Lectures on Lyapunov exponents

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To Tania, Miguel and Anita,
for their understanding.

Contents

Preface

1. The study of characteristic exponents originated from the fundamental work of Aleksandr Mikhailovich Lyapunov [85] on the stability of solutions of differential equations. Consider a linear equation

$$\dot{v}(t) = B(t) \cdot v(t) \tag{1}$$

where $B(\cdot)$ is a bounded function from $\mathbb{R}$ to the space of $d \times d$ matrices. By the general theory of differential equations, there exists a so-called *fundamental matrix* A^t, $t \in \mathbb{R}$ such that $v(t) = A^t \cdot v_0$ is the unique solution of (1) with initial condition $v(0) = v_0$. If the *characteristic exponents*

$$\lambda(v) = \limsup_{t \to \infty} \frac{1}{t} \log \|A^t \cdot v\| \tag{2}$$

are negative, for all $v \neq 0$, then the trivial solution $v(t) \equiv 0$ is asymptotically stable, and even exponentially asymptotically stable. The stability theorem of Lyapunov asserts that, under an additional regularity condition, stability remains valid for nonlinear perturbations

$$\dot{w}(t) = B(t) \cdot w(t) + F(t,w) \quad \text{with } \|F(t,w)\| \leq \text{const}\, \|w\|^{1+\varepsilon}.$$

That is, the trivial solution $w(t) \equiv 0$ is still exponentially asymptotically stable.

The regularity condition of Lyapunov means, essentially, that the limit in (2) does exist, even if one replaces vectors v by l-vectors $v_1 \wedge \cdots \wedge v_l$; that is, elements of the k-exterior power of $\mathbb{R}^d$, for any $0 \leq l \leq d$. This is usually difficult to check in specific situations. But the multiplicative ergodic theorem of Oseledets asserts that Lyapunov regularity holds with full probability, in great generality. In particular, it holds on almost every flow trajectory, relative to any probability measure invariant under the flow.

2. The work of Furstenberg, Kesten, Oseledets, Kingman, Ledrappier, Guivarc'h, Raugi, Gol'dsheid, Margulis and other mathematicians, mostly in the

1960s–80s, built the study of Lyapunov characteristic exponents into a very active research field in its own right, and one with an unusually vast array of interactions with other areas of Mathematics and Physics, such as stochastic processes (random matrices and, more generally, random walks on groups), spectral theory (Schrödinger-type operators) and smooth dynamics (non-uniform hyperbolicity), to mention just a few.

My own involvement with the subject goes back to the late 20th century and was initially motivated by my work with Christian Bonatti and José F. Alves on the ergodic theory of partially hyperbolic diffeomorphisms and, soon afterwards, with Jairo Bochi on the dependence of Lyapunov exponents on the underlying dynamical system. The way these two projects unfolded very much inspired the choice of topics in the present book.

3. A diffeomorphism $f : M \to M$ is called *partially hyperbolic* if there exists a Df-invariant decomposition

$$TM = E^s \oplus E^c \oplus E^u$$

of the tangent bundle such that E^s is uniformly contracted and E^u is uniformly expanded by the derivative Df, whereas the behavior of Df along the *center bundle* E^c lies somewhere in between. It soon became apparent that to improve our understanding of such systems one should try to get a better hold of the behavior of $Df \mid E^c$ and, in particular, of its Lyapunov exponents. In doing this, we turned to the classical linear theory for inspiration.

That program proved to be very fruitful, as much in the linear context (e.g. the proof of the Zorich–Kontsevich conjecture, by Artur Avila and myself) as in the setting of partially hyperbolic dynamics we had in mind originally (e.g the rigidity results by Artur Avila, Amie Wilkinson and myself), and remains very active to date, with important contributions from several mathematicians.

4. Before that, in the early 1980s, Ricardo Mañé came to the surprising conclusion that generic (a residual subset of) volume-preserving C^1 diffeomorphisms on any surface have zero Lyapunov exponents, or else they are globally hyperbolic (Anosov); in fact, the second alternative is possible only if the surface is the torus $\mathbb{T}^2$. This discovery went against the intuition drawn from the classical theory of Furstenberg.

Although Mañé did not write a complete proof of his findings, his approach was successfully completed by Bochi almost two decades later. Moreover, the conclusions were extended to arbitrary dimension, both in the volume-preserving and in the symplectic case, by Bochi and myself.

5. In this monograph I have sought to cover the fundamental aspects of the classical theory (mostly in Chapters 1 through 6), as well as to introduce some of the more recent developments (Chapters 7 through 10).

The text started from a graduate course that I taught at IMPA during the (southern hemisphere) summer term of 2010. The very first draft consisted of lecture notes taken by Carlos Bocker, José Régis Varão and Samuel Feitosa. The unpublished notes [9] and [28], by Artur Avila and Jairo Bochi were important for setting up the first part of the course.

The material was reviewed and expanded later that year, in my seminar, with the help of graduate students and post-docs of IMPA's Dynamics group. I taught the course again in early 2014, and I took that occasion to add some proofs, to reorganize the exercises and to include historic notes in each of the chapters. Chapter 10 was completely rewritten and this preface was also much expanded.

6. The diagram below describes the logical connections between the ten chapters. The first two form an introductory cycle. In Chapter 1 we offer a glimpse of what is going to come by stating three main results, whose proofs will appear, respectively, in Chapters 3, 6 and 10. In Chapter 2 we introduce the notion of linear cocycle, upon which is built the rest of the text. We examine more closely the particular case of hyperbolic cocycles, especially in dimension 2, as this will be useful in Chapter 9.

$$\begin{tikzcd}
\text{Ch. 2} \arrow[d] & & \text{Ch. 1} \arrow[d] \arrow[dll] & & \\
\text{Ch. 3} \arrow[r] & \text{Ch. 4} \arrow[r] & \text{Ch. 6} \arrow[dl] \arrow[dl] \arrow[dr] & \text{Ch. 5} \arrow[l] \arrow[d] & \\
\text{Ch. 7} \arrow[r] & \text{Ch. 8} \arrow[r] & \text{Ch. 9} \arrow[r] & \text{Ch. 10} &
\end{tikzcd}$$

In the next four chapters we present the main classical results, including the Furstenberg–Kesten theorem and the subadditive ergodic theorem of Kingman (Chapter 3), the multiplicative ergodic theorem of Oseledets (Chapter 4), Ledrappier's exponent representation theorem, Furstenberg's formula for exponents of irreducible cocycles and Furstenberg's simplicity theorem in dimension 2 (Chapter 6). The proof of the multiplicative ergodic theorem is based on the subadditive ergodic theorem and also heralds the connection between Lya-

punov exponents and invariant/stationary measures that lies at the heart of the results in Chapter 6. In Chapter 5 we provide general tools to develop that connection, in both the invertible and the non-invertible case.

7. The last four chapters are devoted to more advanced material. The main goal there is to provide a friendly introduction to the existing research literature. Thus, the emphasis is on transparency rather than generality or completeness. This means that, as a rule, we choose to state the results in the simplest possible (yet relevant) setting, with suitable references given for stronger statements.

Chapter 7 introduces the invariance principle and exploits some of its consequences, in the context of locally constant linear cocycles. This includes Furstenberg's criterion for $\lambda_- = \lambda_+$, that extends Furstenberg's simplicity theorem to arbitrary dimension. The invariance principle has been used recently to analyze much more general dynamical systems, linear and nonlinear, whose Lyapunov exponents vanish. A finer extension of Furstenberg's theorem appears in Chapter 8, where we present a criterion for simplicity of the whole Lyapunov spectrum.

Then, in Chapter 9, we turn our attention to the contrasting Mañé–Bochi phenomenon of systems whose Lyapunov spectra are generically *not* simple. We prove an instance of the Mañé–Bochi theorem, for continuous linear cocycles. Moreover, we explain how those methods can be adapted to construct examples of discontinuous dependence of Lyapunov exponents on the cocycle, even in the Hölder-continuous category. Having raised the issue of (dis)continuity, in Chapter 10 we prove that for products of random matrices in $\mathrm{GL}(2)$ the Lyapunov exponents do depend continuously on the cocycle data.

8. Each chapter ends with set of notes and a list of exercises. Some of the exercises are actually used in the proofs. They should be viewed as an invitation for the reader to take an active part in the arguments. Throughout, it is assumed that the reader is familiar with the basic ideas of Measure Theory, Differential Topology and Ergodic Theory. All that is needed can be found, for instance, in my book with Krerley Oliveira, *Fundamentos da Teoria Ergódica* [114]; a translation into English is under way.

I thank David Tranah, of Cambridge University Press, for his interest in this book and for patiently waiting for the writing to be completed. I am also grateful to Vaughn Climenhaga, and David himself, for a careful revision of the manuscript that very much helped improve the presentation.

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1
Introduction

This chapter is a kind of overture. Simplified statements of three theorems are presented that set the tone for the whole text. Much broader versions of these theorems will appear later and several other themes around them will be introduced and developed as we move on. At this initial stage we choose to focus on the following special, yet significant, setting.

Let $A_1, \ldots, A_m$ be invertible 2×2 real matrices and let $p_1, \ldots, p_m$ be positive numbers with $p_1 + \cdots + p_m = 1$. Consider

$$L^n = L_{n-1} \cdots L_1 L_0, \quad n \geq 1,$$

where the L_j are independent random variables with identical probability distributions, such that

$$\text{the probability of } \{L_j = A_i\} \text{ is equal to } p_i$$

for all $j \geq 0$ and $i = 1, \ldots, m$. In brief, our goal is to describe the (almost certain) behavior of L^n as $n \to \infty$.

1.1 Existence of Lyapunov exponents

We begin with the following seminal result of Furstenberg and Kesten [56]:

Theorem 1.1 *There exist real numbers λ_+ and λ_- such that*

$$\lim_n \frac{1}{n} \log \|L^n\| = \lambda_+ \quad \textit{and} \quad \lim_n \frac{1}{n} \log \|(L^n)^{-1}\|^{-1} = \lambda_-$$

with full probability.

The numbers λ_+ and λ_- are called *extremal Lyapunov exponents*. Clearly,

$$\lambda_+ \geq \lambda_- \tag{1.1}$$

because $\|B\| \geq \|B^{-1}\|^{-1}$ for any invertible matrix B. If B has determinant ± 1 then we even have $\|B\| \geq 1 \geq \|B^{-1}\|^{-1}$. Hence,

$$\lambda_+ \geq 0 \geq \lambda_- \tag{1.2}$$

when all matrices A_i, $1 \leq i \leq m$ have determinant ± 1.

1.2 Pinching and twisting

Next, we discuss conditions for the inequalities (1.1) and (1.2) to be strict.

Let $\mathscr{B}$ be the *monoid* generated by the matrices A_i, $i = 1, \ldots, m$; that is, the set of all products $A_{k_1} \cdots A_{k_n}$ with $1 \leq k_j \leq m$ and $n \geq 0$ (for $n = 0$ interpret the product to be the identity matrix). We say that $\mathscr{B}$ is *pinching* if for any constant $\kappa > 1$ there exists some $B \in \mathscr{B}$ such that

$$\|B\| > \kappa \|B^{-1}\|^{-1}. \tag{1.3}$$

This means that the images of the unit circle under the elements of $\mathscr{B}$ are ellipses with arbitrarily large eccentricity. See Figure 1.1.

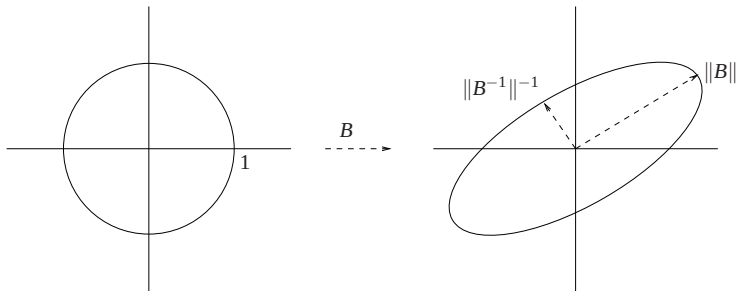

Figure 1.1 Eccentricity and pinching

We say that the monoid $\mathscr{B}$ is *twisting* if given any vector lines F, G_1, ..., $G_n \subset \mathbb{R}^2$ there exists $B \in \mathscr{B}$ such that

$$B(F) \notin \{G_1, \ldots, G_n\}. \tag{1.4}$$

The following result is a variation of a theorem of Furstenberg [54]:

Theorem 1.2 *Assume $\mathscr{B}$ is pinching and twisting. Then $\lambda_- < \lambda_+$. In particular, if $|\det A_i| = 1$ for all $1 \leq i \leq m$ then both extremal Lyapunov exponents are different from zero.*

1.3 Continuity of Lyapunov exponents

The extremal Lyapunov exponents λ_+ and λ_- may be viewed as functions of the data

$$A_1, \dots, A_m, p_1, \dots, p_m.$$

Let the matrices A_j vary in the linear group $\mathrm{GL}(2)$ of invertible 2×2 matrices and the probability vectors $(p_1, \dots, p_m)$ vary in the open simplex

$$\Delta^m = \{(p_1, \dots, p_m) : p_1 > 0, \dots, p_m > 0 \text{ and } p_1 + \cdots + p_m = 1\}.$$

The following result is part of a theorem of Bocker and Viana [35]:

Theorem 1.3 *The extremal Lyapunov exponents $\lambda_\pm$ depend continuously on $(A_1, \dots, A_m, p_1, \dots, p_m) \in \mathrm{GL}(2)^m \times \Delta^m$ at all points.*

Example 1.4 Let $m = 2$, with

$$A_1 = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma^{-1} \end{pmatrix} \quad \text{and} \quad A_2 = R_\theta A_1 R_{-\theta}, \;\; R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

for some $\sigma > 1$ and $\theta \in \mathbb{R}$. By Theorem 1.3, the Lyapunov exponents $\lambda_\pm$ depend continuously on the parameter σ and θ. Moreover, using Theorem 1.2, we have $\lambda_+ = 0$ if and only if $p_1 = p_2 = 1/2$ and $\theta = \pi/2 + n\pi$ for some $n \in \mathbb{Z}$.

1.4 Notes

Theorem 1.1 is a special case of the theorem of Furstenberg and Kesten [56], which is valid in any dimension $d \geq 2$. The full statement and the proof will appear in Chapter 3: we will deduce this theorem from an even more general statement, the subadditive ergodic theorem of Kingman [74]. Kingman's theorem will also be used in Chapter 4 to prove the fundamental result of the theory of Lyapunov exponents, the multiplicative ergodic theorem of Oseledets [92].

It is natural to ask whether the type of asymptotic behavior prescribed by Theorem 1.1 for the norm $\|L^n\|$ and conorm $\|(L^n)^{-1}\|^{-1}$ extends to the individual *matrix coefficients* $L^n_{i,j}$. Furstenberg, Kesten [56] proved that this is so if the coefficients of the matrices A_i, $1 \leq i \leq m$ are all strictly positive. The example in Exercise 1.3 shows that this assumption cannot be removed. On the other hand, the theorem of Oseledets theorem does contain such a description for the matrix *column vectors*.

Theorem 1.2 is also the tip of a series of fundamental results, which are to be discussed in Chapters 6 through 8. The full statement and proof of Furstenberg's theorem for 2-dimensional cocycles (Furstenberg [54]) will be given in

Chapter 6. The extension to any dimension will be stated and proved in Chapter 7: it will be deduced from the invariance principle (Ledrappier [81], Bonatti, Gomez-Mont and Viana [37], Avila and Viana [16], Avila, Santamaria and Viana [13]), a general tool that has several other applications, both for linear and nonlinear systems.

In dimension larger than 2, there is a more ambitious problem: rather than asking when $\lambda_- < \lambda_+$, one wants to know when *all* the Lyapunov exponents are distinct. That will be the subject of Chapter 8, which is based on Avila and Viana [14, 15].

Furstenberg and Kifer [57] proved continuity of the Lyapunov exponents of products of random matrices, restricted to the (almost) irreducible case. A variation of their argument will be given in Section 6.2.2. The reducible case requires a delicate analysis of the random walk defined by the cocycle in projective space. That was carried out by Bocker and Viana [35], in the 2-dimensional case, using certain discretizations of projective space. At the time of writing, Avila, Eskin and Viana [12] are extending the statement of the theorem to arbitrary dimension, using a very different strategy. The proof of Theorem 1.3 that we present in Chapter 10 is based on this more recent approach.

The problem of the dependence of Lyapunov exponents on the data can be formulated in the broader context of linear cocycles that we are going to introduce in Chapter 2. We will see in Chapter 9 that, in contrast, continuity often breaks down in that generality.

1.5 Exercises

The following elementary notions are used in some of the exercises that follow. We call a 2×2 matrix *hyperbolic* if it has two distinct real eigenvalues, *parabolic* if it has a unique real eigenvalue, with a one-dimensional eigenspace, and *elliptic* if it has two distinct complex eigenvalues. Multiples of the identity belong to neither of these three classes.

Exercise 1.1 Show that, in dimension $d = 2$, if $|\det A_i| = 1$ for all $1 \le i \le m$ then $\lambda_+ + \lambda_- = 0$.

Exercise 1.2 Calculate the extremal Lyapunov exponents for $m = 2$ and p_1, $p_2 > 0$ with $p_1 + p_2 = 1$ and

(1) $A_1 = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma^{-1} \end{pmatrix}$ and $A_2 = \begin{pmatrix} \sigma^{-1} & 0 \\ 0 & \sigma \end{pmatrix}$, where $\sigma > 1$;

(2) $A_1 = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma^{-1} \end{pmatrix}$ and $A_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, where $\sigma > 1$.

Exercise 1.3 (Furstenberg and Kesten [56]) Take $m = 2$ with $p_1 = p_2 = 1/2$ and

$$A_1 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Show that $\lim_n (1/n) \log |L^n_{i,j}|$ does not exist for any i, j, with full probability.

Exercise 1.4 Show that if some matrix A_i, $1 \le i \le m$ is either hyperbolic or parabolic then the monoid $\mathscr{B}$ is pinching.

Exercise 1.5 Show that the monoid $\mathscr{B}$ may be pinching even if all the matrices A_i, $1 \le i \le m$ are elliptic.

Exercise 1.6 Suppose that there exists $1 \le i \le m$ such that A_i is conjugate to an irrational rotation. Conclude that $\mathscr{B}$ is twisting.

Exercise 1.7 Suppose that there exist $1 \le i, j \le m$ such that A_i and A_j are either hyperbolic or parabolic and that they have no common eigenspace. Conclude that $\mathscr{B}$ is twisting (and pinching).

Exercise 1.8 Let A_i, $i = 1, 2$ be as in the second part of Exercise 1.2. Check that

$$\lambda_+(A_1, A_2, 1, 0) \neq \lim_{p_2 \to 0} \lambda_+(A_1, A_2, 1 - p_2, p_2).$$

Thus, the hypothesis $p_1 > 0, \ldots, p_m > 0$ cannot be removed in Theorem 1.3.

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