Lyapunov Exponents

M compact space / manifold $\quad \mathbb{R} \quad \mathbb{C}$

$\pi : Z \longrightarrow M$ finite – dim vector bundle

$f : M \longrightarrow M$ cont transf / homeo

Linear cocycle over f

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\;F\;} & \mathcal{E} \\ \pi \downarrow & & \downarrow \pi \\ M & \xrightarrow[\;f\;]{} & M \end{array}$$

continuous and linear (isomorphism) on fibers

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continuous and linear (isomorphism) on fibers

Ex: $\mathcal{E} = M \times \mathbb{R}^d$ or $M \times \mathbb{C}^d$

$$F(x,v) = (f(x)\, ,\, A(x)v)$$

where $A : M \longrightarrow G\, ,\ G \leq GL(d)$

$$F^n(x,v) = (f^n(x)\, ,\, A^n(x)v)$$

with $A^n(x) = A(f^{n-1}x) \ldots A(fx)\, A(x)$

Oseledets thm:

for any f-invariant probability μ
and μ-almost every $x \in M$

$$\exists\ \mathcal{E}_x = E_x^1 \oplus \ldots \oplus E_x^k \qquad k = k(x)$$
$$\exists\ \lambda_1(x) > \ldots > \lambda_k(x)$$

s.t. $$\lim_{n \to \pm\infty} \frac{1}{n} \log \| F_x^n v \| = \lambda_i(x)$$

for all $v \in E_x^i \setminus \{0\}$

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△ S $\quad x \longmapsto E_x^i$ and $x \longmapsto \lambda_i(x)$
are just measurable

$\angle(E_x^i, E_x^j)$ is not bounded
from zero

k and dim E_x^i depend on x

1) How do the λ_i depend on F ?

2) How often do $\lambda_i = 0$?

C^r (M,G) $\quad r \geq 0$

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Ex: (dynamical cocycles)

$f : M \to M$ diffeomorphism

$\mathcal{E} = TM \qquad F = Df$

$\mathrm{Diff}_\mu^r(M) \qquad \mathrm{Symp}_\omega^r(M)$

$r \geq 1$

Ex: (random matrices)

Let $\alpha_0, \alpha_1, \ldots$ be i. i. d. random variables in SL(d), with probability distribution ν. What can be said of

$$(*) \quad \lim_{n \to \infty} \frac{1}{n} \log \| \alpha_{n-1} \ldots \alpha_1 \alpha_0 \| \,?$$

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$M = \mathrm{supp}(\nu)^{\mathbb{Z}} \qquad \mu = \nu^{\mathbb{Z}}$

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$f : M \to M$ shift map

$A : M \to SL(d)$, $(\alpha_n)_n \longmapsto \alpha_0$

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Ex: Let M = S^1

f : M → M expanding map

e.g. $\theta \to k\ \theta \mod 1$

$f(0) = 0$

μ ergodic with supp μ = M

A : M → SL(2,ℝ)

$$A(\theta) = A_0 \cdot \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

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<u>Thm</u>: Assume $\deg(f) \neq 2,3$.
There exists a C^0 neigbourhood $\mathcal{U}$ of A such that

1) for generic (dense G_δ) $B \in \mathcal{U}$

$$\lambda_1 = 0 \ (\Rightarrow\ k = 1) \text{ a.e.}$$

2) for every Hölder continuous $B \in \mathcal{U}$

$$\lambda_1 > 0 > \lambda_2 = -\lambda_1 \text{ a.e.}$$

Thm1 (Bochi, V) $d = 2$: Mañé, Bochi

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$\Downarrow$

- it admits a continuous extension to supp(μ)
- angles are bounded from zero

Ex: $M = S^1$

$f : M \to M$ continuous

μ ergodic with supp $(\mu) = M$

$A : M \to SL(2,\mathbb{R})$

such that

$$\deg(f) - 1 \nmid 2 \deg(A) \quad (*)$$

Then, generically in the homotopy class of A, all Lyapunov exponents are zero a.e.

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Reason: $(*) \Rightarrow$ the cocycle has no continuous invariant sub-bundle

Comments on the proof of thm1:

$\lambda_1(x) > \ldots > \lambda_k(x)$ Lyapunov exps

$\hat{\lambda}_1(x) > \ldots > \hat{\lambda}_k(x)$ Lyapunov exps counted with multiplicity $\dim E_x^i$

$$\hat{\lambda}_i(A) = \int_M \hat{\lambda}_i(x)\, d\mu$$

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Thm2 (Bochi, V)

A is a point of continuity of

$$C^0(M,G) \ni B \to (\hat{\lambda}_1(B), \ldots, \hat{\lambda}_d(B))$$

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$\{$continuity points$\}$ contains dense G_δ

Ex: $M = S^1$

$$f : M \to M, \quad f(\theta) = \theta + \omega, \quad \omega \notin \mathbb{Q}$$

$$A(\theta) = \begin{pmatrix} V(\theta) - E & -1 \\ 1 & 0 \end{pmatrix}$$

Schrödinger cocycle

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Schrödinger cocycle

A is a point of continuity of Lyapunov exponents

$\Updownarrow$

either the exponents are zero

or $E \notin$ spectrum of associated Schrödinger operator

What about dynamical cocycles ?

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<u>Thm3</u> (Bochi, V) d = 2 : Mañé, Bochi

There exists a residual $\mathfrak{R} \subset \mathrm{Diff}^1_\mu(M)$ in the space of volume preserving diffeomorphisms, such that for every $f \in \mathfrak{R}$ and μ- almost every $x \in M$

either a) all Lyapunov exponents are zero at x

or b) the Oseledets decomposition is dominated on the orbit of x

Thm4 (Bochi, V)

There is a residual $\mathfrak{R} \subset \mathrm{Symp}^1_\omega(M)$ in the space of symplectic diffeomorphisms such that for every $f \in \mathfrak{R}$

either a) almost every point has zero as Lyapunov exponent (multiplicity ≥ 2)

or b) f is Anosov

⇓

very strong restrictions on the manifold !

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$f \in \mathrm{Diff}_\mu^{r+1}(M)$ $F = Df$

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for $r > 0$?

Assume (f, μ) is hyperbolic (non-uniformly)

- all exponents of DF non-zero
- μ ergodic, non-atomic, with local product structure

G = SL(d) or Symp (2d)

Thm5

For every $r > 0$ the set of $A \in C^r(M,G)$ with positive Lyapunov exponents contains an open dense set.

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Moreover, its complement has ∞ - codimension.

it is contained in finite unions of closed submanifolds with arbitrary codimension

Bonatti, Gomez-Mont, V:

same conclusion when $f \in$ Axiom A and the cocycle is partially hyperbolic

One key ingredient:

Consider $f : M \rightarrow M$ uniformly expanding and supp $(\mu) = M$ and r = Lipschitz

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Consider $f : M \to M$ uniformly expanding and supp $(\mu) = M$ and $r =$ Lipschitz

The cocycle A is <u>bundle-free</u> if

$\forall \eta \geq 1$ there exists no Lipschitz map

$$M \ni x \mapsto \underbrace{\{\xi_1(x), \dots, \xi_\eta(x)\}}_{\text{distinct points in } \mathbb{R}p^{d-1}} = \xi(x)$$

invariant under A

$$A(x) . \xi(x) = \xi(f(x)) \quad \forall x \in M$$

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Thm

Assume $A \in C^{Lipschitz}$ satisfies

1) A is bundle-free

2) there is $p \in Fix(f^k)$, $k \geq 1$ such that all eigenvalues of $A^k(p)$ have distinct norms

Then $\lambda_1(A) > 0$.

Both conditions contain open and dense set, the complement has ∞-codimension

What are the continuity points of Lyapunov exponents in $C^r(M,G)$?

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ongoing , Avila, Bochi, V

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Lower estimates of exponents ?