THEOREM OF OSELEDETS

We recall some basic facts and terminology relative to linear cocycles and the multiplicative ergodic theorem of Oseledets [1].

0.1. **Cocycles over maps.** Let μ be a probability measure on some space M and $f : M \to M$ be a measurable transformation that preserves μ. Let $\pi : \mathcal{E} \to M$ be a finite-dimensional vector bundle endowed with a Riemannian norm $\|\cdot\|$. A *linear cocycle* (or vector bundle morphism) over f is a map $F : \mathcal{E} \to \mathcal{E}$ such that

$$\pi \circ F = f \circ \pi$$

and the action $A(x) : \mathcal{E}_x \to \mathcal{E}_{f(x)}$ of F on each fiber is a linear isomorphism. It is often possible to assume that the vector bundle is trivial, meaning that $\mathcal{E} = M \times \mathbb{R}^d$, restricting to some full μ-measure subset of M if necessary. Then $A(\cdot)$ takes values in the linear group $\mathrm{GL}(d, \mathbb{R})$ of invertible $d \times d$ matrices. Notice that, in general, the action of the nth iterate is given by $A^n(x) = A(f^{n-1}(x)) \cdots A(f(x)) \cdot A(x)$, for every $n \geq 1$. Given any $y > 0$, we denote $\log^+ y = \max\{\log y, 0\}$.

Theorem 0.1. *Assume the function* $\log^+ \|A(x)\|$ *is* μ*-integrable. Then, for* μ*-almost every* $x \in M$*, there exists* $k = k(x)$*, numbers* $\lambda_1(x) > \cdots > \lambda_k(x)$*, and a filtration* $\mathcal{E}_x = F_x^1 > \cdots > F_x^k > \{0\} = F_x^{k+1}$ *of the fiber, such that*

(1) $k(f(x)) = k(x)$ *and* $\lambda_i(f(x)) = \lambda_i(x)$ *and* $A(x) \cdot F_x^i = F_{f(x)}^i$ *and*
(2) $\displaystyle\lim_{n\to+\infty} \frac{1}{n} \log \|A^n(x)v\| = \lambda_i(x)$ *for all* $v \in F_x^i \setminus F_x^{i+1}$ *and all* $i = 1, \ldots, k$.

The *Lyapunov exponents* λ_i and the subspaces F^i depend in a measurable (but usually not continuous) fashion on the base point. The statement of the theorem, including the values of $k(x)$, the $\lambda_i(x)$, and the $F^i(x)$, is not affected if one replaces $\|\cdot\|$ by any other Riemann norm $|||\cdot|||$ equivalent to it in the sense that there exists some μ-integrable function $c(\cdot)$ such that

$$e^{-c(x)}\|v\| \leq |||v||| \leq e^{c(x)}\|v\| \quad \text{for all } v \in T_x M. \tag{1}$$

When the measure μ is ergodic, the values of $k(x)$ and of each of the $\lambda_i(x)$ are constant on a full measure subset, and so are the dimensions of the subspaces F_x^i. We call $\dim F_x^i - \dim F_x^{i+1}$ the *multiplicity* of the corresponding Lyapunov exponent $\lambda_i(x)$. The *Lyapunov spectrum* of F is the set of all Lyapunov exponents, each counted with multiplicity. The Lyapunov spectrum is *simple* if all Lyapunov exponents have multiplicity 1.

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0.2. **The invertible case.** If the transformation f is invertible then so is the cocycle F. Applying Theorem 0.1 also to the inverse F^{-1} and combining the invariant filtrations of the two cocycles, one gets a stronger conclusion than in the general non-invertible case:

Theorem 0.2. *Let $f : M \to M$ be invertible and both functions $\log^+ \|A(x)\|$ and $\log^+ \|A^{-1}(x)\|$ be μ-integrable. Then, for μ-almost every point $x \in M$, there exists $k = k(x)$, numbers $\lambda_1(x) > \cdots > \lambda_k(x)$, and a decomposition $\mathcal{E}_x = E_x^1 \oplus \cdots \oplus E_x^k$ of the fiber, such that*

(1) $A(x) \cdot E_x^i = E_{f(x)}^i$ *and* $F_x^i = \oplus_{j=i}^k E_x^j$ *and*

(2) $\displaystyle\lim_{n\to\pm\infty} \frac{1}{n} \log \|A^n(x)v\| = \lambda_i(x)$ *for all non-zero* $v \in E_x^i$ *and*

(3) $\displaystyle\lim_{n\to\pm\infty} \frac{1}{n} \log \angle\big(E_{f^n(x)}^i, E_{f^n(x)}^j\big) = 0$ *for all* $i, j = 1, \ldots, k$.

Note that the multiplicity of each Lyapunov exponent λ_i coincides with the dimension $\dim E_x^i = \dim F_x^i - \dim F_x^{i+1}$ of the associated *Oseledets subspace* E_x^i. From the conclusion of the theorem one easily gets that

$$\lim_{n\to\pm\infty} \frac{1}{n} \log |\det A^n(x)| = \sum_{i=1}^k \lambda_i(x) \dim E_x^i. \tag{2}$$

In most cases we deal with, the determinant is constant equal to 1. Then the sum of all Lyapunov exponents, counted with multiplicity, is identically zero.

Remark 0.3. The *natural extension* of a (non-invertible) map $f : M \to M$ is defined on the space $\hat{M}$ of sequences $(x_n)_{n\le 0}$ with $f(x_n) = x_{n+1}$ for $n < 0$, by

$$\hat{f} : \hat{M} \to \hat{M}, \quad (\ldots, x_n, \ldots, x_0) \mapsto (\ldots, x_n, \ldots, x_0, f(x_0)).$$

Let $P : \hat{M} \to M$ be the canonical projection assigning to each sequence $(x_n)_{n\le 0}$ the term x_0. It is clear that $\hat{f}$ is invertible and $P \circ \hat{f} = f \circ P$. Every f-invariant probability μ lifts to a unique $\hat{f}$-invariant probability $\hat{\mu}$ such that $P_*\hat{\mu} = \mu$. Every cocycle $F : \mathcal{E} \to \mathcal{E}$ over f extends to a cocycle $\hat{F} : \hat{\mathcal{E}} \to \hat{\mathcal{E}}$ over $\hat{f}$, as follows: $\hat{\mathcal{E}}_{\hat{x}} = \mathcal{E}_{P(\hat{x})}$ and $\hat{A}(\hat{x}) = A(P(\hat{x}))$, where $\hat{A}(\hat{x})$ denotes the action of $\hat{F}$ on the fiber $\hat{\mathcal{E}}_{\hat{x}}$. Clearly, $\int \log^+ \|\hat{A}\| \, d\hat{\mu} = \int \log^+ \|A\| \, d\mu$ and, assuming the integrals are finite, the two cocycles F and $\hat{F}$ have the same Lyapunov spectrum and the same Oseledets filtration. Moreover, $\int \log^+ \|\hat{A}^{-1}\| \, d\hat{\mu} = \int \log^+ \|A^{-1}\| \, d\mu$ and when the integrals are finite we may apply Theorem 0.2 to the cocycle $\hat{F}$.

Remark 0.4. Any sum $F_x^i = \oplus_{j=i}^k E_x^j$ of Oseledets subspaces corresponding to the smallest Lyapunov exponents depends only on the forward iterates of the cocycle. Analogously, any sum of Oseledets subspaces corresponding to the largest Lyapunov exponents depends only on the backward iterates.

0.3. **Symplectic cocycles.** Suppose there exists some symplectic form, that is, some non-degenerate alternate 2-form ω_x on each fiber $\mathcal{E}_x$, which is preserved by the linear cocycle F:

$$\omega_{f(x)}(A(x)u, A(x)v) = \omega_x(u, v) \quad \text{for all } x \in M \text{ and } u, v \in \mathcal{E}_x.$$

Assume the symplectic form is integrable, in the sense that there exists a μ-integrable function $x \mapsto c(x)$ such that

$$|\omega_x(u, v)| \le e^{c(x)} \|u\| \, \|v\| \quad \text{for all } x \in M \text{ and } u, v \in \mathcal{E}_x.$$

Remark 0.5. We are going to use the following easy observation. Let μ be an invariant ergodic probability for a transformation $f : M \to M$, and let $\phi : M \to \mathbb{R}$ be a μ-integrable function. Then

$$\lim_{n\to\infty} \frac{1}{n}\phi(f^n(x)) = 0 \quad \mu\text{-almost everywhere.}$$

This follows from the Birkhoff ergodic theorem applied to $\psi(x) = \phi(f(x)) - \phi(x)$. Note that the argument remains valid under the weaker hypothesis that the function ψ be integrable.

Proposition 0.6. *If F preserves an integrable symplectic form then its Lyapunov spectrum is symmetric: if λ is a Lyapunov exponent at some point x then so is $-\lambda$, with the same multiplicity.*

This statement can be justified as follows. Consider any i and j such that $\lambda_i(x) + \lambda_j(x) \neq 0$. For all $v^i \in E^i_x$ and $v^j \in E^j_x$,

$$|\omega_x(v^i, v^j)| = |\omega_{f^n(x)}(A^n(x)v^i, A^n(x)v^j)| \leq e^{c(f^n(x))}\|A^n(x)v^i\|\|A^n(x)v^j\|$$

for all $n \in \mathbb{Z}$. Since $c(x)$ is integrable the first factor has no exponential growth: by Remark 0.5,

$$\lim_{n\to\pm\infty} \frac{1}{n}c(f^n(x)) = 0 \quad \text{almost everywhere.}$$

The assumption implies that $\|A^n(x)v^i\|\|A^n(x)v^j\|$ goes to zero exponentially fast, either when $n \to +\infty$ or when $n \to -\infty$. So, the right hand of the previous inequality goes to zero either when $n \to +\infty$ or when $n \to -\infty$. Therefore, in either case, the left hand side must vanish. This proves that

$$\lambda_i(x) + \lambda_j(x) \neq 0 \quad \Rightarrow \quad \omega_x(v^i, v^j) = 0 \;\text{ for all } v^i \in E^i_x \text{ and } v^j \in E^j_x.$$

Since the symplectic form is non-degenerate, it follows that for every i there exists j such that $\lambda_i(x) + \lambda_j(x) = 0$. We are left to check that the multiplicities of such symmetric exponents coincide. We may suppose $\lambda_i(x) \neq 0$, of course. Let s be the dimension of E^i_x. Using a Gram-Schmidt argument, one constructs a basis $v^i_1, \ldots, v^i_s$ of E^i_x and a family of vectors $v^j_1, \ldots, v^j_s$ in E^j_x such that

$$\omega_x(v^i_p, v^j_q) = \begin{cases} 1 & \text{if } p = q \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

Notice that $\omega_x(v^i_p, v^i_q) = 0 = \omega_x(v^j_p, v^j_q)$ for all p and q, since $\lambda_i(x) = -\lambda_j(x)$ is non-zero. The relations (3) imply that the $v^j_1, \ldots, v^j_s$ are linearly independent, and so $\dim E^j_x \geq \dim E^i_x$. The converse inequality is proved in the same way.

0.4. **Adjoint linear cocycle.** Let $\pi^* : \mathcal{E}^* \to M$ be another vector bundle which is *dual* to $\pi : \mathcal{E} \to M$, in the sense that there exists a nondegenerate bilinear form

$$\mathcal{E}^*_x \times \mathcal{E}_x \ni (u, v) \mapsto u \cdot v \in \mathbb{R}, \quad \text{for each } x \in M.$$

The *annihilator* of a subspace $E^* \subset \mathcal{E}^*_x$ is the subspace $E \subset \mathcal{E}_x$ of all $v \in \mathcal{E}_x$ such that $u \cdot v = 0$ for all $u \in E^*$. We also say that E^* is the annihilator of E. Notice that $\dim E + \dim E^* = \dim \mathcal{E}_x = \dim \mathcal{E}^*_x$. The norm $\|\cdot\|$ may be transported from $\mathcal{E}$ to $\mathcal{E}^*$ through the duality:

$$\|u\| = \sup\{|u \cdot v| : v \in \mathcal{E}_x \text{ with } \|v\| = 1\} \quad \text{for } u \in \mathcal{E}^*_x. \tag{4}$$

For $x \in M$, the *adjoint* of $A(x)$ is the linear map $A^*(x) : \mathcal{E}^*_{f(x)} \to \mathcal{E}^*_x$ defined by

$$A^*(x)u \cdot v = u \cdot A(x)v \quad \text{for every } u \in \mathcal{E}^*_{f(x)} \text{ and } v \in \mathcal{E}_x. \tag{5}$$

The inverses $A^{-1*}(x) : \mathcal{E}^*_x \mapsto \mathcal{E}^*_{f(x)}$ define a linear cocycle $F^{-1*} : \mathcal{E}^* \to \mathcal{E}^*$ over f.

Proposition 0.7. *The Lyapunov spectra of F and F^{-1*} are symmetric to one another at each point.*

Indeed, the definitions (4) and (5) imply $\|A^*(x)\| = \|A(x)\|$ and, analogously, $\|A^{-1*}(x)\| = \|A^{-1}(x)\|$ for any $x \in M$. Thus, F^{-1*} satisfies the integrability condition in Theorem 0.2 if and only if F does. Let $\mathcal{E}_x = \oplus_{j=1}^k E^j_x$ be the Oseledets decomposition of F at each point x. For each $i = 1, \dots, d$ define

$$E^{i*}_x = \text{ annihilator of } E^1_x \oplus \cdots \oplus E^{i-1}_x \oplus E^{i+1}_x \oplus \cdots \oplus E^k_x. \tag{6}$$

The decomposition $\mathcal{E}^*_x = \oplus_{j=1}^k E^{j*}_x$ is invariant under F^{-1*}. Moreover, given any $u \in E^{i*}_x$ and any $n \geq 1$,

$$\|A^{-n*}(x)u\| = \max_{\|v\|=1} |A^{-n*}(x)u \cdot v| = \max_{\|v\|=1} |u \cdot A^{-n}(x)v| .$$

Fix any $\varepsilon > 0$. Begin by considering $v \in E^i_{f^n(x)}$. Then $A^{-n}(x)v \in E^i_x$, and so

$$|u \cdot A^{-n}(x)v| \geq c\,\|u\|\,\|A^{-n}(x)v\| \geq c\,\|u\|\,e^{-(\lambda_i(x)+\varepsilon)n}$$

for every n sufficiently large, where $c = c(E^i_x, E^{i*}_x) > 0$. Consequently,

$$\lim_{n\to\infty} \frac{1}{n} \log \|A^{-n*}(x)u\| \geq -(\lambda_i(x) + \varepsilon). \tag{7}$$

Next, observe that a general unit vector $v \in \mathcal{E}_{f^n(x)}$ may be written

$$v = \sum_{j=1}^k v^j \quad \text{with } v^j \in E^j_{f^n(x)}.$$

Using part 3 of Theorem 0.2, we see that every $\|v^j\| \leq e^{\varepsilon n}$ if n is sufficiently large. Therefore, given any $u \in E^{i*}_x$,

$$|u \cdot A^{-n}(x)v| = |u \cdot A^{-n}(x)v^i| \leq \|u\|\,e^{-(\lambda_i(x)-\varepsilon)n}\,\|v^i\| \leq \|u\|\,e^{-(\lambda_i(x)-2\varepsilon)n}$$

for every unit vector $v \in \mathcal{E}_{f^n(x)}$, and so

$$\lim_{n\to\infty} \frac{1}{n} \log \|A^{-n*}(x)u\| \leq -(\lambda_i(x) - 2\varepsilon). \tag{8}$$

Since $\varepsilon > 0$ is arbitrary, the relations (7) and (8) show that the Lyapunov exponent of F^{-1*} along E^{i*}_x is precisely $-\lambda_i(x)$, for every $i = 1, \dots, k$. Thus, $\mathcal{E}^*_x = \oplus_{j=1}^k E^{j*}_x$ must be the Oseledets decomposition of F^{-1*} at x. Observe, in addition, that $\dim E^{i*}_x = \dim E^i_x$ for all $i = 1, \dots, k$.

0.5. **Cocycles over flows.** We call *linear cocycle over a flow* $f^t : M \to M$, $t \in \mathbb{R}$ a flow extension $F^t : \mathcal{E} \to \mathcal{E}$, $\quad t \in \mathbb{R}$ such that $\pi \circ F^t = f^t \circ \pi$ and the action $A^t(x) : \mathcal{E}_x \to \mathcal{E}_{f^t(x)}$ of F^t on every fiber is a linear isomorphism. Notice that $A^{t+s}(x) = A^s(f^t(x)) \cdot A^t(x)$ for all $t, s \in \mathbb{R}$.

Theorem 0.8. *Assume $\log^+ \|A^t(x)\|$ is μ-integrable for all $t \in \mathbb{R}$. Then, for μ-almost every $x \in M$, there exists $k = k(x) \leq d$, numbers $\lambda_1(x) > \cdots > \lambda_k(x)$, and a decomposition $\mathcal{E}_x = E^0_x \oplus E^1_x \oplus \cdots \oplus E^k_x$ of the fiber, such that*

(1) *$A^t(x) \cdot E^i_x = E^i_{f^t(x)}$ and E^0_x is tangent to the flow lines*

(2) $\lim_{t\to\pm\infty} \frac{1}{t} \log \|A^t(x)\| = \lambda_i(x)$ *for all non-zero* $v \in E^i_x$

(3) $\lim_{t\to\pm\infty} \frac{1}{t} \log \angle\big(E^i_{f^t(x)}, E^j_{f^t(x)}\big) = 0$ *for all* $i, j = 1, \dots, k$.

As a consequence, the relation (2) also extends to the continuous time case, as do the observations made in the previous sections for discrete time cocycles.

An important special case is the *derivative cocycle* $Df^t : TM \to TM$ over a smooth flow $f^t : M \to M$. We call Lyapunov exponents and Oseledets subspaces of the flow the corresponding objects for this cocycle Df^t, $t \in \mathbb{R}$.

0.6. **Induced cocycle.** The following construction will be useful later. Let $f : M \to M$ be a transformation, not necessarily invertible, μ be an invariant probability measure, and D be some positive measure subset of M. Let $\rho(x) \geq 1$ be the first return time to D, defined for almost every $x \in D$. Given any cocycle $F = (f, A)$ over f, there exists a corresponding cocycle $G = (g, B)$ over the first return map $g(x) = f^{\rho(x)}(x)$, defined by $B(x)v = A^{\rho(x)}(x)v$.

Proposition 0.9. (1) *The normalized restriction* μ_D *of the measure* μ *to the domain* D *is invariant under the first return map* g.

(2) $\log^+ \|B^{\pm 1}\|$ *are integrable for* μ_D *if* $\log^+ \|A^{\pm 1}\|$ *are integrable for* μ.

(3) *For* μ*-almost every* $x \in D$*, the Lyapunov exponents of* G *at* x *are obtained multiplying the Lyapunov exponents of* F *at* x *by some constant* $c(x) \geq 1$.

Proof. First, we treat the case when the transformation f is invertible. For each $j \geq 1$, let D_j be the subset of points $x \in D$ such that $\rho(x) = j$. The $\{D_j : j \geq 1\}$ are a partition of a full measure subset of D, and so are the $\{f^j(D_j) : j \geq 1\}$. Notice also that $g \mid D_j = f^j \mid D_j$ for all $j \geq 1$. For any measurable set $E \subset D$ and any $j \geq 1$,

$$\mu\big(g^{-1}(E \cap f^j(D_j))\big) = \mu\big(f^{-j}(E \cap f^j(D_j))\big) = \mu\big(E \cap D_j\big),$$

because μ is invariant under f. It follows that

$$\mu\big(g^{-1}(E)\big) = \sum_{j=1}^{\infty} \mu\big(g^{-1}(E \cap f^j(D_j))\big) = \sum_{j=1}^{\infty} \mu\big(E \cap D_j\big) = \mu(E).$$

This implies that μ_D is invariant under g, as claimed in part (1). Next, from the definition $B(x) = A^{\rho(x)}(x)$ we conclude that

$$\int_D \log^+ \|B\| \, d\mu = \sum_{j=1}^{\infty} \int_{D_j} \log^+ \|A^j\| \, d\mu \leq \sum_{j=1}^{\infty} \sum_{i=0}^{j-1} \int_{D_j} \log^+ \|A \circ f^i\| \, d\mu.$$

Since μ is invariant under f and the domains $f^i(D_j)$ are pairwise disjoint for all $0 \leq i \leq j-1$, it follows that

$$\int_D \log^+ \|B\| \, d\mu \leq \sum_{j=1}^{\infty} \sum_{i=0}^{j-1} \int_{f^i(D_j)} \log^+ \|A\| \, d\mu \leq \int \log^+ \|A\| \, d\mu.$$

The corresponding bound for the norm of the inverse is obtained in the same way. This implies part (2) of the proposition. To prove part (3), define

$$c(x) = \lim_{k\to\infty} \frac{1}{k} \sum_{j=0}^{k-1} \rho(f^j(x)).$$

Notice that ρ is integrable relative to μ_D:

$$\int_D \rho\, d\mu = \sum_{j=1}^{\infty} j\mu(D_j) = \sum_{j=1}^{\infty}\sum_{i=0}^{j-1} \mu(f^i(D_j)) \le 1.$$

Thus, by the ergodic theorem, $c(x)$ is well defined at μ_D-almost every x. It is clear from the definition that $c(x) \ge 1$. Now, given any vector $v \in \mathcal{E}_x \setminus \{0\}$ and a generic point $x \in D$,

$$\lim_{k\to\infty} \frac{1}{k} \log \|B^k(x)v\| = c(x) \lim_{n\to\infty} \frac{1}{n} \log \|A^n(x)v\|$$

(we are assuming $\log^+ \|A\|$ is μ-integrable and so Theorem 0.1 ensures that both limits exist). This proves part (3) of the proposition, when f is invertible.

Finally, we extend the proposition to the non-invertible case. Let $\hat{f}$ be the natural extension of f and $\hat{\mu}$ be the lift of μ (Remark 0.3). Denote $\hat{D} = P^{-1}(D)$. It is clear that the $\hat{f}$-orbit of a point $\hat{x} \in \hat{D}$ returns to $\hat{D}$ at some time n if and only if the f-orbit of $x = P(\hat{x})$ returns to D at time n. Thus, the first return map of $\hat{f}$ to the domain $\hat{D}$ is

$$\hat{g}(x) = \hat{f}^{\rho(x)}(\hat{x}), \quad x = P(\hat{x}),$$

and so it satisfies $P \circ \hat{g} = g \circ P$. It is also clear that the normalized restriction $\hat{\mu}_D$ of $\hat{\mu}$ to the domain $\hat{D}$ satisfies $P_*\hat{\mu}_D = \mu_D$. By the invertible case, $\hat{\mu}_D$ is invariant under $\hat{g}$. It follows that μ_D is invariant under g:

$$\mu_D(g^{-1}(E)) = \hat{\mu}_D(P^{-1}g^{-1}(E)) = \hat{\mu}_D(\hat{g}^{-1}P^{-1}(E)) = \hat{\mu}_D(P^{-1}(E)) = \mu_D(E),$$

for every measurable set $E \subset D$. This settles part (1). Now let $\hat{F} = (\hat{f}, \hat{A})$ be the natural extension of the cocycle F (Remark 0.3) and $\hat{G}$ be the cocycle it induces over $\hat{g}$:

$$\hat{G}(\hat{x}, v) = (\hat{g}(\hat{x}), \hat{B}(\hat{x})v), \quad \hat{B}(\hat{x}) = \hat{A}^{\rho(x)}(\hat{x}).$$

By definition, $\hat{A}(\hat{x}) = A(x)$, and so $\hat{B}(\hat{x}) = B(x)$. Consequently,

$$\int \log^+ \|A\|\, d\mu = \int \log^+ \|\hat{A}\|\, d\hat{\mu} \quad \text{and} \quad \int \log^+ \|B\|\, d\mu_D = \int \log \|\hat{B}\|\, d\hat{\mu}_D.$$

By the invertible case, $\log^+ \|\hat{B}\|$ is $\hat{\mu}_D$-integrable if $\log^+ \|\hat{A}\|$ is $\hat{\mu}$-integrable. It follows that $\log^+ \|B\|$ is μ_D-integrable if $\log^+ \|A\|$ is μ-integrable. The same argument applies to the inverses. This settles part (2) of the proposition. Part (3) also extends easily to the non-invertible case: as observed in Remark 0.3, the Lyapunov exponents of $\hat{F}$ at $\hat{x}$ coincide with the Lyapunov exponents of F at x. For the same reasons, the Lyapunov exponents of $\hat{G}$ at $\hat{x}$ coincide with the Lyapunov exponents of G at x. By the invertible case, the exponents of $\hat{G}$ at $\hat{x}$ are obtained multiplying the exponents of $\hat{F}$ at $\hat{x}$ by some positive factor. Consequently, the exponents of G at x are obtained multiplying the exponents of F at x by that same factor. This concludes the proof of the proposition. $\square$

References

[1] V. I. Oseledets. A multiplicative ergodic theorem: Lyapunov characteristic numbers for dynamical systems. *Trans. Moscow Math. Soc.*, 19:197–231, 1968.