Issues in partially hyperbolic dynamics

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Partially hyperbolic maps

A diffeomorphism $f : M \to M$ is partially hyperbolic if there exists a continuous decomposition of the tangent bundle

$$T_x M = E^u_x \oplus E^c_x \oplus E^s_x$$

which is invariant under the dynamics:

$$Df_x(E^*_x) = E^*_{f(x)} \quad \text{for all } * \in \{u, c, s\}.$$

and...

Partially hyperbolic maps

- $Df \mid E^s$ is uniformly contracting:

$$\|Df \mid E^s_x\| \leq \lambda < 1,$$

- $Df \mid E^u$ is uniformly expanding:

$$\|(Df \mid E^s_u)^{-1}\| \leq \lambda < 1,$$

- $Df \mid E^c$ is "in between":

$$\frac{1}{\lambda} \frac{\|Df_x(v^s)\|}{\|v^s\|} \leq \frac{\|Df_x(v^c)\|}{\|v^c\|} \leq \lambda \frac{\|Df_x(v^u)\|}{\|v^u\|}.$$

Lyapunov exponents

The center Lyapunov exponents of f are the rates of growth

$$\lim \frac{1}{n} \log \|Df_x^n(v^c)\| \quad \text{of vectors } v^c \in E_x^c.$$

They are well defined almost everywhere, with respect to any invariant probability measure.

Problem

Can one always perturb f to make the center Lyapunov exponents different from zero ?

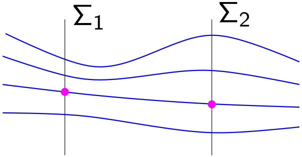
Invariant foliations

Anosov, Sinai, Brin, Pesin, Hirsch, Pugh, Shub

- The stable and unstable bundles are uniquely integrable: there exist (unique) foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ such that

$$T_x \mathcal{F}^s_x = E^s_x \quad \text{and} \quad T_x \mathcal{F}^u_x = E^u_x \quad \text{everywhere.}$$

- These foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ are absolutely continuous: projections along leaves send zero measure sets to zero measure sets.



Absolute continuity

Absolute continuity of invariant foliations is the fundamental ingredient in the proof of

Anosov

The geodesic flow on any compact manifold with negative curvature is ergodic.

Center foliation

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Problem

How frequently is the center foliation absolutely continuous ?

Time averages

We call f conservative if $\mathrm{vol}(f(A)) = \mathrm{vol}(A)$ for any $A \subset M$. Then (Birkhoff ergodic theorem) the time average

$$\lim \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x))$$

exists almost everywhere, for any continuous $\varphi : M \to \mathbb{R}$.

This is not necessarily so if f is dissipative, i.e., non-conservative.

Time averages

An f-invariant probability μ is a physical measure if the set $B(\mu)$ of points for which

$$\lim \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x)) = \int \varphi \, d\mu, \quad \forall \varphi$$

has positive volume.

Problem

Do physical measures exist? Is almost every point in the basin of some physical measure?

A model

Let $f_0 = h \times \mathrm{id} : \mathbb{T}^2 \times S^1 \to \mathbb{T}^2 \times S^1$, where $h : \mathbb{T}^2 \to \mathbb{T}^2$ is the (Anosov) map induced on the torus $\mathbb{T}^2$ by

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

Every map f in a neighborhood of f_0 is partially hyperbolic, with a unique center foliation $\mathcal{F}^c$, whose leaves are smooth circles.

Failure of absolute continuity

Shub, Wilkinson

There exists f arbitrarily close to f_0, volume preserving, ergodic, and such that the Lyapunov exponent

$$\lambda^c(f) = \int \log |D^c f| \, dm \neq 0.$$

The latter implies $\mathcal{F}^c$ is not absolutely continuous.

Atomic disintegration

Ruelle, Wilkinson

For any f close to f_0, if $\lambda^c(f)$ is non-zero then there is $k \geq 1$ and a full volume set Z with $\#(Z \cap F) = k$ for every center leaf F.

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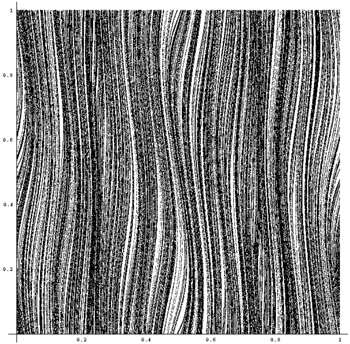

Linear cocycles

A linear cocycle is a skew-product over a measure preserving map $f : M \to M$,

$$L : M \times \mathbb{R}^d \to M \times \mathbb{R}^d, \quad (f(x), L_x(v)),$$

where each $L_x : \mathbb{R}^d \to \mathbb{R}^d$ is a linear isomorphism.

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More generally, one considers morphisms of vector bundles

$$\begin{array}{ccccl} L : \mathcal{V} & \to & \mathcal{V} & & \text{acting linearly on fibers} \\ \pi \downarrow & & \downarrow \pi & & \\ f : M & \to & M & & \text{preserving some probability } \mu. \end{array}$$

Extremal Lyapunov exponents

The extremal Lyapunov exponents are the limits

$$\lambda_+(L,x) = \lim \frac{1}{n} \log \|L_x^n\| \qquad \lambda_-(L,x) = \lim \frac{1}{n} \log \|(L_x^n)^{-1}\|^{-1}.$$

They are defined μ-almost everywhere if $\log \|L_x^{\pm 1}\| \in L^1(\mu)$.

Problem (going back to Furstenberg)

When is $\lambda_-(L,\cdot) < \lambda_+(L,\cdot)$?

Results

Furstenberg, Kesten, in the 1960's, for i.i.d. random matrices. Important contributions by Ledrappier and many others.

More recently, Mañé, Bochi, Viana, for continuous cocycles, and Bonatti, Viana, Avila, Santamaria, for more regular cocycles.

Cocycles over hyperbolic maps

From the latter works one conclude that

Assuming (f, μ) is mildly hyperbolic, the set of cocycles L for which $\lambda_-(L, \cdot) = \lambda_+(L, \cdot)$ has infinite codimension.

This applies if (f, μ) is non-uniformly hyperbolic (Pesin theory) with local product structure and also if f is partially hyperbolic, accessible, center bunched, and $\mu =$ volume.

Non-linear cocycles

Thus, one might try to approach the problem about center Lyapunov exponents of partially hyperbolic maps as follows:

1. develop a theory of Lyapunov exponents for non-linear cocycles over mildly hyperbolic systems
2. and apply it to the fiber bundle morphism

$$\begin{array}{rcl} f : M & \to & M \\ \pi \downarrow & & \downarrow \pi \\ g : M/\mathcal{F}^c & \to & M/\mathcal{F}^c \end{array}$$

(the base map g is a hyperbolic homeomorphism on the space $M/\mathcal{F}^c$ of center leaves).

Codimension issues

There is an important difficulty, however:

- while the linear theory predicts that vanishing Lyapunov exponents are an infinite codimension phenomenon,
- for partially hyperbolic systems $\{\lambda^c = 0\}$ should have codimension one: it separates $\{\lambda^c > 0\}$ from $\{\lambda^c < 0\}$.

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- for partially hyperbolic systems $\{\lambda^c = 0\}$ should have codimension one: it separates $\{\lambda^c > 0\}$ from $\{\lambda^c < 0\}$.

The solution to this apparent paradox lies in the properties of the center foliation, namely, in the failure of absolute continuity!

Rigidity theorem

Let $f_0 : \mathbb{T}^2 \times S^1 \to \mathbb{T}^2 \times S^1$ be defined by $f_0 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \times \mathrm{id}$. Let f be any C^k volume preserving diffeomorphism close to f_0.

Rigidity Theorem (Avila, Viana, Wilkinson)

If the center foliation $\mathcal{F}^c$ of f is absolutely continuous then f is C^k conjugate to a rotation extension

$$\mathbb{T}^2 \times S^1 \to \mathbb{T}^2 \times S^1, \quad (x, \theta) \mapsto (h(x), \theta + \omega(x))$$

of a C^k Anosov diffeomorphism h, and $\mathcal{F}^c$ is actually C^k.

Dichotomy theorem

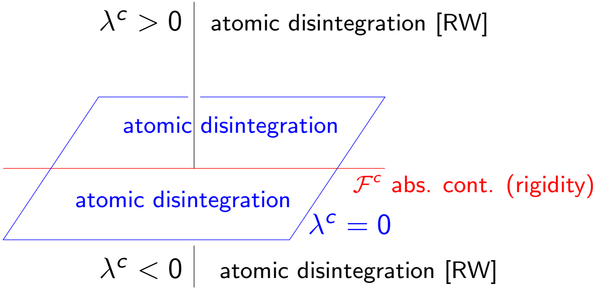
A partially hyperbolic diffeomorphism is accessible if any two points in M can be joined by a piecewise smooth curve whose legs are contained in $\mathcal{F}^s$ and $\mathcal{F}^u$ leaves.

Dichotomy Theorem (Avila, Viana, Wilkinson)

If f is accessible then either the center foliation $\mathcal{F}^c$ is absolutely continuous or there exists $k \geq 1$ and a full measure set which intersects every center leaf on exactly k points. Generically, $k = 1$.

Nitiçă, Török showed that accessibility is an open and dense property on a neighborhood of f_0.

Summary (accessible case)



Time averages

Now let f be a general (possibly dissipative) C^k diffeomorphism close to f_0.

Existence Theorem (Viana, Yang)

Assume f is accessible and the center-stable foliation $\mathcal{F}^{cs}$ is absolutely continuous. Then almost every point belongs to the basin of some physical measure.

The center-stable foliation is tangent to $E^c \oplus E^s$ at every point. Surprisingly, absolute continuity of $\mathcal{F}^{cs}$ is not a rigid property.

Physical measures

Dichotomy Theorem (Viana, Yang)

Assume f is accessible and the center-stable foliation $\mathcal{F}^{cs}$ is absolutely continuous. Then either

- f is conjugate to a rotation extension of an Anosov map and admits a unique physical measure, with zero center Lyapunov exponent, or
- f has a finite number of physical measures, all with negative center Lyapunov exponents.

Further problems

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- However, the case of non-compact center leaves is generally not understood.
- Why is absolute continuity sometimes a rigid property and sometimes not?
- Similar methods are being applied to certain smooth actions of higher rank groups.