps $g$ are strongly mixing. On the other hand, other properties of expanding maps and solenoids are used in apparently less important ways. In view of this, we expect an extension of these arguments to apply when such hyperbolic maps are replaced in our construction by more general systems with fast decay of correlations. A natural example we have in mind is the coupling of nonuniformly hyperbolic maps of the interval, e.g. $\varphi(\theta, x) = (g(\theta), a(\theta) - x^2)$, $g$ a unimodal (or multimodal) map as in [Ja] or [BC1].

Similar methods to the ones we use here should also prove useful for understanding other classes of higher dimensional attractors, including the partially hyperbolic systems in [Sh], [Ma], [BD], whose ergodic properties are mostly unknown.

**Acknowledgements.** I am thankful to J.-C. Yoccoz, L.-S. Young and, specially, L. Carleson, for fruitful conversations. Most of this work was done during visits to the University of Paris-Sud, the Mittag-Leffler Institute, the UCLA, and Princeton University, whose hospitality and support are gratefully acknowledged.

# 2 Proof of Theorem A

First we assume that $\varphi: S^1 \times \mathbb{R} \longrightarrow S^1 \times \mathbb{R}$ has the form

$$(1) \qquad \varphi(\theta, x) = (g(\theta), f(\theta, x)), \quad \text{with} \quad \partial_x f(\theta, x) = 0 \text{ if and only if } x = 0$$

and we prove that the conclusion of the theorem holds as long as $\varphi$ is $C^2$ and

$$(2) \qquad \|\varphi - \varphi_\alpha\|_{C^2} \le \alpha \quad \text{on} \quad S^1 \times I_0.$$

Then, in Section 2.5, we explain how to remove assumption (1).

Our basic strategy goes as follows. We call $\hat{X} \subset S^1 \times I_0$ an *admissible curve* if $\hat{X} = \operatorname{graph}(X)$, for some $X: S^1 \longrightarrow I_0$ satisfying

1. $X$ is $C^2$ except, possibly, for being discontinuous on the left at $\theta = \tilde{\theta}_0$.

2. $|X'(\theta)| \le \alpha$ and $|X''(\theta)| \le \alpha$ at every $\theta \in S^1$.

Here $\tilde{\theta}_0$ denotes the fixed point of $g$ close to $\theta = 0$ and we assume that $S^1 = \mathbb{R}/\mathbb{Z}$ has the orientation induced by the usual order in $\mathbb{R}$. Let $\hat{X}_0 = \operatorname{graph}(X_0)$ be any admissible curve and denote $\hat{X}_j(\theta) = \varphi^j(\theta, X_0(\theta))$, for $j \geq 0$ and $\theta \in S^1$. Clearly, $\|D\varphi^n(\hat{X}_1(\theta))v\| \geq \text{const } |(g^n)'(\theta)|$ grows exponentially fast, whenever $v$ is a non-vertical (meaning, non-colinear to $\partial/\partial x$) tangent vector. On the other hand, we prove that there are positive constants $c$, $C$, and $\gamma$ such that for every sufficiently large $n$ we have

$$\left\| D\varphi^n(\hat{X}_1(\theta)) \frac{\partial}{\partial x} \right\| = \prod_{j=1}^{n} \left| \partial_x f(\hat{X}_j(\theta)) \right| \geq e^{cn}, \tag{3}$$

except for a set $E_n$ of values of $\theta$ with Lebesgue measure $m(E_n) \leq C e^{-\gamma\sqrt{n}}$. We take $E = \cap_{n \geq 1} \cup_{k \geq n} E_k$ and then

$$m\left(\bigcup_{k \geq n} E_k\right) \leq \sum_{k \geq n} C e^{-\gamma\sqrt{k}} \leq \text{const}\, e^{-\gamma\sqrt{n}} \quad \text{for all } n,$$

implying $m(E) = 0$. Moreover, by construction, $\varphi$ has two positive Liapunov exponents at $\hat{X}_1(\theta)$ for every $\theta \in S^1 \backslash E$. Since the admissible curve $\hat{X}_0 \subset S^1 \times I_0$ is arbitrary, this proves the theorem.

Now we come to a detailed exposition of the arguments leading to (3). Except where otherwise stated, all constants to appear below are independent of $\alpha$. Moreover, our statements always assume $\alpha$ to be sufficiently small. For future reference, we remark that these conditions on $\alpha$ never involve the value of $d$.

## 2.1 Admissible curves

We begin by introducing the Markov partitions $\mathcal{P}_n$, $n \geq 1$, of $S^1$ defined by

- $\mathcal{P}_1 = \{[\tilde{\theta}_{j-1}, \tilde{\theta}_j) : 1 \leq j \leq d\}$, where $\tilde{\theta}_0, \tilde{\theta}_1, \ldots, \tilde{\theta}_d = \tilde{\theta}_0$ are the pre-images of $\tilde{\theta}_0$ under $g$ (ordered according to the orientation of $S^1$);
- $\mathcal{P}_{n+1} = \{\text{connected components of } g^{-1}(\omega) : \omega \in \mathcal{P}_n\}$, for each $n \geq 1$.

The following simple fact is to be used several times below. Given $\hat{X} = \text{graph}\,(X)$ and $\omega \subset S^1$ we denote $\hat{X}|\omega = \text{graph}\,(X|\omega)$.

**Lemma 2.1** *If $\hat{X}$ is an admissible curve and $\omega \in \mathcal{P}_n$ then $\varphi^n(\hat{X}|\omega)$ is also an admissible curve.*

**Proof:** The first property in the definition is obvious. As for the second one, it suffices to observe that it is preserved at each iteration. Define $Y: S^1 \longrightarrow I_0$ by $Y(g(\theta)) = f(\theta, X(\theta))$, $\theta \in \omega \in \mathcal{P}_1$. Note that (2) implies (take $\alpha$ small)

$$\begin{array}{lll} |g'| \geq d - \alpha \geq 15; & |g''| \leq \alpha; & \\ |\partial_x f| \leq |2x| + \alpha \leq 4; & |\partial_\theta f| \leq \alpha\,|\phi'| + \alpha \leq 8\alpha; & \\ |\partial_{xx} f| \leq 2 + \alpha \leq 3; & |\partial_{\theta\theta} f| \leq \alpha\,|\phi''| + \alpha \leq 50\alpha; & |\partial_{\theta x} f| \leq \alpha. \end{array}$$

Then a direct calculation gives

$$|Y'| = \left|\left(\tfrac{1}{g'}\right)(\partial_\theta f + \partial_x f X')\right| \le \tfrac{1}{15}(8\alpha + 4\alpha) \le \alpha \text{ and, analogously,}$$
$$|Y''| = \left|\left(\tfrac{1}{g'}\right)^2 (\partial_{\theta\theta} f + 2\partial_{\theta x} f X' + \partial_{xx} f (X')^2 + \partial_x f X'' - Y' g'')\right| \le \alpha$$

and the lemma follows. □

**Lemma 2.2** *Let* $\hat{X} = \operatorname{graph}(X)$ *be an admissible curve and denote* $\hat{X}(\theta) = (\theta, X(\theta))$, $\hat{Z} = \varphi(\hat{X})$ *and* $\hat{Z}(\theta) = \varphi(\hat{X}(\theta)) = (g(\theta), Z(\theta))$. *Then, given any interval* $I \subset I_0$, *we have*

$$m\left(\{\theta \in S^1 \colon \hat{Z}(\theta) \in S^1 \times I\}\right) \le \frac{4\,|I|}{\alpha} + 2\sqrt{\frac{|I|}{\alpha}}.$$

**Proof:** Let $\mathcal{A}_1 = \{\theta \in S^1 \colon |\sin 2\pi\theta| \le 1/3\}$ and $\mathcal{A}_2 = S^1 \backslash \mathcal{A}_1$. Note that each of $\mathcal{A}_1$ and $\mathcal{A}_2$ has exactly two connected components. Suppose first that $\theta \in \mathcal{A}_1$. Then $|\cos 2\pi\theta| \ge 11/12$, which implies

$$(4) \quad \left|\partial_\theta f(\hat{X}(\theta))\right| \ge \alpha\,|\phi'(\theta)| - \alpha \ge \frac{9\alpha}{2} \quad \text{and so} \quad |Z'(\theta)| \ge \left(\frac{9\alpha}{2} - 4\alpha\right) = \frac{\alpha}{2}.$$

It follows that

$$m\left(\{\theta \in \mathcal{A}_1 \colon Z(\theta) \in I\}\right) \le 2\,|I|\,/(\alpha/2) = 4\,|I|\,/\alpha.$$

On the other hand, if $\theta \in \mathcal{A}_2$ then $|\partial_{\theta\theta} f(\hat{X}(\theta))| \ge \alpha\,|\phi''(\theta)| - \alpha \ge 10\alpha$, implying $|Z''(\theta)| \ge 4\alpha$. Hence,

$$m\left(\{\theta \in \mathcal{A}_2 \colon Z(\theta) \in I\}\right) \le 4\sqrt{|I|\,/(4\alpha)} = 2\sqrt{|I|\,/\alpha},$$

which completes the proof. □

**Corollary 2.3** *There is* $C_1 > 0$ *such that, given* $\hat{X}_0 = \operatorname{graph}(X_0)$ *an admissible curve and* $I \subset I_0$ *an interval with* $|I| \le \alpha$ *we have*

$$m\left(\{\theta \in S^1 \colon \hat{X}_j(\theta) \in S^1 \times I\}\right) \le C_1\sqrt{\frac{|I|}{\alpha}} \quad \text{for every } j \ge 1.$$

**Proof:** Let $\omega \in \mathcal{P}_{j-1}$ (resp. $\omega = S^1$ in case $j = 1$), $\hat{X}_\omega = \varphi^{j-1}(\hat{X}_0|\omega)$ and $\hat{Z}_\omega = \varphi(\hat{X}_\omega)$. By the previous lemma, the measure of $\{\theta \in S^1 \colon \hat{Z}_\omega(\theta) \in S^1 \times I\}$ is bounded above by $4\,|I|\,/\alpha + 2\sqrt{|I|\,/\alpha} \le 6\sqrt{|I|\,/\alpha}$. Now, $\hat{X}_j(\theta) = \hat{Z}_\omega(g^{j-1}(\theta))$ for every $\theta \in \omega$ and so $m(\{\theta \in \omega \colon \hat{X}_j(\theta) \in S^1 \times I\}) \le 6C_*\sqrt{(|I|\,/\alpha)}m(\omega)$ where $C_*$ is some uniform bound for the metric distortion of iterates of $g$. □

## 2.2 Building expansion

Given $(\theta, x) \in S^1 \times I_0$ and $j \geq 0$ we let $(\theta_j, x_j) = \varphi^j(\theta, x)$. We also introduce positive constants $0 < \eta \leq 1/3$ and $0 < \kappa < 1$, whose value will be made precise below (in terms of the map $h(x) = a_0 - x^2$ alone).

**Lemma 2.4** *There are $\delta_1 > 0$ and $\sigma_1 > 1$ satisfying*

*a) For each small $\alpha > 0$ there is $N = N(\alpha) \geq 1$ such that $\prod_{j=0}^{N-1} |\partial_x f(\theta_j, x_j)| \geq |x| \, \alpha^{-1+\eta}$ whenever $|x| < 2\sqrt{\alpha}$.*

*b) For each $(\theta, x) \in S^1 \times I_0$ with $\sqrt{\alpha} \leq |x| < \delta_1$ there is $p(x) \leq N$ such that $\prod_{j=0}^{p(x)-1} |\partial_x f(\theta_j, x_j)| \geq \frac{1}{\kappa} \sigma_1^{p(x)}$.*

**Proof:** Throughout the proof we use $C$ to denote any large constant depending only on the quadratic map $h$. Take $l \geq 1$ minimum such that $q = h^l(0)$ is a periodic point of $h$, let $k \geq 1$ be its period, and denote $\rho^k = |(h^k)'(q)|$. Note that it must be $\rho > 1$, by [Si]. We fix $\rho_1 < \rho < \rho_2$ with $\rho_1 > \rho_2^{1-\eta/2}$ and then take $\delta_0 > 0$ small enough so that

$$\rho_1^k < \prod_{j=0}^{k-1} \left| \partial_x f(\varphi^j(\tau, y)) \right| < \rho_2^k \quad \text{whenever} \quad |y - q| < \delta_0$$

(and $\alpha$ is sufficiently small). Given $(\theta, x) \in S^1 \times I_0$ we denote $d_i = |x_{l+ki} - q|$, for $i \geq 0$. We suppose $\delta_1 > 0$ and $\alpha$ small so that $|x| < \delta_1 \Rightarrow d_0 \leq Cx^2 + C\alpha < \delta_0$. Now let $(\theta, x)$ and $i \geq 1$ be such that $|x| < \delta_1$ and $d_0, \ldots, d_{i-1} < \delta_0$. Then $d_i \leq (\rho_2^k d_{i-1} + C\alpha)$ and so, by induction,

$$d_i \leq (1 + \rho_2^k + \cdots + \rho_2^{k(i-1)}) C\alpha + \rho_2^{ki} d_0 \leq \rho_2^{ki} (C\alpha + Cx^2). \tag{5}$$

Suppose first $|x| < 2\sqrt{\alpha}$: then (5) becomes $d_i \leq \rho_2^{ki} C\alpha$. We set $\tilde{N} = \tilde{N}(\alpha) \geq 1$ to be the minimum integer such that $\rho_2^{k\tilde{N}} C\alpha \geq \delta_0$ and then define $N = l + k\tilde{N}$. The previous argument implies that $d_i < \delta_0$ for every $0 \leq i \leq \tilde{N} - 1$ and using

$$\prod_{j=0}^{N-1} |\partial_x f(\theta_j, x_j)| = \prod_{j=0}^{l-1} |\partial_x f(\theta_j, x_j)| \cdot \prod_{i=0}^{\tilde{N}-1} \left( \prod_{j=0}^{k-1} |\partial_x f(\theta_{l+ki+j}, x_{l+ki+j})| \right)$$

we get

$$\prod_{j=0}^{N-1} |\partial_x f(\theta_j, x_j)| \geq \frac{1}{C} |x| \, \rho_1^{k\tilde{N}} \geq \frac{1}{C} |x| \, \rho_2^{(1-\eta/2)k\tilde{N}} \geq \frac{1}{C} |x| \, \alpha^{-1+\eta/2} \geq |x| \, \alpha^{-1+\eta},$$

which proves the first part of the lemma. Suppose now that $|x| \geq \sqrt{\alpha}$: then (5) gives $d_i \leq \rho_2^{ki} C x^2$. We let $\tilde{p}(x) \geq 1$ be minimum such that $\rho_2^{k\tilde{p}(x)} C x^2 \geq \delta_0$ and we define $p(x) = l + k\tilde{p}(x)$. Then, in the same way as before,

$$\prod_{j=0}^{p(x)-1} |\partial_x f(\theta_j, x_j)| \geq \frac{1}{C} |x| \rho_1^{k\tilde{p}(x)} \geq \frac{1}{C} \left(\frac{\rho_1}{\sqrt{\rho_2}}\right)^{k\tilde{p}(x)} \geq \frac{1}{C} \rho_2^{(1/2-\eta/2)k\tilde{p}(x)} \geq \frac{1}{\kappa} \rho^{p(x)/4},$$

where, for the last inequality, we use the fact that $\tilde{p}(x) \gg 1$ (uniformly) as long as $\delta_1 \ll \delta_0$. We conclude the proof by taking $\sigma_1 = \rho^{1/4}$. $\square$

**Lemma 2.5** *There are $\sigma_2 > 1$, $C_2 > 0$ such that $\prod_{j=0}^{k-1} |\partial_x f(\theta_j, x_j)| \geq C_2 \sqrt{\alpha} \sigma_2^k$ for all $(\theta, x) \in S^1 \times I_0$ with $|x_0|, \ldots, |x_{k-1}| \geq \sqrt{\alpha}$. If, in addition, $|x_k| < \delta_1$ then we even have $\prod_{j=0}^{k-1} |\partial_x f(\theta_j, x_j)| \geq C_2 \sigma_2^k$.*

**Proof:** We fix $\delta_1$ as above and keep the notations from the previous lemma. Since $h$ has negative schwarzian derivative, there are $\sigma_0 > 1$ and $m \geq 1$ such that $|(h^m)'(y)| > \sigma_0^m$ whenever $|y|, \ldots, |h^{m-1}(y)| \geq \delta_1$, see e.g. [MS, Theorem III.3.3]. Then, by continuity (suppose $\alpha$ small enough), a similar fact holds for $\partial_x f$:

(6) $$\prod_{j=0}^{m-1} |\partial_x f(\tau_j, y_j)| \geq \sigma_0^m \text{ whenever } (\tau, y) \in S^1 \times I_0 \text{ has } |y_0|, \ldots, |y_{m-1}| \geq \delta_1.$$

As a consequence, there is $A > 0$ such that

(7) $$\prod_{j=0}^{n-1} |\partial_x f(\tau_j, y_j)| \geq A \sigma_0^n \text{ for all } n \geq 1 \text{ and } (\tau, y) \text{ with } |y_0|, \ldots, |y_{n-1}| \geq \delta_1.$$

Moreover, there is a constant $0 < \kappa < 1$ such that, reducing $\delta_1 > 0$ and $\sigma_0 > 1$ if necessary, $\left|(h^l)'(y)\right| > \kappa \sigma_0^l$ whenever $|y|, \ldots, \left|h^{l-1}(y)\right| \geq \delta_1 > \left|h^l(y)\right|$: this follows from [No], together with (6) and a continuity argument. Then we restrict to $l < m$ and invoke continuity once more to conclude a similar statement for $\partial_x f$. Combining this with (6) we get

(8) $$\prod_{j=0}^{n-1} |\partial_x f(\tau_j, y_j)| \geq \kappa \sigma_0^n \text{ whenever } |y_0|, \ldots, |y_{n-1}| \geq \delta_1 > |y_n|.$$

Now let $(\theta, x)$ be as in the statement and let $j_1 < \cdots < j_s$ be the values of $j \in \{0, \ldots, k-1\}$ for which $|x_j| < \delta_1$. Clearly, we may suppose $s > 0$ for otherwise the lemma follows immediately from (7), (8). When $|x_k| < \delta_1$ we also set $j_{s+1} = k$. On the other hand, we denote $p_i = p(x_{j_i})$, $i = 1, \dots, s$, and then Lemma 2.4 gives

$$\prod_{j=j_i}^{j_i+p_i-1} |\partial_x f(\theta_j, x_j)| \geq \frac{1}{\kappa}\sigma_1^{p_i}, \tag{9}$$

for all $i < s$. Moreover, (9) holds also for $i = s$ if $j_s + p_s \leq k$; note that this is necessarily the case if $|x_k| < \delta_1$, as our definition of $p(x)$ implies $j_i + p_i \leq j_{i+1}$ for all $i$, see above. On the other hand, by (8),

$$\prod_{j=0}^{j_1-1} |\partial_x f(\theta_j, x_j)| \geq \kappa\sigma_0^{j_1} \quad \text{and} \quad \prod_{j=j_i+p_i}^{j_{i+1}-1} |\partial_x f(\theta_j, x_j)| \geq \kappa\sigma_0^{j_{i+1}-j_i-p_i}, \tag{10}$$

for all $i < s$ and, again, the second inequality remains valid for $i = s$ when $|x_k| < \delta_1$. At this point we take $\sigma_2 = \min\{\sigma_0, \sigma_1\}$ and get

$$\prod_{j=0}^{k} |\partial_x f(\theta_j, x_j)| \geq \kappa\sigma_0^{j_1} \prod_{i=1}^{s} \left(\sigma_1^{p_i}\sigma_0^{j_{i+1}-j_i-p_i}\right) \geq \kappa\sigma_2^n$$

whenever $|x_k| < \delta_1$. This proves the second part of the lemma. As for the first one, it follows from a similar calculation and the remark that in general (that is, even if (9), (10) are not valid for $i = s$)

$$\prod_{j=j_s}^{k-1} |\partial_x f(\theta_j, x_j)| \geq (2-\alpha)\,|x_{j_s}|\, A\sigma_0^{k-j_s-1} \geq A\sqrt{\alpha}\sigma_0^{k-j_s-1},$$

as a consequence of (7). □

## 2.3 A technical lemma

We are now in a position to explicit our choice of $\eta$: having in mind the proof of Lemma 2.6 below we take $\eta = \log\sigma_2/(4\log 32)$. On the other hand, we introduce $M = M(\alpha)$ to be the maximum integer such that $32^M\alpha \leq 1$; note that $M < N$, since $\rho \leq \sup|h'| \leq 4$, recall also the proof of Lemma 2.4. Finally, for $r \geq 0$ we denote $J(r) = \{x \in \mathbb{R}\colon |x| < \sqrt{\alpha}e^{-r}\}$.

**Lemma 2.6** *There are $C_3 > 0$ and $\beta > 0$ such that, given any admissible curve $\hat{Y}_0 = \operatorname{graph}(Y_0)$ and any $r \geq (\frac{1}{2} - 2\eta)\log\frac{1}{\alpha}$,*

$$m\left(\{\theta \in S^1 : \hat{Y}_M(\theta) \in S^1 \times J(r-2)\}\right) \leq C_3 e^{-5\beta r}$$

Before proving this lemma let us state and prove the following auxiliary result. We take $\hat{X} = \operatorname{graph}(X)$ to be an admissible curve and for $1 \le j \le d$ we denote $\hat{Z}_j = \varphi(\hat{X} | [\tilde{\theta}_{j-1}, \tilde{\theta}_j)) = \operatorname{graph}(Z_j)$.

**Lemma 2.7** *There are $H_1, H_2 \subset \{1, \ldots, d\}$ with $\# H_1, \# H_2 \ge [d/16]$ such that $|Z_{j_1}(\theta) - Z_{j_2}(\theta)| \ge \alpha/100$ for all $\theta \in S^1$, $j_1 \in H_1$, and $j_2 \in H_2$.*

**Proof:** Let $\hat{Z}(\theta) = (g(\theta), Z(\theta)) = \varphi(\hat{X}(\theta))$ and $\chi_1 < \chi_2$ be the two critical points of $Z$, recall the proof of Lemma 2.2. We set $l = [d/16]$ and define $k_i$ by $\chi_i \in [\tilde{\theta}_{k_i - 1}, \tilde{\theta}_{k_i})$, $i = 1, 2$. If neither of $\chi_1$, $\chi_2$ belongs to $[1/4, 3/4]$ then we take $H_1 = \{k_1 + 1, \ldots, k_1 + l\}$ and $H_2 = \{k_2 - l, \ldots, k_2 - 1\}$. Observe that $\tilde{\theta}_{k_1 + l} < \frac{1}{4} + \frac{l+1}{d-\alpha} < \frac{1}{2} - \frac{1}{2\pi} \arcsin \frac{1}{3}$ and, analogously, $\tilde{\theta}_{k_2 - l} > \frac{1}{2} + \frac{1}{2\pi} \arcsin \frac{1}{3}$. Moreover, $Z$ is monotone decreasing on $[\tilde{\theta}_{k_1}, \tilde{\theta}_{k_2 - 1})$. Hence, using also $|Z' | \mathcal{A}_1| \ge \frac{\alpha}{2}$, we get $\inf Z | [\tilde{\theta}_{j_1 - 1}, \tilde{\theta}_{j_1}) - \sup Z | [\tilde{\theta}_{j_2 - 1}, \tilde{\theta}_{j_2}) \ge \frac{\alpha}{2\pi} \arcsin \frac{1}{3} \ge \alpha/100$ for every $j_1 \in H_1$, $j_2 \in H_2$. Clearly, this proves the lemma in this case. On the other hand, if $\chi_1 \ge 1/4$ (resp. $\chi_2 \le 3/4$), we take $H_1 = \{k_1 - l, \ldots, k_1 - 1\}$, $H_2 = \{1, \ldots, l\}$ (resp. $H_1 = \{d - l + 1, \ldots, d\}$, $H_2 = \{k_2 + 1, \ldots, k_2 + l\}$) and then similar estimates yield the same conclusion as in the previous case. □

**Proof of Lemma 2.6:** Let us begin by giving a brief sketch of the proof. By Lemma 2.1, $\hat{Y}_M$ is the union of $d^M$ admissible curves. We fix a constant $\gamma_1 > 0$ and organize the set of these admissible curves into subsets, each of which containing $d^{\gamma_1 r}$ elements, in such a way that curves belonging to a same subset are spread along the $x$-direction: at most $(d - [d/16])^{\gamma_1 r}$ of them are within $|J(r-2)| \approx \operatorname{const} e^{-r} \sqrt{\alpha}$ from each other. We obtain this by combining the previous result with the expansion given by Lemma 2.5. Then at most that many curves intersect each $\{\theta\} \times J(r-2)$, hence $m(\{\theta : Y_M(\theta) \in J(r-2)\}) \le \operatorname{const} ((d - [d/16])/d)^{\gamma_1 r}$, from which the lemma follows.

Now we come to the details. Let $\hat{Y}_j(\theta) = \varphi^j(\theta, Y_0(\theta)) = (g^j(\theta), Y_j(\theta))$. We also use $C$ to represent any large positive constant depending only on $h$. Note first that $\operatorname{osc}(Y_0) \le \alpha$ and $\operatorname{osc}(Y_j) \le 4 \operatorname{osc}(Y_{j-1}) + 2\alpha$, where $\operatorname{osc}(Y_j) = \sup Y_j - \inf Y_j$. As a consequence, $\operatorname{osc}(Y_j) \le 2\alpha 4^j \le 2\left(32^{-M} 4^j\right)$, $\operatorname{osc}(Y_M) \le 2\alpha^{3/5} < \sqrt{\alpha}$. Clearly, we may suppose that $|Y_M(\tau)| < \sqrt{\alpha}$ at some $\tau \in S^1$ (otherwise the conclusion of the lemma is obvious) and then

$$|Y_M(\theta)| \le 2\sqrt{\alpha} \ (< \delta_1) \quad \text{for every } \theta \in S^1. \tag{11}$$

Let us denote $\mathcal{O} = \{h^i(0) : i \ge 1\}$ and $\delta_j(\theta) = \operatorname{dist}(Y_j(\theta), \mathcal{O})$. The same argument as in (5) yields, for all $\theta \in S^1$, $0 \le j \le M-1$, and $1 \le i \le M - j$,

$$\delta_{j+i}(\theta) \le C 4^i \left(\alpha + |Y_j(\theta)|^2\right). \tag{12}$$

We claim that $|Y_j(\theta)| \ge \sqrt{\alpha}$ for every $\theta \in S^1$ and $0 \le j \le M-1$. Indeed, if it were not so then $\delta_M(\theta) \le C 4^{M-j} \alpha \le C 4^M \alpha \le C\sqrt{\alpha}$ for some $\theta \in S^1$. Up to assuming $\alpha$ sufficiently small, this would contradict (11) (recall that $\mathcal{O}$ is a finite set not containing zero) and so our claim is justified. Note that this argument proves somewhat more: (taking $C \gg 1/\operatorname{dist}(0, \mathcal{O})$)

$$4^{M-j}\left|Y_j(\theta)\right|^2 \geq \frac{1}{C} \quad \text{for all } \theta \in S^1 \text{ and } 0 \leq j \leq M-1. \tag{13}$$

Now we derive a uniform bound for the distortion of $\partial_x f$ on iterates of $\hat{Y}_0$. Note that by (1), (2), we may write $\partial_x f(\theta, x) = x\psi(\theta, x)$ with $|\psi + 2| \leq \alpha$ at every point. Then given $0 \leq j \leq M-1$, $(\theta_j, x_j), (\tau_j, y_j) \in \hat{Y}_j$, and $1 \leq i \leq M-j$,

$$\left|\frac{\partial_x f^i(\theta_j, x_j)}{\partial_x f^i(\tau_j, y_j)}\right| = \prod_{m=j}^{j+i-1}\left|\frac{x_m}{y_m}\right| \cdot \prod_{m=j}^{j+i-1}\left|\frac{\psi(\theta_m, x_m)}{\psi(\tau_m, y_m)}\right|. \tag{14}$$

Our previous estimates imply, recall (13),

$$\left|\frac{x_m}{y_m} - 1\right| \leq \frac{2\alpha 4^m}{\sqrt{\frac{1}{C}4^{m-M}}} \leq C\alpha 4^M < \sqrt{\alpha} \text{ and } \left|\frac{\psi(\theta_m, x_m)}{\psi(\tau_m, y_m)} - 1\right| \leq \frac{2\alpha}{2-\alpha} < \sqrt{\alpha}.$$

Hence, (14) is bounded by $(1+\sqrt{\alpha})^{2i} \leq e^{2M\sqrt{\alpha}} \leq 2$ (we use $M \approx \log\frac{1}{\alpha}$ and also assume $\alpha$ small enough). Altogether, this proves that, given any $0 \leq j \leq M-1$ and $1 \leq i \leq M-j$,

$$\left|\frac{\partial_x f^i(\theta_j, x_j)}{\partial_x f^i(\tau_j, y_j)}\right| \leq 2 \quad \text{for every } (\theta_j, x_j), (\tau_j, y_j) \in \hat{Y}_j.$$

We fix an arbitrary $\hat{y} \in \hat{Y}_0$ and let $\lambda_j = |\partial_x f^{M-j}(\varphi^j(\hat{y}))|$. Lemma 2.5, together with (11), gives $\lambda_j \geq C_2\sigma_2^{M-j}$ for $0 \leq j \leq M-1$. On the other hand, the previous inequality gives

$$\frac{1}{2}\frac{\lambda_j}{\lambda_{i+j}} \leq \left|\partial_x f^i(\theta_j, x_j)\right| \leq 2\frac{\lambda_j}{\lambda_{i+j}} \quad \text{for all } (\theta_j, x_j) \in \hat{Y}_j. \tag{15}$$

We fix $K = 400e^2$ and consider $t_1 < t_2 < \cdots \leq M$ given by $t_1 = 1$ and

$$t_{i+1} = \min\{s\colon t_i < s \leq M \text{ and } \lambda_{t_i} \geq 2K\lambda_s\} \quad \text{(if it exists)}.$$

Moreover, given $r \geq \left(\frac{1}{2} - 2\eta\right)\log\frac{1}{\alpha}$ we set $k = k(r) = \max\{i\colon \lambda_{t_i} \geq 2Ke^{-r}/\sqrt{\alpha}\}$ and then we claim that $k(r) \geq \gamma_1 r$ for some constant $\gamma_1 > 0$. Indeed, we have $\lambda_{t_i} \leq 2K\lambda_{t_{i+1}-1} \leq 8K\lambda_{t_{i+1}}$ for all $i$ and so $\lambda_{t_{k+1}} \geq C_2\sigma_2^{M-1}(8K)^{-k}$. Also, by definition, $\lambda_{t_{k+1}} \leq 2Ke^{-r}/\sqrt{\alpha}$. Putting these two inequalities together we get (recall also the definition of $\eta$ and $M$)

$$\begin{aligned} k\log(8K) &\geq r + M\log\sigma_2 - \tfrac{1}{2}\log\tfrac{1}{\alpha} + C \geq r - \left(\tfrac{1}{2} - 4\eta\right)\log\tfrac{1}{\alpha} + C \\ &\geq r\left(1 - \tfrac{1/2-4\eta}{1/2-2\eta}\right) + C \geq \eta r, \end{aligned}$$

which proves our claim.

For each $\bar{l} = (l_1, \ldots, l_M) \in \{1, \ldots, d\}^M$ we denote by $\omega(\bar{l})$ the only element $\omega \in \mathcal{P}_M$ satisfying $g^{i-1}(\omega) \subset [\tilde{\theta}_{l_i-1}, \tilde{\theta}_{l_i})$, $i = 1, \ldots, M$. Given $1 \le j \le M$ we let $\hat{Y}_j(\bar{l}) = \operatorname{graph}(Y_j(\bar{l})) = \varphi^j(\hat{Y}_0|\omega(\bar{l}))$. We call $\bar{l}$ and $\bar{m}$ *incompatible* if

$$\left|Y_M(\bar{l}, \theta) - Y_M(\bar{m}, \theta)\right| \ge 4e^{2-r}\sqrt{\alpha} \text{ for all } \theta \in S^1 :$$

observe that this implies that $\hat{Y}_M(\bar{l})$ and $\hat{Y}_M(\bar{m})$ can not both intersect a same vertical segment $\{\theta\} \times J(r-2)$. By Lemma 2.7 there are $H_1', H_1'' \subset \{1, \ldots, d\}$ with $\#H_1', \#H_1'' \ge [d/16]$ such that given any $l_1' \in H_1'$ and $l_1'' \in H_1''$

$$|Y_1(l_1', l_2, \ldots, l_M, \theta) - Y_1(l_1'', l_2, \ldots, l_M, \theta)| \ge \frac{\alpha}{100}$$

for all $\theta \in g(\omega(l_1', l_2, \ldots, l_M)) = g(\omega(l_1'', l_2, \ldots, l_M))$ and $l_2, \ldots, l_M$. Then, by (15),

$$|Y_M(l_1', l_2, \ldots, l_M, \theta) - Y_M(l_1'', l_2, \ldots, l_M, \theta)| \ge \frac{\lambda_1}{2}\frac{\alpha}{100} \ge 4e^{2-r}\sqrt{\alpha} \text{ for } \theta \in S^1$$

(because $1 \le k(r)$), that is $(l_1', l_2, \ldots, l_M)$ and $(l_1'', l_2, \ldots, l_M)$ are incompatible for every $l_2, \ldots, l_M$. In fact, we claim that all pairs $(l_1', l_2, \ldots, l_{t_2-1}, l_{t_2}', \ldots, l_M')$, $(l_1'', l_2, \ldots, l_{t_2-1}, l_{t_2}'', \ldots, l_M'')$ are incompatible. Observe that,

$$|Y_{t_2}(l_1', l_2, \ldots, l_M, \theta) - Y_{t_2}(l_1'', l_2, \ldots, l_M, \theta)| \ge \frac{\lambda_1}{2\lambda_{t_2}}\frac{\alpha}{100} \ge 4e^2\alpha,$$

as a consequence of (15) and the definition of $t_2$. On the other hand,

$$\begin{aligned}&\left|Y_{t_2}(l_1', l_2, \ldots, l_{t_2-1}, l_{t_2}', \ldots, l_M')(\theta) - Y_{t_2}(l_1', l_2, \ldots, l_{t_2-1}, l_{t_2}, \ldots, l_M)(\theta)\right| \le \\ &\quad \le \operatorname{osc}(\varphi(\hat{Y}_{t_2-1}(l_1', l_2, \ldots, l_{t_2-1}))) \le 8\alpha\end{aligned}$$

for all $\theta \in S^1$, and a similar fact holds for $Y_{t_2}(l_1'', \ldots)$. Therefore,

$$\begin{aligned}&\left|Y_M(l_1', l_2, \ldots, l_{t_2-1}, l_{t_2}', \ldots, l_M', \theta) - Y_M(l_1'', l_2, \ldots, l_{t_2-1}, l_{t_2}'', \ldots, l_M'', \theta)\right| \ge \\ &\quad \ge \tfrac{1}{2}\lambda_{t_2}(4e^2 - 16)\alpha \ge 4e^{2-r}\sqrt{\alpha}\end{aligned}$$

(because $2 \le k(r)$), proving our claim. Now we just repeat this argument for each of the successive $t_i$: at the $i$th step and for each fixed $L_i = (l_1, \ldots, l_{t_i-1})$, we find $H_i', H_i''$ with $\#H_i', \#H_i'' \ge [d/16]$ such that given any $l_{t_i}' \in H_i'$ and $l_{t_i}'' \in H_i''$ then all pairs $(L_i, l_{t_i}', l_{t_i+1}, \ldots, l_M)$, $(L_i, l_{t_i}'', l_{t_i+1}, \ldots, l_M)$ are incompatible and, in fact, the same is true for every pair $(L_i, l_{t_i}', l_{t_i+1}, \ldots, l_{t_{i+1}-1}, l_{t_{i+1}}', \ldots, l_M')$, $(L_i, l_{t_i}'', l_{t_i+1}, \ldots, l_{t_{i+1}-1}, l_{t_{i+1}}'', \ldots, l_M'')$ as long as $i+1 \le k(r)$. In this way we conclude that each segment $\{\theta\} \times J(r-2)$ intersects at most $d^{M-k(r)} \cdot (d - [d/16])^{k(r)}$ admissible curves $\hat{Y}(\bar{l})$. Using also $|(g^M)'| \ge (d-\alpha)^M \ge \text{const}\, d^M$ (because $M \le \text{const} \log \frac{1}{\alpha}$), we conclude

$$m\left(\{\theta : \hat{Y}_M(\theta) \in S^1 \times J(r-2)\}\right) \le \frac{d^M((d-[d/16])/d)^{k(r)}}{(d-\alpha)^M} \le \text{const}\left(\frac{99}{100}\right)^{\gamma_1 r}$$

and the lemma follows by taking $\beta = \frac{\gamma_1}{5}\log\left(\frac{100}{99}\right)$. $\square$

## 2.4 Large deviations

Now we use the previous lemmas to complete the proof of Theorem A. In all that follows we let $n \geq 1$ be fixed, sufficiently large. We define $m \geq 1$ by $m^2 \leq n < (m+1)^2$ and take also $l = m - M$, where $M = M(\alpha)$ is as above. Note that $l \approx m \approx \sqrt{n}$ as long as $n \gg \log \frac{1}{\alpha}$. Recall also that we are considering an arbitrary admissible curve $\hat{X}_0$. Given $1 \leq \nu \leq n$ and $\omega_{\nu+l} \in \mathcal{P}_{\nu+l}$, we set $\gamma = \varphi^\nu(\hat{X}_0|\omega_{\nu+l})$. Then we say that $\nu$ is

- a $\mathrm{I}_n$-*situation* for $\theta \in \omega_{\nu+l}$ if $\gamma \cap (S^1 \times J(0)) \neq \emptyset$ but $\gamma \cap (S^1 \times J(m)) = \emptyset$;
- a $\mathrm{II}_n$-*situation* for $\theta \in \omega_{\nu+l}$ if $\gamma \cap (S^1 \times J(m)) \neq \emptyset$.

Note that, by Lemma 2.1, $\gamma$ is the graph of a function defined on $g^\nu(\omega_{\nu+l}) \in \mathcal{P}_l$ and whose derivative is bounded above by $\alpha$. Therefore, its diameter in the $x$-direction is bounded by $\alpha(d-\alpha)^{-l} \ll \sqrt{\alpha}e^{-m}$. This means that whenever $\nu$ is a $\mathrm{II}_n$-situation for $\omega_{\nu+l}$ then $\gamma \subset (S^1 \times J(m-1))$. At this point we introduce $B_2(n) = \{\theta \in S^1\colon \text{some } 1 \leq \nu \leq n \text{ is a } \mathrm{II}_n\text{-situation for } \theta\}$ and then Corollary 2.3 gives

$$m(B_2(n)) \leq nC_1\sqrt{\frac{|J(m-1)|}{\alpha}} \leq \text{const}\,\alpha^{-1/4}ne^{-m/2} \leq \text{const}\,e^{-\sqrt{n}/4}. \tag{16}$$

From now on we consider only values of $\theta \in S^1 \backslash B_2(n)$ that is, having no $\mathrm{II}_n$-situations in $[1, n]$. Let $1 \leq \nu_1 < \cdots < \nu_s \leq n$ be the $\mathrm{I}_n$-situations of $\theta$. Note that our definition of $N$ (recall the proof of Lemma 2.4) implies $\nu_{i+1} \geq \nu_i + N$ for every $i$; in particular $(s-1)N \leq n$. For each $\nu = \nu_i$ we fix $r = r_i \in \{1, \ldots, m\}$ minimum such that $\gamma \cap (S^1 \times J(r)) = \emptyset$. Then, by Lemma 2.4,

$$\prod_{\nu_i}^{\nu_i+N-1} \left|\partial_x f(\hat{X}_j(\theta))\right| \geq e^{-r_i}\alpha^{-1/2+\eta},$$

for each $1 \leq i < s$. On the other hand, Lemma 2.5 gives

$$\prod_{1}^{\nu_1-1} \left|\partial_x f(\hat{X}_j(\theta))\right| \geq C_2\sigma_2^{\nu_1-1} \quad \text{and} \quad \prod_{\nu_i+N}^{\nu_{i+1}-1} \left|\partial_x f(\hat{X}_j(\theta))\right| \geq C_2\sigma_2^{\nu_{i+1}-\nu_i-N},$$

for every $1 \leq i < s$ and also

$$\prod_{\nu_s}^{n} \left|\partial_x f(\hat{X}_j(\theta))\right| \geq (2-\alpha)\,|x_{\nu_s}|\,C_2\sqrt{\alpha}\sigma_2^{n-\nu_s} \geq \text{const}\,\alpha e^{-r_s}\sigma_2^{n-\nu_s}.$$

Altogether, this yields the following lower bound for $\log \prod_{j=1}^{n} \left|\partial_x f(\hat{X}_j(\theta))\right|$:

$$(n-(s-1)N)\log\sigma_2 + \sum_{i=1}^{s}\left(\left(\frac{1}{2}-\eta\right)\log\frac{1}{\alpha} - r_i\right) - s\,\text{const} - \frac{3}{2}\log\frac{1}{\alpha}.$$

We consider $G = \{i \colon r_i \geq (\frac{1}{2}-2\eta)\log\frac{1}{\alpha}\}$ (note that it depends on $\theta$) and then

$$\sum_{i=1}^{s}\left(\left(\frac{1}{2}-\eta\right)\log\frac{1}{\alpha} - r_i\right) \geq -\sum_{i\in G} r_i + \eta s\log\frac{1}{\alpha} \geq -\sum_{i\in G} r_i + \gamma_2 N s$$

for some $\gamma_2 > 0$ independent of $\alpha$ or $n$ (because $N \approx \log\frac{1}{\alpha}$). Replacing above we get

$$\log\prod_{1}^{n}\left|\partial_x f(\hat{X}_j(\theta))\right| \geq 3cn - \sum_{i\in G} r_i - s\,\text{const} - \frac{3}{2}\log\frac{1}{\alpha} \geq 2cn - \sum_{i\in G} r_i$$

where $c = \frac{1}{3}\min\{\gamma_2, \log\sigma_2\}$ and we use $n \gg \log\frac{1}{\alpha} \approx N \gg 1$. Now we introduce $B_1(n) = \{\theta \in S^1 \colon \sum_{i\in G} r_i \geq cn\}$ and set $E_n = B_1(n) \cup B_2(n)$. Then

$$\log\prod_{1}^{n}\left|\partial_x f(\hat{X}_j(\theta))\right| \geq cn \quad \text{for every } \theta \in S^1 \backslash E_n.$$

In view of (16), we are left to prove that $m(B_1(n)) \leq \text{const}\, e^{-\gamma\sqrt{n}}$ for some $\gamma > 0$. We deduce this from Lemma 2.6 by means of a large deviations argument. First we let $0 \leq q \leq m-1$ be fixed and denote

$$G_q = \{i \in G \colon \nu_i \equiv q \bmod m\}.$$

We also take $m_q = \max\{j \colon mj + q \leq n\}$ (note $m_q \approx m \approx \sqrt{n}$) and for each $0 \leq j \leq m_q$ we let $\hat{r}_j = r_i$ if $mj + q = \nu_i$, some $i \in G_q$, and $\hat{r}_j = 0$ otherwise. Observe that $G_q$ and the $\hat{r}_j$ are, in fact, functions of $\theta$. Then we introduce

$$\Omega_q(\rho_0, \ldots, \rho_{m_q}) = \{\theta \in S^1 \backslash B_2(n) \colon \hat{r}_j = \rho_j \text{ for } 0 \leq j \leq m_q\}$$

where for each $j$ either $\rho_j = 0$ or $\rho_j \geq \left(\frac{1}{2}-2\eta\right)\log\frac{1}{\alpha}$; we also assume the $\rho_j$ not to be simultaneously zero. Consider $0 \leq j \leq m_q$ and $\omega_{mj+q+l} \in \mathcal{P}_{mj+q+l}$. Recall that our construction is such that the value of $\hat{r}_j$ is constant on $\omega_{mj+q+l}$. Now $\hat{Y}_0 = \varphi^{mj+q+l}(\hat{X}_0|\omega_{mj+q+l})$ is an admissible curve and we have defined $l$ in such a way that $mj + q + l = m(j+1) + q - M$. Therefore, we are in a position to apply Lemma 2.6 to this curve and obtain in this way

$$m\left(\{\theta \in \omega_{mj+q+l} \colon \hat{r}_{j+1} = \rho\}\right) \leq C_* C_3 e^{-5\beta\rho} \quad \text{for all } \rho \geq \left(\frac{1}{2}-2\eta\right)\log\frac{1}{\alpha}.$$

Here, as before, $C_*$ is a uniform upper bound for the metric distortion of the iterates of $g$. Repeating this reasoning for each $0 \le j \le m_q$ we conclude that

$$m\left(\Omega_q(\rho_0, \ldots, \rho_{m_q})\right) \le C_4^\tau e^{-5\beta \sum \rho_j},$$

where $C_4 = C_* C_3$ and $\tau = \#\{j : \rho_j \ne 0\}$. As a consequence,

$$\int e^{2\beta \sum_{i \in G_q} r_i} \, d\theta \le \sum_{(\rho_0, \ldots, \rho_{m_q})} C_4^\tau e^{-3\beta \sum \rho_j} \le \sum_{\tau, R} C_4^\tau \zeta(\tau, R) e^{-3\beta R},$$

where the integral is taken over the union of all the sets $\Omega_q(\rho_0, \ldots, \rho_{m_q})$ for all possible $(\rho_0, \ldots, \rho_{m_q})$, and $\zeta(\tau, R)$ is the number of integer solutions of the equation $x_1 + \cdots + x_\tau = R$ satisfying $x_j \ge \left(\frac{1}{2} - 2\eta\right) \log \frac{1}{\alpha}$ for all $j$. Now, for some absolute constant $K > 0$,

$$\zeta(\tau, R) \le \frac{(R+\tau)!}{R!\,\tau!} \le K \frac{(R+\tau)^{R+\tau}}{R^R\,\tau^\tau} = \left(K^{\frac{\tau}{R}} \left(1 + \frac{\tau}{R}\right) \left(1 + \frac{R}{\tau}\right)^{\frac{\tau}{R}}\right)^R \le e^{\beta R}.$$

For the last inequality we use the fact that $R/\tau \ge$ const $\log \frac{1}{\alpha}$, which ensures that all three factors can be made arbitrarily close to 1 by taking $\alpha$ sufficiently small. For this same reason we may also suppose $C_4^\tau \le e^{\beta R}$. It follows that

$$\int e^{2\beta \sum_{i \in G_q} r_i} \, d\theta \le \sum_{\tau, R} e^{-\beta R} \le \sum_R R e^{-\beta R} \le 1,$$

since $\tau \le R$ and $R \ge \left(\frac{1}{2} - 2\eta\right) \log \frac{1}{\alpha} \gg 1$ (because $\tau \ge 1$). Therefore,

$$m\left(\left\{\theta : \sum_{i \in G_q} r_i \ge \frac{cn}{m}\right\}\right) \le e^{-2c\beta n/m} \int e^{2\beta \sum_{i \in G_q} r_i} \, d\theta \le e^{-2c\beta n/m}.$$

Now, clearly, $\theta \in B_1(n) \Rightarrow \sum_{i \in G_q} r_i \ge \frac{cn}{m}$ for some $0 \le q \le m-1$ and so

$$m\left(B_1(n)\right) \le m e^{-2c\beta n/m} \le e^{-\gamma\sqrt{n}}, \qquad \text{for } \gamma = c\beta. \tag{17}$$

This concludes the proof of the theorem, under the simplifying assumption (1).

## 2.5 Conclusion of the proof and extensions

Now, we prove the theorem in full generality: we take $\varphi$ to be any $C^3$ map of the form $\varphi(\theta, x) = (g(\theta, x), f(\theta, x))$ satisfying $\|\varphi - \varphi_\alpha\| \le \varepsilon$, where $\varepsilon > 0$ is small with respect to $\alpha$, and we explain how the conclusion of theorem may be obtained for $\varphi$ by a variation of the previous argument.

The first step is to show that such a $\varphi$ always admits an invariant foliation $\mathcal{F}^c$ by nearly vertical smooth curves. This is a direct consequence of the fact that the vertical straight lines $\{\theta = \text{ const }\}$ constitute a *normally expanding invariant foliation* for $\varphi_\alpha$, see [HPS], but we sketch the main points in the proof, since results on persistence of normally hyperbolic objects are somewhat less standard in this setting of non-invertible dynamics. Let $\mathcal{X}$ be the space of continuous maps $\xi: S^1 \times I_0 \longrightarrow [-1,1]$, endowed with the sup-norm, and define $F: \mathcal{X} \longrightarrow \mathcal{X}$ by

$$F\xi(z) = \frac{\partial_x f(z)\xi(\varphi(z)) - \partial_x g(z)}{-\partial_\theta f(z)\xi(\varphi(z)) + \partial_\theta g(z)}, \qquad z = (\theta, x) \in S^1 \times I_0.$$

Note that $F$ is indeed well defined

$$|F\xi(z)| \le \frac{(4+\varepsilon)+\varepsilon}{-(\text{ const }\alpha + \varepsilon) + (d-\varepsilon)} < 1$$

and, moreover, it is a contraction on $\mathcal{X}$: $|F\xi - F\eta|$ is bounded above by

$$\frac{|\det D\varphi|\, |\xi - \eta|}{|(-\partial_\theta f\xi + \partial_\theta g)(-\partial_\theta f\eta + \partial_\theta g)|} \le \frac{((d+\varepsilon)(4+\varepsilon)+\varepsilon)\, |\xi-\eta|}{(d - \text{ const }\alpha)^2} \le \frac{1}{2}\, |\xi - \eta|\, .$$

Let $\xi^c \in \mathcal{X}$ be the fixed point of $\mathcal{X}$. Then we take $\mathcal{F}^c$ to be the integral foliation of the vector field $z \mapsto (\xi^c(z), 1)$. Note that we defined $F$ in such a way that, for every $z \in S^1 \times I_0$, $D\varphi(z)$ maps $E^c(z) = \text{span}\{(\xi^c(z),1)\}$ to $E^c(\varphi(z))$ and this implies that $\mathcal{F}^c$ is invariant. It also follows from the methods of [HPS] that the leaves of $\mathcal{F}^c$ are as smooth as the map $\varphi$, i.e. they are $C^3$-embedded intervals; moreover, they approach vertical segments, uniformly in the $C^3$ topology, as $\varepsilon \to 0$. Note that, in general, $\mathcal{F}^c$ is not a smooth foliation (its holonomy maps may not even be Lipschitz continuous).

Existence of such an invariant foliation replaces the skew-product assumption of (1) in the general case of the theorem. More precisely, what we do now is to show that almost every $z = (\theta, x) \in S^1 \times I_0$ has positive Liapunov exponent *along the direction of* $E^c(z)$. In order to do this we introduce $\Delta(z)$ defined by $D\varphi(z)(\xi^c(z),1) = \Delta(z)(\xi^c(\varphi(z)),1)$, i.e. $\Delta(z) = \partial_\theta f(z)\xi^c(z) + \partial_x f(z)$. We also need an analog of the second part of (1). Let the *critical set* $\mathcal{C}$ of $\varphi$ be defined by $\mathcal{C} = \{z \in S^1 \times I_0 \colon \Delta(z) = 0\}$. We claim that $\mathcal{C} = \text{ graph}\,(\eta)$ for some $C^2$ map $\eta: S^1 \longrightarrow I_0$ and, moreover, $\eta$ is $C^2$-close to zero if $\varepsilon > 0$ is small. Indeed, it is clear that $z \in \mathcal{C}$ implies $\det D\varphi(z) = 0$ and the converse is also easy to deduce: if $z$ is such that $\det D\varphi(z) = 0$ then the image of $D\varphi(z)$ is a one-dimensional subspace with slope $|\partial_\theta f(z)/\partial_\theta g(z)| \ll 1$; on the other hand, $D\varphi(z)(\xi^c(z),1)$ is colinear to $(\xi^c(\varphi(z)),1)$, a nearly vertical vector; hence, it must be $D\varphi(z)(\xi^c(z),1) = 0$. Therefore, our claim follows directly from an implicit function argument applied to $\det D\varphi(\theta,x) = 0$. Now, this means that up to a $C^2$ change of coordinates $C^2$-close to the identity we may suppose $\eta \equiv 0$ and, hence, write $\Delta(\theta,x) = x\psi(\theta,x)$

with $|\psi + 2|$ close to zero if $\varepsilon$ and $\alpha$ are small. We define *admissible curve* in just the same way as before but a few words are required concerning the definition of the partitions $\mathcal{P}_n$ in the present setting. Indeed, since $\mathcal{F}^c$ is usually not a smooth foliation, there is no natural smooth structure (let alone smooth expanding action of the dynamics) on the space of its leaves, as happened in the previous case. Instead, we let $\Theta_0$ denote the leaf of $\mathcal{F}^c$ close to $\{\theta = 0\}$ which is fixed under $\varphi$ and we define $\mathcal{P}_n$ to be the set of all intervals $[\theta', \theta'')$ such that $(\theta', \theta'')$ is a connected component of $\hat{X}_n^{-1}((S^1 \times I_0) \backslash \Theta_0)$. Note that this depends on the admissible curve $\hat{X}_0$ (in an unimportant way). On the other hand, it is easy to check that $(d + \operatorname{const} \alpha)^{-n} \leq |\omega| \leq (d - \operatorname{const} \alpha)^{-n}$ for every $\omega \in \mathcal{P}_n$. At this point we may use the same argument as before, with $\partial_x f(\theta, x)$ replaced by $\Delta(\theta, x)$, to show that $\prod_{j=1}^{n} \Delta(\varphi^j(x, \theta))$ grows exponentially almost surely. The proof of the theorem is complete.

Concluding this section, we mention two easy extensions of the arguments we have presented. To start with, we note that the quadratic map $h(x) = a_0 - x^2$ may be replaced by any unimodal or multimodal map with negative schwarzian derivative and having all critical points nondegenerate (quadratic) and preperiodic. In particular, one may take such a map defined on the circle, for instance $h(x) = x + a_0 \sin 2\pi x \bmod 1$, $a_0 = 3/4$. Then the same arguments as before yield a $C^3$ open set of maps of the torus $T^2 = S^1 \times S^1$ exhibiting multidimensional expanding behaviour on the whole manifold $U = T^2$.

Next, we explain how these arguments can be easily adapted to give a higher-dimensional version of our construction. We consider $\varphi_\alpha \colon T^m \times \mathbb{R} \longrightarrow T^m \times \mathbb{R}$, $(\theta, x) \mapsto (\hat{g}(\theta), \hat{f}_\alpha(\theta, x))$, where $\hat{g}$ is an expanding map on the $m$-torus $T^m$ and $\hat{f}_\alpha(\theta, x) = a_0 + \alpha\phi(\theta) - x^2$, $a_0$ as before. For simplicity, we take $\hat{g}$ to be linear and to have a unique largest eigenvalue $\lambda_u$. Then we suppose the function $\phi$ to vary in a Morse fashion along the corresponding eigendirection $\omega_u$. In this setting we take *admissible curve* to mean a curve of the form $\{(\Theta(t) = \theta_0 + t\omega_u, X(t))\} \subset T^m \times \mathbb{R}$ with $|X'|$, $|X''|$ small. Then, up to assuming $\lambda_u$ sufficiently large (depending only on the Morse function $\phi$), the same arguments as before prove that for small enough $\alpha$ the map $\varphi_\alpha$ has $m+1$ positive Liapunov exponents at $\varphi(\Theta(t), X(t))$, for almost every $t$. Moreover, the same remains true for all small perturbations of $\varphi_\alpha$, as long as every eigenvalue of $\hat{g}$ is larger than 4 (this is to assure that the invariant foliation $\{\theta = \operatorname{const}\}$ is normally expanding, recall argument above).

# 3 Proof of Theorem B

In proving Theorem B we follow a similar global strategy as for Theorem A, but we have to deal with several additional difficulties arising, fundamentally, from the higher-dimensional nature of the $X$–variable. We fully present the new ingredients required to bypass such difficulties and refer the reader to the previous section for many details which are common to both proofs. First we derive the conclusion of the theorem for $\varphi(\Theta, X) = \varphi_{\alpha,b}(\Theta, X) = (\hat{g}(\Theta), \hat{f}_{\alpha,b}(\Theta, X)$. Extension to all maps in a neighbourhood of $\varphi_{\alpha,b}$ follows precisely the same lines as before, as we comment in Section 3.5.

For the sake of notational simplicity we write $g = \hat{g}$ and $f = \hat{f}_{\alpha,b}$. In all that follows we let $\alpha$ and $b$ be small, more precisely $0 < b \leq \alpha \leq c_0^2$ for some $c_0 \ll 1$. The constant $c_0$ is determined by a number of conditions which we state along the way. We point out that none of these conditions involves the value of $d$, cf. also remark preceding Lemma 2.1. In addition, for fixed $d$ we assume that $b$ is large enough with respect to $1/\sqrt{d}$, say $b\sqrt{d} \geq 100$. Clearly, this last condition is compatible with the previous one, provided $d$ be large enough.

## 3.1 Admissible curves

Recall that $\Theta = (\theta, T) \in T_3 = S^1 \times B^2$ and $X = (x, y) \in I_0^2$. Here we call *$X$-vector* to any tangent vector of $T_3 \times I_0^2$ which is a linear combination of $\partial/\partial x$ and $\partial/\partial y$. An $X$-vector $\dot{x}\, \partial/\partial x + \dot{y}\, \partial/\partial y$ will also be denoted, simply, $(\dot{x}, \dot{y})$. By *admissible curve* we now mean any subset $\hat{\mathcal{X}} \subset T_3 \times I_0^2 \times S^1$ which is the graph of some $\mathcal{X}: S^1 \longrightarrow B^2 \times I_0^2 \times S^1$, $\mathcal{X}(\theta) = (T(\theta), X(\theta), \Psi(\theta))$, satisfying

1. $T$, $X$, $\Psi$ are $C^2$ except, possibly, for being left-discontinuous at $\tilde{\theta}_0 = 0$;
2. $|X'| \leq \alpha$, $|X''| \leq \alpha$, $|\Psi'| \leq b\alpha/d$ and $|\Psi''| \leq b\alpha/d^2$.

We think of $T_3 \times I_0^2 \times S^1$ as the bundle over $T_3 \times I_0^2$ whose fibers are the unit balls of $X$-vectors and we let $\Phi$ denote the action induced by $D\varphi$ on this bundle,

$$\Phi(\Theta, X, \Psi) = \left( g(\Theta), f(\Theta, X), \frac{\partial_X f(\Theta, X)\Psi}{\|\partial_X f(\Theta, X)\Psi\|} \right).$$

In the same way as we did for Theorem A, we reduce the proof of Theorem B to a main claim stated in terms of admissible curves. We let $\hat{\mathcal{X}}_0 = \operatorname{graph}(\mathcal{X}_0)$, $\mathcal{X}_0 = (T_0, X_0, \Psi_0)$, be any admissible curve with $\Psi_0(\theta) \equiv \partial/\partial y$. For every $j \geq 0$ we denote $\hat{\mathcal{X}}_j(\theta) = (g^j(\theta), T_j(\theta), X_j(\theta), \Psi_j(\theta)) = \Phi^j(\theta, \mathcal{X}_0(\theta))$ and also $W_j(\theta) = \partial_X f^j(\hat{\mathcal{X}}_1(\theta))\Psi_1(\theta)$. Note that $W_0(\theta) = \Psi_1(\theta) \equiv \partial/\partial x$ and $W_j(\theta) = \|W_j(\theta)\|\, \Psi_{j+1}(\theta)$. Now, clearly, $\|D\varphi^n(\hat{\mathcal{X}}_1(\theta))\frac{\partial}{\partial\theta}\| \geq d^n$ for every $\theta \in S^1$ and $n \geq 1$. We claim that

(C) for convenient constants $c, C, \gamma > 0$ and all large $n$ we have $\|W_n(\theta)\| \geq e^{cn}$, except on a set $E_n$ of values of $\theta$ with $m(E_n) \leq Ce^{-\gamma\sqrt{n}}$.

The proof of this statement occupies most of what follows. On the other hand, the theorem is an easy consequence. Markov partitions $\mathcal{P}_n$ for $\theta \mapsto g(\theta) = d\theta \bmod 1$, to be used in the sequel, are defined in just the same way as in Section 2.

**Lemma 3.1** *If $\hat{\mathcal{X}}$ is an admissible curve and $\omega \in \mathcal{P}_n$ then $\Phi^n(\hat{\mathcal{X}}|\omega)$ is also an admissible curve.*

**Proof:** Clearly, it is sufficient to consider the case $n=1$. Observe that if $\hat{\mathcal{X}} = \operatorname{graph}(T, X, \Psi)$ then $\Phi(\hat{\mathcal{X}}|\omega) = \operatorname{graph}(T_*, X_*, \Psi_*)$ where $T_*(d\theta) = G(\theta, T(\theta))$, $X_*(d\theta) = (a(\theta) - x(\theta)^2 + by(\theta), -bx(\theta))$ and $\Psi_*(d\theta)$ is the unit vector obtained dividing

$$\begin{pmatrix} -2x(\theta) & b \\ -b & 0 \end{pmatrix} \Psi(\theta)$$

by its norm. Property 1 is clear and so we proceed to check 2. Note first that

$$|X'_*| = \left|\frac{1}{d}(\alpha\phi' - 2xx' + by', -bx')\right| \leq \left|\frac{1}{d}(2\pi\alpha + 4\alpha + b\alpha, b\alpha)\right| \leq \frac{12\alpha}{d} \leq \alpha$$

and a similar calculation gives $|X''_*| \leq (48\alpha/d^2) \leq \alpha$. On the other hand, we write $\Psi = (\cos\psi, \sin\psi)$ and $\Psi_* = (\cos\psi_*, \sin\psi_*)$ and then

$$\operatorname{tg}\psi_*(d\theta) = \frac{b\cos\psi(\theta)}{2x(\theta)\cos\psi(\theta) - b\sin\psi(\theta)}$$

which leads to

$$\psi'_* = \frac{1}{d}\frac{b^2\psi' - 2bx'\cos^2\psi}{(2x\cos\psi - b\sin\psi)^2 + (b\cos\psi)^2}. \tag{18}$$

Note that the denominator is bounded from below by $\|\partial_X f^{-1}\|^{-2} \geq (b^2/5)^2$ and, clearly, also by $b^2\cos^2\psi$. In this way we find

$$|\psi'_*| \leq \frac{1}{db^2}(25\,|\psi'| + 2b\,|x'|) \leq \frac{1}{db^2}(25\frac{b\alpha}{d} + 2b\frac{12\alpha}{d}) \leq \frac{b\alpha}{d}, \tag{19}$$

recall that $b^2 d \geq 50$. On the other hand, taking derivatives in (18) and performing the same kind of estimations as before, we get

$$\begin{aligned} |\psi''_*| \leq\ & \tfrac{1}{d^2b^4}(\operatorname{const} b^2\,|\psi''| + \operatorname{const} b^3\,|x''| + \operatorname{const} b^2\,|x'|\,|\psi'|) + \\ & + \tfrac{1}{d^2b^4}(\operatorname{const}\,|\psi'| + \operatorname{const} b\,|x'|)(\operatorname{const}\,|x'| + \operatorname{const} b^{-1}\,|\psi'|), \end{aligned}$$

where const always replaces some numerical constant. (For the deduction of this inequality note also that $(2x\cos\psi - b\sin\psi)^2 + (b\cos\psi)^2 \geq \operatorname{const} b^3\,|\sin\psi|\,|\cos\psi|$.) It follows that $|\psi''_*| \leq (\operatorname{const} b^2 + \operatorname{const}\alpha)b\alpha/d^2 \leq b\alpha/d^2$. $\square$

## 3.2 Expansion and contracting directions

Given $(\Theta, X) \in T_3 \times I_0^2$ and $j \geq 0$ we denote $(\Theta_j, X_j) = \varphi^j(\Theta, X)$ and also $\Theta_j = (\theta_j, T_j)$, $X_j = (x_j, y_j)$. In what follows $\eta > 0$, $\delta_1 > 0$, $N = N(\alpha) \geq 1$, and $\sigma_2 > 1$ have the same meaning as in Section 2. Given an $X$-vector $V = (\dot{x}, \dot{y})$ we denote $\operatorname{slope} V = \dot{y}/\dot{x}$.

**Lemma 3.2** *There is $C_5 > 0$ and, given $(\Theta, X) \in T_3 \times I_0^2$ and $k \geq 1$ with $|x_0| < 2\sqrt{\alpha}$, $|x_j| \geq \sqrt{\alpha}$ for $1 \leq j \leq k$ and $|x_{k+1}| < \delta_1$, there exists $\hat{S} = \hat{S}_k(Z)$ an $X$-vector at $Z = (\Theta_1, X_1)$ satisfying*

*a) For any $X$-vector $U$ at $Z$ with $|\operatorname{slope} U| \leq c_0$ and every $1 \leq j \leq k$, we have $|\operatorname{slope} \partial_X f^j(Z)U| \leq c_0$ and $\|\partial_X f^j(Z)U\| \geq C_5 \sigma_2^j \|U\|$; in addition, $\left\|\partial_X f^k(Z)U\right\| \geq C_5 \alpha^{-1+\eta} \sigma_2^{k-N} \|U\|$.*

*b) $\hat{S} = (\hat{s}, 1)$ with $|\hat{s}| \leq b$ and $\partial_X f^k(Z)\hat{S} = (0, \hat{r})$ with $|\hat{r}| \leq b^{2k}/C_5$; moreover $\left|\operatorname{slope} \partial_X f^j(Z)\hat{S}\right| \geq 1/c_0$ for all $1 \leq j \leq k$.*

**Proof:** For simplicity we denote $L^j = \partial_X f^j(Z)$, $1 \leq j \leq k$. Observe first that, given any point $Y \in \{|x| \geq \sqrt{\alpha}\}$ and any $X$-vector $V = (\dot{x}, \dot{y})$ at $Y$ with $|\operatorname{slope} V| \leq c_0$, then $\|\partial_X f(Y)V\| \geq |-2x\dot{x} + b\dot{y}| \geq (1 - c_0)\,|-2x|\,\|V\|$ and

$$|\operatorname{slope} \partial_X f(Y)V| = \left|\frac{b}{2x - b\operatorname{slope} V}\right| \leq \frac{b}{2\sqrt{\alpha} - bc_0} \leq \frac{b}{\sqrt{\alpha}} \leq c_0. \tag{20}$$

In view of this, and up to assuming $c_0$ small, the arguments in Lemma 2.4 give

$$\left\|L^j U\right\| \geq \operatorname{const} \rho_1^j \|U\| \text{ for } 1 \leq j < N \quad \text{and} \quad \left\|L^N U\right\| \geq \alpha^{-1+\eta} \|U\|. \tag{21}$$

An analog of Lemma 2.4b) also follows from those arguments. Moreover, we take $m \geq 1$, $\sigma_0 > 1$, $\kappa > 0$ as in Lemma 2.5 and then, by continuity (take $c_0$ small),

1. $\|\partial_X f^m(Y)V\| \geq \sigma_0^m \|V\|$ if $\varphi^j(Y) \in \{|x| \geq \delta_1\}$ for $0 \leq j \leq m-1$
2. $\|\partial_X f^l(Y)V\| \geq \kappa\sigma_0^l \|V\|$ if $\varphi^j(Y) \in \{|x| \geq \delta_1\}$ for $0 \leq j \leq l-1$ and $\varphi^l(Y) \in \{|x| < \delta_1\}$, with $l < m$.

Now the same reasoning as in Lemma 2.5 yields

$$\left\|L^k U\right\| \geq C_2 \sigma_2^{k-N} \alpha^{-1+\eta} \|U\| \quad \text{and} \quad \left\|L^j U\right\| \geq C_2 \sigma_2^{j-N} \alpha^{-\frac{1}{2}+\eta} \|U\| \tag{22}$$

for all $N < j < k$. Recall (Lemmas 2.4, 2.5) that $\alpha^{-1} \approx \operatorname{const} \rho_2^N \geq \operatorname{const} \sigma_2^{4N}$. This, together with (20)–(22), completes the proof of a).

In order to prove b) we introduce $\hat{s}_j, \hat{r}_j$ defined by $L^j(\hat{s}_j, 1) = (0, \hat{r}_j)$, every $j \geq 1$, and then we take $\hat{s} = \hat{s}_k$, $\hat{r} = \hat{r}_k$. Note first that

$$\begin{pmatrix} -2x_1 & b \\ -b & 0 \end{pmatrix} \begin{pmatrix} \hat{s}_1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \hat{r}_1 \end{pmatrix} \text{ gives } \left|\hat{s}_1 = \frac{b}{2x_1}\right| \leq \frac{b}{2} \text{ and } \left|\hat{r}_1 = -\frac{b^2}{2x_1}\right| \leq b^2,$$

since $x_1 \approx a_0 > 1$. In general, we write $L^j = \begin{pmatrix} A_j & B_j \\ C_j & D_j \end{pmatrix}$ and then $\hat{s}_j = -B_j/A_j$ and $\hat{r}_j = D_j - B_j C_j/A_j = \det L^j/A_j$. Note that $A_{j+1} = -2x_{j+1}A_j + bC_j$ and

$B_{j+1} = -2x_{j+1}B_j + bD_j$ and so $\hat{s}_{j+1} - \hat{s}_j = -b \det L^j/(A_{j+1}A_j)$. On the other hand, the estimates in part a) imply $|A_j| \geq (1-c_0)\,\|L^j(1,0)\| \geq$ const . In this way we get $|\hat{r}_j| \leq$ const $b^{2j}$ and $|\hat{s}_{j+1} - \hat{s}_j| \leq$ const $b^{2j+1}$ for every $j \geq 1$ and this last inequality also gives $|\hat{s}_k| \leq (b/2) + \sum_{j\geq 1}$ const $b^{2j+1} \leq b$. Finally, if it were $|\,\text{slope}\, L^j \hat{S}_k| < 1/c_0$ for some $j < k$ then the same calculation as in (20) would give $|\,\text{slope}\, L^k \hat{S}_k| \leq c_0$, contradicting our definition of $\hat{S}_k$. $\square$

**Remark 1:** If we drop the assumptions on $x_0$, $x_{k+1}$ (keeping only $|x_j| \geq \sqrt{\alpha}$ for $1 \leq j \leq k$) then the same arguments yield the following slightly weaker conclusions, which will be of use below. For any $X$-vector $U$ with $|\,\text{slope}\, U| \leq c_0$ and any $1 \leq j \leq k$, we have $|\,\text{slope}\, \partial_X f^j(Z)U| \leq c_0$ and $\|\partial_X f^j(Z)U\| \geq$ const $\sqrt{\alpha}\sigma_2^j\,\|U\|$ (cf. Lemma 2.5). A vector $\hat{S}_k = (\hat{s}_k, 1)$ is defined, satisfying $|\hat{s}_k| \leq (b/\sqrt{\alpha})$, $\partial_X f^k(Z)\hat{S}_k = (0, \hat{r}_k)$ with $|\hat{r}_k| \leq$ const $b^{2k}/\sqrt{\alpha}$, and $|\,\text{slope}\, \partial_X f^k(Z)\hat{S}| \geq 1/c_0$ for $1 \leq j \leq k$. Moreover, if $|x_j| \geq \sqrt{\alpha}$ for all $j \geq 1$ then the corresponding sequence $(\hat{s}_j, 1)$ converges to some $X$-vector $\hat{S}_\infty = (\hat{s}_\infty, 1)$ such that $|\hat{s}_\infty| \leq c_0$, $|\,\text{slope}\, \partial_X f^j(Z)\hat{S}_\infty| \geq 1/c_0$, and $\|\partial_X f^j(Z)\hat{S}_\infty\| \leq$ const $b^{2j}/\sqrt{\alpha}$ for all $j \geq 1$.

We also need to consider pairs of points $(\Theta^{(i)}, X^{(i)}) \in T_3 \times I_0^2$, $i = 1, 2$, and the corresponding objects $\Theta_j^{(i)} = (\theta_j^{(i)}, T_j^{(i)})$, $X_j^{(i)} = (x_j^{(i)}, y_j^{(i)})$, and $Z^{(i)} = (\Theta_1^{(i)}, X_1^{(i)})$. Let $[Z^{(1)}, Z^{(2)}]$ be the straight line segment connecting $Z^{(1)}$, $Z^{(2)}$. Given any $Z \in [Z^{(1)}, Z^{(2)}]$, we define $\Theta_j = (\theta_j, T_j)$ and $X_j = (x_j, y_j)$ by $(\Theta_j, X_j) = \varphi^{j-1}(Z)$, every $j \geq 1$.

**Lemma 3.3** *There exists $C_6 > 0$ such that the following holds. Let $\theta_1^{(1)} = \theta_1^{(2)}$ and $k \geq 1$ be such that $|x_j| \geq \sqrt{\alpha}$ for any $1 \leq j \leq k$ and any $Z \in [Z^{(1)}, Z^{(2)}]$. Let $\xi = \min\{|x_1^{(1)}|, |x_1^{(2)}|\}$. Then*

*a)* $\left|\hat{s}_k(Z^{(1)}) - \hat{s}_k(Z^{(2)})\right| \geq \frac{b}{8}\left|x_1^{(1)} - x_1^{(2)}\right| - C_6 \frac{b^3}{\xi\sqrt{\alpha}}$

*b)* $\left|\hat{s}_k(Z^{(1)}) - \hat{s}_k(Z^{(2)})\right| \leq \frac{b}{2\xi^2}\left|x_1^{(1)} - x_1^{(2)}\right| + C_6 b \left\|X_1^{(1)} - X_1^{(2)}\right\|$.

**Proof:** We keep the notations of the previous lemma. First, we observe

$$\frac{b}{8}\left|x_1^{(1)} - x_1^{(2)}\right| \leq \left|\hat{s}_1(Z^{(1)}) - \hat{s}_1(Z^{(2)})\right| = \frac{b|x_1^{(1)} - x_1^{(2)}|}{|2x_1^{(1)}x_1^{(2)}|} \leq \frac{b}{2\xi^2}\left|x_1^{(1)} - x_1^{(2)}\right|.$$

Recall that $\hat{s}_{j+1} - \hat{s}_j = -b^{2j+1}/(A_{j+1}A_j)$. Using $|A_1| \geq \xi$ and $|A_j| \geq$ const $\sqrt{\alpha}$ for all $j$ (see Remark 1), $|\hat{s}_k - \hat{s}_1| \leq$ const $(b^3/\xi\sqrt{\alpha}) + \sum_{j\geq 2}$ const $(b^{2j+1}/\alpha) \leq$ const $(b^3/\xi\sqrt{\alpha})$. This proves a). Now, let $D$ denote derivative with respect to the $X_1$–variable. A simple induction argument shows that $\|Dx_j\|$, $|A_j| \leq 4^j$ and $\|DA_j\| \leq 4^{2j}$. Hence, $\|D(\hat{s}_{j+1} - \hat{s}_j)\| \leq$ const $b^{2j+1}4^{2j+2}\alpha^{-3/2}$ and so $\|D(\hat{s}_k - \hat{s}_1)\| \leq$ const $\sum_{j\geq 1} b^{2j+1}4^{2j+2}\alpha^{-3/2} \leq$ const $b$. Then b) follows from our first estimate and the mean value theorem (note that the $\hat{s}_j$ are independent of $T_1$, and $\theta_1$ is constant on $[Z^{(1)}, Z^{(2)}]$). $\square$

### 3.3 Estimating expansion losses

Recall that $\hat{\mathcal{X}}_0 = \operatorname{graph}(T_0, X_0, \Psi_0)$ is an arbitrary admissible curve with $\Psi_0 \equiv \partial/\partial y$. Let $n$ be any large integer and $M = M(\alpha)$, $J(r)$, be as in Section 2. Let $\Theta_j(\theta) = (g^j(\theta), T_j(\theta))$, $X_j(\theta) = (x_j(\theta), y_j(\theta))$, and $\Psi_j(\theta) = (\cos\psi_j(\theta), \sin\psi_j(\theta))$. Given $1 \le \nu \le n$ and $\omega_{\nu-M} \in \mathcal{P}_{\nu-M}$ we say that $\nu$ is a *return* for (every) $\theta \in \omega_{\nu-M}$ if $x_\nu(\omega_{\nu-M}) \cap (-\sqrt{\alpha}, \sqrt{\alpha}) \ne \emptyset$. Note that this implies $x_\nu(\omega_{\nu-M}) \subset (-2\sqrt{\alpha}, 2\sqrt{\alpha})$, since $\operatorname{osc}(x_\nu|\omega_{\nu-M}) \le \sqrt{\alpha}$, recall (11). For completeness we set $\mathcal{P}_i = \{S^1\}$ for all $i \le 0$; observe also that $\nu < M$ can only occur for, at most, one return.

Our goal at this point is to introduce a function $\Delta_\nu(\theta)$, defined on $\theta \in \omega_{\nu-M}$, bounding the amount of expansion lost by $\partial_X f$ at $(\Theta_\nu(\theta), X_\nu(\theta))$. In what follows $\omega_i(\theta)$ denotes the element of the partition $\mathcal{P}_i$ containing $\theta$. First we take $k = k_\nu(\theta) \ge 1$ minimum such that $\mu = \nu+k+1$ satisfies $x_\mu(\omega_{\mu-M}(\theta)) \cap (-\sqrt{\alpha}, \sqrt{\alpha}) \ne \emptyset$ (in other words, $\mu$ is the next return of $\theta$). Then we let $S_\nu(\theta) = (s_\nu(\theta), 1) = \hat{S}_k(\Theta_{\nu+1}(\theta), X_{\nu+1}(\theta))$, given by Lemma 3.2 (use Remark 1 if $k = \infty$), and we decompose $W_\nu = \|W_{\nu-1}\| \, (h_\nu(1,0) + v_\nu(s_\nu, 1))$ and $W_{\nu-1} = \|W_{\nu-1}\| \, (\cos\psi_\nu, \sin\psi_\nu)$. Now we define

$$\Delta_\nu(\theta) = \frac{h_\nu(\theta)}{\cos\psi_\nu(\theta)} = -2x_\nu(\theta) + b \operatorname{tg}\psi_\nu(\theta) + b s_\nu(\theta) \tag{23}$$

(note that $W_\nu / \|W_{\nu-1}\| = \partial_X f(\Theta_\nu, X_\nu)\Psi_\nu = (-2x_\nu \cos\psi_\nu + b\sin\psi_\nu, -b\cos\psi_\nu)$). A main property of these $\Delta_\nu(\cdot)$ is to be stated in Lemma 3.7, which is an analog of Lemma 2.6 in this context. First we need a few auxiliary results.

For $1 \le q \le k_\nu(\theta)$ we define $\Delta_{\nu,q}(\theta) = -2x_\nu(\theta) + b\operatorname{tg}\psi_\nu(\theta) + b s_{\nu,q}(\theta)$, with $s_{\nu,q}(\theta) = \hat{s}_q(\Theta_{\nu+1}(\theta), X_{\nu+1}(\theta))$ as in Lemmas 3.2, 3.3. We denote by $A_j$, $B_j$, $C_j$, $D_j$ the entries of $\partial_X f^j(\Theta_{\nu+1}(\theta), X_{\nu+1}(\theta))$. Finally, as in Lemma 2.2, we let $\mathcal{A}_1 = \{\theta\colon |\sin 2\pi\theta| \le 1/3\}$ and $\mathcal{A}_2 = S^1 \setminus \mathcal{A}_1$.

**Lemma 3.4** *Let $\theta \in \omega_{\nu-M}$ be such that $|b\operatorname{tg}\psi_\nu(\theta)| \le 10\sqrt{\alpha}$. Then for every $1 \le q \le k_\nu(\theta)$,*

*a)* $\left|\Delta'_{\nu,q}(\theta)\right| \ge \frac{\alpha}{10}\frac{b^{2q}}{|A_q A_{q-1}|} d^{\nu+q-1}$ *if* $d^{\nu+q-1}\theta \in \mathcal{A}_1$ *and*

*b)* $\left|\Delta''_{\nu,q}(\theta)\right| \ge \frac{\alpha}{10}\frac{b^{2q}}{|A_q A_{q-1}|} d^{2(\nu+q-1)}$ *if* $d^{\nu+q-1}\theta \in \mathcal{A}_2$.

**Proof:** We detail the proof of a): statement b) is proved along precisely the same lines and we omit the corresponding calculations. Note first that our assumption on $\psi_\nu(\theta)$ implies (recall also Lemma 3.1)

$$|(-2x_\nu + b\operatorname{tg}\psi_\nu)'| \le \left(\frac{24\alpha}{d} + b\left(1 + \frac{100\alpha}{b^2}\right)\frac{b\alpha}{d}\right) d^\nu \le 25\alpha d^{\nu-1}.$$

Moreover, $25\alpha d^{\nu-1} \leq (\alpha/10)b^{2q}4^{1-2q}d^{\nu+q-1} \leq (\alpha/10)(b^{2q}/|A_qA_{q-1}|)d^{\nu+q-1}$ for every $q \geq 1$, since $b^2d \geq 1000$ and $|A_j| \leq 4^j$. Therefore, it suffices to prove that

$$\left|s'_{\nu,q}(\theta)\right| \geq \frac{\alpha}{5}\frac{b^{2q-1}}{|A_qA_{q-1}|}d^{\nu+q-1} \text{ whenever } d^{\nu+q-1}\theta \in \mathcal{A}_1.$$

Recall, from the proof of Lemma 3.2, that $s_{\nu,1} = b/(2x_{\nu+1}) = -b/A_1$ and also $s_{\nu,j+1} - s_{\nu,j} = -b^{2j+1}/(A_{j+1}A_j)$ for all $j$. Hence,

$$s'_{\nu,1} = \frac{b}{A_1}\frac{A'_1}{A_1} \quad \text{and} \quad s'_{\nu,j+1} = s'_{\nu,j} + \frac{b^{2j+1}}{A_{j+1}A_j}\left(\frac{A'_{j+1}}{A_{j+1}} + \frac{A'_j}{A_j}\right)$$

and so, by induction, $s'_{\nu,q}$ is equal to

$$s'_{\nu,1} + \sum_{j=1}^{q-1}\frac{b^{2j+1}}{A_{j+1}A_j}\left(\frac{A'_{j+1}}{A_{j+1}} + \frac{A'_j}{A_j}\right) = \frac{b^{2q-1}}{A_qA_{q-1}}\frac{A'_q}{A_q} + \sum_{j=1}^{q-1}\frac{A'_j}{A_j}\left(\frac{b^{2j+1}}{A_{j+1}A_j} + \frac{b^{2j-1}}{A_jA_{j-1}}\right)$$

(convention $A_0 = 1$). In order to estimate the right-hand side we introduce $E_j = A_j/A_{j-1}$ and note that $E_1 = -2x_1$ and $E_j = -2x_{\nu+j} - b^2/E_{j-1}$. It follows, by induction,

(i) $\sqrt{\alpha} \leq |E_j| \leq 4$ for all $j \geq 1$ ; moreover, no two consecutive values of $|E_j|$ can be less than 1 (because no two consecutive $|x_i|$ are less than $3/5$).

(ii) $\left|E'_j\right| \leq 25\alpha d^{\nu+j-1}$ for $j \geq 1$ (note that $\left|x'_{\nu+j}\right| \leq 12\alpha d^{\nu+j-1}$, cf. Lemma 3.1).

Now, suppose $|\sin(2\pi d^{\nu+q-1}\theta)| \leq 1/3$. Then (compare the proof of Lemma 2.2) $\left|x'_{\nu+q}\right| = \left|a'(d^{\nu+q-1}\theta)d^{\nu+q-1} - 2x_{\nu+q-1}x'_{\nu+q-1} - by'_{\nu+q-1}\right| \geq (3\alpha/2)d^{\nu+q-1}$ and so $\left|E'_q\right| \geq 3\alpha d^{\nu+q-1} - (b^2/\alpha)25\alpha d^{\nu+q-2} \geq (5\alpha/2)d^{\nu+q-1}$. As a consequence,

$$\left|\frac{A'_q}{A_q}\right| \geq \left|\frac{E'_q}{E_q}\right| - \sum_{i=1}^{q-1}\left|\frac{E'_i}{E_i}\right| \geq \alpha d^{\nu+q-1}\left(\frac{5}{8} - \frac{25}{\sqrt{\alpha}}\sum_{i=1}^{q-1}d^{i-q}\right) \geq \frac{\alpha}{2}d^{\nu+q-1}. \tag{24}$$

On the other hand,

$$\left|\frac{b^{2j-1}}{A_jA_{j-1}}\frac{A'_j}{A_j} \Big/ \frac{b^{2q-1}}{A_qA_{q-1}}\frac{A'_q}{A_q}\right| \leq b^{2(j-q)}\prod_{i=j}^{q-1}|E_iE_{i+1}|\frac{\sum_{s=1}^{j}|E'_s/E_s|}{(\alpha/2)d^{\nu+q-1}} \leq 20\left(\frac{db^2}{4^2}\right)^{j-q}.$$

In order to check this, use $|E_i| \leq 4$ and (i), (ii) above if $s < j$, and

$$\prod_{i=j}^{q-1}|E_iE_{i+1}|\left|E'_j/E_j\right| = |E_{j+1}|\prod_{i=j+1}^{q-1}|E_iE_{i+1}|\left|E'_j\right| \leq 4^{2(q-j)-1}\cdot 25\alpha d^{\nu+s-1}$$

otherwise. In a similar fashion,

$$\left| \frac{b^{2j+1}}{A_{j+1}A_j} \frac{A'_j}{A_j} \Big/ \frac{b^{2q-1}}{A_qA_{q-1}} \frac{A'_q}{A_q} \right| \leq 5 \left( \frac{db^2}{4^2} \right)^{j-q} .$$

Using $b^2 d \geq 1000$ once more, we get

$$\left| s'_{\nu,q} \right| \geq \left| \frac{b^{2q-1}}{A_qA_{q-1}} \frac{A'_q}{A_q} \right| \left( 1 - \sum_{j=1}^{q-1} 25 \left( \frac{4^2}{b^2 d} \right)^{q-j} \right) \geq \frac{b^{2q-1}}{|A_qA_{q-1}|} \frac{\alpha}{4} d^{\nu+q-1}, \tag{25}$$

and our argument is complete. □

**Remark 2:** The same type of estimates (cf. (25), (24), and (ii) above) also gives the following upper bound which will be useful below:

$$\left| s'_{\nu,q} \right| \leq 2 \left| \frac{b^{2q-1}}{A_qA_{q-1}} \frac{A'_q}{A_q} \right| \leq 100\sqrt{\alpha} \frac{b^{2q-1}}{|A_qA_{q-1}|} d^{\nu+q-1} \leq \text{ const } \sqrt{\alpha} b^{2q-1} d^{\nu+q-1}. \tag{26}$$

**Lemma 3.5** *Let $1 \leq \nu \leq n$ and $\omega_{\nu-M} \in \mathcal{P}_{\nu-M}$ be as above and let $\omega_\nu$ be any element of $\mathcal{P}_\nu$ contained in $\omega_{\nu-M}$.*

*a) If $|b \operatorname{tg} \psi_\nu(\tau)| \geq 10\sqrt{\alpha}$ for some $\tau \in \omega_\nu$ then $|\Delta_\nu(\theta)| \geq 2\sqrt{\alpha}$ for every $\theta \in \omega_\nu$.*

*b) $m\left(\{\theta \in \omega_\nu \colon \Delta_\nu(\theta) \in J(r)\}\right) \leq b^{-1/4} \left( \frac{|J(r)|}{b^2} \right)^{1/10} m(\omega_\nu)$ for every $r \geq 0$.*

**Proof:** Suppose first that the assumption of a) is satisfied. Since $|\omega_\nu| = d^{-\nu}$ and (by Lemma 3.1) $|\psi'_\nu| \leq (b\alpha/d)d^\nu$, we must have $|b \operatorname{tg} \psi_\nu(\theta)| \geq 8\sqrt{\alpha}$ for all $\theta \in \omega_\nu$ and then a) follows immediately: $|\Delta_\nu(\theta)| \geq -4\sqrt{\alpha} + 8\sqrt{\alpha} - b^2 \geq 2\sqrt{\alpha}$. Now we prove b). Note that, in view of a), it is no restriction to assume $|b \operatorname{tg} \psi_\nu| \leq 10\sqrt{\alpha}$ and we do so in what follows. We start with a few simple remarks concerning the $A_j$, $E_j$ introduced above. Let $1 \leq j \leq q$. It follows from the proof of Lemma 3.4, namely from (ii), that $|E_j(\theta_1) - E_j(\theta_2)| \leq 25\alpha d^{j-q}$ whenever $\theta_1, \theta_2$ belong to a same $\omega_{\nu+q-1} \in \mathcal{P}_{\nu+q-1}$. Hence, under this assumption we have

$$\frac{|A_q(\theta_1)|}{|A_q(\theta_2)|} = \prod_{j=1}^{q} \frac{|E_j(\theta_1)|}{|E_j(\theta_2)|} \leq \prod_{j=1}^{q} \left( 1 + 25\sqrt{\alpha} d^{j-q} \right) \leq \sqrt{2}. \tag{27}$$

We also observe that

$$|\Delta_{\nu,q} - \Delta_\nu| = \left| \sum_{j=q}^{k-1} \frac{b^{2j+2}}{A_{j+1}A_j} \right| \leq \frac{2b^{2q+2}}{|A_{q+1}A_q|} \leq \frac{2b^2}{\sqrt{\alpha}} \frac{b^{2q}}{|A_qA_{q-1}|}, \tag{28}$$

since (recall (i) above)

$$\left| \frac{b^{2j+2}}{A_{j+1}A_j} \middle/ \frac{b^{2j}}{A_j A_{j-1}} \right| = \frac{b^2}{|E_{j+1}E_j|} \le \frac{b^2}{\sqrt{\alpha}} \left( < \frac{1}{2} \right) \quad \text{for all} \quad j. \tag{29}$$

We define $\kappa = \kappa(r)$ to be the maximum integer satisfying $\sqrt{\alpha}e^{-r} \le 16(b/4)^{2\kappa}$. In what follows we assume $\kappa \ge 1$, as the bound in b) is trivial when $\kappa = 0$. Observe also that we defined $k_\nu$ in such a way that $k_\nu(\theta) = k$ implies $k_\nu(\tau) = k$ for every $\tau \in \omega_{\nu+k+1-M}(\theta)$, in particular for every $\tau \in \omega_{\nu+k-1}(\theta)$. For $1 \le k \le \kappa - 1$ we let $\mathcal{F}_k$ be the family of all intervals $\omega_{\nu+k-1} \in \mathcal{P}_{\nu+k-1}$, with $\omega_{\nu+k-1} \subset \omega_\nu$, satisfying

- $k_\nu(\theta) = k$ for (every) $\theta \in \omega_{\nu+k-1}$ and $\Delta_\nu(\tau) \in J(r)$ for some $\tau \in \omega_{\nu+k-1}$.

Then we also let $\mathcal{F}_\kappa$ be the set of all $\omega_{\nu+\kappa-1} \in \mathcal{P}_{\nu+\kappa-1}$, $\omega_{\nu+\kappa-1} \subset \omega_\nu$, such that

- $k_\nu(\theta) \ge \kappa$ for (every) $\theta \in \omega_{\nu+\kappa-1}$ and $\Delta_\nu(\tau) \in J(r)$ for some $\tau \in \omega_{\nu+\kappa-1}$.

We claim that

1) Given $1 \le q \le \kappa$, each $\omega_{\nu+q-1} \in \mathcal{P}_{\nu+q-1}$ contains at most $(100b^{1/4}d)^{\kappa-q}$ elements of $\mathcal{F}_\kappa$. In particular (taking $q = 1$) $\#\mathcal{F}_\kappa \le (100b^{1/4}d)^{\kappa-1}$.

We prove this statement by induction on $\kappa - q$. Note that case $q = \kappa$ is trivial. Let $q < \kappa$ and assume that 1) holds for $q + 1$ (i.e. for all $\omega_{\nu+q} \in \mathcal{P}_{\nu+q}$). Given $\omega_{\nu+q-1}$ an arbitrary element of $\mathcal{P}_{\nu+q-1}$, we count the intervals $\omega_{\nu+q} \in \mathcal{P}_{\nu+q}$ with $\omega_{\nu+q} \subset \omega_{\nu+q-1}$ and containing some element of $\mathcal{F}_\kappa$. Let $\omega_{\nu+q}$ be any such interval: then there exists $\tau \in \omega_{\nu+q}$ with $|\Delta_\nu(\tau)| \le \sqrt{\alpha}e^{-r}$ and hence, in view of (28), (29) and the fact that $q + 1 \le \kappa$,

$$|\Delta_{\nu,q}(\tau)| \le \sqrt{\alpha}e^{-r} + \frac{2b^{2q+2}}{|A_{q+1}A_q(\tau)|} \le \frac{6b^{2q+2}}{|A_{q+1}A_q(\tau)|} \le \frac{6b^2}{\sqrt{\alpha}} \max \left( \frac{b^{2q}}{|A_q A_{q-1}|} \right),$$

where the maximum is taken on $\omega_{\nu+q-1}$. In other words, $\omega_{\nu+q}$ must intersect

$$\mathcal{Z} = \left\{ \theta \in \omega_{\nu+q-1} \colon |\Delta_{\nu,q}(\theta)| \le \frac{6b^2}{\sqrt{\alpha}} \max \left( \frac{b^{2q}}{|A_q A_{q-1}|} \right) \right\}.$$

Now, using Lemma 3.4, the same arguments as in Lemma 2.2, Corollary 2.3, give

$$m(\mathcal{Z}) \le 6 \left( \frac{(12b^2/\sqrt{\alpha}) \max \left( b^{2q}/ |A_q A_{q-1}| \right)}{(\alpha/10) \min \left( b^{2q}/ |A_q A_{q-1}| \right) d^{2(\nu+q-1)}} \right)^{1/2}.$$

Therefore, recall (27), $m(\mathcal{Z}) \le 6 \left( 240b^2/(\alpha\sqrt{\alpha}) \right)^{1/2} d^{-\nu-q+1} \le 95b^{1/4}d^{-\nu-q+1}$. On the other hand, Lemma 3.4 implies that $(\Delta_{\nu,q}|\omega_{\nu+q-1})$ is at most 3-to-1 and so $\mathcal{Z}$ has no more than 3 connected components. Hence, there are at most $95b^{1/4}d + 6 \le 100b^{1/4}d$ intervals $\omega_{\nu+q}$ as above. This, together with the induction hypothesis, implies that 1) holds for $\omega_{\nu+q-1}$ and so the proof of our claim is complete. Now we observe that the same argument also proves

2) Given $1 \leq q \leq k < \kappa$, each $\omega_{\nu+q-1} \in \mathcal{P}_{\nu+q-1}$ contains at most $(100b^{1/4}d)^{k-q}$ elements of $\mathcal{F}_k$. In particular, $\#\mathcal{F}_k \leq (100b^{1/4}d)^{k-1}$.

Moreover, using Lemma 3.4 (with $q = k$), along with the calculations of Lemma 2.2 and Corollary 2.3, we get (the minimum is over $\omega_{\nu+k-1}$)

$$m(\{\theta \in \omega_{\nu+k-1} \colon \Delta_\nu(\theta) \in J(r)\}) \leq 6\left(\frac{2\sqrt{\alpha}e^{-r}}{(\alpha/10)\min(b^{2k}/|A_k A_{k-1}|)\,d^{2(\nu+k-1)}}\right)^{1/2}.$$

Now, $\sqrt{\alpha}e^{-r} \leq 16(b^{2\kappa}/4^{2\kappa}) \leq 4(b^{2\kappa}/|A_\kappa A_{\kappa-1}|) \leq 4(b^2/\sqrt{\alpha})^{\kappa-k}(b^{2k}/|A_k A_{k-1}|)$, recall (29). It follows that the Lebesgue measure of $\{\theta \in \omega_{\nu+k-1} \colon \Delta_\nu(\theta) \in J(r)\}$ is bounded above by $6((80/\alpha)(b^2/\sqrt{\alpha})^{\kappa-k})^{1/2}\,d^{-\nu-q+1} \leq (60b^{1/4})^{\kappa-k}d^{-\nu-k+1}$. Altogether, this gives

$$\begin{aligned} m(\{\theta \in \omega_\nu \colon \Delta_\nu(\theta) \in J(r)\}) &\leq \sum_{k=1}^{\kappa}(100b^{1/4}d)^{k-1}(60b^{1/4})^{\kappa-k}d^{-\nu-k+1} \\ &\leq 3(100b^{1/4})^{\kappa-1}d^{-\nu} \leq b^{-1/4}(100b^{1/4})^{\kappa}\,m(\omega_\nu). \end{aligned}$$

Finally, up to assuming $c_0$ small enough, the maximality condition in the definition of $\kappa$ implies $(100b^{1/4})^\kappa \leq (b/4)^{\kappa/5} \leq (\sqrt{\alpha}e^{-r}/b^2)^{1/10}$. □

## 3.4 A technical lemma

Let $\hat{\mathcal{X}} = \operatorname{graph}(\mathcal{X})$ be an admissible curve and $\hat{Z}(\theta) = \varphi(\hat{\mathcal{X}}(\theta))$. We write $\hat{Z} = (g, U, Z, \Sigma)$, $Z = (z, w)$ and $\Sigma = (\cos\sigma, \sin\sigma)$. For $1 \leq i \leq d$ denote $\hat{\mathcal{Z}}_i = \Phi(\hat{\mathcal{X}}|[\tilde{\theta}_{i-1}, \theta_i)) = \operatorname{graph}(\mathcal{Z}_i)$, with $\mathcal{Z}_i = (U_i, Z_i, \Sigma_i)$, $Z_i = (z_i, w_i)$, and $\Sigma_i = (\cos\sigma_i, \sin\sigma_i)$. Observe that $z(\theta) = z_i(g(\theta))$ for $\theta \in [\tilde{\theta}_{i-1}, \theta_i)$, and similarly for $w$, $w_i$, and $\sigma$, $\sigma_i$. Given $j \geq 1$ we let $A_j$, $B_j$, $C_j$, $D_j$, be the entries of $\partial_X f^j$ and (recall Lemmas 3.2, 3.3), we write $\hat{s}_j = -B_j/A_j$ and $\hat{r}_j = \det \partial_X f/A_j$.

**Lemma 3.6** *Let $|z(\theta)| \geq \sqrt[4]{\alpha}$ for all $\theta \in S^1$. Then there are $H_1, H_2 \subset \{1, \ldots, d\}$ with $\#H_1, \#H_2 \geq [d/100]$, such that*

*a) $|z_{i_1}(\theta) - z_{i_2}(\theta)| \geq \alpha/250$ for all $\theta \in S^1$, $i_1 \in H_1$ and $i_2 \in H_2$,*

*b) $|\operatorname{cotg}\sigma_i(\theta) - \hat{s}_k(\theta, U_i(\theta), Z_i(\theta))| \geq b^2\sqrt{\alpha}$ for all $\theta \in S^1$ and $i \in H_1 \cup H_2$,*

*for any $k \geq 1$ such that $\varphi^j(\hat{\mathcal{X}}(\theta)) \in \{|x| \geq \sqrt{\alpha}\}$ for all $1 \leq j \leq k$ and all $\theta \in S^1$.*

**Proof:** Let $l = [d/100]$ and define $\tilde{H}_s = \{2sl+1, \ldots, 2sl+l\}$ for $s = 0, 1, 2$. Observe that $[\tilde{\theta}_{i-1}, \tilde{\theta}_i) \subset [0, 1/20] \subset \mathcal{A}_1$ for all $1 \leq i \leq 5l$ (recall that $\tilde{\theta}_i = i/d$). Moreover, as in (4), $|z'(\theta)| \geq \alpha/2$ for all $\theta \in \mathcal{A}_1$. Hence, given any $i_1 \in \tilde{H}_r$, $i_2 \in \tilde{H}_s$ with $r \neq s$, we have $\inf_{S^1}|z_{i_1} - z_{i_2}| \geq (\alpha/2)(l/d) \geq \alpha/250$, which gives us a). Now we claim that given any such $i_1$, $i_2$, at least one of them satisfies b). Note that the lemma is a direct consequence: one obtains sets $H_1$, $H_2$ as in the statement just by choosing appropriate elements from $\tilde{H}_0$, $\tilde{H}_1$, $\tilde{H}_2$. We prove the claim by contradiction. Suppose there is $\tau \in S^1$ such that $|\operatorname{cotg}\sigma_i(\tau) - s_i(\tau)| \le b^2\sqrt{\alpha}$ for both $i = i_1$, $i = i_2$ (we write $s_i(\tau) = \hat{s}_j(\tau, U_i(\tau), Z_i(\tau))$ ). Since $|\hat{s}_k| \le b/\sqrt{\alpha}$ (recall Remark 1), it follows $|\operatorname{cotg}\sigma_i(\tau)| \le (b/\sqrt{\alpha}) + (b^2\sqrt{\alpha}) \le 1$ for $i = i_1, i_2$ (if $c_0$ is small). Using (19), $\sigma_i = \sigma \circ (g|[\tilde{\tilde{\theta}}_{i-1}, \tilde{\tilde{\theta}}_i))^{-1}$, and the mean value theorem, we get

$$|\operatorname{cotg}\sigma_{i_1}(\tau) - \operatorname{cotg}\sigma_{i_2}(\tau)| \le \frac{2}{db^2} 50 \frac{b\alpha}{d} 5l \le \frac{b\alpha}{2000},$$

recall that $b\sqrt{d} \ge 100$. It follows that $|s_{i_1}(\tau) - s_{i_2}(\tau)| \le 2b^2\sqrt{\alpha} + (b\alpha/2000) \le (b\alpha/1500)$, for small enough $c_0$. But Lemma 3.3a) gives (note that our assumptions imply $\xi \ge \sqrt[4]{\alpha}$)

$$|s_{i_1}(\tau) - s_{i_2}(\tau)| \ge \frac{b}{8}|z_{i_1}(\tau) - z_{i_2}(\tau)| - \text{const}\,\frac{b^3}{\alpha^{3/4}} \ge \frac{b\alpha}{800} - \text{const}\, b\alpha^{5/4} \ge \frac{b\alpha}{1000},$$

if $c_0$ is small. We have reached a contradiction, thus proving our claim. □

Now we are in a position to prove our last lemma. Given $\hat{\mathcal{Y}}_0 = \operatorname{graph}(\mathcal{Y}_0)$ an admissible curve and $j \ge 0$, we write $\hat{\mathcal{Y}}_j = \Phi^j(\hat{\mathcal{Y}}_0)$ and $\hat{\mathcal{Y}}_j(\theta) = \Phi^j(\theta, \mathcal{Y}_0(\theta)) = (g^j(\theta), S_j(\theta), Y_j(\theta), \Gamma_j(\theta))$, with $Y_j = (\xi_j, \eta_j)$ and $\Gamma_j = (\cos\gamma_j, \sin\gamma_j)$. We suppose that $|\xi_M(\tau)| < \sqrt{\alpha}$ for some $\tau \in S^1$. Then $|\xi_M(\theta)| < 2\sqrt{\alpha}$ for every $\theta \in S^1$: this is because $\operatorname{osc}(\xi_j) \le 2\alpha 4^j$ for all $j \ge 0$, in particular $\operatorname{osc}(\xi_M) < \sqrt{\alpha}$, cf. (11). To the curve $\hat{\mathcal{Y}}_M$ we associate $\Delta_M(\theta) = -2\xi_M(\theta) + b\operatorname{tg}\gamma_M(\theta) + bs_M(\theta)$ with $s_M(\theta) = \hat{s}(g^{M+1}(\theta), S_{M+1}(\theta), Y_{M+1}(\theta))$, as in (23).

**Lemma 3.7** *There are $C_7 > 0$ and $\beta > 0$ such that, given any admissible curve $\hat{\mathcal{Y}}_0 = \operatorname{graph}(\mathcal{Y}_0)$ as above and any $r \ge (\frac{1}{2} - \eta)\log\frac{1}{\alpha}$,*

$$m\left(\{\theta \in S^1 \colon \Delta_M(\theta) \in J(r-2)\}\right) \le C_7 e^{-5\beta r}.$$

**Proof:** The argument has two parts, which can be sketched as follows. The first step is parallel to the proof of Lemma 2.6. For each $\bar{l} = (l_1, \ldots, l_M)$ in $\{1, \ldots, d\}^M$ we let $\hat{\mathcal{Y}}_j(\bar{l}) = \operatorname{graph}(\mathcal{Y}_j(\bar{l})) = \Phi^j(\hat{\mathcal{Y}}_0|\omega(\bar{l}))$ and also introduce the objects $S_j(\bar{l}, \theta)$, $Y_j(\bar{l}, \theta) = (\xi_j(\bar{l}, \theta), \eta(\bar{l}, \theta))$, $\Gamma_j(\bar{l}, \theta) = (\cos\gamma_j(\bar{l}, \theta), \sin\gamma_j(\bar{l}, \theta))$, $s_M(\bar{l}, \theta)$, and $\Delta_M(\bar{l}, \theta)$ corresponding to it. We fix $r \ge (\frac{1}{2} - 2\eta)\log\frac{1}{\alpha}$ and say that $\bar{l}$ and $\bar{m}$ are *incompatible* if

$$|\xi_M(\bar{l}, \theta) - \xi_M(\bar{m}, \theta)| \ge 4e^{2-r}\sqrt{\alpha} \text{ for every } \theta \in S^1. \tag{30}$$

Using Lemma 3.6a) we prove that each $\bar{l}$ is incompatible with all but, at most, $d^M((d - [d/100])/d)^{\text{const}\, r}$ elements $\bar{m} \in \{1, \ldots, d\}^M$. Then, in a second step, we prove that the incompatible pairs $\bar{l}$, $\bar{m}$ obtained in this way also satisfy

$$|\Delta_M(\bar{l}, \theta) - \Delta_M(\bar{m}, \theta)| \ge 2e^{2-r}\sqrt{\alpha} \text{ for every } \theta \in S^1, \tag{31}$$

thus ensuring that $\Delta_M(\bar{l},\theta)$ and $\Delta(\bar{m},\theta)$ can not both belong in $J(r-2)$. This is done by checking that the two last terms in $\Delta_M$ have a negligible effect, which relies on the property in Lemma 3.6b). The lemma follows directly from the combination of these conclusions.

First we note that $|\xi_j(\theta)| \geq \sqrt[4]{\alpha}$ for all $\theta \in S^1$ and $0 \leq j \leq M-1$. Indeed, let $\delta_j(\theta) = \operatorname{dist}(\xi_j(\theta), \mathcal{O}) + |\eta_j(\theta)|$, recall that $\mathcal{O} \subset \mathbb{R}$ is the post-critical orbit of $h(x) = a_0 - x^2$. In the same way as in (12), $\delta_{j+i}(\theta) \leq C4^i(\alpha + |\xi_j(\theta)|^2)$ for $0 \leq j \leq M-1$ and $1 \leq i \leq M-j$ (throughout, $C$ denotes any positive constant depending only on $h$). Then, by the same argument as in (13), one gets $|\xi_j(\theta)|^2 \geq \text{ const } 4^{j-M} \geq \sqrt{\alpha}$.

Let $\hat{y} \in \hat{\mathcal{Y}}_0$ be fixed and write $\hat{y}_j = \Phi^j(\hat{y}_0) = (\hat{\theta}_j, \hat{S}_j, \hat{Y}_j, \hat{\Gamma}_j)$, with $\hat{Y}_j = (\hat{\xi}_j, \hat{\eta}_j)$ and $\hat{\Gamma}_j = (\cos\hat{\gamma}_j, \sin\hat{\gamma}_j)$. Define also $\lambda_j = \prod_{i=j}^{M-1} |-2\hat{\xi}_i|$. Then the same continuity argument as in Lemmas 2.5 and 3.2, gives $\lambda_j \geq \text{ const } \sigma_2^{M-j}$ for all $0 \leq j \leq M-1$. We let $K = 1000e^2$ and define $t_1 < t_2 < \cdots \leq M$ by $t_1 = 1$ and

$$t_{i+1} = \min\{s\colon t_i < s < M-5 \text{ and } \lambda_{t_i} \geq 2K\lambda_s\} \quad \text{(if it exists)}.$$

Then we set $k = k(r) = \max\{i\colon \lambda_{t_i} \geq 2Ke^{-r}/\sqrt{\alpha}\}$. In precisely the same way as in the proof of Lemma 2.6, we deduce that $k(r) \geq \gamma_1 r$ where $\gamma_1 = \eta/\log 8K$.

We already remarked that $|\xi_j(\theta)| \geq \sqrt[4]{\alpha}$ for all $\theta \in S^1$ and $0 \leq j \leq M-1$. Hence, we are in a position to apply Lemma 3.6 to obtain $H_1', H_1'' \subset \{1,\ldots,d\}$ with $\#H_1', \#H_1'' \geq [d/100]$, such that

$$|\xi_1(\bar{l}_1',\theta) - \xi_1(\bar{l}_1'',\theta)| \geq \frac{\alpha}{250} \quad \text{for all } \theta \in S^1,$$

for all $\bar{l}_1' = (l_1', l_2, \ldots, l_M)$ and $\bar{l}_1'' = (l_1'', l_2, \ldots, l_M)$ with $l_1' \in H_1'$, $l_1'' \in H_1''$, and arbitrary $l_2, \ldots, l_M$. Since $\operatorname{osc}(\eta_1) = b \operatorname{osc}(\xi_0) \leq b\alpha$, we also have

$$|\eta_1(\bar{l}_1',\theta) - \eta_1(\bar{l}_1'',\theta)| \leq b\alpha < \sqrt{\alpha}\frac{\alpha}{250} \quad \text{for all } \theta.$$

By induction (using the same kind of calculations as in (20)),

$$|\eta_j(\bar{l}_1',\theta) - \eta_j(\bar{l}_1'',\theta)| \leq \sqrt{\alpha}|\xi_j(\bar{l}_1',\theta) - \xi_j(\bar{l}_1'',\theta)|$$

for every $j \geq 1$ and $\theta$ in the corresponding domain. Combining this with

$$\frac{\operatorname{osc}(\xi_j)}{\left|\hat{\xi}_j\right|} \leq \frac{2\alpha 4^j}{\sqrt{C4^{j-M}}} \leq C\alpha 2^{j+M} \leq C\alpha 4^M \leq \sqrt{\alpha}$$

(take $c_0$ small for the last inequality), we find

$$\begin{aligned} &|\xi_{j+1}(\bar{l}_1',\theta) - \xi_{j+1}(\bar{l}_1'',\theta)| \geq \\ &\geq |-2\hat{\xi}_j|(1-\sqrt{\alpha})|\xi_j(\bar{l}_1',\theta) - \xi_j(\bar{l}_1'',\theta)| - b|\eta_j(\bar{l}_1',\theta) - \eta_j(\bar{l}_1'',\theta)| \\ &\geq |-2\hat{\xi}_j|(1-2\sqrt{\alpha})|\xi_j(\bar{l}_1',\theta) - \xi_j(\bar{l}_1'',\theta)|. \end{aligned} \tag{32}$$

Hence, using $\log(1-2\sqrt{\alpha})^M \approx$ const $\sqrt{\alpha}\log\alpha \geq -\log 2$ (if $c_0$ is small),

$$\begin{aligned}|\xi_M(\bar{l}'_1,\theta)-\xi_M(\bar{l}''_1,\theta)| &\geq \lambda_1(1-2\sqrt{\alpha})^{M-1}|\xi_1(\bar{l}'_1,\theta)-\xi_1(\bar{l}''_1,\theta)| \\ &\geq \tfrac{1}{2}\lambda_1\tfrac{\alpha}{250} \geq 4e^{2-r}\sqrt{\alpha}\end{aligned}$$

(because $k(r)\geq 1$), which means that any such $\bar{l}'_1, \bar{l}''_1$ are incompatible. Moreover,

$$|\xi_{t_2}(\bar{l}'_1,\theta)-\xi_{t_2}(\bar{l}''_1,\theta)| \geq \frac{1}{2}\frac{\lambda_1}{\lambda_{t_2}}\frac{\alpha}{250} \geq 4e^2\alpha,$$

by the definition of $t_2$. Also, given any $\bar{m}'_1 = (l'_1, l_2, \ldots, l_{t_2-1}, l'_{t_2}, \ldots, l'_m)$ and $\bar{m}''_1 = (l''_1, l_2, \ldots, l_{t_2-1}, l''_{t_2}, \ldots, l''_m)$, we have

$$|\xi_{t_2}(\bar{l}'_1,\theta)-\xi_{t_2}(\bar{m}'_1,\theta)| \leq \operatorname{osc}(\xi_{t_2}(l'_1, l_2, \ldots, l_{t_2-1})) \leq 8\alpha$$

and analogously for $\bar{l}''_1$, $\bar{m}''_1$. Therefore, $\bar{m}'_1$, $\bar{m}''_1$ are incompatible:

$$|\xi_M(\bar{m}'_1,\theta)-\xi_M(\bar{m}''_1,\theta)| \geq \frac{1}{2}\lambda_{t_2}(4e^2-16)\alpha \geq 4e^{2-r}\sqrt{\alpha}$$

(if $k(r)\geq 2$). Now, proceeding as in the proof of Lemma 2.6, we conclude that each $\bar{l} \in \{1,\ldots,d\}^M$ is compatible with not more than $d^M((d-[d/100])/d)^{\gamma_1 r}$ elements $\bar{m} \in \{1,\ldots,d\}^M$, as claimed above.

Starting the second part of the proof, we note that it is sufficient to consider the case $e^r \leq 4\alpha b^{-9}$. Indeed, otherwise the lemma is an immediate consequence of Lemma 3.5b) (take $\beta \leq 1/20$). Let $\bar{l}$, $\bar{m}$ be a pair of incompatible sequences as constructed above. More precisely, for some $1 \leq i \leq k(r)$ we have $l_j = m_j$ for $j < t_i$ and $l_{t_i} \in H'_i$, $m_{t_i} \in H''_i$. For notational simplicity, we shall write $t = t_i$ and let $\tilde{s}_i(\bar{l},\theta)$, $\tilde{r}_i(\bar{l},\theta)$, $\tilde{A}_i(\bar{l},\theta)$, $\tilde{C}_i(\bar{l},\theta)$, be the values of $\hat{s}_{M-t}$, $\hat{r}_{M-t}$, $A_{M-t}$, $C_{M-t}$, respectively, at the point $(\theta, S_t(\bar{l},\theta), Y_t(\bar{l},\theta)) \in T_3 \times I_0^2$. We write $\Gamma_t(\bar{l},\theta) = (\cos\gamma_t(\bar{l},\theta), \sin\gamma_t(\bar{l},\theta)) = h_i(1,0) + v_i(\tilde{s}_i(\bar{l},\theta),1)$ and then $\Gamma_M(\bar{l},\theta) = (\cos\gamma_M(\bar{l},\theta), \sin\gamma_M(\bar{l},\theta)) = h_i(\tilde{A}_i(\bar{l},\theta), \tilde{C}_i(\bar{l},\theta)) + v_i(0, \tilde{r}_i(\bar{l},\theta))$, which yields

$$b\,\mathrm{tg}\,\gamma_M(\bar{l},\theta) = b\frac{\tilde{C}_i}{\tilde{A}_i}(\bar{l},\theta) + b\frac{v_i}{h_i}\frac{\tilde{r}_i}{\tilde{A}_i}(\bar{l},\theta). \tag{33}$$

Note that

(i) $|(\tilde{r}_i/\tilde{A}_i)(\bar{l},\theta)| \leq$ const $b^{2(M-t)}$, by Lemma 3.4;

(ii) $|h_i/v_i| = |\operatorname{cotg}\gamma_t(\bar{l},\theta) - \tilde{s}_i(\bar{l},\theta)| \geq b^2\sqrt{\alpha}$, by Lemma 3.6b).

Hence, the last term in (33) is bounded by

$$\text{const}\, b^{2(M-t)+1}/(b^2\sqrt{\alpha}) \leq \text{const}\, b^{11}/\sqrt{\alpha} \leq e^{2-r}\sqrt{\alpha}$$

(using $M - t > 5$ and $e^r \le 4\alpha b^{-9}$ and assuming $c_0$ small enough). Clearly, the same arguments and estimates hold also for $\bar{m}$. In order to control the first term in (33) we introduce $u_j(\bar{l},\theta) = \operatorname{slope} \partial_X f^{j-t}(\theta, S_t(\bar{l},\theta), Y_t(\bar{l},\theta))(1,0)$, and let $u_j(\bar{m},\theta)$ be the analogous object for $\bar{m}$. Note that $u_t = 0$, $u_M = \tilde{C}_i/\tilde{A}_i$, and $|u_j| \le c_0$ for all $t \le j \le M$ (by Lemma 3.2a) and Remark 1). Note also that $u_{j+1} = b/(2\xi_j - bu_j)$ for all $j$. Hence, using $|\xi_j| \ge \sqrt[4]{\alpha} \gg bc_0 \ge |bu_j|$,

$$\left|u_{j+1}(\bar{l},\theta) - u_{j+1}(\bar{m},\theta)\right| \le \sqrt{b}|\xi_j(\bar{l},\theta) - \xi(\bar{m},\theta)| + \sqrt{b}|u_j(\bar{l},\theta) - u_j(\bar{m},\theta)|.$$

By recurrence, $\left|u_M(\bar{l},\theta) - u_M(\bar{l},\theta)\right| \le \sum_{s=1}^{t} (\sqrt{b})^s |\xi_{M-s}(\bar{l},\theta) - \xi_{M-s}(\bar{m},\theta)|$. Moreover (recall (32)), $|\xi_{M-s}(\bar{l},\theta) - \xi_{M-s}(\bar{m},\theta)| \le (2/\lambda_s)|\xi_M(\bar{l},\theta) - \xi_M(\bar{m},\theta)|$ and $2/\lambda_s$ is uniformly bounded from above. This gives

$$\begin{aligned}\left|\frac{\tilde{C}_i}{\tilde{A}_i}(\bar{l},\theta) - \frac{\tilde{C}_i}{\tilde{A}_i}(\bar{m},\theta)\right| &\le \sum_{s=1}^{t} \text{const}\,(\sqrt{b})^s |\xi_M(\bar{l},\theta) - \xi_M(\bar{m},\theta)| \\ &\le \tfrac{1}{8}|\xi_M(\bar{l},\theta) - \xi_M(\bar{m},\theta)|.\end{aligned}$$

Altogether, we conclude that

$$\left|b \operatorname{tg} \gamma_M(\bar{l},\theta) - b \operatorname{tg} \gamma_M(\bar{m},\theta)\right| \le \frac{1}{8}|\xi_M(\bar{l},\theta) - \xi_M(\bar{m},\theta)| + 2e^{2-r}\sqrt{\alpha}. \tag{34}$$

Now we estimate $\left|bs_M(\bar{l},\theta) - bs_M(\bar{m},\theta)\right|$. First we recall that by definition $s_M(\bar{l},\theta) = \hat{s}_k(g(\theta), S_{M+1}(\bar{l},\theta), Y_{M+1}(\bar{l},\theta))$ where $k = k(\bar{l},\theta)$ is the number of iterates before the next return. Clearly, up to taking $c_0$ small enough, we may suppose $k(\bar{l},\theta) \ge 4$ for all $\theta \in S^1$. Then the proof of Lemma 3.2 gives

$$\left|bs_M(\bar{l},\theta) - b\hat{s}_4(g(\theta), S_{M+1}(\bar{l},\theta), Y_{M+1}(\bar{l},\theta))\right| \le \text{const}\, b^{10} \le e^{2-r}\sqrt{\alpha}$$

(we use $e^r \le 4\alpha b^{-9}$ once more). Of course, the same holds for $\bar{m}$. On the other hand, Lemma 3.3b) gives (here we have $\xi \ge 1$)

$$\begin{aligned}&\left|b\hat{s}_4(g(\theta), S_{M+1}(\bar{l},\theta), Y_{M+1}(\bar{l},\theta)) - b\hat{s}_4(g(\theta), S_{M+1}(\bar{m},\theta), Y_{M+1}(\bar{m},\theta))\right| \\ &\quad \le \tfrac{b^2}{2}\left|\xi_{M+1}(\bar{l},\theta) - \xi_{M+1}(\bar{m},\theta)\right| + \text{const}\, b^2 \left|Y_{M+1}(\bar{l},\theta) - Y_{M+1}(\bar{m},\theta)\right| \\ &\quad \le \tfrac{1}{8}\left|\xi_M(\bar{l},\theta) - \xi_M(\bar{m},\theta)\right|,\end{aligned}$$

using the inequalities $\left|\xi_{M+1}(\bar{l},\theta) - \xi_{M+1}(\bar{m},\theta)\right| \le \left|Y_{M+1}(\bar{l},\theta) - Y_{M+1}(\bar{m},\theta)\right| \le \text{const}\left|\xi_M(\bar{l},\theta) - \xi_M(\bar{m},\theta)\right|$. We conclude that

$$\left|bs_M(\bar{l},\theta) - bs_M(\bar{m},\theta)\right| \le \frac{1}{8}\left|\xi_M(\bar{l},\theta) - \xi_M(\bar{m},\theta)\right| + 2e^{2-r}\sqrt{\alpha}. \tag{35}$$

Altogether, (30), (34), (35) imply (31) and so our argument is complete. □

### 3.5 Conclusion of the proof

Finally, we derive Theorem B from the previous lemmas. In the rest of the paper $\hat{\mathcal{X}}_0 = \operatorname{graph}(T_0, X_0, \Psi_0)$ is any admissible curve with $\Psi_0 = \partial/\partial y$ and the notations are as introduced at the beginning of Section 3. We take $l, m \approx \sqrt{n}$, $l = m - M$, as before. For any return $1 \le \nu \le n$ (recall the definition above) and $\theta \in \omega_{\nu-M}$ we define $r = r(\nu, \theta) \ge 0$ to be the smallest integer such that $\Delta_\nu(\omega_{\nu+l}(\theta)) \cap J(r) = \emptyset$. Recall that $\omega_i(\theta)$ denotes the element of $\mathcal{P}_i$ containing $\theta$. Then we say that $\nu$ is a 0-*situation*, resp. a $\mathrm{I}_n$-*situation*, resp. a $\mathrm{II}_n$-*situation*, for $\theta$ if $r = 0$, resp. $1 \le r \le m$, resp. $r > m$. Clearly, Lemma 3.5a) asserts that $|b \operatorname{tg} \psi_\nu| \ge 10\sqrt{\alpha}$ can only occur in a 0-situation. On the other hand, if $\nu$ is a $\mathrm{II}_n$-situation then $\Delta_\nu(\omega_{\nu+l}(\theta)) \subset J(m-1)$, because

(i) $\operatorname{osc}(x_\nu | \omega_{\nu+l}(\theta)) \le \alpha d^{-l} \ll \sqrt{\alpha} e^{-m}$;

(ii) $\operatorname{osc}(b \operatorname{tg} \psi_\nu | \omega_{\nu+l}(\theta)) \le b(1 + (100\alpha/b^2))(b\alpha/d) d^{-l} \ll \sqrt{\alpha} e^{-m}$;

(iii) $\operatorname{osc}(b s_\nu | \omega_{\nu+l}(\theta)) \le \sqrt{\alpha} b^l \ll \sqrt{\alpha} e^{-m}$.

Indeed, (i) and (ii) are direct consequences of the definition of admissible curve and Lemma 3.1, and (iii) can be justified as follows. If $k_\nu(\tau) \le l/2 (< m)$ for some $\tau \in \omega_{\nu+l}(\theta)$ then, by definition, $k_\nu$ is constant on $\omega_{\nu+l}(\theta)$ and then (26) gives $\operatorname{osc}(b s_\nu | \omega_{\nu+l}(\theta)) \le \operatorname{const} \sqrt{\alpha} b^{2k} d^{k-1-l} \le \operatorname{const} \sqrt{\alpha} d^{-l/2-1} \le \sqrt{\alpha} b^l$. If $k_\nu(\tau) > l/2$ for all $\tau \in \omega_{\nu+l}(\theta)$, we take $q = [l/2]$ and, using also $|s_\nu - s_{\nu,q}| \le \operatorname{const} b^{2q+1}$, we get $\operatorname{osc}(b s_\nu | \omega_{\nu+l}(\theta)) \le \operatorname{const} \sqrt{\alpha} b^{2q} d^{q-1-l} + \operatorname{const} b^{2q+2} \le \sqrt{\alpha} b^l$. Claim (iii) is proved. Now we let $B_2(n) = \{\theta \in S^1\colon \text{some } 1 \le \nu \le n \text{ is a } \mathrm{II}_n\text{-situation for } \theta\}$ and then Lemma 3.5b) gives

$$m(B_2(n)) \le n b^{-1/2} \left(\sqrt{\alpha} e^{-m+1}\right)^{1/10} \le \operatorname{const} e^{-\sqrt{n}/20}$$

for all $n$ sufficiently large (with respect to $\alpha$ and $b$).

From now on we may restrict ourselves to those values of $\theta \in S^1$ having no $\mathrm{II}_n$-situations in $[1, n]$. Let $1 \le \nu_1 \le \cdots \le \nu_s \le n$ be the returns of $\theta$ and denote $r_i = r(\nu_i, \theta)$. We write $W_\nu(\theta) = \|W_{\nu-1}(\theta)\| \left(h_\nu(1,0) + v_\nu(s_\nu(\theta), 1)\right)$, for each $\nu = \nu_i$. Then Lemma 3.2 and the definition of $\Delta_\nu$ in (23) give

$$\begin{aligned} \|W_{\nu+j}(\theta)\| &\ge (1 - c_0) \|W_{\nu-1}(\theta)\| \, |h_\nu| \, \|\partial_X f(Z_{\nu+1}(\theta))(1,0)\| \\ &\ge (1 - c_0) \|W_{\nu-1}(\theta)\| \, |\cos \psi_\nu(\theta)| \, |\Delta_\nu(\theta)| \operatorname{const} \sigma_2^j, \end{aligned}$$

for all $0 \le j < \nu_{i+1} - \nu_i$, where $Z_{\nu+1}(\theta) = \varphi^{\nu+1}(\theta, T_0(\theta), X_0(\theta))$. Moreover, taking $j = \nu_{i+1} - \nu_i - 1$ and writing $\mu = \nu_{i+1}$,

$$\begin{aligned} \|W_{\mu-1}(\theta)\| &\ge \|W_{\mu-1}(\theta)\| \, |\cos \psi_\mu(\theta)| \\ &\ge (1 - c_0) \|W_{\nu-1}(\theta)\| \, |\cos \psi_\nu(\theta)| \, |\Delta_\nu(\theta)| \operatorname{const} \alpha^{-1+\eta} \sigma_2^{\mu-\nu-N}. \end{aligned}$$

By recurrence (recall that $W_0 = \frac{\partial}{\partial x}$),

$$\|W_n(\theta)\| \geq \sigma_2^{n-(s-1)N} \alpha^{3/2-\eta} \prod_{i=1}^{s} (\text{ const } |\Delta_{\nu_i}(\theta)| \alpha^{-1+\eta})$$

and so $\log \|W_n(\theta)\|$ is bounded from below by

$$(n-(s-1)N) \log \sigma_2 + \sum_{i=1}^{s} \left( \left( \frac{1}{2} - \eta \right) \log \frac{1}{\alpha} - r_i \right) - s \text{ const } - \frac{3}{2} \log \frac{1}{\alpha}$$

From here on the argument is completely analogous to the one in Section 2, with Lemma 3.7 replacing Lemma 2.6. We get $\log \|W_n(\theta)\| \geq 2cn - \sum_{i \in G} r_i$, where $c > 0$ and $G = \{i : r_i \geq (\frac{1}{2} - 2\eta) \log \frac{1}{\alpha}\}$, and we prove that the Lebesgue measure of $B_1(n) = \{\theta \in S^1 : \sum_{i \in G} r_i \geq cn\}$ is bounded by $\text{const}\, e^{\text{const} \sqrt{n}}$ (if $\nu_1 < M$ then Lemma 3.7 can not be used at time $\nu_1$ but this is irrelevant for the conclusion; recall that $\nu_i > M$ for all $i > 1$). Then $E_n = B_1(n) \cup B_2(n)$ satisfies the claim (C) stated at the beginning of Section 3. This completes the proof of the theorem in the case $\varphi = \varphi_{\alpha,b}$.

Moreover, it is not difficult to see that all these arguments remain valid for arbitrary diffeomorphisms in a sufficiently small neighbourhood of $\varphi_{\alpha,b}$ (depending on $\alpha$ and $b$), if one uses the same approach as for Theorem A. Indeed, since $\mathcal{F}_0 = \{\Theta = \text{ const }\}$ is a normally hyperbolic invariant foliation for $\varphi_{\alpha,b}$, we have that any nearby diffeomorphism $\varphi$ also admits such a foliation $\mathcal{F}(\varphi)$. Moreover, the leaves of $\mathcal{F}(\varphi)$ converge to those of $\mathcal{F}_0$ as $\varphi$ approaches $\varphi_{\alpha,b}$. Hence, we may reproduce the previous calculations for $\varphi$, just taking $X$-vector to mean any vector tangent to a leaf of $\mathcal{F}(\varphi)$ and making straightforward adjustments in the notations. As our argument is based on analysing pieces of orbits whose length is bounded independent of $n$ (this remark is particularly clear e.g. in the context of Lemma 3.2), all our estimates remain valid, by continuity, if $\varphi$ is close enough to $\varphi_{\alpha,b}$. Therefore, the proof of Theorem B is complete.

# References


[BC1] M. Benedicks, L. Carleson, *On iterations of* $1 - ax^2$ *on* $(-1, 1)$, Ann. Math. 122 (1985), 1-25.

[BC2] M. Benedicks, L. Carleson, *The dynamics of the Hénon map*, Ann. Math. 133 (1991), 73-169.

[BY] M. Benedicks, L.-S. Young, *SBR-measures for certain Hénon maps*, Invent. Math. 112-3 (1993), 541-576.

[BD] C. Bonatti, L. J. Díaz, *Persistent nonhyperbolic transitive diffeomorphisms*, Ann. Math. 143 (1996), 357-396.

[CE] P. Collet, J.-P. Eckmann, *On the abundance of aperiodic behaviour*, Comm. Math. Phys. 73 (1980), 115-160.

[HPS] M. Hirsch, C. Pugh, M. Shub, *Invariant Manifolds*, Lect. Notes Math. 583 (1977), Springer Verlag.

[Ja] M. Jakobson, *Absolutely continuous invariant measures for one-parameter families of one-dimensional maps*, Comm. Math. Phys. 81 (1981), 39-88.

[Ma] R. Mañé, *Contributions to the stability conjecture*, Topology 17 (1978), 383-396.

[MS] W. de Melo, S. van Strien, *One-Dimensional Dynamics*, Springer Verlag, 1993.

[MV] L. Mora, M. Viana, *Abundance of strange attractors*, Acta Math. 171 (1993), 1-71.

[No] T. Nowicki, *A positive Lyapunov exponent for the critical value of an S-unimodal mapping implies uniform hyperbolicity*, Ergod. Th. & Dynam. Sys. 8 (1988), 425-435.

[Sh] M. Shub, *Topologically transitive diffeomorphisms on* $T^4$, Lect. Notes in Math. 206 (1971), 39, Springer Verlag.

[Si] D. Singer, *Stable orbits and bifurcations of maps of the interval*, SIAM J. Appl. Math. 35 (1978), 260-267.

[Sm] S. Smale, *Differentiable dynamical systems*, Bull. Am. Math. Soc. 73 (1967), 747-817.

[Vi] M. Viana, *Strange attractors in higher dimensions*, Bull. Braz. Math. Soc. 24 (1993), 13-62.

[Yo] L.-S. Young, *Some open sets of nonuniformly hyperbolic cocycles*, Ergod. Th. & Dynam. Sys. 13 (1993), 409-415.

Marcelo Viana
IMPA, Est. D. Castorina 110, Jardim Botânico, 22460 Rio de Janeiro, Brazil
Tel: +55(21)529-5114 Fax: +55(21)512-4115 E-mail: viana@impa.br