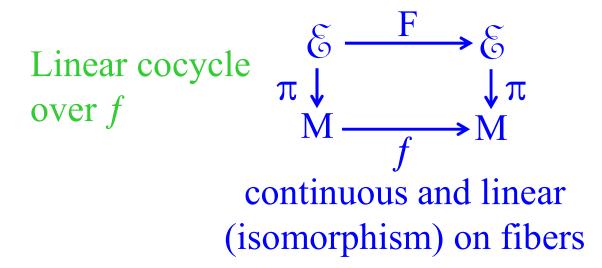
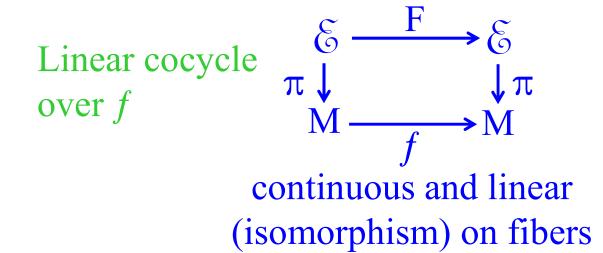
## Lyapunov Exponents

M compact space / manifold  $\mathbb{R} \mathbb{C}$  $\pi: \mathbb{Z} \longrightarrow \mathbb{M}$  finite – dim vector bundle  $f: \mathbb{M} \longrightarrow \mathbb{M}$  cont transf / homeo



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Ex:  $\mathcal{E} = M \times \mathbb{R}^d$  or  $M \times \mathbb{C}^d$ F(x,v) = (f(x), A(x)v)

where  $A: M \longrightarrow G$ ,  $G \le GL(d)$  $F^n(x,v) = (f^n(x), A^n(x)v)$ 

with  $A^{n}(x) = A(f^{n-1}x) \dots A(fx) A(x)$ 

# **Oseledets thm:**

for any *f*-invariant probability  $\mu$ and  $\mu$ -almost every  $x \in M$  $\exists \&_x = E_x^1 \oplus \dots \oplus E_x^k$  $\exists \lambda_1(x) > \dots > \lambda_k(x)$  k = k(x)s.t.  $\lim_{n \to \pm \infty} \frac{1}{n} \log || F_x^n v || = \lambda_i(x)$ for all  $v \in E_x^i \setminus \{0\}$ 

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for any *f*-invariant probability  $\mu$ and  $\mu$ -almost every  $x \in M$  $\exists \mathcal{E}_{x} = E_{x}^{1} \oplus \ldots \oplus E_{x}^{k}$  $\mathbf{k} = \mathbf{k}(\mathbf{x})$  $\exists \lambda_1(x) > \ldots > \lambda_{l}(x)$  $\lim_{n \to \pm \infty} \frac{1}{n} \log \| \mathbf{F}_x^n \mathbf{v} \| = \lambda_i(x)$ s.t. for all  $v \in E_{v}^{i} \setminus \{0\}$  $x \mapsto E'_{x}$  and  $x \mapsto \lambda_{i}(x)$ S are just measurable  $\langle (\mathbf{E}_{\mathbf{x}}^{i}, \mathbf{E}_{\mathbf{x}}^{j})$  is not bounded from zero k and dim  $E_x^i$  depend on x

1) How do the  $\lambda_i$  depend on F? 2) How often do  $\lambda_i = 0$ ?  $C^r (M,G) \quad r \ge 0$ 

1) How do the  $\lambda_i$  depend on F? 2) How often do  $\lambda_i = 0$  ?  $C^r(M,G)$   $r \ge 0$ Ex: (dynamical cocycles)  $f: M \rightarrow M$  diffeomorphism  $\mathcal{E} = TM$  $\mathbf{F} = \mathbf{D}f$  $\operatorname{Symp}_{\omega}^{r}(M)$  $\operatorname{Diff}_{\mu}^{r}(M)$  $r \ge 1$ 

Let  $\alpha_0, \alpha_1, \dots$  be i. i. d. random variables in SL(d), with probability distribution v. What can be said of

(\*)  $\lim_{n\to\infty} \frac{1}{n} \log \|\alpha_{n-1} \dots \alpha_1 \alpha_0\|$ ?

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 $M = \operatorname{supp}(v)^{\mathbb{Z}} \qquad \mu = v^{\mathbb{Z}}$   $\mathcal{E} = M \times \mathbb{R}^{d} \quad \text{or} \quad M \times \mathbb{C}^{d}$   $f : M \to M \quad \text{shift map}$   $A : M \to SL(d) , (\alpha_{n})_{n} \longmapsto \alpha_{0}$ then (\*) =  $\lambda_{1}$  for the corresponding cocycle

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$$A^n(x), x = (\alpha_n)_n$$

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<u>Thm</u>: Assume  $deg(f) \neq 2,3$ . There exists a C<sup>0</sup> neigbourhood  $\mathfrak{A}$  of *A* such that

- 1) for generic (dense  $G_{\delta}$ )  $B \in \mathbb{N}$  $\lambda_1 = 0 \ (\Rightarrow k = 1)$  a.e.
- 2) for every Hölder continuous *B* ∈ 𝔄  $\lambda_1 > 0 > \lambda_2 = -\lambda_1$  a.e.

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- or b) the Oseledets splitting is dominated
  - it admits a continuous extension to supp(μ)
  - angles are bounded from zero

Ex:  $M = S^1$   $f: M \rightarrow M$  continuous  $\mu$  ergodic with supp ( $\mu$ ) = M  $A: M \rightarrow SL(2, \mathbb{R})$ 

such that

 $\deg(f) - 1 \not\mid 2 \deg(A)$  (\*) Then, generically in the homotopy class of *A*, all Lyapunov exponents are zero a.e. Ex:  $M = S^1$   $f: M \rightarrow M$  continuous  $\mu$  ergodic with supp ( $\mu$ ) = M  $A: M \rightarrow SL(2, \mathbb{R})$ 

such that

 $\deg(f) - 1 \not\mid 2 \deg(A) \quad (*)$ 

Then, generically in the homotopy class of *A*, all Lyapunov exponents are zero a.e.

Reason: (\*) ⇒ the cocycle has no continuous invariant sub-bundle

Comments on the proof of thm1:  

$$\lambda_1(x) > ... > \lambda_k(x)$$
 Lyapunov exps  
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Thm2 (Bochi, V)  
*A* is a point of continuity of  
 $\mathbb{C}^0(M, \mathbb{G}) \ni B \rightarrow (\hat{\lambda}_1(B), ..., \hat{\lambda}_d(B))$   
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{continuity points} contains dense  $G_{\delta}$ 

#### Ex: $M = S^1$

 $f: \mathbf{M} \to \mathbf{M}, \ f(\theta) = \theta + \omega, \ \omega \notin \mathbb{Q}$  $\mathbf{A}(\theta) = \begin{pmatrix} \mathbf{V}(\theta) - \mathbf{E} & -1 \\ 1 & 0 \end{pmatrix}$ Schrödinger cocycle

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A is a point of continuity of Lyapunov exponents
↓
either the exponents are zero or E ∉ spectrum of associated Schrödinger operator

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<u>Thm3</u> (Bochi, V) d = 2 : Mañé, Bochi

There exists a residual  $\Re \subset \text{Diff}_{\mu}^{T}(M)$ in the space of volume preserving diffeomorphisms, such that for every  $f \in \Re$  and  $\mu$ - almost every  $x \in M$ 

either a) all Lyapunov exponents are zero at *x* 

or b) the Oseledets decomposition is dominated on the orbit of x

# <u>Thm4</u> (Bochi, V)

There is a residual  $\Re \subset \text{Symp}_{\omega}^{1}$  (M) in the space of symplectic diffeomorphisms such that for every  $f \in \Re$ 

either a) almost every point has zero as Lyapunov exponent (multiplicity ≥ 2)

or b) f is Anosov

very strong restrictions on the manifold !

# What about $A \in C^{r}(M,G)$ $f \in Diff_{\mu}^{r+1}(M)$ F = Dffor r > 0?

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Assume  $(f, \mu)$  is hyperbolic (non-uniformly)

- all exponents of DF non-zero
- μ ergodic, non-atomic, with local product structure

G = SL(d) or Symp(2d)

## Thm5

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For every r > 0 the set of  $A \in C^{r}(M,G)$ with positive Lyapunov exponents contains an open dense set.

Moreover, its complement has  $\infty$  - codimension.

it is contained in finite unions of closed submanifolds with arbitrary codimension

Bonatti, Gomez-Mont, V:

same conclusion when  $f \in Axiom A$  and the cocycle is partially hyperbolic

One key ingredient:

Consider  $f : M \rightarrow M$  uniformly expanding and supp  $(\mu) = M$  and r = Lipschitz One key ingredient:

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The cocycle A is <u>bundle-free</u> if

 $\forall \eta \geq 1$  there exists no Lipschitz map

$$M \ni x \mapsto \{ \xi_1(x), \dots, \xi_\eta(x) \} = \xi(x)$$
  
distinct points in  $\mathbb{R}p^{d-1}$ 

invariant under A

 $A(x) \cdot \xi(x) = \xi(f(x)) \quad \forall x \in \mathbf{M}$ 

Ex:  $M = S^1$ 

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<u>Thm</u>

Assume  $A \in C^{Lipschitz}$  satisfies

- 1) A is bundle-free
- 2) there is p ∈ Fix(f<sup>k</sup>), k ≥ 1 such that all eigenvalues of A<sup>k</sup>(p) have distinct norms
  Then λ<sub>1</sub>(A) > 0.

Both conditions contain open and dense set, the complement has ∞-codimension

# What are the continuity points of Lyapunov exponents in $C^r(M,G)$ ?

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ongoing, Avila, Bochi, V

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Lower estimates of exponents ?