Dynamics: a long century

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The n-body problem

$$\frac{d^2\overrightarrow{r_i}}{dt^2} = \sum_{i \neq j} G m_i m_j \frac{\overrightarrow{r_j} - \overrightarrow{r_i}}{\| \overrightarrow{r_j} - \overrightarrow{r_i} \|^3} \quad i = 1, 2, \ldots, n$$

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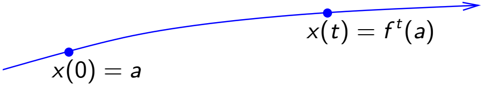
Poincaré proposes to use tools from geometry, probability, analysis, algebra, to describe the qualitative behavior of solutions without actually computing them.

Continuous time and discrete time

Flow (associated to a vector field F)

$\{f^t : M \to M : t \in \mathbb{R}\}$ defined by $f^t(a)$ = value at time t of the solution to

$$\frac{dx}{dt} = F(x) \text{ and } x(0) = a.$$



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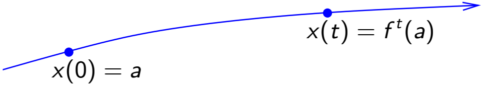

Iteration of an (invertible) transformation

$f : M \to M$ invertible map; one denotes by f^n the n-fold iterate.

Return maps and suspensions

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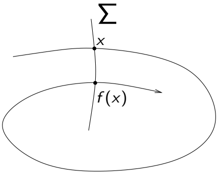

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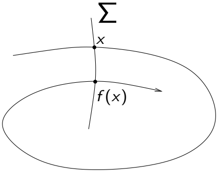

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Every invertible transformation f can be realized as a first return map of some flow, that we call suspension flow of f.

Poincaré recurrence

Theorem (Poincaré)

If $f : M \to M$ preserves a probability μ then, given any measurable set $B \subset M$, the orbit of μ-almost any $x \in B$ returns to B.

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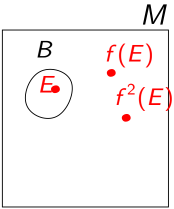


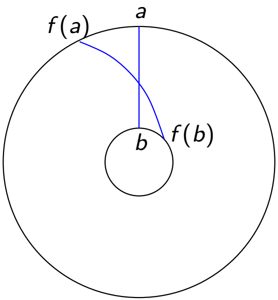
Let $E = \{$points whose orbits never return to $B\}$. Then $\mu(E) = 0$:

$$\sum_n \mu(E) = \sum_n \mu(f^n(E)) = \mu\left(\bigcup_n f^n(E)\right) \le \mu(M) = 1.$$

Twist maps

Theorem (Poincaré-Birkhoff)

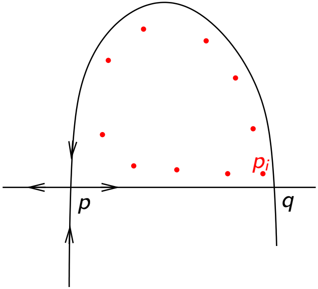
Assume $f : A \to A$ preserves the boundary and the area measure and twists the radii. Then there is $p \in A$ such that $f(p) = p$.



Transverse homoclinic intersections

Theorem (Birkhoff)

Any transverse homoclinic point q associated to a saddle point p is accumulated by periodic points, $f^{k_i}(p_i) = p_i$, with periods $k_i \to \infty$.



The ergodic hypothesis

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Reformulation: Over long periods of time, the time spent by the system in some region of the phase space of microstates with the same energy is proportional to the volume of that region.

In other words, all micro-states with the same energy should be equally probable, in the long run. Is this so ?

The ergodic theorem

Theorem (von Neumann, Birkhoff)

Assume $f : M \to M$ preserves a probability μ. Then, for any integrable function $\psi : M \to \mathbb{R}$, the time-average

$$\tilde{\psi}(x) = \lim_{n\to\infty} \frac{1}{n}\left(\psi(f(x)) + \cdots + \psi(f^n(x))\right)$$

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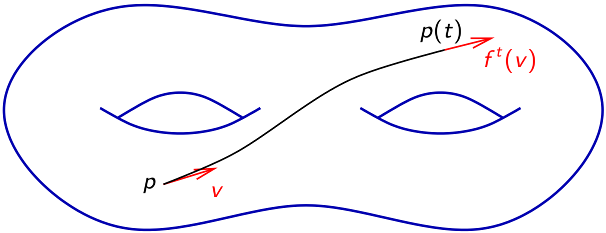
The system is ergodic if the time-averages are constant μ-almost everywhere.

Geodesic flows on surfaces



Theorem (Hedlund, Hopf)

The geodesic flow on a surface with negative curvature is ergodic, relative to the Liouville measure.

KAM theory

Theorem (Kolmogorov, Arnold, Moser)

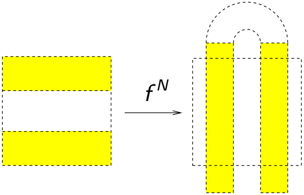
If $f : A \longrightarrow A$ is a C^∞ area preserving twist map, there exists a subset of A with positive area consisting of closed curves that are fixed by f. In particular, f is not ergodic.

This is a manifestation of a very general phenomenon that applies, in particular, to volume preserving (symplectic) transformations and flows in any dimension.

The horseshoe

Theorem (Smale)

If f has some transverse homoclinic point then some iterate f^N exhibits a horseshoe.



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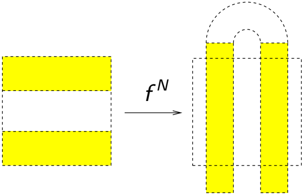

So, f has periodic points with arbitrarily large periods (Birkhoff). This explains how an infinite number of periodic points can coexist in a robust fashion.

Hyperbolicity

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The transformation is hyperbolic if the tangent space admits a splitting

$$T_x M = E_x^s \oplus E_x^u$$

at every point $x \in L(f)$, invariant under the derivative and such that Df contracts E^s and Df^{-1} contracts E^u, at uniform rates.

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When $L(f) = M$, we say f is globally hyperbolic or Anosov.

Geodesic flows on manifolds

Theorem (Anosov)

1. *The geodesic flow on any manifold with negative sectional curvature is globally hyperbolic (and preserves the Liouville volume measure).*
2. *Every globally hyperbolic system that preserves the volume measure is ergodic.*

Dynamical decomposition

Theorem (Smale, Newhouse)

If f is hyperbolic then $L(f) = \Lambda_1 \cup \cdots \cup \Lambda_N$ where the Λ_i are invariant, disjoint, and indecomposable (contain dense orbits). The forward/backward accumulation set of the orbit of every $x \in M$ is contained in some Λ_i.

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Λ_i is an attractor if the basin of attraction

$$B(\Lambda_i) = \{x \in M \text{ whose forward orbit accumulates in } \Lambda_i\}$$

has positive volume (then it is a neighborhood of Λ_i).

The basins of the attractors of a hyperbolic transformation cover a full volume set.

Physical measures

Theorem (Sinai, Ruelle, Bowen)

Given any hyperbolic attractor Λ*, there exists a probability* μ_{SRB} *supported on* Λ *such that every time-average*

$$\lim_n \left(\psi(f(x)) + \cdots + \psi(f^n(x))\right) = \int \psi \, d\mu_{SRB}$$

at volume-almost every point x *in the basin of attraction of* Λ*.*

Beyond hyperbolicity ?

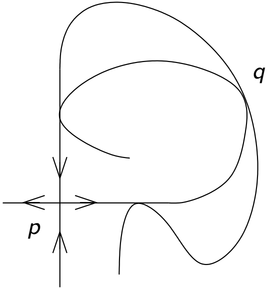
However, many important dynamical systems are not hyperbolic.

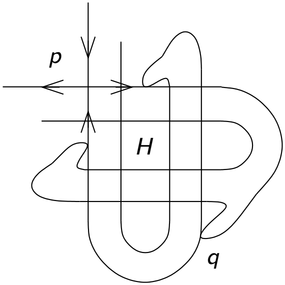
To what extent can we develop a similarly rich theory for very general dynamical systems?

Dynamics beyond uniform hyperbolicity, Bonatti, Díaz, Viana, Springer Verlag.

Homoclinic tangencies

Only two mechanisms are known that yield robustly non-hyperbolic behavior:





Heterodimensional cycles

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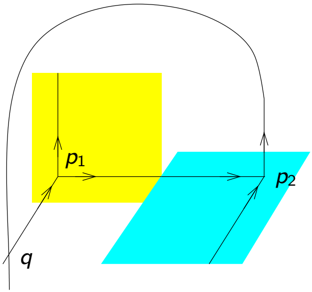


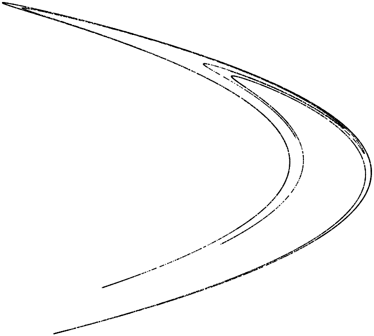
Strong density conjecture

Conjecture (Palis)

Every dynamical system can be approximated by one which either is hyperbolic, has a homoclinic tangency, or a heterodimensional cycle.

Hénon strange attractor

The Hénon map is defined by $f_{a,b}(x,y) = (1 - ax^2 + by, x)$ where a and b are parameters. Hénon observed that it seems to have a "strange" attractor:



Abundance of strange attractors

Theorem (Benedicks, Carleson, Young, Viana)

For a positive measure set of parameters a and b,

- BC *the Hénon map has a strange (non-hyperbolic) attractor* Λ
- BY *the attractor* Λ *supports some physical measure* μ_{phys}
- BV $\tilde{\psi}(x) = \int \psi \, d\mu_{phys}$ *for volume-almost all* $x \in B(\Lambda)$ *and all* ψ
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Theorem (Mora, Viana)

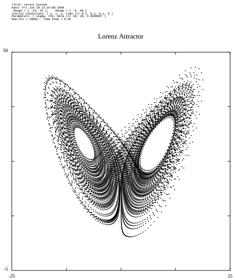
Hénon-like strange attractors occur, in a persistent way, whenever a dissipative homoclinic tangency is unfolded.

There is considerable recent progress by Wang, Young.

Lorenz strange attractors

The Lorenz flow is defined by the system of differential equations

$$
\begin{aligned}
\dot{x} &= -\sigma x + \sigma y & \qquad \sigma &= 10\\
\dot{y} &= \rho x - y - xz & \qquad \rho &= 28\\
\dot{z} &= xy - \beta z & \qquad \beta &= 8/3
\end{aligned}
$$



Robust strange attractors

Theorem (Morales, Pujals, Pacifico)

Every robust attractor Λ of a flow in 3 dimensions is either hyperbolic or Lorenz-like (singular-hyperbolic): the latter occurs precisely if the attractor contains some equilibrium point.

Robust means that there exists a neighborhood U of Λ such that the set of points whose orbits never leave U

- is indecomposable (dense orbits), for any perturbation of the original flow
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Theorem (Tucker)

The Lorenz original equations do exhibit a strange attractor.

Partial hyperbolicity

A transformation is partially hyperbolic if the tangent space admits an invariant splitting

$$T_x = E^s_x \oplus E^c_x \oplus E^u_x$$

at every point, such that $Df \mid E^s$ is a contraction, $Df \mid E^u$ is an expansion, and $Df \mid E^c$ is "in between" the two, with uniform rates.

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This notion turns out to be crucial for understanding

- robust indecomposability: Bonatti, Díaz, Pujals, Ures, Viana, Arbieto, Matheus, Horita, Tahzibi
- stable ergodicity: Grayson, Pugh, Shub, Wilkinson, Burns, Dolgopyat, Nitika, Torok, Xia, Bonatti, Matheus, Viana, Rodriguez-Herz (F & J), Ures

Two conjectures

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Hyperbolicity and physical measures (Viana)

If there is an invariant splitting $T_x M = E_x^1 \oplus E_x^2$ at volume-almost every point, such that $Df^n \mid E^1$ is eventually contracting, and $Df^n \mid E^2$ is eventually expanding, then the system admits some physical measure.