Lorenz Strange Attractors

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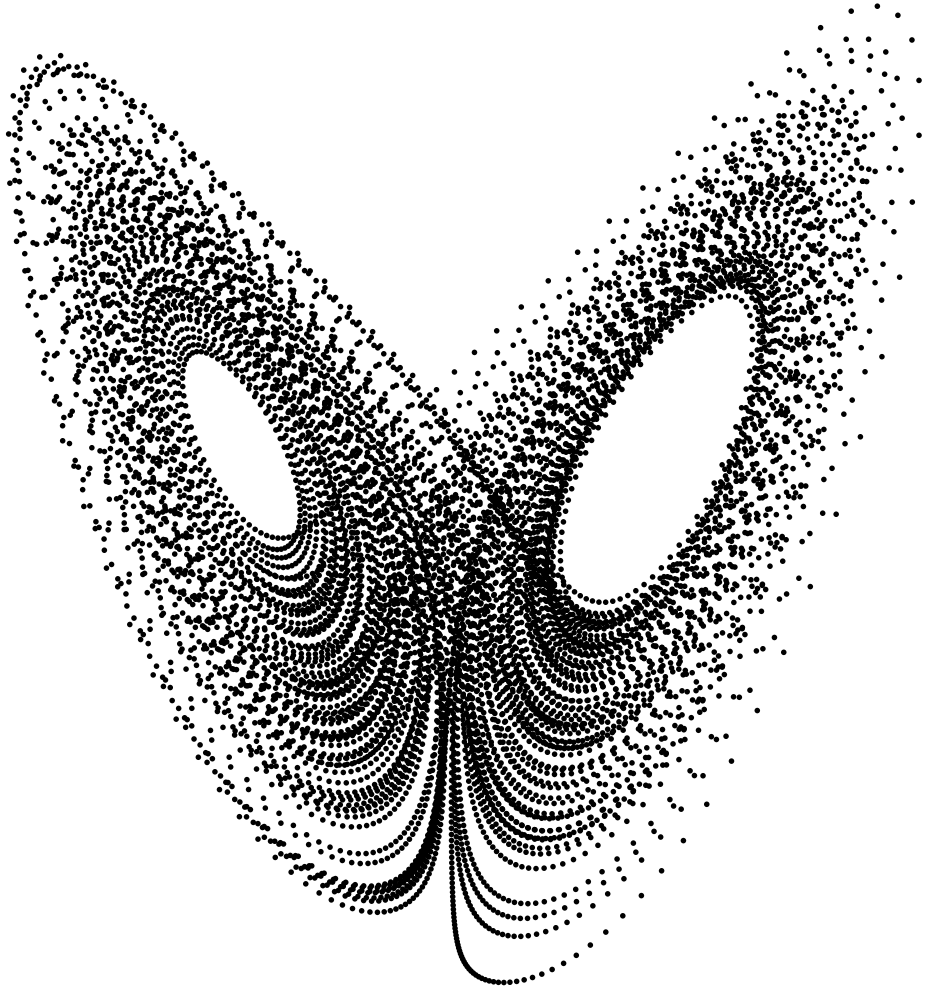
Lorenz Equations

E. N. Lorenz, Journal of Atmospheric Sciences, 1963:

$$\frac{dx}{dt} = -\sigma x + \sigma y$$
$$\frac{dy}{dt} = \rho x - y - xz$$
$$\frac{dz}{dt} = xy - \beta z$$

where $\sigma = 10, \quad \rho = 28, \quad \beta = 8/3.$

Lorenz Attractor



Plan of the lecture

1. Origins/Sensitivity
2. Strange attractors
3. Geometric Lorenz models
4. Theory of robustness
5. Lorenz attractors do exist
6. What comes next ?

Weather Prediction

Lorenz was challenging the theoretical foundations of the methods for weather forecast used in 1950's, that were mostly based on linear regression.

He felt that the evolution laws governing the weather are much too non-linear for such methods to yield consistent predictions.

He tested his thesis by a thorough numerical analysis of the most popular methods applied to certain simplified models.

Weather Prediction

The complexity of the model was a critical issue, because contemporary computers were very slow:

Lorenz had just got a substantial grant that allowed him to acquire (much to the envy of his colleagues at MIT . . .) a Royal McBee LGP-30. This machine had 16k RAM and could run up to 60 multiplications per second.

After several attempts, Lorenz finally selected a system with 3 differential equations that had been recently introduced by another meteorologist, B. Saltzmann, and which came to be known as the Lorenz system.

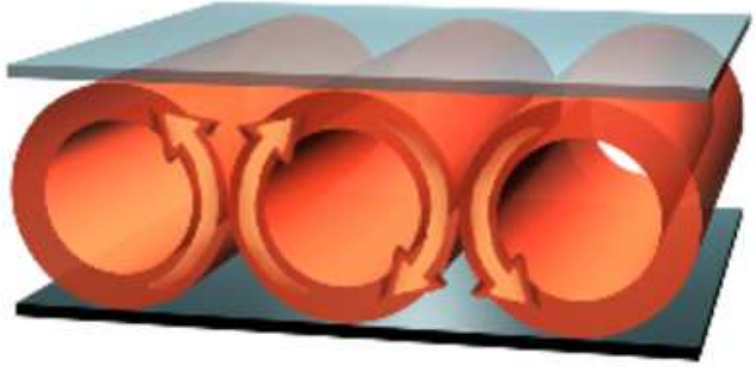
Thermal Convection

This was a (drastic) simplification of Lord Rayleigh's model

$$\frac{\partial \nabla^2 \Psi}{\partial t} = -\frac{\partial(\Psi, \nabla^2 \Psi)}{\partial(\xi, \eta)} + \nu \nabla^4 \Psi + g\alpha \frac{\partial \Theta}{\partial \xi}$$

$$\frac{\partial \Theta}{\partial t} = -\frac{\partial(\Psi, \Theta)}{\partial(\xi, \eta)} + \frac{\Delta T}{H} \frac{\partial \Psi}{\partial \xi} + \kappa \nabla^2 \Theta$$

for thermal convection:



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for thermal convection: here

ξ and η are spatial coordinates

t represents time

Ψ is a current function

Θ is the temperature difference function.

Thermal Convection

Writing these functions Ψ and Θ as Fourier series on the spatial coordinates,

$$\Psi(\xi,\eta,t) = \sum_{m,n} X_{m,n}(t) \exp\left(im\frac{\pi a}{H}\xi\right) \exp\left(im\frac{\pi a}{H}\eta\right)$$

Lord Rayleigh's model becomes a system of infinitely many ordinary differential equations on the variables $X_{m,n}$.

Keeping only the most representative modes (variables) and discarding all the others, we get a finite system of differential equations.

Saltzmann argued that one could keep as few as 3 modes.

Sensitivity

Lorenz carried his calculations to 6 decimal digits, but only 3 were printed:

$$0.707107$$

$$0.121320$$

$$0.363961$$

$$0.091883$$

$$0.275649$$

$$0.826948$$

$$0.480843$$

$$0.442530$$

Sensitivity

Lorenz carried his calculations to 6 decimal digits, but only 3 were printed. One day, just before leaving for lunch, he introduced an intermediate result as new initial data:

0.707107	
0.121320	0.121
0.363961	0.363
0.091883	0.089
0.275649	0.267
0.826948	0.801
0.480843	0.403
0.442530	0.209

Sensitivity

Upon returning, Lorenz realized, with great surprise, that the new calculation diverged much from the original one: the weather predictions for the 4th day were already completely different.

(his first reaction was to have the Royal McBee LGP-30 checked for mechanical failure...)

This discovery, that even very simple mathematical models may be sensitive to the initial data, had profound practical and philosophical consequences.

Sensitivity

Why have meteorologists such difficulty in predicting the weather with any certainty ? Why is it that showers and even storms seem to come by chance, so that many people think it is quite natural to pray for them, though they would consider it ridiculous to ask for an eclipse by prayer ?

Sensitivity

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. . . a tenth of a degree more or less at any given point, and the cyclone will burst here and not there, and extend its ravages over districts that it would otherwise have spared. If they had been aware of this tenth of a degree, they could have known it beforehand, but the observations were neither sufficiently comprehensive nor sufficiently precise, and that is the reason why it all seems due to the intervention of chance.

H. Poincaré

Strange Attractors

Just a few years later (1971), D. Ruelle e F. Takens were questioning the current mathematical interpretation of the phenomenon of turbulence:

E. Hopf had suggested that turbulence corresponds to the existence in the space of configurations of invariant tori (quasi-periodic motions) with very large dimension (very large number of frequencies), and this interpretation had been backed by no other than L. Landau and E. Lifshitz.

Strange Attractors

But Ruelle and Takens proved that this is mathematically flawed: typical dissipative systems such as viscous fluids do not exhibit invariant tori.

Instead, they suggested, turbulence should correspond to the presence in the configuration space of some "strange attractor".

Strange Attractors

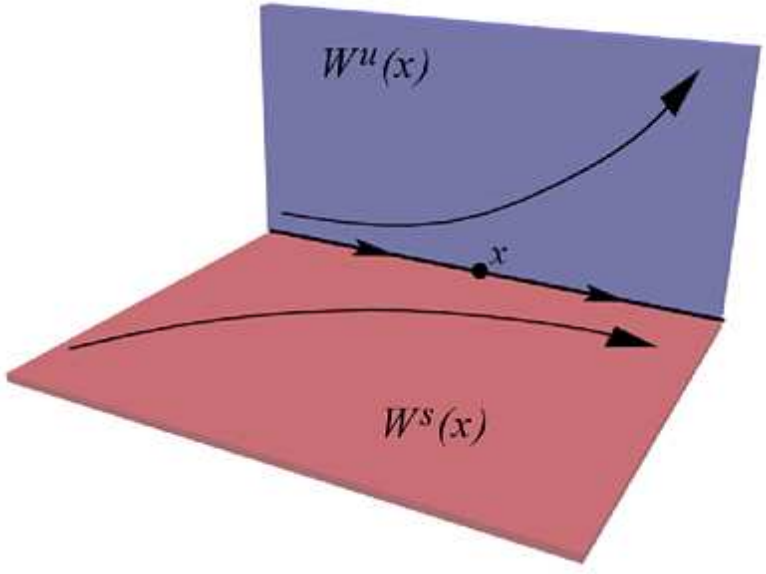
An attractor is a region in the configuration space that is invariant under time evolution and attracts most, if not all, nearby configurations.

Ruelle and Takens did not offer a definition of strange. It came to mean that typical trajectories approaching the attractor depend sensitively on the initial state.

The Lorenz attractor was a spectacular example, but the work of Lorenz remained unknown to the mathematical community. The only examples available to them were the hyperbolic attractors of S. Smale.

Hyperbolic Attractors

An attractor is called hyperbolic if all nearby trajectories behave in a saddle-type way relative to the trajectories inside the attractor:



Geometric Lorenz Attractors

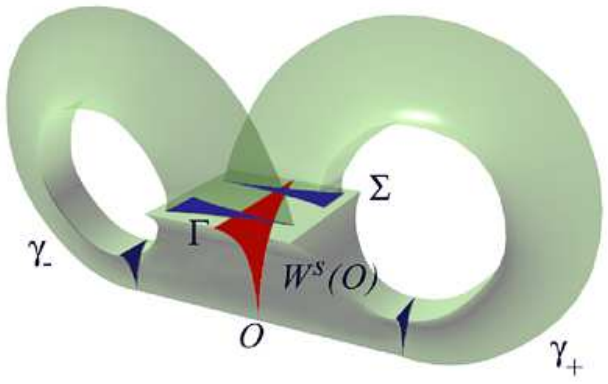
In the decades that followed, there was great interest in understanding the behavior of the Lorenz equations and, most specially, in proving that those equations do exhibit a strange attractor.

A very successful strategy was to develop mathematical models for the equations observed in the equations. Such geometric Lorenz models were proposed and studied by

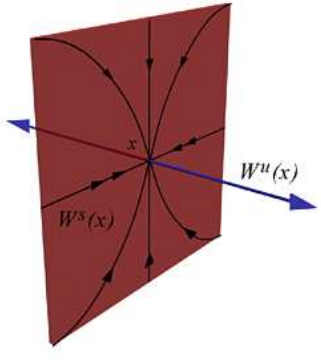
- Afraimovich, Bykov, Shilnikov (URSS)
- Guckenheimer, Williams (USA)

in the mid 1970's.

Geometric Lorenz Attractors



The geometric models are vector fields in 3-dimensional space such that there exists some stationary point O.



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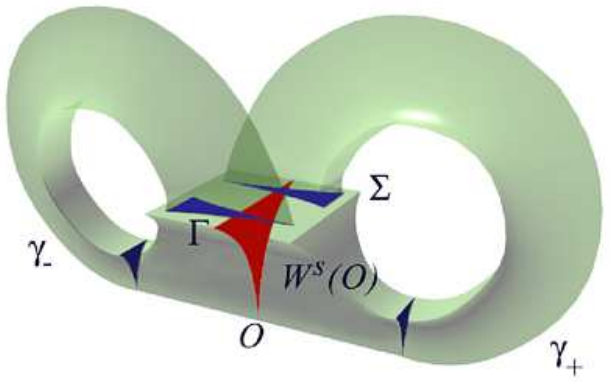
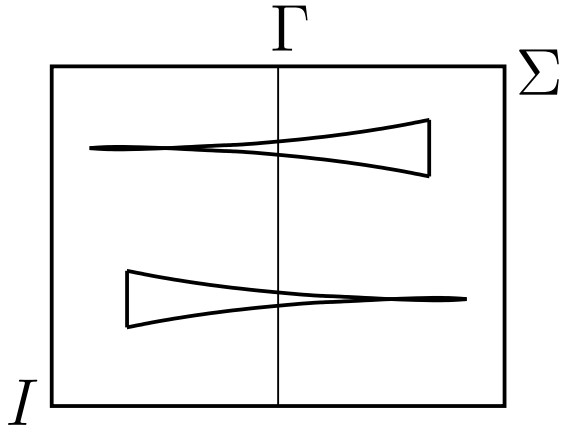
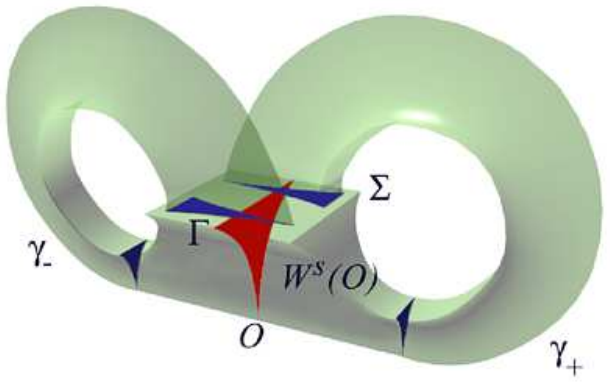


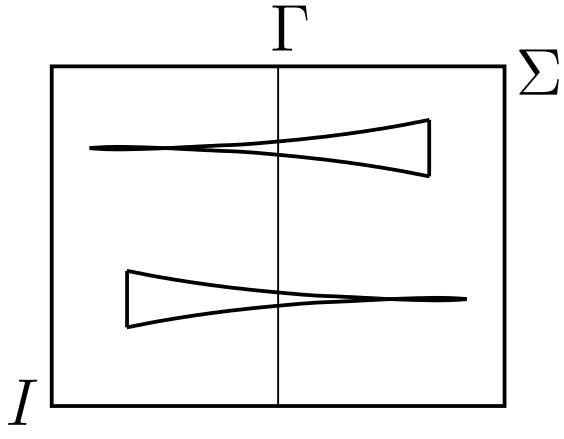
The geometric models are vector fields in 3-dimensional space such that there exists some cross-section to the vector field.



Geometric Lorenz Attractors



The geometric models are vector fields in 3-dimensional space such that the return map to the cross-section has some invariant foliation.



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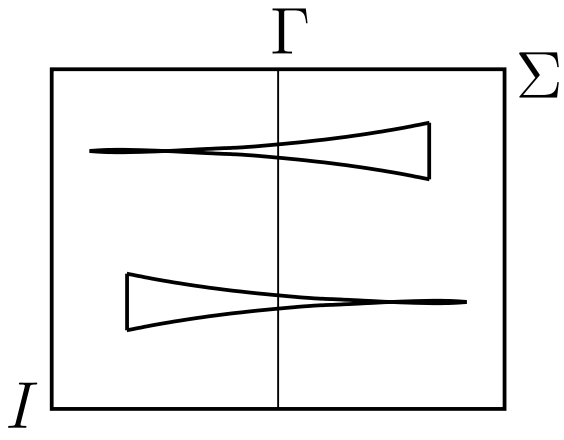
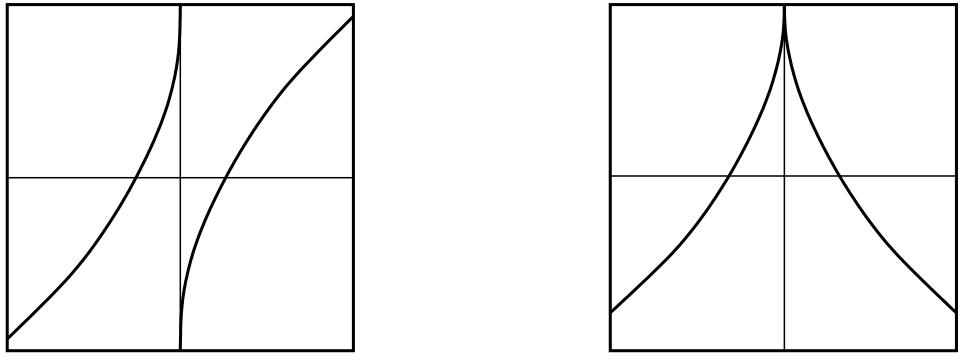


The geometric models are vector fields in 3-dimensional space such that the map induced in the space of leaves is uniformly expanding:



Geometric Lorenz Attractors

Guckenheimer, Williams and Afraimovich, Bykov, Shilnikov proved that

- such vector fields do exist and
- they have strange (sensitive) attractors.

One of the most surprising facts about this construction is that it is robust: the attractor remains if one modifies the vector field slightly.

This is all the more striking, because the attractor contains a stationary point accumulated by non-stationary ones.

Robustness

Then, around the mid 1990's, there were some remarkable developments. Among the most spectacular, there was the following result obtained by researchers in Rio de Janeiro:

Theorem (C. Morales, M. J. Pacifico, E. Pujals)**.** *Every robust attractor of a vector field in* 3 *dimensions is either hyperbolic or Lorenz-like.*

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Theorem (V. Araújo, M. J. Pacifico, E. Pujals, M. Viana)**.** *Lorenz-like attractors of vector fields in 3 dimensions are chaotic: expansive, sensitive, fast mixing, ...*

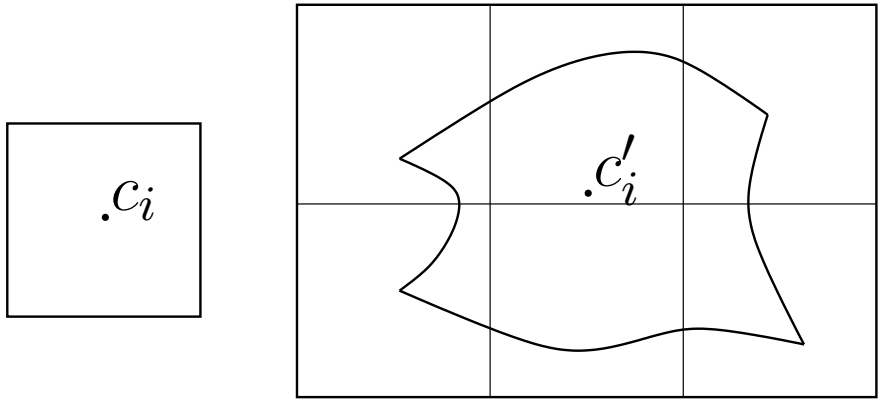
Existence of Lorenz Attractors

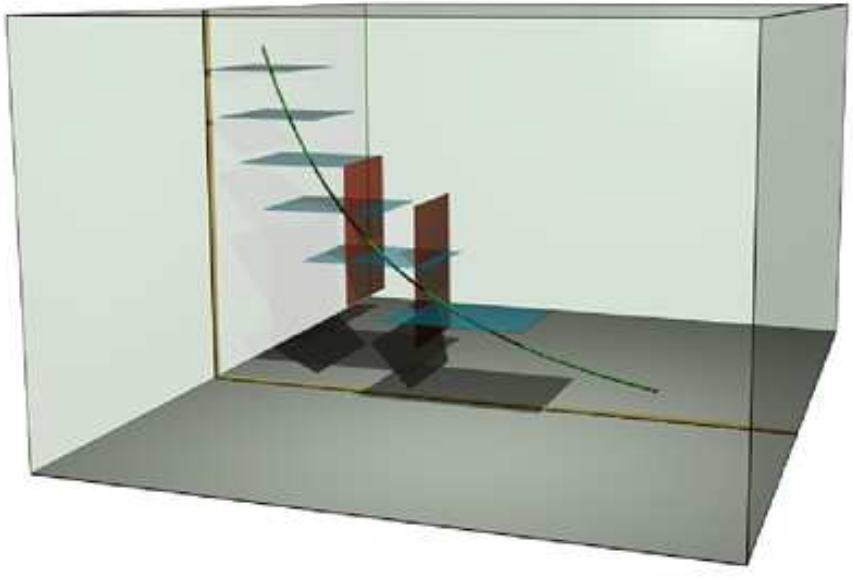
It remained an open problem whether the original equations of Lorenz do really have a strange attractor. This was finally settled in 1998 by a graduate student at KTH-Stockholm:

Theorem (W. Tucker)**.** *The Lorenz equations admit a strange attractor, for the values of the parameters σ, ρ, β, originally considered by Lorenz.*

The proof is computer assisted: a careful analysis of the equations near the equilibrium point O together with rigorous computations:

Existence of Lorenz Attractors





Higher Dimensions

What about systems with more degrees of freedom ?

The number of external degrees of freedom is the dimension of the configuration space. For the classical Lorenz system this is 3.

It is not difficult to construct and analyze models with any number (larger than 2) of external degrees of freedom.

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It is not difficult to construct and analyze models with any number (larger than 2) of external degrees of freedom.

The number of internal degrees of freedom is the (topological) dimension of the attractor. For the classical Lorenz system this is 1.

This is a much harder problem, that remained open for a long time: can robust attractors containing equilibria have more than 1 internal degree of freedom ?

Higher Dimensions

This problem was settled by C. Bonatti, A. Pumariño, Viana: there are mathematical models for such attractors with any given number of internal degrees of freedom. Three main challenges remain:

- to understand the properties of these and other models
- to identify them in concrete situations
- to characterize robust attractors in any dimension

Conclusion

1. The study of Lorenz strange attractors originated from concrete problems related to dynamics of fluids, especially thermal convection.
2. It led to the development of a rich mathematical theory, in which the Brazilian/Latin American school of Dynamical Systems played a very prominent role.
3. This development combined new ideas and methods in Dynamics with the use of the computer, both for simulations and for rigorous computations.
4. This topic still contains many important challenges, especially in what concerns systems with large number of degrees of freedom.