The statistics of attractors

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The observation of many experimental systems shows that, even when the time-evolution is described by some deterministic process (e.g. a smooth transformation or an ordinary differential equation), the behaviour of the system may be very hard to understand in deterministic terms, and a stochastic analysis may be a more fruitful approach. Here we restrict ourselves to discrete-time dynamical systems, namely, smooth or piecewise smooth transformations $f : M \to M$ on some compact manifold M (possibly with boundary). However, most of the questions and results have natural extensions for flows or semi-flows.

A main goal is to describe the evolution of "observable" quantities of the system, that is, of real (or complex) functions φ defined on the phase space M and having some degree of regularity. Quite often, the sequence $\varphi(f^j(x))$ of observations for a typical trajectory $f^j(x)$ behaves rather erratically as time j varies. This can be illustrated by the following table where the values of $\varphi(f^j(x))$, corresponding to $f : [0,1] \to [0,1]$, $f(x) = 3 * x \bmod 1$, and $\varphi(z) = z$, are listed for $x = 1/\sqrt{2}$ and $0 \le j < 60$

0.707107,	0.121320,	0.363961,	0.091883,	0.275649,	0.826948,
0.480843,	0.442530,	0.327591,	0.982774,	0.948322,	0.844967,
0.534901,	0.604702,	0.814105,	0.442315,	0.326945,	0.980836,
0.942508,	0.827525,	0.482574,	0.447722,	0.343166,	0.029497,
0.088491,	0.265472,	0.796417,	0.389252,	0.167755,	0.503266,
0.509797,	0.529390,	0.588171,	0.764512,	0.293537,	0.880612,
0.641835,	0.925504,	0.776511,	0.329534,	0.988603,	0.965809,
0.897426,	0.692277,	0.076830,	0.230491,	0.691473,	0.074419,
0.223257,	0.669772,	0.009317,	0.027950,	0.083851,	0.251553,
0.754660,	0.263979,	0.791938,	0.375814,	0.127441,	0.382322

Our approach is to regard such a sequence as essentially random and to focus on studying its statistical properties. Of particular interest are those properties which are intrinsic to the dynamical system (i.e. independent of the choice of x), even more if they are robust under small modifications of the system.

Physical measures

A first, basic question concerns the existence of asymptotic time averages

$$E_x(\varphi) = \lim_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x)) \tag{1}$$

for "many" points $x \in M$. Clearly, $E_x(\varphi)$ exists whenever x is a periodic point of f, i.e. whenever $f^k(x) = x$ for some $k \geq 1$. More generally, Birkhoff's ergodic theorem asserts that asymptotic time averages exist for almost every point, with respect to any f-invariant probability measure. This is most relevant if f is volume-preserving, that is, leaves invariant some smooth (Lebesgue) measure on the manifold M. However, arbitrary invariant measures may lack physical meaningfulness. In general, we take "many" above to mean "positive measure set" with respect to some Lebesgue measure.

Furthermore, one wants to understand if and when time averages can be independent of the initial point. Suppose that, for every continuous function $\varphi : M \to \mathrm{R}$, the average $E_x(\varphi)$ exists and is independent of the point x taken in some positive measure set $B \subset M$. Then

$$\varphi \mapsto E(\varphi) = E_x(\varphi) \qquad (\text{any } x \in B)$$

defines a non-negative linear operator on the space $C^0(M, \mathrm{R})$ of real continuous functions which, by the representation theorem, can be thought of as a Borel measure μ on M:

$$\int \varphi \, d\mu = E(\varphi) = \lim_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x)) \qquad (\text{any } x \in B).$$

Observe that such a measure μ can be "physically observed" by computing time averages of continuous functions for randomly chosen points $x \in M$ (positive probability of getting $x \in B$).

This motivates the following definition. An f-invariant probability measure μ is a *physical*, or *SRB-* (for *Sinai-Ruelle-Bowen-*) measure for f if there exists a positive Lebesgue measure set of points $x \in M$ such that

$$\int \varphi \, d\mu = E(\varphi) = \lim_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x)) \qquad \text{for every } \varphi \in C^0(M, \mathrm{R}). \tag{2}$$

The set of points $x \in M$ satisfying (2) is called the *(ergodic) basin* of μ, and is denoted $B(\mu)$. The previous considerations can then be summarized in

Problem 1. Given $U \subset M$ such that $f(U) \subset U$, investigate the existence of some SRB-measure μ with $B(\mu) \subset U$. Study the uniqueness and the ergodicity of μ. Describe its basin $B(\mu)$.

SRB-measures are believed to exist in great generality (the assumption that such a measure exists is usually implicit in numerical studies of experimental systems), but actual constructions are known only for certain classes of systems, which we refer below. Also, the following simple counterexample, due to Bowen, shows that this is a matter of some subtlety.

Example 1. The example consists of a vector field in the plane with two saddle-points A_1, A_2 exhibiting a double saddle-connection. The two saddle-connections bound an open, lens-shaped region L containing another equilibrium point B, which is a source. The trajectory $X^t(z)$ of any point $z \in L \setminus \{B\}$ accumulates on the boundary of L as time $t \to +\infty$. However, given any continuous φ with $\varphi(A_1) \neq \varphi(A_2)$, the time average

$$\lim_{T\to+\infty} \frac{1}{T}\int_0^T \varphi(X^t(z))\,dt$$

does not exist (for any such z).

It is an important open question whether examples such as these can be made generic (Bowen's counterexample has codimension 2 in the space of flows).

Independence

From now on we let $U \subset M$ be some open set with $f(U) \subset U$. We suppose that f admits a unique SRB-measure μ with $B(\mu) \subset U$, and we analyse the system $(f|U, \mu)$.

The next step is to try and understand how fast memory of the past is lost by the system as time evolves. In more precise terms, one wants to know to what extent observations $\varphi(f^n(x))$ made at some instant $n \gg 1$ are affected by initial values $\psi(x)$ of some given observable ψ (possibly with $\psi = \varphi$). This is naturally expressed by means of the *correlation functions*

$$C_n(\varphi, \psi) = \int (\varphi \circ f^n)\psi\,d\mu - \int \varphi\,d\mu \cdot \int \psi\,d\mu. \tag{3}$$

Note that $C_n(\varphi, \psi) = 0$ corresponds, in probabilistic terms, to $\varphi \circ f^n$ and ψ being independent random variables. We say that (f, μ) is *mixing* if $C_n(\varphi, \psi) \to 0$ for every pair (φ, ψ): the value of $\varphi \circ f^n$ becomes less and less dependent of the value of ψ as time goes to infinity. We say that (f, μ) is *exponentially mixing* (or, *has exponential decay of correlations*) if this "loss of memory" occurs exponentially fast: there is $\tau < 1$ and for each (φ, ψ) there is $C > 0$ such that

$$|C_n(\varphi, \psi)| \leq C\tau^n \qquad \text{for all } n \geq 1. \tag{4}$$

The following very simple examples are meant to illustrate these ideas. First, a word of warning: one usually takes φ, ψ varying in some convenient Banach space of observables $\mathcal{F}$, and then the previous definitions are relative to that space (e.g. the existence and the value of τ may depend on $\mathcal{F}$). The particular choice of the Banach space varies with the context, $\mathcal{F}$ may not contain characteristic functions.

Example 2. Let $M = [0,1]$, f be given by $f(x) = 1 - |2x-1|$, and μ be Lebesgue measure. It is not difficult to find $\tau < 1$ such that given any pair of intervals $I, J \subset M$ there is $C > 0$ such that $\varphi = \chi_I$, $\psi = \chi_J$ satisfy (4).

Example 3. On the other hand, if $M = S^1$, f is a rigid rotation, and μ is Lebesgue measure, then $C_n(\chi_I, \chi_J)$ does not converge to zero, for any intervals $I, J \subset S^1$.

An important difference between these two examples concerns hyperbolicity: in the first case f is uniformly expanding, while in the second one f completely lacks hyperbolicity. In fact, an important theme in what follows is that a small amount of hyperbolicity (together with topological mixing, say) suffices for exponential decay of correlations. The next example shows that this theme should be taken with some precaution.

Example 4. Let $M = [0,1]$ and f be continuous and satisfy i) f is monotone increasing and smooth on $[0,1/2)$ and monotone decreasing and smooth on $(1/2,1]$; ii) $f(0) = f(1) = 0$ and $f'(0) = 1$ but $|f'(x)| > 1$ for every $x \notin 0, 1/2$. One can show that f does not admit *finite* SRB-measures but does have an infinite SRB-measure μ. Moreover, (f,μ) has polynomial decay of correlations (i.e. (4) holds if the righthand side is replaced by Cn^{-d} for some $d \geq 1$), but it is not exponentially mixing.

Another important characterization of (almost) independence of successive observations is through the central limit theorem, which describes the oscillations of finite-time averages

$$\frac{1}{n}\sum_{j=0}^{n-1}\varphi(f^j(x))$$

around their expected value $\int \varphi\, d\mu$. The usual central limit theorem from probability theory asserts that if $X_0, \dots, X_n, \dots$ are independent, identically distributed random variables then, for any open interval $I \subset \mathrm{R}$, the probability of

$$\sqrt{n}\left(\frac{1}{n}\sum_{j=0}^{n-1}X_j - E\right) \in I, \qquad E = E(X_0),$$

converges to

$$\frac{1}{\sqrt{2\pi}\sigma}\int_I e^{\frac{-t^2}{2\sigma^2}}\, dt, \qquad \sigma^2 = E(X_0^2),$$

as $n \to +\infty$.

Going back to our dynamical context, we say that an observable φ *satisfies the central limit theorem* for (f, μ) if there is $\sigma > 0$ such that, for every interval $I \subset \mathrm{R}$,

$$\mu \left\{ x \in M : \sqrt{n} \left(\frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x)) - \int \varphi \, d\mu \right) \in I \right\} \to \frac{1}{\sqrt{2\pi}\sigma} \int_I e^{\frac{-t^2}{2\sigma^2}} \, dt, \tag{5}$$

as $n \to +\infty$. It should come as no surprise that this holds when φ has sufficiently fast decay of correlations.

We summarize the discussion in this section in

Problem 2. Determine whether (f, μ) satisfies the mixing properties and or the central limit theorem, for all the observables in some appropriate Banach space. Estimate the rate of decay of the correlation functions.

Stochastic stability

Very often, the mathematical formulation $f : M \to M$ of a given physical process involves simplifications, where a “main” part of the process is isolated (this is what f is meant to describe) and external influences are discarded as to complex to be taken in consideration and, hopefully, too small to be relevant. Clearly, this procedure requires a justification, specially if, as often happens, the simplified system f turns out to be structurally unstable (meaning that arbitrarily close transformations g may have completely different dynamical behaviour).

In many instances where such external influences are not completely known, or are too complex to be effectivelly expressed in deterministic terms, one can think of them as a kind of random “noise”. One then speaks of stochastic stability if the adjunction of small noise has only a small effect on the asymptotic behaviour of f. In more precise terms, for each small $\varepsilon > 0$ one considers iterates

$$x_j = f_j \circ \cdots \circ f_1(x), \qquad x \in U, \quad j \geq 0,$$

where the f_i are chosen randomly and independently from each other in the ε-neighbourhood of f, according to some given distribution law. It is convenient to assume that $f(\bar{U}) \subset U$, to ensure that $f_i(U) \subset U$ for every i. Then, under general conditions there exist probability measures μ_ε such that

$$\lim \frac{1}{n} \sum_{j=0}^{n-1} \varphi(x_j) = \int \varphi \, d\mu_\varepsilon \tag{6}$$

for all continuous $\varphi : M \to \mathrm{R}$ and “many” (positive probability) random trajectories $(x_j)_{j \geq 0}$ with $x_0 \in U$. We say that $(f|U, \mu)$ is *stochastically stable* if μ_ε

converges to μ in the weak*-sense, that is,

$$\int \varphi \, d\mu_\varepsilon \to \int \varphi \, d\mu \quad \text{for all } \varphi \in C^0(M, \mathrm{R}) \tag{7}$$

as $\varepsilon \to 0$ (if μ_ε is not unique then we require convergence to μ for all such stationary measures with $B(\mu_\varepsilon) \subset U$).

Unlike structural stability, stochastic stability is likely to hold for quite general systems. In fact, another informal theme inspired by the results we mention below is that systems with exponential decay of correlations tend to be stochastically stable: known counterexamples, such as the next one, are non-generic in some way or the other.

Example 5. (see [BaY]) The example is a continuous piecewise affine and expanding map $f : [0,1] \to [0,1]$: there are $c_1 = 0 < c_2 < c_3 = 1/2 < c_4 < c_5 = 1$ and $\sigma_i > 1$, $i = 1, 2, 3, 4$, such that $f'(x) = (-1)^i \sigma_i > 1$ for all $x \in [c_i, c_{i+1}]$. Moreover, $f(1/2) = 1/2$ and, due to the presence of this periodic turning point, the map f is not stochastically stable.

Problem 3. Obtain general conditions ensuring stochastic stability.

Further understanding of the dynamics (resonances, distribution of periodic points, ...) can be obtained from other important invariants, such as the *correlation spectrum* or *dynamical zeta functions*. Although we do not treat these invariants explicitly here, their study is closely related to that of the problems we stated above.

Fairly complete answers to those problems are now available for uniformly hyperbolic systems and for certain classes of nonuniformly hyperbolic systems. Here we focus on some recent results in the latter context. See e.g. [Bo], [Ki1], [Ki2], [BaY], and references therein for the rich theory concerning the uniformly hyperbolic case as well as for general background.

Unimodal maps of the interval

In this section we briefly discuss our joint work with V. Baladi [BaV] on the ergodic properties of certain nonuniformly hyperbolic maps of the interval $f : I \to I$.

For simplicity, let us take f to be quadratic i.e. $f(x) = a - x^2$ (all the arguments hold for general unimodal maps with negative schwarzian derivative and nondegenerate critical point). We formulate the nonuniform hyperbolicity property in terms of the orbit of the critical point $c = 0$: let us assume that

1. $\left|(f^k)'(f(c))\right| \geq \lambda_c^k$ (positive Lyapunov exponent);
2. $\left|f^k(c) - c\right| \geq e^{-\alpha k}$ (exponential recurrence bound)

for every $k \geq 1$ and for some constants $0 < \alpha \ll 1 < \lambda_c$. We also suppose that f is topologically mixing (on the interval $f^2(I)$).

This formulation is motivated by [BC1], [BC2], where it is proved that 1 and 2 above are satisfied by quadratic maps for a positive measure set of values of the parameter a. It follows from condition 1 and [Si] that f can not have attracting periodic orbits. On the other hand, maps $f_s(x) = f(x) + s$ with small s may exhibit such periodic attractors and so, in particular, f is structurally unstable. Quite in contrast, f is stochastically stable (in a strong sense), as we shall see.

It is now well-known that condition 1 implies the conclusion of [Ja]: f admits an invariant probability measure μ_0 which is absolutely continuous with respect to the Lebesgue measure m on I. Moreover, μ_0 is unique, ergodic, and equivalent to m restricted to $f^2(I)$. As a consequence of the ergodic theorem, μ_0 is an SRB-measure:

$$\frac{1}{n}\sum_{j=0}^{n-1}\varphi(f^j(x)) \to \int \varphi\, d\mu_0 \quad \text{as } t \to +\infty$$

for every continuous function φ and m-almost all $x \in I$.

Now we want to consider the effect of adding random noise to the iteration of f: we want to compare the asymptotic behaviour of f^j with that of $f_{s_j} \circ \cdots \circ f_{s_1}$, where $s_1, \dots, s_j, \dots$ are chosen randomly and independently in some small interval $[-\varepsilon, \varepsilon]$. We denote by θ_ε the corresponding probability distribution. In this context there exists a unique stationary measure μ_ε, satisfying

$$\frac{1}{n}\sum_{j=0}^{n-1}\varphi(f_{s_j} \circ \cdots f_{s_1}(x)) \to \int \varphi\, d\mu_\varepsilon \quad \text{as } t \to +\infty$$

for every continuous function φ, m-almost all $x \in I$, and almost all choices of $(s_j)_{j\geq 1}$. Moreover, μ_ε is absolutely continuous with respect to Lebesgue measure.

Remark. All these facts we have been listing, concerning the measures μ_0 and μ_ε, are recovered as by-products of the proof of Theorems A and B below.

Theorem A *[BaV]*

$$\frac{d\mu_\varepsilon}{dm} \to \frac{d\mu_0}{dm} \quad \textit{in } L^1(m) \quad \textit{as } \varepsilon \to 0 \quad \textit{(strong stochastic stability).}$$

In particular (f, μ) is stochastically stable.

We also state a result concerning decay of correlations, both for the unpertubed system (f, μ) and for its random perturbations. Correlation functions $C_{n,\varepsilon}(\varphi, \psi)$ for the random perturbation scheme are defined by

$$C_{n,\varepsilon}(\varphi, \psi) = \int (U_\varepsilon^n \varphi)\psi\, d\mu_\varepsilon - \int \varphi\, d\mu_\varepsilon \cdot \int \psi\, d\mu_\varepsilon \tag{8}$$

where $(U_\varepsilon\varphi)(x) = \int \varphi(f_t(x))\theta_\varepsilon(t)\, dt$ (compare $(U\varphi)(x) = \varphi(f(x))$.

Theorem B *[BaV] Both f and its random perturbation schemes $(f_s)_{|s|\le\varepsilon}$, are exponentially mixing, with mixing rates τ, τ_ε, uniformly bounded away from 1.*

Not all the content of Theorems A and B is new in [BaV]. Weak stochastic stability for quadratic maps was first proved by [KK], for uncountably many parameters, and by [BY1], for a positive measure set of parameters (but see also [Co], where strong stability was already considered). Exponential decay of correlations for (unperturbed) quadratic maps was proved independently by [KN] and [Yo].

Hénon-like maps

To conclude, let us comment on the ergodic properties of attractors of dissipative diffeomorphims. Theorems C and D below are part of an ongoing work in collaboration with M. Benedicks.

The kind of systems we want to consider is inspired by the Hénon model

$$f(x, y) = (1 - ax^2 + y, bx),$$

which combines hyperbolic dynamics, for x far from zero, with "folding" behaviour near $x = 0$. In [BC2] it was proved that given any small enough $b > 0$ there exists a positive measure set of values of a for which f has a "strange attractor": a compact f-invariant set Λ containing dense orbits on which the norm of the derivative grows exponentially fast. Then [MV] showed that attractors with similar properties occur in very general contexts of bifurcations of dynamical systems. See also [DRV] for a more global construction. Henceforth, *Hénon-like strange attractors* will always refer to nonuniformly attractors such as those constructed in these papers.

It was proved in [BY2] that Hénon-like strange attractors support a unique invariant probability measure μ which is ergodic, has a positive Lyapunov exponent and, most important, induces absolutely continuous conditional measures along unstable manifolds (absolute continuity is with respect to the riemannian measure on the unstable manifold). Then standard arguments show that μ is an SRB-measure:

$$\frac{1}{n}\sum_{j=0}^{n-1}\varphi(f^j(x)) \to \int \varphi\, d\mu \quad \text{as } t \to +\infty$$

for a positive (two-dimensional) Lebesgue measure subset of points x in the (topological) basin $B(\Lambda)$ of Λ. Recall that $B(\Lambda)$ is the set of points whose trajectories accumulate on Λ as time goes to $+\infty$.

Of course, one would like to know whether this property holds for a *full measure* subset of $B(\Lambda)$ and this is part of the content of the next theorem.

Theorem C *Through Lebesgue almost every point in $B(\Lambda)$ there is a local stable manifold which intersects Λ. Moreover,*

$$\frac{1}{n}\sum_{j=0}^{n-1}\varphi(f^j(x)) \to \int \varphi\, d\mu \quad \textit{as } t \to +\infty,$$

for every continuous function φ and for Lebesgue almost every $x \in B(\Lambda)$.

As before, Lebesgue refers to the two-dimensional Lebesgue measure. Also, by a stable manifold we mean a curve which is exponentially contracted under all positive iterates of f.

Recently, Benedicks-Young have announced a proof of exponential mixing and the central limit theorem for Hénon-like strange attractors. These results, together with our next statement, joint with Benedicks, conclude the study of Problems 1-3 in this context of strongly dissipative diffeomorphisms.

Theorem D *Hénon-like strange attractors are stochastically stable.*

More precisely, we consider (f, μ), where μ is an SRB-measure as in [BY2], and we prove that this system is stable under random perturbations $f_s(x, y) = f(x, y) + s$, where s takes values in a small neighbourhood of $0 \in \mathrm{R}^2$. Note that Theorems C and D are somewhat related: since Λ is not invariant under the perturbed maps f_s, some control of the basin of μ is necessary to prove stochastic stability.

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