Infinite-modal maps with global chaotic behavior

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Dedicated to the memory of Ricardo Mañé

Abstract

We prove that certain parametrized families of one-dimensional maps with infinitely many critical points exhibit global chaotic behavior in a persistent way: for a positive Lebesgue measure set of parameter values the map is transitive and almost every orbit has positive Lyapunov exponent. An application of these methods yields a proof of existence and even persistence of global spiral attractors for smooth flows in three dimensions, to be given in [PRV].

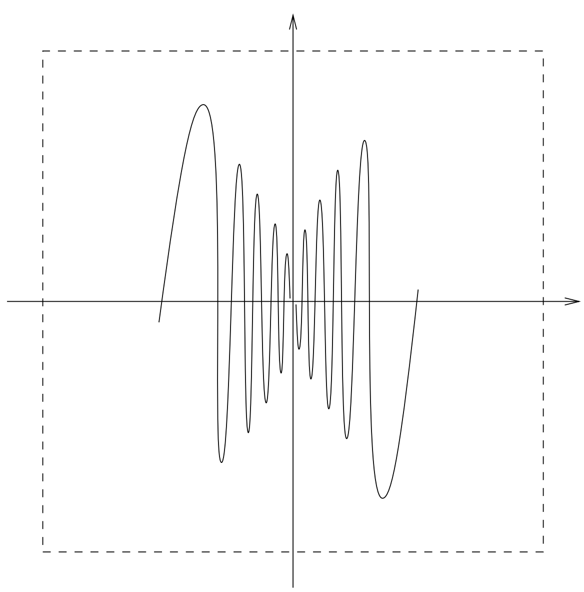

Figure 1: An infinite-modal map

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1 Introduction

Our main goal in this paper is to study the dynamics of certain parametrized families $(f_\mu)_\mu$ of one-dimensional maps with infinite critical set. In this Introduction we outline and motivate the objects and results involved, precise definitions and statements are postponed to Section 2.

The kind of maps we want to consider is described in Figure 1: they are smooth everywhere, except at some distinguished point 0; most important, this point is accumulated by critical points of the map, exponentially fast and from both sides. More precisely, the figure corresponds to the initial map f_0: the graph of f_μ for $\mu \neq 0$ is obtained by translating the lefthand side and the righthand side, vertically, in opposite directions; in particular, 0 becomes a discontinuity.

This class of systems is motivated by a problem in the dynamics of flows in three dimensions: the unfolding of saddle-focus homoclinic connections, see [Sh] and Figure 2. Roughly speaking, one-dimensional maps f_μ as we treat here can be obtained by considering first-return maps of the flow to appropriate cross-sections and "forgetting" one of the variables. This last step results in considerable simplification of the dynamics, nevertheless the maps f_μ retain a large share of the complexity of the corresponding flow and, thus, provide important insight to its behavior, as we shall see.

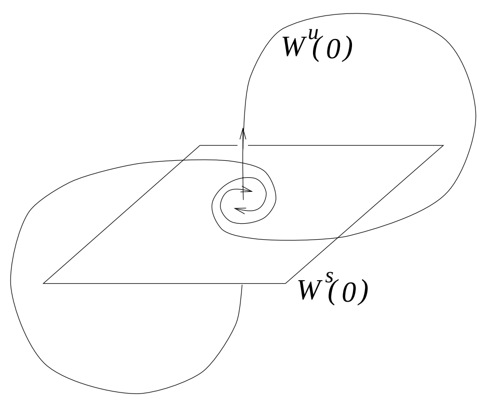

Figure 2: Double saddle-focus homoclinic connections

Our main result asserts that, despite the existence of infinitely many regions of contraction, chaotic (expanding) dynamics is persistent among the maps f_μ: *for a positive Lebesgue measure set of values of μ, f_μ has positive Lyapunov exponent at every critical value and, indeed, at Lebesgue almost all points in its domain; moreover, f_μ is transitive, i.e., has dense orbits*. This will be restated in a more precise form as Theorem A, in Section 2.

Persistence of chaotic dynamics for one-dimensional maps was fist proved by Jakobson [Ja], who considered quadratic maps $q_\mu(x) = 1 - \mu x^2$ for values of the parameter close to $\mu = 2$. Since then, other approaches have been introduced, and extensions to various settings of smooth maps with finitely many critical points have been obtained. On the other hand, to the best of our knowledge, this is the first time such a statement of global "chaotic" behavior is given for infinite-modal maps.

Another new ingredient here concerns the behavior of the critical orbits for the unperturbed map f_0. Indeed, in virtually every case where chaotic dynamics has been described for smooth unimodal or multimodal maps, one starts by assuming closeness to a system where all the critical points are nonrecurrent. This plays a key role in ensuring a fair amount of expansion during early iterates, thus providing a starting point for the recursive argument. Clearly, the present situation does not provide such a control of the critical orbits: the origin 0, where critical points accumulate, is a fixed (and so recurrent) point for f_0.

As a consequence, the proof of Theorem A has two main steps. First, we bypass the recurrent behavior of the critical orbits, to ensure that all of them do exhibit initial expansion. We do this by making parameter exclusions right from the first iterates, which has no analog in the previously mentioned situations. In a second stage, we control the way critical trajectories return to the vicinity of critical points, to guarantee that the expansion is preserved in all subsequent iterates. This is done through additional exclusions of the parameters and is inspired by the arguments in [BC, Section 2], extended to deal with the presence of infinitely many criticalities. We believe that the present methods are interesting by themselves and will find applications in a broader context.

As already mentioned, an application we had in mind when introducing this class of systems was the study of flows unfolding saddle-focus homoclinic connections. A large amount of, mostly numerical, analysis of this bifurcation mechanism suggests that it often leads to the formation of a "strange" attractor with rather complex spiral geometry, see e.g. [ACT] and [CKR]. The possible existence of such attractors seems to have been first mentioned by Ya. Sinai. A further extension of the methods in the present paper allows us to give rigorous support to those numerical observations and prove that such attractors do exist. In fact we obtain a stronger statement: *spiral attractors are measure-theoretically persistent in certain smooth families of flows unfolding saddle-focus homoclinic connections*. The proof of this fact is quite long and will appear in [PRV].

Acknowledgements and personal note: This work was started during a visit to the ICTP/Trieste and, for the most part, carried out at IMPA/Rio de Janeiro. Sometimes R. Mañé would be around, trying to find out what we were doing, and once he warned us: "Be careful, you might end up writing a paper together !". We all miss you, Ricardo.

2 Statement of results

In this section we define the class of systems we are interested in. We consider certain symmetric vector fields X_0 exhibiting a double homoclinic connection associated to a saddle-focus singularity, and we derive expressions for first-return maps of the corresponding flows. Then, to each generic one-parameter family of vector fields X_μ through X_0, we associate a certain one-parameter family of maps f_μ, which may be thought of as partial models for the dynamical behavior of X_μ. These f_μ turn out to be naturally defined on the circle and constitute the main object of study in the present paper.

In what follows X_0 is a vector field in R^3, symmetric with respect to $w \mapsto -w$ and having a singularity at the origin with eigenvalues θ, $\lambda \pm i\omega$, such that $\theta > 0$, $\lambda < 0$, $\omega \neq 0$ and $\alpha = -\lambda/\theta < 1$. We assume that X_0 is smoothly linearizable in some neighborhood of 0. Let (x, y, z) denote linearizing coordinates. Up to rescaling, we may suppose that $Q = \{|x| \leq 2, |y| \leq 2, |z| \leq 2\}$ is contained in the domain of linearization, that the local unstable manifold W^u_{loc} of the origin is contained in $\{x = y = 0\}$, and that the local stable manifold W^s_{loc} of the origin is contained in $\{z = 0\}$. We also assume that X_0 has (two) homoclinic connections associated to zero, i.e., that its unstable separatrices W^u_+, respectively, W^u_-, contain points $(q_0, 0, 0)$, respectively $(-q_0, 0, 0)$, in W^s_{loc}. We rescale the coordinates once more so that $q_0 = 1$. See Figure 3.

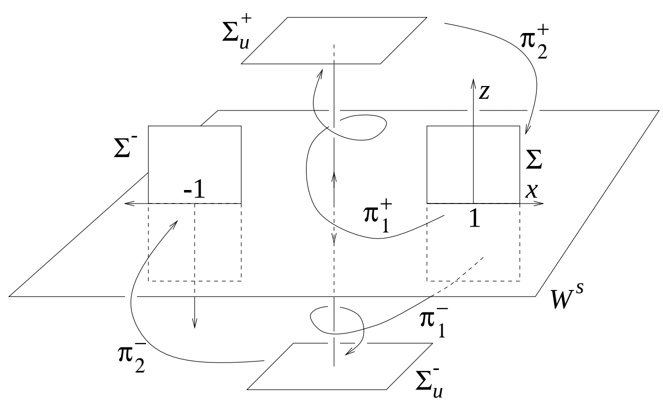

Figure 3: The first-return map

Then we fix small positive constants δ_1 and ϵ_1 and define cross-sections

$$\Sigma = \{1 - \delta_1 \leq x \leq 1 + \delta_1,\ y = 0,\ |z| \leq \epsilon_1\},\ \Sigma^- = -\Sigma,$$
$$(0, 0, 1) \in \Sigma_u^+ \subset \{z = 1\},\ \text{and}\ (0, 0, 1) \in \Sigma_u^- \subset \{z = -1\}.$$

We fix δ_1 and ϵ_1 small enough so that we have well-defined Poincaré maps

$$\pi_1^+ : \Sigma \cap \{z > 0\} \to \Sigma_u^+ \text{ and } \pi_1^- : \Sigma \cap \{z < 0\} \to \Sigma_u^-,$$

given by, see e.g. [Sh],

$$\pi_1^+(x+1,0,z) = (z^\alpha x\cos(\beta\log 1/z), z^\alpha x\sin(\beta\log 1/z), 1),$$
$$\pi_1^-(x+1,0,z) = (|z|^\alpha x\cos(\beta\log 1/|z|), |z|^\alpha x\sin(\beta\log 1/|z|), -1),$$

where $\alpha = -\lambda/\theta$ and $\beta = \omega/\theta$. By symmetry, $\Sigma^-\cap\{z>0\}$ is mapped into Σ_u^+ by the map $w\mapsto -\pi_1^-(-w)$, and $\Sigma^-\cap\{z<0\}$ is mapped into Σ_u^- by $w\mapsto -\pi_1^+(-w)$.

Due to the existence of homoclinic connections we also have Poincaré maps

$$\pi_2^+ : \Sigma_u^+ \to \tilde{\Sigma} \text{ and } \pi_2^- : \Sigma_u^- \to \tilde{\Sigma}^-,$$

where $\pi_2^\pm(0,0,\pm1) = (\pm1,0,0)$, and $\tilde{\Sigma}\supset\Sigma$ and $\tilde{\Sigma}^-\supset\Sigma^-$ are also cross-sections to the flow. Note that π_2^+ and π_2^- are diffeommorphisms and $\pi_2^-(w) = -\pi_2^+(-w)$. Their exact expression is not important for what follows and so, for the sake of simplicity, we suppose

$$\pi_2^+(x,y,1) = (1+bx, 0, ay)$$

where b and a are nonzero constants. This means that we have a Poincaré map $F:\Sigma\cup\Sigma^-\to\tilde{\Sigma}\cup\tilde{\Sigma}^-$, given by

$$\begin{aligned} F(x+1,z) &= \pi_2^+\circ\pi_1^+(x+1,0,z) \\ &= (1+bz^\alpha x\cos(\beta\log(1/z)), az^\alpha x\sin(\beta\log(1/z))) \end{aligned}$$

for $(x+1,0,z)\in\Sigma\cap\{z>0\}$,

$$\begin{aligned} F(x+1,z) &= \pi_2^-\circ\pi_1^-(x+1,0,z) \\ &= (-1+b|z|^\alpha x\cos(\beta\log(1/|z|))), a|z|^\alpha x\sin(\beta\log(1/|z|))) \end{aligned}$$

for $(x+1,0,z)\in\Sigma\cap\{z<0\}$, and

$$F(w) = -F(-w) \text{ for } w\in\Sigma^-.$$

We are particularly interested in the case when b is small. Then, for all $w\in\Sigma\cup\Sigma^-$, the first coordinate of $F(w)$ is close to ±1. Therefore, in a first approximation, the dynamics of F may be described by the one-dimensional map $\hat{h}:\{+1,-1\}\times[-\epsilon_1,\epsilon_1]\to\{+1,-1\}\times[-1,1]$ given by

$$\begin{aligned} &\hat{h}(1,z) = (1, az^\alpha\sin(\beta\log(1/z))) \text{ for } z>0, \\ &\hat{h}(1,z) = (-1, a|z|^\alpha\sin(\beta\log(1/|z|))) \text{ for } z<0, \text{ and} \\ &\hat{h}(-1,z) = -\hat{h}(1,-z) \text{ for all } z\in[-\epsilon_1,\epsilon_1]. \end{aligned} \tag{1}$$

Moreover, since our purpose is to study expanding behavior, $\hat{h}$ may be replaced by the interval map $\hat{f}:[-\epsilon_1,\epsilon_1]\to[-1,1]$ given by

$$\hat{f}(z) = \begin{cases} az^\alpha\sin(\beta\log(1/z))) & \text{if } z>0 \\ -a|z|^\alpha\sin(\beta\log(1/|z|))) & \text{if } z<0. \end{cases} \tag{2}$$

Indeed, the symmetry of $\hat{h}$, given by the third expression in (1), implies

$$\hat{h}^n(1,z) = \pm(1, \hat{f}^n(z)) \text{ for all } n \geq 1$$

and so the two maps $\hat{f}$ and $\hat{h}$ have the same Lyapunov exponents. Maps $\hat{f}$ as above have infinitely many critical points, of the form

$$x_k = \hat{x}\exp(-k\pi/\beta) \text{ and } x_{-k} = -x_k \text{ for each large } k > 0 \tag{3}$$

where $\hat{x} > 0$ is independent of k. Let $k_0 \geq 1$ be the smallest integer such that x_k is defined for all $|k| \leq k_0 - 1$, and x_{k_0} is a local minimum.

So far we have made use of the local features of X_0 in a neighborhood of the singularity, to derive the expression, near the origin, of the one-dimensional maps $\hat{f}$ we are interested in. Now we extend this expression to the whole circle $S^1 = [-1,1]/(-1 \sim 1)$ in the following way. Let $\tilde{f}$ be an orientation-preserving expanding map of S^1 such that $\tilde{f}(0) = 0$ and $\tilde{f}' > \sigma_0$ for some constant $\sigma_0 >> 1$. Define

$$\epsilon = \frac{2}{1 + e^{-\pi/\beta}}\, x_{k_0},$$

so that x_{k_0} is the middle point of the interval $(e^{-\pi/\beta}\epsilon, \epsilon)$, cf. the first paragraph of Section 3. Fix two points $x_{k_0} < \hat{y} < \tilde{y} < \epsilon$, with $|\hat{f}'(\hat{y})| >> 1$. Then take f to be any smooth map on S^1 coinciding with $\hat{f}$ on $[-\hat{y}, \hat{y}]$, coinciding with $\tilde{f}$ on $S^1 \setminus [-\tilde{y}, \tilde{y}]$, and monotone on each $\pm[\hat{y}, \tilde{y}]$. We point out that the extended maps f obtained in this way are naturally associated to the first-return map $F_\mu : \hat{\Sigma} \cup \hat{\Sigma}^- \to \hat{\Sigma} \cup \hat{\Sigma}^-$ for convenient cross-sections $\hat{\Sigma} \supset \tilde{\Sigma}$, $\hat{\Sigma}^- \supset \tilde{\Sigma}^-$, as will be explained in [PRV].

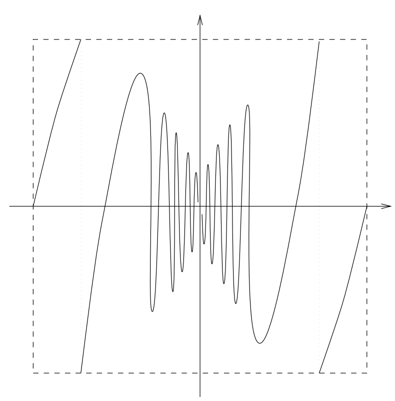

Figure 4: Graph of the circle map f

Finally, let f_μ be any one-parameter family of circle maps unfolding the dynamics of $f = f_0$ in the sense that

$$f_\mu(z) = \begin{cases} f(z) + \mu & \text{for } z \in (0, \epsilon] \\ f(z) - \mu & \text{for } z \in [-\epsilon, 0). \end{cases} \tag{4}$$

The reason why we take f_μ depending on μ in this way is that we want to model the unfolding of the homoclinic connections of the initial vector field X_0 by a generic one-parameter family of vector fields X_μ. Dependence on the parameter for large $|z|$ is irrelevant for all our purposes.

Theorem A *There exists a positive Lebesgue measure set S and a constant $\sigma > 1$ such that for every $\mu \in S$*

1. $\left|\left(f_\mu^n\right)'(z_k^\pm(\mu))\right| \geq \sigma^n$ *for all $n \geq 1$ and all $k_0 \leq |k| \leq \infty$;*
2. $\liminf_{n\to+\infty} n^{-1}\log|(f_\mu^n)'(z)| > 0$ *for Lebesgue almost every point $z \in S^1$;*
3. *there exists $z \in S^1$ whose orbit $\{f_\mu^n(z) : n \geq 0\}$ is dense in S^1.*

The proof of this theorem occupies the remaining sections of the paper. It involves a few constants chosen as follows. We fix $\gamma \in (\alpha, 1)$ and $\sigma \in (1, \sqrt{\sigma_0})$, recall that $\sigma_0 > 1$ is a strict lower bound for the derivative of $\tilde{f}$. We also use small positive constants $\hat{\tau}$, ρ, and τ: these are subject to certain conditions which we state along the way. Finally, $\epsilon > 0$ is supposed small with respect to all these constants.

Before proceeding, let us observe that the theorem is valid in more generality, in particular, it holds for any family of maps $(g_\mu)_\mu$ close enough to $(f_\mu)_\mu$. This statement requires some explanation, since our maps are not smooth. In proving Theorem A we use the fact that $f_0(x) = \pm|x|^\alpha\xi(\log|x|)$ with $\alpha \in (0,1)$ and ξ a smooth real map (we also take some advantage of ξ being periodic, but this is not really necessary). Let us say that a map g_0 is C^r *close to* f_0, $r \geq 0$, if it can be written $g_0(x) = \pm|x|^\gamma\eta(\log|x|)$ with γ close to α and η a C^r map C^r close to ξ. For instance, if $g_0 = f_0 + \psi$, where $\psi(0) = 0$ and the C^{r+1} norm of ψ is small, then g_0 is C^r close to f_0. Also, if g_0 derives from a vector field Y_0 having a double homoclinic connection, in the same way as we derived f_0 from X_0, then g_0 is C^r close to f_0 if Y_0 is C^r close to X_0. Therefore, this topology is rather natural in the present context. Then we say that a parametrized family $(g_\mu)_\mu$ is C^r close to $(f_\mu)_\mu$ if it is of the form $g_\mu(x) = g_0(x) + \psi(x,\mu)$ where $\psi(x,\mu)$ is a C^r map on $\{x \neq 0\}$, uniformly C^r close to $(x,\mu) \mapsto \mu$ on $\{x > 0\}$ and to $(x,\mu) \mapsto -\mu$ on $\{x < 0\}$. Up to straightforward adaptations, all our arguments in Sections 2–9 extend to such a $(g_\mu)_\mu$, if it is sufficiently C^2 close to $(f_\mu)_\mu$. One may ask whether the theorem is also valid for other classes of families, e.g.,

$$g_\mu(x) = \varphi_\mu(\pm|x|^{\gamma(\mu)}\eta_\mu(\log|x|)), \quad \text{with } \varphi_0 \approx \text{id}, \quad |\partial_\mu\varphi_\mu| \approx 1$$

that come up naturally in the context of homoclinic bifurcations of flows. Such an extension does seem possible, albeit technically involved, but we have not attempted to check all the details.

3 Preliminary lemmas

This section is devoted to proving some simple properties of the map f introduced above. We begin by fixing a few notations. For each $k \geq k_0$ we define

$$y_k = \frac{2}{1+e^{-\pi/\beta}}\, x_k,$$

so that x_k is the middle point of the interval (y_{k+1}, y_k). In particular $y_{k_0} = \epsilon$. We call x_k the *closest critical point* to any $y \in (y_{k+1}, y_k)$. Finally, we introduce similar notations for $k \leq -k_0$.

Lemma 3.1 *Given $\alpha_1, \alpha_2, \beta_1, \beta_2$ with $\frac{\alpha_1}{\alpha_2} \neq \frac{\beta_1}{\beta_2}$, there exists $\delta > 0$ such that, for every x, at least one of the following assertions hold:*

1. $|\alpha_1 \sin x + \beta_1 \cos x| \geq \delta$

2. $|\alpha_2 \sin x + \beta_2 \cos x| \geq \delta$

Proof: Define $\phi_i(x) = \alpha_i \sin x + \beta_i \cos x$. These are linearly independent solutions of $\phi'' = -\phi$, hence cannot have common zeroes. Let V_1, V_2 be disjoint neighborhoods of the zeroes of ϕ_1, ϕ_2 and take $\delta = \min\{\phi_1/V_1^c, \phi_2/V_2^c\}$. $\square$

We use $C > 0$ to denote any large constant depending on the map f, but not on $\epsilon > 0$.

Lemma 3.2 *For every $x \in (y_{l+1}, y_l)$ and $l \geq k_0$, respectively, $x \in (y_l, y_{l-1})$ and $l \leq -k_0$, we have*

1. $C^{-1}|x_l|^{\alpha-2}|x - x_l|^2 \leq |f(x) - f(x_l)| \leq C|x_l|^{\alpha-2}|x - x_l|^2$

2. $C^{-1}|x_l|^{\alpha-2}|x - x_l| \leq |f'(x)| \leq C|x_l|^{\alpha-2}|x - x_l|$

Proof: It suffices to prove the second item because then the first will follow by integration. Moreover, we may suppose $l > 0$, since the case $l < 0$ is entirely analogous. If $l > k_0$ and $x \in (y_{l+1}, y_l)$ then

$$f'(x) = a|x|^{\alpha-1}[\alpha \sin(\beta \log |x|^{-1}) - \beta \cos(\beta \log |x|^{-1})]$$

$$f''(x) = a|x|^{\alpha-2}[A \sin(\beta \log |x|^{-1}) + B \cos(\beta \log |x|^{-1})]$$

for some A and B depending only on α and β. The conclusion follows applying the previous lemma to the factors inside brackets and noting that $|f'(x)| \leq C|x|^{\alpha-1}$ and $|f''(x)| \leq C|x|^{\alpha-2}$. For $l = k_0$, precisely the same arguments apply when $x \in (y_{k_0+1}, \hat{y})$. On the other hand, on $[\hat{y}, y_{k_0})$ both $|x - x_{k_0}|$ and $|f'(x)|$ are bounded away from zero and from infinity, and so the statement holds just by taking the constant C sufficiently large. $\square$

Lemma 3.3 *Let $s,t \in [y_{l+1}, y_l]$ with $l \geq k_0$, respectively, $s,t \in [y_l, y_{l-1}]$ with $l \leq -k_0$. Then*

$$\left|\frac{f'(s)-f'(t)}{f'(t)}\right| \leq K_1 \frac{|s-t|}{|t-x_l|}$$

where $K_1 > 0$ is independent of l, s, t, ϵ.

Proof: Once again, we only need to treat the case $l > 0$. We begin by supposing $l > k_0$. Recall from the proof of the previous lemma that

$$|f'(x)| = |x|^{\alpha-1}\phi_1(\log|x|) \quad \text{and} \quad |f''(x)| = |x|^{\alpha-2}\phi_2(\log|x|)$$

for every $x \in [-\hat{y}, \hat{y}] \supset [y_{l+1}, y_l]$, where ϕ_1 and ϕ_2 are smooth periodic functions. Using Lemma 3.1 we find $\delta > 0$ such that for any y either $\phi_1(y) \geq \delta$ or $\phi_2(y) \geq \delta$. Recall that x_l is the unique point in $[y_l, y_{l-1}]$ such that $\phi_1(\log|x_l|) = 0$. Then we can find $\epsilon_0 > 0$ small and $0 < \delta_2 < \delta_1 < \delta$ so that, given any $t \in [y_l, y_{l-1}]$,

$$|t-x_l| \leq \epsilon_0|x_l| \Rightarrow \phi_1(\log|t|) \leq \delta_1 \quad \text{and} \quad |t-x_l| \geq \epsilon_0|x_l| \Rightarrow \phi_1(\log|t|) \geq \delta_2.$$

By the mean value theorem, there is $u \in (s,t) \subset [y_l, y_{l-1}]$

$$|f'(s)-f'(t)| = |f''(u)|\,|s-t| = |u|^{\alpha-2}\phi_2(\log|u|)\,|s-t|\,. \tag{5}$$

Now we consider two cases, according to the position of t.

If $|t-x_l| \leq \epsilon_0|x_l|$ then we use the mean value theorem to get

$$|f'(t)| = |f''(v)|\,|t-x_l| = |v|^{\alpha-2}\phi_2(\log|v|)\,|t-x_l|\,, \tag{6}$$

for some $v \in (t, x_l) \subset [y_l, y_{l-1}]$. Our choices imply $\phi_1(\log|v|) \leq \delta_1 < \delta$, and so $\phi_2(\log|v|) \geq \delta$. Using also the fact that ϕ_2 is bounded, we get

$$\frac{|f'(s)-f'(t)|}{|f'(t)|} \leq \frac{C\,|u|^{\alpha-2}\,|s-t|}{\delta\,|v|^{\alpha-2}\,|t-x_l|} \leq \frac{Ce^{(2-\alpha)(\pi/\beta)}}{\delta}\,\frac{|s-t|}{|t-x_l|}\,,$$

which proves the lemma in this case.

If, on the contrary, $|t-x_l| \geq \epsilon_0|x_l|$, then we use

$$|f'(t)| = |t|^{\alpha-1}\phi_1(\log|t|) \geq \delta_1\,|t|^{\alpha-1}.$$

Noting that $|t-x_l| \leq e^{(\pi/\beta)}|t|$ for some large $C > 0$ (all these points are in $[y_l, y_{l-1}]$),

$$\frac{|f'(s)-f'(t)|}{|f'(t)|} \leq \frac{C\,|u|^{\alpha-2}\,|s-t|}{\delta_1|t|^{\alpha-2}e^{-(\pi/\beta)}\,|t-x_l|} \leq \frac{Ce^{(3-\alpha)(\pi/\beta)}}{\delta_1}\,\frac{|s-t|}{|t-x_l|}\,.$$

This proves the lemma also in the second case.

Finally, we consider $l = k_0$. On the one hand, we may fix $C_1 > 0$ such that $|f'(t)| \geq C_1|t - x_{k_0}|$. Indeed, for t in a neighbourhood of x_{k_0} this follows from the same estimates as in (6), noting that $|f''|$ is bounded away from zero close to x_{k_0}, and for t far from x_{k_0} it is trivial since $|f'|$ is bounded away from zero. On the other hand, letting C_2 be an upper bound for $|f''|$ on $[y_{k_0+1}, y_{k_0}]$, we have $|f'(s) - f'(t)| \leq C_2|s - t|$. Hence, the statement of the lemma holds, as long as $K_1 \geq C_2/C_1$. $\square$

4 Initial expansion

As already discussed in the Introduction, a main step in the proof of Theorem A is to ensure that every critical value $z_k(\mu) = f_\mu(x_k)$ in the ϵ-neighborhood of 0 exhibits expansion right from early iterates. In rough terms, the way we do this is by imposing, as a condition on the parameter μ, that all such iterates move away from the origin and, while doing this, avoid the regions where $|f'_\mu|$ is less than 1.

In fact, given a point $x \in [-\epsilon, \epsilon]$ and a parameter value $\mu \in [\epsilon, \epsilon]$, we denote by $j_0(x, \mu)$ the minimum value of $j \geq 1$ for which $|f^j_\mu(x)| > \epsilon$. Recall that the critical points of f are denoted x_k, with $|k| \geq k_0$. Then, we introduce a set G of parameters μ for which all the critical values $z_k(\mu)$ satisfy

1. the sequence $|f^j_\mu(z_k(\mu))|$ is an increasing function of $j \leq j_0(z_k(\mu), \mu)$;
2. the iterates $f^j_\mu(z_k(\mu))$ with $j < j_0(z_k(\mu), \mu)$ are all contained in the region where f_μ has derivative larger than 1.

Then we show that *for every $\delta_0 > 0$, there exists $\epsilon_0 > 0$ such that if $\epsilon < \epsilon_0$ then the relative measure of G in $[-\epsilon, \epsilon]$ is larger than $1 - \delta_0$.*

Let us state this in a somewhat more precise form. We fix some $\alpha < \gamma < 1$ and define $\phi(x) = |x|^\gamma$. Let

$$B_\mu = B_\mu(\epsilon) = \{x \in [-\epsilon, \epsilon] : |f_\mu(x)| \leq \phi(x) \text{ or } |x - x_k| \leq \hat{\tau}|x_k|\} \tag{7}$$

where $\hat{\tau}$ is a small positive constant: in particular, $(1 + \hat{\tau})x_{k_0} < \hat{y}$, so that $\{|x - x_k| \leq \hat{\tau}|x_k|\}$ is contained in the region where f coincides with $\hat{f}$, for every $|k| \geq k_0$. Define also

$$G_\mu = G_\mu(\epsilon) = \{x \in [-\epsilon, \epsilon] : f^j_\mu(x) \notin B_\mu \text{ for } 0 \leq j \leq j_0(x, \mu)\}. \tag{8}$$

It is easy to see that if $x \in G_\mu$ then $j_0(x, \mu)$ is finite. Indeed, by definition, $|f_\mu(x)| > |x|^\gamma \geq 0$ and

$$|f^j_\mu(x)| > |f_\mu(x)|^{\gamma^{j-1}} \text{ for all } j \leq j_0(x, \mu)$$

and so $|f_\mu^j(x)|$ must, eventually, be larger than ϵ. In what follows we restrict to $|\mu| > \epsilon^2$ and then even have that $j_0(x,\mu)$ admits a uniform upper bound independent of ϵ, μ, and $x \in G_\mu$. To prove this observe that for $|x| \geq (\epsilon^2/2a)^{1/\alpha}$ we have

$$|f_\mu(x)| \geq |x|^\gamma \geq (\epsilon^2/2a)^{\gamma/\alpha}$$

and for $|x| < (\epsilon^2/2a)^{1/\alpha}$

$$|f_\mu(x)| = |\mu + f(x)| > |\mu| - \epsilon^2/2 \geq \epsilon^2/2 \geq (\epsilon^2/2a)^{\gamma/\alpha}$$

(recall that $\gamma > \alpha$ and take ϵ small enough). It follows that

$$\epsilon \geq |f_\mu^j(x)| \geq (\epsilon^2/2a)^{(\gamma/\alpha)\gamma^{j-1}} \tag{9}$$

for all $j < j_0(x,\mu)$, and this implies our claim.

Finally, we define

$$G = G(\epsilon) = \{\mu : |\mu| \in [\epsilon^2, \epsilon] \text{ and } z_k(\mu) \in G_\mu \text{ for all } k_0 \leq k \leq \infty\}. \tag{10}$$

For any parameter $\mu \in G$ all the critical values leave the interval $(-\epsilon, \epsilon)$ in a finite, uniformly bounded number of iterates. In addition, as will be seen later, one has strong expansion during this initial part of the orbit. Let $\hat{\tau}$ be the constant introduced in (7).

Theorem 4.1 *We have*

$$\lim_{\epsilon \to 0} \frac{1}{2\epsilon} m(G) = 1 - o(\hat{\tau}),$$

where $o(\hat{\tau}) \to 0$ *when* $\hat{\tau} \to 0$.

The proof of this result has three main steps. First, in Lemma 4.2, we show that $B_\mu(\epsilon)$ is a small subset of the interval $[-\epsilon, \epsilon]$. Then it will follow, Lemma 4.3, that $G_\mu(\epsilon)$ is a large set. The final step is to pass this information to the parameter space. We use $C > 0$ to denote any large constant depending only on f, and not on ϵ.

Lemma 4.2 *There exists a function* $b(\epsilon)$ *such that*

$$m(B_\mu) \leq b(\epsilon) \text{ for all } 0 < \mu < \epsilon \quad \text{and} \quad \frac{b(\epsilon)}{\epsilon} \to 0 \text{ as } \epsilon \to 0.$$

Proof: It is clear that B_μ is the union of the sets B_k and $C_{k,\mu}$ defined by

$$B_k = [x_k - \hat{\tau}|x_k|, x_k + \hat{\tau}|x_k|]$$

and by

$$C_{k,\mu} = \{x\in[x_k, x_{k-1}] \,:\, |f_\mu(x)| \le \phi(x)\}.$$

Then

$$\sum_{|k|\ge k_0} m(B_k) \le \sum_{|k|\ge k_0} \hat\tau |x_k| \le C\hat\tau \sum_{|k|\ge k_0} \exp(-k\pi/\beta) \le C\hat\tau\epsilon. \tag{11}$$

Now we estimate the measure of $C_{k,\mu}$. Observe that if $x \in C_{k,\mu}$ then $f_\mu(x)$ belongs to the interval $[-\phi(x_{k-1}), \phi(x_{k-1})]$ whose length is bounded from above by $C|x_k|^\gamma$. On the other hand, from Lemma 3.2 it follows immediately that there exist open sets D_1 and D_2 covering $[-\epsilon, \epsilon] \setminus \{0\}$, such that

$$|f'_\mu(x)| \ge \frac{1}{C}|x|^{\alpha-1} \text{ for } x\in D_1 \quad \text{and} \quad |f''_\mu(x)| \ge \frac{1}{C}|x|^{\alpha-2} \text{ for } x\in D_2.$$

Therefore,

$$m(C_{k,\mu} \cap D_1) \le C\frac{|x_k|^\gamma}{|x_k|^{\alpha-1}} \le C|x_k|^{\gamma-\alpha+1}.$$

Analogously, using the lower bound on f''_μ,

$$m(C_{k,\mu} \cap D_2) \le C\sqrt{\frac{|x_k|^\gamma}{|x_k|^{\alpha-2}}} \le C|x_k|^{1+(\gamma-\alpha)/2}.$$

From these estimates we get

$$m(C_{k,\mu}) \le C|x_k|^{1+(\gamma-\alpha)/2} \tag{12}$$

and the lemma follows from (11) and (12). □

Lemma 4.3 *There exists a function $g(\epsilon)$ such that*

$$m(G_\mu) \ge g(\epsilon) \text{ for all } \mu \le \epsilon \quad \text{and} \quad \lim_{\epsilon\to 0} \frac{g(\epsilon)}{\epsilon} = 1 - o(\hat\tau),$$

where $o(\hat\tau) \to 0$ when $\hat\tau \to 0$

Proof: Let Π denote any finite sequence $\Pi = (p_0, p_1, \ldots, p_{s-1}, k)$, where $s > 0$, and p_i, k are integers with $|p_i| \ge k_0$ and $|k| \ge k_0$. We define $B_\mu(\Pi)$ to be the set of points $x \in [-\epsilon, \epsilon]$ such that

$$f^j_\mu(x) \in (x_{p_j}, x_{p_j-1}) \setminus (B_{p_j} \cup B_{p_{j-1}}) \text{ for } 0 \le j < s, \text{ and } f^s_\mu(x) \in B_k$$

and $C_\mu(\Pi)$ to be the set of points $x \in [-\epsilon, \epsilon]$ such that

$$f^j_\mu(x) \in (x_{p_j}, x_{p_j-1}) \setminus (B_{p_j} \cup B_{p_{j-1}}) \text{ for } 0 \le j < s, \text{ and } f^s_\mu(x) \in C_{k,\mu} \setminus (B_k \cup B_{k-1})$$

(throughout, replace (x_{p_j}, x_{p_j-1}) by (x_{p_j+1}, x_{p_j}) whenever $p_j < 0$). Recall that $B_k = [x_k - \hat{\tau}|x_k|, x_k + \hat{\tau}|x_k|]$. We claim that if $y \in (x_p, x_{p-1}) \setminus (B_p \cup B_{p-1})$ then

$$|f'_\mu(y)| \geq |x_p|^{\gamma-1}. \tag{13}$$

Indeed, for $y \in D_1$ we have $|f'_\mu(y)| \geq \frac{1}{C}|y|^{\alpha-1} \geq |x_p|^{\gamma-1}$ (use $\alpha < \gamma$ and $|y|$ is close to $|x_p| \leq \epsilon$, and take ϵ small) and for $y \in D_2$ we may argue

$$|f'_\mu(y)| \geq \frac{1}{C}|y|^{\alpha-2}\min\{|y - x_p|, |y - x_{p-1}|\} \geq \frac{\hat{\tau}}{C}|x_p|^{\alpha-2}|x_p| \geq |x_p|^{\gamma-1}$$

(supposing ϵ small with respect to $\hat{\tau}$). Now, using the claim, the remark that $B_\mu(\Pi)$ and $C_\mu(\Pi)$ are intervals, and (11), (12), we get

$$m(B_\mu(\Pi)) \leq \frac{2\hat{\tau}|x_k|}{\prod_{j=0}^{s-1}|x_{p_j}|^{\gamma-1}} \tag{14}$$

$$m(C_\mu(\Pi)) \leq C\frac{|x_k|^{1+(\gamma-\alpha)/2}}{\prod_{j=0}^{s-1}|x_{p_j}|^{\gamma-1}} \leq \frac{2\hat{\tau}|x_k|}{\prod_{j=0}^{s-1}|x_{p_j}|^{\gamma-1}} \tag{15}$$

(use $\gamma > \alpha$ and $|x_k| \leq \epsilon$, then suppose ϵ small with respect to $\hat{\tau}$). Since the sets $B_\mu(\Pi)$ and $C_\mu(\Pi)$ cover $[-\epsilon, \epsilon] \setminus G_\mu$, it follows that

$$\begin{aligned} m([-\epsilon,\epsilon] \setminus G_\mu) &\leq 2\hat{\tau} \sum_{|k|\geq k_0} \sum_{s\geq 0} \sum_{p_0,\ldots,p_{s-1}} |x_k| \prod_0^{s-1} |x_{p_j}|^{1-\gamma} \\ &\leq 2\hat{\tau} \sum_{|k|\geq k_0} |x_k| \left(\sum_{s\geq 0} \Big(\sum_{|p|\geq k_0} |x_p|^{1-\gamma} \Big)^s \right). \end{aligned}$$

Now,

$$\sum_{|p|\geq k_0} |x_p|^{1-\gamma} \leq C|x_{k_0}|^{1-\gamma} \leq C\epsilon^{1-\gamma}$$

and so

$$m([-\epsilon,\epsilon]\setminus G_\mu) \leq 2\hat{\tau} \sum_{|k|\geq k_0} |x_k| \sum_{s\geq 0} (C\epsilon^{1-\gamma})^s \leq C\hat{\tau} \sum_{|k|\geq k_0} \leq C\hat{\tau}\epsilon$$

This completes the proof of the lemma. □

Now we start the third, and final, step in the proof of Theorem 4.1. The key ingredient is to show that the critical values $z_l(\mu)$ vary faster than the endpoints of connected components of G_μ when the parameter μ varies. Combining this with the fact that the measure of G_μ is large, see the previous lemma, one concludes that *each* $z_l(\mu)$ is outside G_μ only for a small set of values of μ. In fact, we need to ensure that *all* the infinitely many critical values are in G_μ, for most values of μ.

In order to get that we take advantage of the fact that critical points accumulate at the origin.

We use dots to denote derivatives with respect to the parameter μ. First, we observe that $|\dot{z}_l(\mu)| = 1$. Next, we prove that if y_μ is an endpoint of some $C_\mu(\Pi)$, respectively some $B_\mu(\Pi)$, then $|\dot{y}_\mu| \leq 1/2$. Indeed, by construction, for such a y_μ there exists $j \leq s$ such that $x_\mu = f_\mu^j(y_\mu)$ is, either an endpoint of some B_l, or an endpoint of some $C_{l,\mu}$ that does not belong in any B_k. In the second case, $f_\mu(x_\mu) = \pm\phi(x_\mu)$ and so

$$\pm 1 + f_\mu'(x_\mu)\dot{x}_\mu = \pm\phi'(x_\mu)\dot{x}_\mu,$$

which implies

$$|\dot{x}_\mu| = \frac{1}{|f_\mu'(x_\mu) \pm \phi'(x_\mu)|}\ .$$

Since $x_\mu \notin (B_l \cup B_{l-1})$, the bound in (13) applies and hence $|f_\mu'(x_\mu)| \geq |x_\mu|^{\gamma-1}$. As $\phi'(x_\mu) = \gamma|x_\mu|^{\gamma-1}$, we find $|\dot{x}_\mu| \leq C|x_\mu|^{1-\gamma}$. Observe that this last estimate holds also in the first case, actually, $\dot{x}_\mu = 0$. Now, as $f_\mu^j(y_\mu) = x_\mu$, we get

$$\begin{aligned} \dot{x}_\mu &= \pm 1 + f_\mu'(f_\mu^{j-1}(y_\mu))\partial_\mu(f_\mu^{j-1}(y_\mu)) \\ &= \pm 1 \pm f_\mu'(f_\mu^{j-1}(y_\mu)) \pm (f_\mu^2)'(f_\mu^{j-2}(y_\mu)) \pm \ldots \pm (f_\mu^j)'(y_\mu)\dot{y}_\mu \end{aligned}$$

yielding

$$\dot{y}_\mu = \frac{\pm\dot{x}_\mu}{(f_\mu^j)'(y_\mu)} + \sum_{i=1}^{j} \pm \frac{1}{(f_\mu^i)'(y_\mu)}.$$

Using (13) once more, we obtain, for $1 \leq i \leq j$,

$$|(f_\mu^i)'(y_\mu)| \geq \prod_{h=0}^{i-1} |f_\mu^h(y_\mu)|^{\gamma-1} \geq \epsilon^{(1-\gamma)i}$$

(because each $f_\mu^h(y_u)$ belongs to some (x_p, x_{p-1}) with $|x_{p-1}| \leq \epsilon$). Therefore,

$$|\dot{y}_\mu| \leq \frac{C|x_\mu|^{1-\gamma}}{\epsilon^{j(\gamma-1)}} + \sum_{i=1}^{j} \epsilon^{i(1-\gamma)} \leq C\epsilon^{1-\gamma} \leq 1/2,$$

for any y_μ which is an endpoint of $B_\mu(\Pi)$ or $C_\mu(\Pi)$. Combining with $|\dot{z}_l(\mu)| = 1$ and (14), (15), we have (see Figure 5)

$$\begin{aligned} m(\{\mu : z_l(\mu) \in B_\mu(\Pi) \cup C_\mu(\Pi)\}) &\leq C\ m(B_\mu(\Pi) \cup C_\mu(\Pi)) \\ &\leq C \frac{\hat{\tau}|x_k|}{\prod_{j=0}^{s-1} |x_{p_j}|^{\gamma-1}}, \end{aligned} \tag{16}$$

for every l.

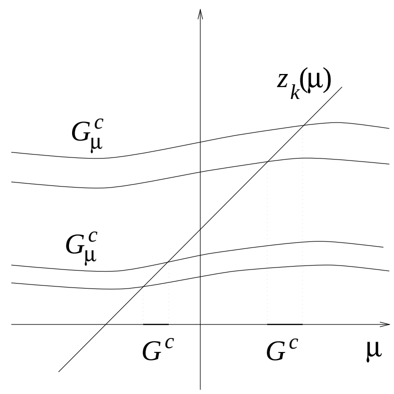

Figure 5: Dependence of G_μ and $z_k(\mu)$ on the parameter

This shows that, for each l, the set of parameters excluded from G because $z_l(\mu)$ does not belong to G_μ is small. Now we must deal with the fact that there are infinitely many critical values. To do that, we enlarge each $B_\mu(\Pi)$ to the interval $\hat{B}_\mu(\Pi)$ such that $B_\mu(\Pi)$ is the middle third of $\hat{B}_\mu(\Pi)$. We define $\hat{C}_\mu(\Pi)$ analogously. Given an interval I, we define

$$l_0(I) = \frac{-\beta}{\alpha\pi}\log(\frac{m(I)}{a\hat{x}^\alpha})$$

where $\hat{x}$ was introduced in (3). We also write

$$l_1 = \frac{-\beta}{\alpha\pi}\log\left(\frac{2\epsilon}{|f(\hat{x})|}\right)$$

and then we define $\hat{B}$ to be the union of the sets

$$\{\mu : z_l(\mu) \in \hat{B}_\mu(\Pi) \text{ for some } l_1 < |l| < l_0(B_\mu(\Pi)) \text{ or } l = \pm\infty\} \quad \text{and}$$

$$\{\mu : z_l(\mu) \in \hat{C}_\mu(\Pi) \text{ for some } l_1 < |l| < l_0(C_\mu(\Pi)) \text{ or } l = \pm\infty\}$$

over all possible Π. Observe, first of all, that $\mu \notin G$ implies $\mu \in \hat{B}$. Indeed, given $\mu \notin G$ let l and Π be such that $z_l(\mu) \in B_\mu(\Pi)$, respectively $z_l(\mu) \in C_\mu(\Pi)$. If $l_1 \leq |l| < l_0$ or $l = \pm\infty$ then $\mu \in \hat{B}$, by definition. If $|l| \geq l_0(B_\mu(\Pi))$ then

$$|z_l(\mu) - z_{\pm\infty}(\mu)| \leq a(\hat{x}e^{-|l|\pi/\beta})^\alpha \leq m(B_\mu(\Pi)).$$

This implies that $z_\infty(\mu) \in \hat{B}_\mu(\Pi)$, and so $\mu \in \hat{B}$. Analogously for $|l| \geq l_0(C_\mu(\Pi))$. Finally, if $|l| < l_1$ then $|z_l(0)| \geq 2\epsilon$ so that $|z_l(\mu)| \geq 2\epsilon - |\mu| \geq \epsilon$ and so $z_l(\mu) \notin B_\mu(\Pi) \cup C_\mu(\Pi)$, as claimed. On the other hand, recalling (16),

$$m(\hat{B}) \quad \leq \quad \sum_\Pi 2(l_0(B_\mu(\Pi)) - l_1) \cdot C\, m(B_\mu(\Pi))$$

$$+\sum_{\Pi} 2l_0(C_\mu(\Pi))\cdot C\, m(C_\mu(\Pi))$$

$$\leq \sum_{\Pi} C\left|\log\left(\frac{m(B_\mu(\Pi))}{\epsilon}\frac{|f(\hat{x})|}{2a\hat{x}^\alpha}\right)\right| m(B_\mu(\Pi))$$

$$+\sum_{\Pi} C\left|\log\left(\frac{m(C_\mu(\Pi))}{a\hat{x}^\alpha}\right)\right| m(C_\mu(\Pi)). \qquad (17)$$

Fix $\delta > 0$ small enough so that $(1+(\gamma-\alpha)/2)(1-\delta) > 1$. Using $x|\log x| \leq x^{1-\delta}$ and (15) we can bound the second sum by

$$C\sum_{\Pi}\left(\frac{|x_k|^{1+(\gamma-\alpha)/2}}{\prod_{j=0}^{s-1}|x_{p_j}|^{\gamma-1}}\right)^{1-\delta} \leq C\epsilon^{(1+(\gamma-\alpha)/2)(1-\delta)}$$

(because $|x_j| \leq \epsilon < 1$ and $\gamma < 1$). Moreover, using (14) we can bound the first sum in (17) by

$$\epsilon\sum_{\Pi} C\frac{m(B_\mu(\Pi))|f(\hat{x})|}{2a\hat{x}^\alpha\epsilon}\left|\log\left(\frac{m(B_\mu(\Pi))|f(\hat{x})|}{2a\hat{x}^\alpha\epsilon}\right)\right| \leq C\epsilon\sum_{\Pi}\left(\frac{\hat{\tau}|x_k|}{\epsilon\prod_{j=0}^{s-1}|x_{p_j}|^{\gamma-1}}\right)^{1-\delta}$$

$$\leq C\epsilon\left(\frac{\hat{\tau}|x_{k_0}|}{\epsilon}\right)^{1-\delta} \leq C\epsilon\hat{\tau}^{1-\delta}$$

Replacing these estimates in (17) we conclude that

$$\frac{m(G)}{2\epsilon} \geq 1-\frac{m(\hat{B})}{2\epsilon} \geq 1-C\epsilon^{(1+(\gamma-\alpha)/2)(1-\delta)-1}-C\hat{\tau}^{1-\delta} \geq 1-C\hat{\tau}^{1-\delta}$$

if ϵ is small with respect to $\hat{\tau}$ (recall the choice of δ). This concludes the proof of Theorem 4.1. □

Corollary 4.4 *Given $|k| \geq k_0$ and $\mu \in G(\epsilon)$, we have $|(f_\mu^j)'(z_k(\mu))| \geq \epsilon^{(\gamma-1)j}$ for every $1 \leq j \leq j_0(z_k(\mu),\mu)$.*

Proof: This follows directly from (13). □

From now on we shall be considering only parameter values in each of the connected components of G. Our arguments require that we restrict to connected components which are not too small, in the following sense. Given any connected component L of G, let

$$j_0(L,k) = \min\{j_0(z_k(\mu),\mu) : \mu \in L\}.$$

Observe that $|f_\mu^{j_0(L,k)}(z_k(\mu))| \geq |x_{k_0}|$ for all $\mu \in L$. In particular, $j_0(z_k(\mu),\mu) \leq j_0(L,k)+1$ for every μ in L. A connected component L of G is called *large* if

$$m(\{f_\mu^{j_0(L,k)}(z_k(\mu)) : \mu \in L\}) \geq \epsilon \quad \text{for all} \quad k_0 \leq |k| \leq \infty.$$

Our next step is to prove that the union of all the large connected components of G has Lebesgue measure close to 1 if $\hat{\tau}$ is small. We need the following auxiliary lemma.

Lemma 4.5 *Let* $c_1 = (1-\alpha)\sum_{j>o}\gamma^j$.

(i) If $x \in G_\mu$ *and* $s \leq j_0(x,\mu)$ *then*

$$|(f_\mu^s)'(x)| \leq [a(\alpha+\beta)]^s \prod_{j=0}^{s-1} |x|^{(\alpha-1)\gamma^j} \leq [a(\alpha+\beta)]^s |x|^{-c_1}$$

(ii) There exists $c_2 > 0$ *such that if* $\mu \in G$ *and* $|k| \geq c_2 \log\frac{1}{|\mu|}$ *then*

$$[z_{\pm\infty}(\mu), z_k(\mu)] \subset G_\mu \quad (\textit{ where } \pm \textit{ is the sign of } k).$$

Proof: Given $x \in G_\mu$ and $l < j_0(x,\mu)$, we have $|f_\mu^l(x)| \geq \phi^l(x) \geq |x|^{\gamma^l}$ and so

$$|f_\mu'(f_\mu^l(x))| \leq a|x|^{\gamma^l(\alpha-1)}(\alpha+\beta)$$

Part (i) follows immediately. Now we prove (ii), by contradiction. Supposing it is not true, let $0 \leq j < j_0(\mu, x_k)$ be the first integer for which $f_\mu^j([z_{\pm\infty}(\mu), z_k(\mu)])$ intersects B_μ. Observe that the endpoints of this interval are not in B_μ, by the definition of G, and so we must have

$$f_\mu^j([z_{\pm\infty}(\mu), z_k(\mu)]) \supset B_l \quad \text{or} \quad f_\mu^j([z_{\pm\infty}(\mu), z_k(\mu)]) \supset C_{l,\mu} \tag{18}$$

for some l. Let us consider the first case. Note that $|z_{\pm\infty}| = |\mu|$ and

$$|z_\infty(\mu) - z_k(\mu)| \leq C|x_k|^\alpha \leq Ce^{-k\pi\alpha/\beta} \leq Ce^{c_2\pi\alpha\log|\mu|/\beta} \leq \frac{|\mu|}{2},$$

as long as c_2 is large enough. It follows that $[z_{\pm\infty}(\mu), z_k(\mu)] \subset \{y : |y| \geq |\mu|/2\}$. Using our choice of j, we conclude that $f_\mu^j([z_{\pm\infty}(\mu), z_k(\mu)])$ is also contained in $\{y : |y| \geq \mu/2\}$. In particular, recall the definition of B_l, we must have

$$|f_\mu^j(z_{\pm\infty}(\mu)) - f_\mu^j(z_k(\mu))| \geq m(B_l) \geq \frac{1}{C}\hat{\tau}|\mu| > \mu^2 \tag{19}$$

(if $|\mu| \leq \epsilon$ is small with respect to $\hat{\tau}$). On the other hand, by part (i) together with the fact that $[z_{\pm\infty}(\mu), z_k(\mu)] \subset \{y : |y| \geq \mu/2\}$,

$$|f_\mu^j(z_{\pm\infty}(\mu)) - f_\mu^j(z_k(\mu))| \leq [a(\alpha+\beta)]^j |\mu|^{-c_1} Ce^{-k\pi\alpha/\beta}. \tag{20}$$

Since $z_{\pm\infty}(\mu) \in G_\mu$,

$$|\mu|^{\gamma^j} = |z_{\pm\infty}(\mu)|^{\gamma^j} \leq |f_\mu^j(z_{\pm\infty}(\mu))| \leq \epsilon.$$

Therefore, $\gamma^j \geq (\log \epsilon / \log |\mu|)$, which implies

$$[a(\alpha+\beta)]^j \leq \left(\frac{\log|\mu|}{\log \epsilon}\right)^{\frac{\log[a(\alpha+\beta)]}{\log(1/\gamma)}}.$$

Replacing in (20), we find

$$|f_\mu^j(z_\infty(\mu)) - f_\mu^j(z_k(\mu))| \leq \left(\frac{\log|\mu|}{\log \epsilon}\right)^{\frac{\log[a(\alpha+\beta)]}{\log \gamma^{-1}}} \frac{1}{C}|\mu|^{-c_1}|\mu|^{c_2\frac{\alpha\pi}{2\beta}} < \mu^2 \tag{21}$$

if $c_2 > 0$ is taken sufficiently large. This contradicts (19) and so we have discarded the first case in (18). Now we consider the second possibility. Note that we may suppose that $C_{l,\mu}$ is disjoint from all the B_i for otherwise we fall in the previous case. Let y stand for any of the extreme points of $C_{l,\mu}$; by the previous arguments $|y| \geq C^{-1}|\mu|$. Then $m(f_\mu(C_{l,\mu})) \geq C^{-1}y^\gamma$ (because of the disjointness assumption) and $|f'_\mu|C_{l,\mu}| \leq C|y|^{\gamma-1}$. Therefore,

$$m(C_{l,\mu}) \geq C^{-1}|y|^{1+\gamma-\alpha} \geq C^{-1}|\mu|^{1+\gamma-\alpha}$$

From this point on the argument proceeds in precisely the same way as in the first case. □

Now we prove that the measure of the union $\widetilde{G} = \widetilde{G}(\epsilon)$ of all the large connected components of G admits the same kind of lower estimate as we obtained for G in Theorem 4.1.

Theorem 4.6 *We have*

$$\lim_{\epsilon \to 0} \frac{1}{2\epsilon} m(\widetilde{G}) = 1 - o(\hat{\tau})$$

with $o(\hat{\tau}) \to 0$ *as* $\hat{\tau} \to 0$.

Proof: Fix $|k| \geq k_0$ and, given any component L of G, associate to (L,k) the sequence $(s, p_0, ..., p_{s-1}, \pm)$ defined as follows:

1. $s = j_0(L,k)$
2. $f_\mu^j(z_k(\mu)) \in (x_{p_j}, x_{p_{j-1}})$ for each $0 \leq j < s$
3. $\pm$ is the sign of $f_\mu^s(z_k(\mu))$.

Clearly, the map $(L,k) \to (s, p_0, ...p_{s-1}, \pm)$ is injective. Note also that

$$|\partial_\mu(f_\mu^s(z_k(\mu)))| \geq C^{-1} \prod_0^{s-1} |x_{p_i}|^{\gamma-1} \quad \text{forall} \mu \in L. \tag{22}$$

Indeed, by (13),

$$|(f_\mu^j)'(x)| \geq \prod_{i=0}^{j-1} |x_{p_i}|^{\gamma-1} \geq \epsilon^{-j(1-\gamma)} \tag{23}$$

for all $1 \leq j \leq s$ and then a standard calculation, see e.g. Lemma 5.4 below, gives that $|\partial_\mu(f_\mu^s(z_k(\mu)))|$ is comparable to $|(f_\mu^j)'(x)|$. Now, if L is not a large component then we must have

$$m(\{f_\mu^s(z_k(\mu)) : \mu \in L\}) \leq 3\epsilon \tag{24}$$

for some k with $|k| < c_2 \log(1/\epsilon)$ or $|k| = \infty$. To see this, it suffices to note that if $m(\{f_\mu^s(z_l(\mu)) : \mu \in L\}) < \epsilon$ for some l with $c_2 \log(1/\epsilon) \leq |l| < \infty$ then (24) holds for $k = \pm\infty$, since

$$|f_\mu^j(z_l(\mu)) - f_\mu^j(z_\infty(\mu))| \leq \mu^2 \leq \epsilon,$$

by (19). Then, using (22) we get

$$m(L) \leq \frac{3\epsilon}{C \prod_{i=0}^{s-1} |x_{p_i}|^{\gamma-1}} \leq C\epsilon \prod_{i=0}^{s-1} |x_{p_i}|^{1-\gamma}.$$

Summing over all possible sequences $\Pi = (s, p_0, ...p_{s-1}, \pm)$, and all the values of k as in (24), we conclude that

$$m(G \setminus \tilde{G}) \leq 2c_2 \log(1/\epsilon) \sum_\Pi C\epsilon \prod_{i=0}^{s-1} |x_{p_i}|^{1-\gamma}.$$

Thus, reasoning as in Lemma 4.3,

$$m(G \setminus \tilde{G}) \leq C\epsilon \log \epsilon^{-1} \sum_{s \geq 1} \Big(\sum_{p \geq k_0} |x_p|^{1-\gamma} \Big)^s \leq C\epsilon^{2-\gamma} \log \epsilon^{-1} \tag{25}$$

and the theorem follows. □

5 The inductive step: the partition algorithm

From now on we consider only parameter values $\mu \in \tilde{G}$. For every such μ all the critical values $z_k \in [-\epsilon, \epsilon]$ exhibit expanding behaviour up to the iterate when they leave $[-\epsilon, \epsilon]$. Afterwards, the orbit of z_k remains (for a period of time which can be made arbitrarily large by diminishing ϵ and μ) in the region $S^1 \setminus [-\epsilon, \epsilon]$, where the map is expanding. At a later iterate n, the orbit of z_k may approach some critical point x_l and then loose expansion. We adapt the "binding period" argument of [BC, Section 2] to our class of maps, to prove that this loss is completely compensated for in iterates subsequent to n, during which the orbits of $f^n(z_k)$ and x_l remain close to each other. In fact, this already involves an assumption on the parameters: those for which $f^n_\mu(z_k)$ is too close to x_l are excluded.

In doing these parameter exclusions we try to think of each critical value z_k as being independent from the others. More precisely, we take $\tilde{G}$ as the parameter space for every z_k, and at each time n we try to exclude a subset of $\tilde{G}$ depending only on the behaviour of the trajectory of z_k up to time n. There are two types of exclusions, corresponding to two conditions that we state in (31) and (30). We shall denote $E'_{k,n}$ and $E''_{k,n}$ the corresponding subsets of excluded parameters. Then the total excluded set is

$$E = \bigcup_k \bigcup_n (E'_{k,n} \cup E''_{k,n}) \tag{26}$$

and we show that

$$m(E) \le \sum_k \sum_n (m(E'_{k,n}) + m(E''_{k,n})) < m(\tilde{G}). \tag{27}$$

Consequently, a positive Lebesgue measure set $S = \tilde{G} \setminus E$ of parameters remains after all the exclusions, and we prove that conclusions 1., 2., 3. of Theorem A hold for every μ in S.

However, there is one instance where the behaviour of z_k can not be considered separately from the other critical points, and that is when an iterate $f^n_\mu(z_k)$ comes close to a different critical point x_l, for some parameter μ. Indeed, for the binding period argument mentioned above, we need to have information on the orbit of $z_l = f_\mu(x_l)$ up to time $n-1$, for that same parameter value μ. If μ happened to be excluded for z_l at some previous time, that is, if

$$\mu \in E'_{l,m} \cup E''_{l,m} \qquad \text{for some } m < n,$$

then such information is not available at stage n, and the binding period argument can not be carried out. In this situation, we are forced to exclude μ also for z_k at time n. We shall denote $E'''_{k,n}$ the set of parameters excluded in this fashion. This means that the total excluded set is, in fact, given by

$$E = \bigcup_k \bigcup_n (E'_{k,n} \cup E''_{k,n} \cup E'''_{k,n}).$$

It is important to point out that this last relation is totally coherent with (26). Indeed, by construction, each set $E'''_{k,n}$ must be contained in $\cup_l \cup_{m<n} (E'_{l,m} \cup E''_{l,m})$, and so

$$\bigcup_k \bigcup_n (E'_{k,n} \cup E''_{k,n}) = \bigcup_k \bigcup_n (E'_{k,n} \cup E''_{k,n} \cup E'''_{k,n}).$$

In particular, the bounds in (27) remain valid. In other words, *this third type of exclusions need not be taken in consideration explicitly when estimating the measure of the total excluded set.*

There is another point worth emphasizing here, related to the existence of infinitely many critical points: although distances close to each critical point x_l are naturally scaled (exponentially) according to the value of l, the bound period argument forces us to use a same scale for all the x_l. Observe also that one can not use the fact that the critical points accumulate on zero to try to reduce this situation to the case of a finite number of critical points: due to the behaviour of our maps near the origin, all the x_l must be taken into account.

The whole argument is done by induction, and in the present section we describe the inductive step. Beforehand, we construct a convenient partition $\{R(l,s,j)\}$ of the phase space into subintervals, with a bounded distortion property: trajectories with the same itinerary with respect to this partition have derivatives which are comparable, up to a multiplicative constant. This is done as follows. Let $l \geq k_0$ and take $y_l \in (x_l, x_{l-1})$ as defined in Section 3: x_l is the middle point of (y_{l+1}, y_l). Partition (x_l, y_l) into subintervals

$$R(l,s) = (x_l + e^{-(\pi/\beta)s}(y_l - x_l), x_l + e^{-(\pi/\beta)(s-1)}(y_l - x_l)), \qquad s \geq 1.$$

Then denote $R(l,-s)$ the subinterval of (y_{l+1}, x_l) symmetrical to $R(l,s)$ with respect to x_l. Now, subdivide $R(l,\pm s)$ into $(l+s)^3$ intervals $R(l,\pm s, j)$, $1 \leq j \leq (l+s)^3$ with equal length. Moreover, perform entirely symmetric constructions for $l \leq -k_0$. Let $R(\pm k_0, 1, 1)$ be the intervals having $\pm\epsilon$ in their boundaries. Clearly, we may suppose that $R(\pm k_0, 1, 1)$ are contained in the region $S^1 \setminus [-\tilde{y}, \tilde{y}]$ where f coincides with $\tilde{f}$, and so $|f'| > \sigma_0 > 1$. Finally, for completeness, set $R(0,0,0) = R(0,0) = S^1 \setminus [-\epsilon, \epsilon]$.

Then this induces a sequence of partitions $\mathcal{P}_{k,n}$, $n \geq 0$, in the parameter space of each critical value z_k: the orbits of every $z_k(\mu)$ have a same itinerary up to time n, for all μ in a same atom ω of $\mathcal{P}_{k,n}$. Most important, the derivative with respect to the parameter of $f^n_\mu(z_k(\mu))$, $\mu \in \omega \in \mathcal{P}_{k,n}$, has a uniformly bounded distortion property. The precise definition of these $\mathcal{P}_{k,n}$ is given by the algorithm presented in Steps 1–29 below, which also defines a number of other important objects, such as $E_{k,n}$ = set of parameters excluded for z_k at time n, $S_{k,n}$ = set of parameters remaining nonexcluded for z_k after n iterates, and $p(n,\omega,k)$ = binding time for z_k after time n for parameters μ in an interval $\omega \subset \widetilde{G}$. We fix $1 < \sigma < \sqrt{\sigma_0}$ and we also introduce small constants $\rho > 0$ and $\tau > 0$: just how small they are is determined by conditions stated after equations (38), (39), (49), (50), (53), and (54).

1. For each k with $|k| \geq k_0$ let $E_{k,0} = \emptyset$, $S_{k,0} = \widetilde{G}$, and $\mathcal{P}_{k,0}$ be the family of connected components of $\widetilde{G}$. Moreover, for each $L \in \mathcal{P}_{k,0}$, set $p(0, L, k) = 0$.

2. For each $n \geq 1$ perform Steps 3–29 below.

3. For each k with $|k| \geq k_0$ perform Steps 4–29 below.

4. Set $E_{k,n} = \emptyset$, $S_{k,n} = S_{k,n-1}$, $\mathcal{P}_{k,n} = \emptyset$. Moreover, define $\Psi_{k,n} : \widetilde{G} \to S^1$ by

$$\Psi_{k,n} : \mu \mapsto f_\mu^n(z_k(\mu)).$$

   The values of $E_{k,n}$, $S_{k,n}$, and $\mathcal{P}_{k,n}$ will be modified in the course of the algorithm.

5. For each interval $\omega \in \mathcal{P}_{k,n-1}$ let L be the connected component of $\widetilde{G}$ containing ω, then perform Steps 6–29 below.

6. **Question:** Is n smaller that $j_0(L, k)$?

   If YES go to Step 7. If NO go to Step 8.

7. Define $p(n, \omega, k) = 0$ and replace $\mathcal{P}_{k,n}$ by $\mathcal{P}_{k,n} \cup \{\omega\}$.

   In other words, ω is also an atom of the new partition.

   Go back to Step 5, for another $\omega \in \mathcal{P}_{k,n-1}$.

8. **Question:** Is there any $0 < m < n$ such that $n \leq m + p(m, \omega, k)$?

   I.e., does n belong in a binding period initiated at some previous iterate ?

   If YES go to Step 9. If NO go to Step 10.

9. Define $p(n, \omega, k) = 0$ and replace $\mathcal{P}_{k,n}$ by $\mathcal{P}_{k,n} \cup \{\omega\}$. Go back to Step 5, for another $\omega \in \mathcal{P}_{k,n-1}$.

10. **Question:** Does $\Psi_{k,n}(\omega)$ fully contain some partition interval $\{R(l, s, j)\}$?

    If YES go to Step 19. If NO go to Step 11.

    The YES case means that the image of ω under $\Psi_{k,n}$ is too big for the bounded distortion property to be valid also up to time n, thus ω must be further decomposed into subintervals. Lemma 5.3(b) implies that this first case must eventually happen.

    Decomposition of ω is not necessary in the NO case. $\Psi_{k,n}(\omega)$ intersects at most two intervals $R(l, s, j)$. Lemma 5.2 implies that the parameter condition (31) is automatically satisfied by ω.

11. **Question:** Does $\Psi_{k,n}(\omega)$ intersect $R(0,0,0)$?

    If YES go to Step 12. If NO go to Step 13.

    In the YES case $\Psi_{k,n}(\omega)$ *is contained in* $R(0,0,0) \cup R(\pm k_0, 1, 1)$.

12. Define $p(n,\omega,k) = 0$ and replace $\mathcal{P}_{k,n}$ by $\mathcal{P}_{k,n} \cup \{\omega\}$. Go back to Step 5, for another $\omega \in \mathcal{P}_{k,n-1}$.

13. Fix (l,s,j) such that $\Psi_{k,n}(\omega)$ intersects $R(l,s,j)$, with s maximum.

14. **Question:** Is

$$e^{-(\pi/\beta)s}\frac{1-e^{-(\pi/\beta)}}{1+e^{-(\pi/\beta)}} \geq \hat{\tau} \quad ? \tag{28}$$

    If YES go to Step 15. If NO go to Step 16.

    The YES case means that the nth iterate of $z_k(\mu)$ *is far away from the critical point* x_l*: in view of the definitions of* y_l *and* $R(l,s)$*, the inequality (28) implies*

$$|f_\mu^n(z_k(\mu)) - x_l| \geq e^{-(\pi/\beta)s}(y_l - x_l) \geq e^{-(\pi/\beta)s}\frac{1-e^{-(\pi/\beta)}}{1+e^{-(\pi/\beta)}}|x_l| \geq \hat{\tau}|x_l|.$$

    In this case there is no expansion loss at time n, and so no binding is required.

15. Define $p(n,\omega,k) = 0$ and replace $\mathcal{P}_{k,n}$ by $\mathcal{P}_{k,n} \cup \{\omega\}$. Go back to Step 5, for another $\omega \in \mathcal{P}_{k,n-1}$.

16. **Question:** Does there exist $\mu_0 \in \omega$ satisfying

    (i) $|(f_{\mu_0}^h)'(f_{\mu_0}(x_l))| \geq \sigma^h$ for all $1 \leq h \leq n-1$ and

    (ii) for every $j_0(x_l,\mu_0) \leq h \leq n-1$, either

$$|f_{\mu_0}^h(f_{\mu_0}(x_l))| > \epsilon \quad \text{or} \quad |f_{\mu_0}^h(f_{\mu_0}(x_l)) - x_{m(h)}| \geq e^{-\rho h},$$

    where $x_{m(h)}$ is the critical point nearest to $f_{\mu_0}^h(f_{\mu_0}(x_l))$?

    If YES go to Step 18. If NO go to Step 17.

    The NO case means that all the parameters in ω *were already excluded for* x_l *at some previous time. As we already commented at the beginning of this section, binding may not be possible, and we are forced to exclude the whole interval* ω *for* z_k *at time n.*

17. Replace $E_{k,n}$ by $E_{k,n} \cup \omega$ and $S_{k,n}$ by $S_{k,n} \setminus \omega$. Go back to Step 5, for another $\omega \in \mathcal{P}_{k,n-1}$.

18. Define the binding period $p(n, \omega, k)$ of z_k at time n to be the largest integer $p \geq 0$ such that

$$
\begin{array}{c}
|f_{\mu_0}^h(x_l)| \leq \epsilon \quad \text{and} \quad |f_\mu^{h+n}(z_k(\mu)) - f_{\mu_0}^h(x_l)| \leq |f_{\mu_0}^h(x_l) - x_{m(h-1)}| e^{-\tau h} \\
\text{or} \\
|f_{\mu_0}^h(x_l)| > \epsilon \quad \text{and} \quad |f_\mu^{h+n}(z_k(\mu)) - f_{\mu_0}^h(x_l)| \leq \epsilon^{1+\tau} e^{-\tau h}
\end{array}
\tag{29}
$$

for all $1 \leq h \leq p$ and $\mu \in \omega$. Replace $\mathcal{P}_{k,n}$ by $\mathcal{P}_{k,n} \cup \{\omega\}$. Go back to Step 5, for another $\omega \in \mathcal{P}_{k,n-1}$.

19. Define

$$B(n, \omega, k) = \sum_{j=1}^{n-1} p(j, \omega, k)$$

and consider the following condition on the parameter μ:

$$B(n, \omega, k) < \frac{n}{2}. \tag{30}$$

Question: Does (30) hold ?

If YES go to Step 21. If NO go to Step 20.

The NO case may be thought of as meaning that the first n iterates of $z_k(\mu)$ are too often close to critical points. In this case ω is excluded for z_k at time n.

20. Replace $E_{k,n}$ by $E_{k,n} \cup \omega$ and $S_{k,n}$ by $S_{k,n} \setminus \omega$. Go back to Step 5, for another $\omega \in \mathcal{P}_{k,n-1}$.

21. Decompose ω into subintervals ω' such that each $\Psi_{k,n}(\omega')$ contains some $R(l, s, j)$ with $(l, s, j) \neq (0, 0, 0)$, and is contained in the union of at most three such partition intervals.

22. For each interval $\omega' \subset \omega$ as before, define $p(m, \omega', k) = p(m, \omega, k)$ for every $m < n$, fix (l, s, j) as above with s maximum, and perform Steps 23-29 below.

23. Consider the following condition on the parameter μ:

$$\text{either} \quad e^{-(\pi/\beta)(|l|+|s|)} \geq e^{-\rho n} \quad \text{or} \quad (l, s, j) = (\pm k_0, 1, 1). \tag{31}$$

Question: Does (31) hold ?

If YES go to Step 25. If NO go to Step 24.

In the YES case we perform Steps 25–29, which are analogs of Steps 14–18 with ω' in the place of ω.

The NO case means that the nth iterate of $z_k(\mu)$ is very close to the critical point x_l, for $\mu \in \omega$. Then ω is excluded for z_k at time n.

24. Replace $E_{k,n}$ by $E_{k,n} \cup \omega'$ and $S_{k,n}$ by $S_{k,n} \setminus \omega'$. Go back to Step 22, for another $\omega' \subset \omega$.

25. **Question:** Does (28) hold ?

    If YES go to Step 26. If NO go to Step 27.

26. Define $p(n, \omega', k) = 0$ and replace $\mathcal{P}_{k,n}$ by $\mathcal{P}_{k,n} \cup \{\omega'\}$. Go back to Step 22, for another $\omega' \subset \omega$.

27. **Question:** Does there exist $\mu_0 \in \omega'$ satisfying conditions (i) and (ii) as in Step 16 ?

    If YES go to Step 29. If NO go to Step 28.

28. Replace $E_{k,n}$ by $E_{k,n} \cup \omega'$ and $S_{k,n}$ by $S_{k,n} \setminus \omega'$. Go back to Step 22, for another $\omega' \subset \omega$.

29. Define the binding period $p(n, \omega', k)$ of z_k at time n to be the largest integer $p \geq 0$ such that (29) holds for all $1 \leq h \leq p$ and $\mu \in \omega'$. Replace $\mathcal{P}_{k,n}$ by $\mathcal{P}_{k,n} \cup \{\omega'\}$. Go back to Step 22, for another $\omega' \subset \omega$.

The next lemmas contain crucial estimates for our inductive argument. We shall prove them, by induction on n, in the forthcoming sections. As we shall see, the proofs are closely intertwined: at each step in the proof of one of the lemmas we assume that all of them hold for all previous times.

Lemma 5.1 *For all $\mu \in \omega \in \mathcal{P}_{k,n}$, we have $|(f_\mu^n)'(z_k(\mu))| \geq \sigma^n$.*

Lemma 5.2 *In the context of Step 10, let (l, s, j) be such that $\Psi_{k,n}(\omega)$ intersects $R(l, s, j)$. Then, either*

$$e^{-(\pi/\beta)(|l|+|s|)} \geq e^{-\rho n} \quad \text{or} \quad (l, s, j) \in \{(-k_0, 1, 1), (0, 0, 0), (k_0, 1, 1)\}.$$

Lemma 5.3 *Suppose $p = p(n, \omega, k) > 0$. Then, for some large constant $A_0 > 0$ to be fixed in (32) below,*

(a) $p \leq \frac{2\rho}{\log \sigma} n < n$.

(b) $|(f_\mu^{p+1})'(f_\mu^n(z_k(\mu)))| \geq A_0 \sigma^{(p+1)/3} > 1$.

Lemma 5.4 *There exists an absolute constant A_1 such that, for every $\mu \in \omega$,*

$$\frac{1}{A_1} \leq \left| \frac{\dot{\Psi}_{k,n}(\mu)}{(f_\mu^n)'(z_k(\mu))} \right| \leq A_1.$$

Lemma 5.5 *There exists an absolute constant A_2 such that, for all $\mu_1, \mu_2 \in \omega$,*

$$\frac{1}{A_2} \le \frac{|(f^n_{\mu_1})'(z_k(\mu_1))|}{|(f^n_{\mu_2})'(z_k(\mu_2))|} \le A_2 \quad \text{and} \quad \frac{1}{A_2} \le \frac{|\dot{\Psi}_{k,n}(\mu_1)|}{|\dot{\Psi}_{k,n}(\mu_2)|} \le A_2.$$

Recall that $(f^n_\mu)'$ is the derivative of the map f^n_μ with respect to the phase-space variable, whereas $\dot{\Psi}_{k,n}$ denotes the derivative of $\Psi_{k,n}$ with respect to μ.

As we announced before, there are three instances were parameters are excluded. We denote by $E'_{k,n}$ the set of parameters excluded in Step 24, by $E''_{k,n}$ the set of parameters excluded in Step 20, and by $E'''_{k,n}$ the set of parameters excluded in Steps 17 and 28. Recall also that for the reasons explained before, this third type of exclusions may be disregarded when estimating the measure of the total excluded set.

Lemma 5.6 *There are absolute constants $B_1 > 0$ and $b_1 > 0$, independent of n and k, such that*

$$m(E'_{k,n} \cap L) \le B_1 e^{-b_1 n} m(L)$$

for any connected component L of $\tilde{G}$.

Lemma 5.7 *There are absolute constants $B_2 > 0$ and $b_2 > 0$, independent of n and k, such that*

$$m(E''_{k,n} \cap L) \le B_2 e^{-b_2 n} m(L)$$

for any connected component L of $\tilde{G}$.

In Sections 6, 7, and 8 we prove Lemmas 5.1 through 5.7. We fix k, and write $\Psi_n = \Psi_{k,n}$. We also write $p(n) = p(n, \omega, k)$, since it is never ambiguous to which partition interval ω we are referring.

As already mentioned, the proof is by induction: in getting each lemma we assume that all of them have already been established at all previous iterates, and for each critical value z_l. The inductive step relies on the control of the distances of orbits to the set of critical points provided by assumptions (30) and (31). One consequence is that these methods, to be exposed in the next sections, are robust under modifications of the initial point leaving distances to the critical set essentially unchanged. Let us explain this feature in more precise terms, since it will be useful in the last section of the paper. Take $n \ge 1$, z_l a critical point, and w any point such that, for each $1 \le j \le n$, the distance from $f^j_\mu(w)$ to $f^j_\mu(z_l)$ is much smaller than the distance from $f^j_\mu(z_l)$ to the critical set (for instance, the former is smaller than the square of the latter). Then the whole construction described in the algorithm above can be carried out for w, up to time n. Condition (31) for z_l implies (31) for w, with the right hand side multiplied by an inocuous factor $1/2$. Defining the binding periods of w to coincide with those of z, the estimates in Lemma 5.3 remain true for w. Moreover, (30) for w is then an immediate consequence of (30) for z_l. This ensures that all the conclusions of the previous lemmas remain valid for such a point w, up to time n.

6 Returns and binding times

In this section we prove Lemmas 5.2 and 5.3. Let us start by introducing some notation, and also a useful estimate given in Lemma 6.1. By the definition of $R(l,s,j)$ in the previous section,

$$|R(l,s,j)| = a_1 \frac{e^{-(\pi/\beta)(|l|+|s|)}}{(|l|+|s|)^3} \quad \text{and} \quad \text{dist}(R(l,s,j),x_l) \geq a_2 e^{-(\pi/\beta)(|l|+|s|)}$$

where $|R|$ denotes the length of the interval R,

$$a_1 = \hat{x}\frac{e^{(\pi/\beta)}-1)^2}{e^{(\pi/\beta)}+1} \quad \text{and} \quad a_2 = \hat{x}\frac{e^{(\pi/\beta)}-1}{e^{(\pi/\beta)}+1}.$$

The next lemma asserts that orbits leaving $[-\epsilon,\epsilon]$ remain expanding during a number of iterates m_0 which can be made arbitrarily large by reducing ϵ and μ. Recall that $|\tilde{f}'| > \sigma_0 > 1$ and that f_μ is C^1-close to $\tilde{f}$ outside $[-\tilde{y},\tilde{y}]$ if μ is small.

Lemma 6.1 *There exist $c>0$ and $m_0 \geq c\log(1/\epsilon)$ such that*

$$f_\mu^{j_0+i}(z_k(\mu)) \notin [-\tilde{y},\tilde{y}] \quad \text{and} \quad |f'_\mu(f_\mu^{j_0+i}(z_k(\mu)))| \geq \sigma_0,$$

for all $1 \leq i \leq m_0$ and all $\mu \in [-\epsilon,\epsilon]$.

Proof: Observe that $|f(z)| = |\hat{f}(z)| \leq C\epsilon^\alpha$ for all $z \in [-\hat{y},\hat{y}]$. Increasing the constant C if necessary, we also have $|f(z)| = |\tilde{f}(z)| \leq C\epsilon \leq C\epsilon^\alpha$ for all $z \in [-\epsilon,\epsilon] \setminus [-\tilde{y},\tilde{y}]$, recall that we took $\tilde{f}(0)=0$. Then $|f(z)| \leq C\epsilon^\alpha$ also for points in $[-\tilde{y},\tilde{y}] \setminus [-\hat{y},\hat{y}]$, because our map is monotone on the connected components of this last set. Recalling the definition of j_0, near the beginning of Section 4, we conclude that

$$\tilde{y} \leq |f_\mu^{j_0}(z_k(\mu))| \leq C\epsilon^\alpha.$$

Moreover, by construction, there exists $\delta_0 > 0$ such that $f_\mu(\pm[\tilde{y},\delta_0]) \cap [-\tilde{y},\tilde{y}] = \emptyset$. This δ_0 depends only on $\tilde{f}$ and so we may suppose $\epsilon << \delta_0$. Then we take $m_0 \geq 0$ to be maximum value of i for which $|f_\mu^{j_0+i}(z_k(\mu))| \leq \delta_0$. Let $\widetilde{K}$ be a strict upper bound for the derivative outside $[-\tilde{y},\tilde{y}]$. In view of the previous estimates, we get

$$(\widetilde{K})^{m_0+1}C\epsilon^\alpha \geq \delta_0 \quad \Rightarrow \quad m_0 \geq \frac{\log(\epsilon^{-\alpha}\delta_0/C)}{\log\widetilde{K}} - 1 \geq c\,\log\frac{1}{\epsilon}$$

for some $c > 0$ (since ϵ is much smaller than δ_0). The remaining claims in the lemma are now automatic. □

The main step in the proof of Lemma 5.2 is to find a convenient lower bound for the length of $\Psi_n(\omega)$. It will follow that $\Psi_n(\omega)$ can not intersect intervals $R(l, s, j)$ close to a critical point, that is, with $|l| + |s|$ large, without intersecting several such intervals, which would contradict the hypothesis.

Suppose first that ω is not a connected component of $\tilde{G}$. Then, it must have been created through interval decomposition as in Step 21 at some iterate n_i prior to n. Equivalently,

$$n_i = \min\{m : \omega \in \mathcal{P}_{k,m}\}.$$

By construction, $\Psi_{n_i}(\omega)$ contains some $R(l_i, s_i, j_i)$ with $e^{-(\pi/\beta)(|l_i|+|s_i|)} \geq e^{-\rho n_i}$ or else $(l_i, s_i, j_i) = (\pm k_0, 1, 1)$. Then

$$|\Psi_{n_i}(\omega)| \geq |R(l_i, s_i, j_i)| \geq a_1 \frac{e^{-(\pi/\beta)(|l_i|+|s_i|)}}{(|l_i| + |s_i|)^3} = a_1 g(e^{-(\pi/\beta)(|l_i|+|s_i|)})$$

where $g(t) = (\pi/\beta)^3 (t/\log^3(1/t))$ is an increasing function of t. Now we claim that

$$|\Psi_n(\omega)| \geq (1 + 2e^{(\pi/\beta)})|\Psi_{n_i}(\omega)|.$$

In order to prove this we note that, by construction, n does not belong to any binding period. Therefore, the time interval $[n_i, n)$ may be written as a union of (complete) binding periods $[l, l + p(l)]$, $n_i \leq l < n$, together with iterates outside $[-\epsilon, \epsilon]$. If $p(n_i) > 0$ then Lemma 5.3, together with the fact that f_μ is expanding in $S^1 \setminus [-\epsilon, \epsilon]$, yields

$$|(f_\mu^{n-n_i})'(f_\mu^{n_i}(z_k(\mu)))| \geq A_0 \sigma^{(n-n_i)/3} \geq A_0.$$

Now, using Lemma 5.4

$$\frac{|\Psi_n(\omega)|}{|\Psi_{n_i}(\omega)|} = \frac{|\Psi_n(\omega)|/|\omega|}{|\Psi_{n_i}(\omega)|/|\omega|} = \frac{|\dot{\Psi}_n(\mu)|}{|\dot{\Psi}_{n_i}(\nu)|} \geq \frac{1}{A_1^2} \frac{|(f_\mu^n)'(z_k(\mu))|}{|(f_\nu^{n_i})'(z_k(\nu))|}$$

for some μ, ν in ω. Combining with Lemma 5.5 we find

$$\frac{|\Psi_n(\omega)|}{|\Psi_{n_i}(\omega)|} \geq \frac{1}{A_1^2 A_2} \frac{|(f_\mu^n)'(z_k(\mu))|}{|(f_\mu^{n_i})'(z_k(\mu))|} \geq \frac{1}{A_1^2 A_2} |(f_\mu^{n-n_i})'(f_\mu^{n_i}(z_k(\mu)))|.$$

We fix

$$A_0 = (1 + 2e^{(\pi/\beta)}) A_1^2 A_2 \tag{32}$$

and the claim follows. If $p(n_i) = 0$ but $\Psi_{n_i}(\omega) \subset (-\hat{y}, \hat{y})$ then (steps 14-15 of the algorithm in Section 5)

$$|(f_\mu^{n-n_i})'(f_\mu^{n_i}(z_k(\mu)))| \geq \epsilon^{\alpha-1} >> A_0$$

and the argument proceeds as before. In the remaining case, $p(n_i) = 0$ and $\Psi_{n_i}(\omega)$ is not contained in $(-\hat{y}, \hat{y})$, necessarily, $n \geq n_i + m_0$ and so, by Lemma 6.1,

$$|(f_\mu^{n-n_i})'(f_\mu^{n_i}(z_k(\mu)))| \geq \sigma_0^{m_0} >> A_0$$

as long as ϵ is small enough. Hence, the claim holds also in this case.

Now the conclusion of lemma is easily deduced: let $(l, s, j) \neq (\pm k_0, 1, 1)$ be such that $\Psi_n(\omega)$ intersects $R(l, s, j)$. The hypothesis implies that $\Psi_n(\omega)$ is contained in the union of $R(l, s, j)$ with some adjacent interval of the partition. Hence,

$$\begin{aligned} g(e^{-(\pi/\beta)(|l|+|s|)}) &= \frac{1}{a_1}|R(l,s,j)| \geq \frac{1}{a_1(1+2e^{(\pi/\beta)})}|\Psi_n(\omega)| \geq \frac{1}{a_1}|\Psi_{n_i}(\omega)| \\ &\geq g(e^{-(\pi/\beta)(|l_i|+|s_i|)}). \end{aligned}$$

Recall that g is monotone. If $(l_i, s_i) = (\pm k_0, 1)$ then it must be $(l, s) = (\pm k_0, 1)$. On the other hand, if $(l_i, s_i) \neq (\pm k_0, 1)$ then $e^{-(\pi/\beta)(|l_i|+|s_i|)} \geq e^{-\rho n_i}$ and it follows that $e^{-(\pi/\beta)(|l|+|s|)} \geq e^{-\rho n_i} \geq e^{-\rho n}$.

We are left to consider the case when ω does coincide with a connected component L of $\widetilde{G}$. In this case we set $n_i = j_0(\omega, k)$. Then, by the definition of $\widetilde{G}$, $|\Psi_{n_i}(\omega)| \geq \epsilon$. In view of the structure of our algorithm, $n \geq n_i + m_0$. Then, using Lemma 6.1 as before,

$$|(f_\mu^{n-n_i})'(f_\mu^{n_i}(z_k(\mu)))| \geq \sigma_0^{m_0} >> A_0$$

and so $|\Psi_n(\omega)| >> \epsilon$. This implies that $(l, s, j) = (\pm k_0, 1, 1)$ in this case. The proof of Lemma 5.2 is complete. □

Now we start the proof of Lemma 5.3. In Steps 18 and 29, we defined the binding time $p(n)$ for partition intervals ω containing some $\mu_0 \in \omega$ for which $z_l(\mu_0)$ is σ-expanding and satisfies (31) up to time n. As before, we take (l, s, j), with s maximum, such that $\Psi_n(\omega)$ intersects $R(l, s, j)$. First we fix $\mu = \mu_0$ and claim that there exists $A_3 > 1$ such that

$$\frac{1}{A_3} \leq \left|\frac{(f_\mu^j)'(\xi)}{(f_\mu^j)'(f_\mu(x_l))}\right| \leq A_3 \tag{33}$$

for every $1 \leq j \leq \min\{p, n\}$ and every $\xi \in [f_\mu(x_l), f_\mu^{n+1}(z_k(\mu)]$. To prove this claim we let $\eta = f_\mu(x_l)$ and consider $0 \leq i < j$. There are two cases to treat, corresponding to the two possibilities in (29). If $|f_\mu^i(\eta)| \leq \epsilon$ then, by Lemma 3.3,

$$\left|\frac{f'(f_\mu^i(\xi)) - f'(f_\mu^i(\eta))}{f'(f_\mu^i(\eta))}\right| \leq C\frac{|f_\mu^i(\xi) - f_\mu^i(\eta)|}{|f_\mu^i(\eta) - x_{m(i-1)}|} \leq Ce^{-\tau i}.$$

If $|f_\mu^i(\eta)| > \epsilon$, then $|f_\mu^i(\xi) - f_\mu^i(\eta)| \le \epsilon^{1+\tau} e^{-\tau i} << \epsilon$ and so the interval bounded by $f_\mu^i(\xi)$ and $f_\mu^i(\eta)$ is contained in the region $S^1 \setminus [-\tilde{y}, \tilde{y}]$, where $f = \tilde{f}$. Thus,

$$\left| \frac{f'(f_\mu^i(\xi)) - f'(f_\mu^i(\eta))}{f'(f_\mu^i(\eta))} \right| \le C |f_\mu^i(\xi) - f_\mu^i(\eta)| \le C\epsilon^{1+\tau} e^{-\tau i} \le C e^{-\tau i}.$$

All in all,

$$\sum_{i=0}^{j-1} \left| \frac{f'(f_\mu^i(\xi)) - f'(f_\mu^i(\eta))}{f'(f_\mu^i(\eta))} \right| \le C \sum_{i=0}^{j} e^{-\tau i} \le C,$$

and (33) follows immediately.

To prove part (a) of the lemma, we use condition (31). As $p > 0$, we have $|f_\mu^n(z_k(\mu)) - x_l| \le \hat{\tau}|x_l| < \epsilon$. In particular, $(l, s, j) \ne (\pm k_0, 1, 1)$ and so

$$|f_\mu^n(z_k(\mu)) - x_l| \ge a_2 e^{-(\pi/\beta)(|l|+|s|)} \ge a_2 e^{-\rho n}. \tag{34}$$

Using second-order Taylor approximation we get

$$|f_\mu^{n+1}(z_k(\mu)) - f_\mu(x_l)| \ge \frac{1}{C} |f''(x_l)| (a_2 e^{-\rho n})^2 \ge \frac{1}{C} \epsilon^{\alpha-2} e^{-2\rho n}, \tag{35}$$

recall that $|f''(x_l)| \ge C^{-1}|x_l|^{\alpha-2} \ge C^{-1}\epsilon^{\alpha-2}$. Then, for each $j \le \min\{p, n\}$, there is some ξ between $f_\mu(x_l)$ and $f_\mu^{n+1}(z_k(\mu))$ such that

$$\begin{aligned} |f_\mu^{j+n+1}(z_k(\mu)) - f_\mu^{j+1}(x_l)| &= |(f_\mu^j)'(\xi)| |f_\mu^{n+1}(z_k(\mu)) - f_\mu(x_l)| \\ &\ge C^{-1}\epsilon^{\alpha-2} e^{-2\rho n} |(f_\mu^j)'(\xi)| \ge 2e^{-2\rho n}\sigma^j, \end{aligned} \tag{36}$$

as a consequence of (33). We use the induction information $|(f_\mu^j)'(f_\mu(x_l))| \ge \sigma^j$, cf. Lemma 5.1, and we suppose $\epsilon > 0$ small enough so that $\epsilon^{\alpha-2} > 2CA_3$. Hence,

$$2e^{-2\rho n}\sigma^j \le |f_\mu^{j+n+1}(z_k(\mu)) - f_\mu^{j+1}(x_l)| \le 2 \tag{37}$$

and then $e^{-2\rho n}\sigma^j \le 1$ for all $1 \le j \le \min\{p, n\}$. In particular,

$$-2\rho n + \log\sigma \, \min\{p, n\} \le 0, \quad \text{implying} \quad \min\{p, n\} \le \frac{2\rho n}{\log \sigma} < n,$$

thus proving (a).

Next we prove (b). First, for some ξ,

$$\begin{aligned} |f_\mu^{p+1}(x_l) - f_\mu^{p+n+1}(z_k(\mu))| &= |(f_\mu^p)'(\xi)| \cdot |f_\mu(x_l) - f_\mu^{n+1}(z_k(\mu))| \\ &\le C |(f_\mu^p)'(f_\mu^{n+1}(z_k(\mu)))| \, |f_\mu''(x_l)| \, |f_\mu^n(z_k(\mu)) - x_l|^2. \end{aligned}$$

Using the claim once again,

$$|(f_\mu^{p+1})'(f_\mu^n(z_k(\mu))|^2 = |(f_\mu^p)'(f_\mu^{n+1}(z_k(\mu)))|^2 \cdot |f_\mu'(f_\mu^n(z_k(\mu)))|^2$$

$$\geq C^{-1}|(f_\mu^p)'(f_\mu(x_l))| \cdot |(f_\mu^p)'(f_\mu^{n+1}(z_k(\mu)))| \cdot C^{-1}|f_\mu''(x_l)|^2 \cdot |f_\mu^n(z_k(\mu)) - x_l|^2$$

$$\geq C^{-1}|(f_\mu^p)'(f_\mu(x_l))| \cdot |f_\mu''(x_l)| \cdot |f_\mu^{p+1}(x_l) - f_\mu^{n+p+1}(z_k(\mu))|$$

$$\geq C^{-1}\sigma^p \epsilon^{\alpha-2}|f_\mu^{p+1}(x_l) - f_\mu^{n+p+1}(z_k(\mu))|.$$

We must distinguish two cases. If $|f_\mu^{p+1}(x_l)| > \epsilon$ then, by the definition of p in (29), we must have $|f_\mu^{p+1}(x_l) - f_\mu^{n+p+1}(z_k(\mu))| > \epsilon^{1+\tau}e^{-\tau(p+1)}$. Therefore,

$$|(f_\mu^{p+1})'(f_\mu^n(z_k(\mu))|^2 \geq C^{-1}\sigma^{p+1}\epsilon^{\alpha-1+\tau}e^{-\tau(p+1)} \geq A_0^2\sigma^{2(p+1)/3}, \tag{38}$$

as long as we fix $\tau < \min\{1-\alpha, \log\sigma/3\}$, and suppose ϵ small enough. On the other hand, if $|f_\mu^{p+1}(x_l)| \leq \epsilon$ then, by (29),

$$|f_\mu^{p+1}(x_l) - f_\mu^{n+p+1}(z_k(\mu))| > |f_\mu^{p+1}(x_l) - x_{m(p)}|e^{-\tau(p+1)}.$$

Since $p < n$, we may also invoke (31). Note that this gives two possibilities: either $|f_\mu^{p+1}(x_l)| - x_{m(p)}| \geq C^{-1}\epsilon$ or $|f_\mu^{p+1}(x_l)| - x_{m(p)}| \geq C^{-1}e^{-\rho p}$. In either case,

$$|f_\mu^{p+1}(x_l) - f_\mu^{n+p+1}(z_k(\mu))| > C^{-1}\epsilon\, e^{-(\rho+\tau)(p+1)}$$

and so

$$|(f_\mu^{p+1})'(f_\mu^n(z_k(\mu))|^2 \geq C^{-1}\sigma^{p+1}\epsilon^{\alpha-1}e^{-(\rho+\tau)(p+1)} \geq A_0^2\sigma^{2(p+1)/3} \tag{39}$$

as long as we take ϵ, ρ, and τ small enough. This proves the lemma for $\mu = \mu_0$.

Finally, consider an arbitrary $\mu \in \omega$. In (a) we have nothing to prove: by definition the binding period $p(n)$ is constant on ω. To prove (b), we begin by claiming that

$$\sum_{i=0}^{p} \left| \frac{f'(f_\mu^i(\xi_1)) - f'(f_{\mu_0}^i(\xi_2))}{f'(f_{\mu_0}^i(\xi_2))} \right| \tag{40}$$

is uniformly bounded, where $\xi_1 = f_\mu^n(z_k(\mu))$ and $\xi_2 = f_{\mu_0}^n(z_k(\mu_0))$. To prove this claim, we estimate each of the terms in the sum. Given any $0 < i \leq p$, there are two possibilities. If $|f_{\mu_0}^i(x_l)| > \epsilon$ then (29) implies

$$\left| \frac{f'(f_\mu^i(\xi_1)) - f'(f_{\mu_0}^i(\xi_2))}{f'(f_{\mu_0}^i(\xi_2))} \right| \leq C|f_\mu^i(\xi_1) - f_{\mu_0}^i(\xi_2)| \leq C\epsilon^{1+\tau}e^{-\tau i} \leq Ce^{-\tau i}.$$

If $|f_{\mu_0}^i(x_l)| \leq \epsilon$ then Lemma 3.3 together with (29) give

$$\left|\frac{f'(f_\mu^i(\xi_1)) - f'(f_{\mu_0}^i(\xi_2))}{f'(f_{\mu_0}^i(\xi_2))}\right| \leq C\frac{|f_\mu^i(\xi_1) - f_{\mu_0}^i(\xi_2)|}{|f_{\mu_0}^i(\xi_2) - x_{m(i-1)}|} \leq C\frac{2\,e^{-\tau i}}{1-e^{-\tau i}} \leq Ce^{-\tau i}.$$

Finally, if $i = 0$ then Lemma 3.3, together with the assumption that ξ_1 and ξ_2 are in the union of, at most, three intervals $R(l, s, j)$, yields

$$\left|\frac{f'(\xi_1) - f'(\xi_2)}{f'(\xi_2)}\right| \leq C\frac{|\xi_1 - \xi_2|}{|\xi_1 - x_l|} \leq C.$$

This proves that the sum in (40) is bounded by $\sum_{i=0}^{\infty} Ce^{-\tau i}$, thus proving our claim. As a consequence,

$$\frac{1}{C} \leq \left|\frac{(f_\mu^{p+1})'(f_\mu^n(z_k(\mu)))}{(f_{\mu_0}^{p+1})'(f_{\mu_0}^n(z_k(\mu_0)))}\right| \leq C$$

for some $C > 0$, which completes the proof of Lemma 5.3. $\square$

Let us remark the following useful fact about the binding period that also follows from the arguments in the proof of the previous lemma:

$$p \leq \frac{2\pi}{\beta \log \sigma}(|l| + |r|). \tag{41}$$

This can be deduced as follows. On the one hand, from (34) and (35),

$$|f_\mu^{n+1}(z_k(\mu)) - f_\mu(x_l)| \geq \frac{1}{C}\,\epsilon^{\alpha-2}|f_\mu^n(z_k(\mu)) - x_l|^2 \geq \frac{1}{C}\,\epsilon^{\alpha-2}e^{-2(\pi/\beta)(|l|+|s|)}.$$

On the other hand, as in (36) and (37)

$$2 \geq |f_\mu^{j+n+1}(z_k(\mu)) - f_\mu^{j+1}(x_l)| \geq \frac{1}{C}\,\sigma^j|f_\mu^{n+1}(z_k(\mu)) - f_\mu(x_l)|.$$

Combining the two bounds and supposing ϵ small, we get $e^{-2(\pi/\beta)(|l|+|s|)} \leq \sigma^{-j}$ for all $j \leq p$. The particular case $j = p$ is precisely (41).

7 Distortion bounds

This section is devoted to the proof of the distortion bounds in Lemmas 5.4 and 5.5. Obtaining these properties is the main reason for introducing the partitions $\mathcal{P}_{k,i}$ we defined in Section 5.

Beginning the proof of Lemma 5.4, we recall that $\partial_\mu f_\mu = \pm 1$ and so

$$\begin{aligned} \left| \frac{\dot{\Psi}_n(\mu)}{(f_\mu^n)'(z_k(\mu))} \right| &= \left| \dot{z}_k(\mu) + \sum_{i=1}^{n} \frac{\partial_\mu f_\mu (f_\mu^{i-1}(z_k(\mu)))}{(f_\mu^i)'(z_k(\mu))} \right| \\ &\leq 1 + \sum_{i=1}^{n} \frac{1}{|(f_\mu^i)'(z_k(\mu))|} \\ &\leq 1 + \sum_{i=1}^{n} \sigma^{-i} < \infty. \end{aligned} \tag{42}$$

This gives the upper bound in the statement of Lemma 5.4. The lower bound is somewhat more delicate: we must show that z_k spends a long time in a region where the derivative is very large (larger than 4, say). Let $j_0 = j_0(z_k, \mu)$. Then, since $\mu \in G$,

$$|(f_\mu^i)'(z_k(\mu))| \geq \epsilon^{-i(1-\gamma)} \geq 4^{-i}$$

for every $i \leq j_0$, recall Section 4. Depending on k, the value of j_0 may not be large enough for our purposes. On the other hand, we have seen in Lemma 6.1 that after time j_0 the trajectory of z_k remains for a long period m_0 in the region $S^1 \setminus [-\tilde{y}, \tilde{y}]$, where the derivative is large. We take $\tilde{f}$ such that $\sigma_0 \geq 4$, and ϵ small enough so that $\sum_{i > j_0 + m_0} \sigma^{-i} < 1/4$. Then (42) gives

$$\begin{aligned} \left| \frac{\dot{\Psi}_n(\mu)}{(f_\mu^n)'(z_k(\mu))} \right| &\geq 1 - \sum_{i=1}^{j_0} \epsilon^{i(1-\gamma)} - \sum_{i=1}^{m_0} \epsilon^{j_0(1-\gamma)} \sigma_0^{-i} - \sum_{i > j_0 + m_0} \sigma^{-i} \\ &\geq 1 - \sum_{i \geq 1} 4^{-i} - \sum_{i > j_0 + m_0} \sigma^{-i} \geq \frac{1}{2}. \end{aligned}$$

This ends the proof of Lemma 5.4 (and we may take $A_1 = 2$, say). □

Next we prove Lemma 5.5. The previous result asserts that, for the parameter values under consideration, the derivatives with respect to the parameter and the phase space variable are comparable. Therefore, we only have to prove the first statement in the lemma, the second one following as an immediate consequence (for some, possibly larger, $A_2 > 0$).

In order to bound

$$|(f_\mu^n)'(z_k(\mu))| = \prod_{h=0}^{n-1} |f'(\Psi_h(\mu))|$$

away from zero and infinity, it suffices to find a uniform bound for

$$\sum_{h=0}^{n-1} \frac{|f'(\Psi_h(\mu_1)) - f'(\Psi_h(\mu_2))|}{|f'(\Psi_h(\mu_2))|}. \tag{43}$$

We start by estimating the contribution of the initial iterates $h < j_0$, where $j_0 = j_0(z_k(\mu), \mu)$, $\mu \in \omega$, is as defined at the beginning of Section 4. Using Lemma 3.3,

$$\begin{aligned}\sum_{h=0}^{j_0-1} \frac{|f'(\Psi_h(\mu_1)) - f'(\Psi_h(\mu_2))|}{|f'(\Psi_h(\mu_2))|} &\le C \sum_{h=0}^{j_0-1} \frac{|\Psi_h(\mu_1) - \Psi_h(\mu_2)|}{|\Psi_h(\mu_2) - x_{m(h)}|} \\ &\le C \sum_{h=0}^{j_0-1} \frac{|x_{m(h)}|}{\hat{\tau}|x_{m(h)}|} \le C\, j_0 \le C,\end{aligned}$$

recall the remarks preceding (9). From now on, we consider $h \ge j_0$ and let $\Delta_h(\omega) = \inf_{\mu\in\omega} |\Psi_h(\mu) - x_l|$, and $R(l, s, j)$ be the partition element associated to the iterate h. We claim that if $p(h) > 0$ then the sum over $[h, h + p(h)]$ is bounded by $C|\Psi_h(\omega)|/\Delta_h(\omega)$. Using Lemma 3.3 we find

$$\frac{|f'(\Psi_h(\mu_1)) - f'(\Psi_h(\mu_2))|}{|f'(\Psi_h(\mu_2))|} \le C\frac{|\Psi_h(\mu_1) - \Psi_h(\mu_2)|}{|\Psi_h(\mu_2) - x_l|} \le C\frac{|\Psi_h(\omega)|}{\Delta_h(\omega)}.$$

Next, we consider $1 \le i \le p(h)$ and proceed as follows. Applying the mean value theorem to $\Psi_h(\mu) \mapsto \Psi_{h+i}(\mu)$ we obtain, for some $\xi \in \omega$,

$$\begin{aligned}|\Psi_{h+i}(\mu_1) - \Psi_{h+i}(\mu_2)| &\le C|(f_\xi^i)'(f_\xi^h(z_k(\xi)))||\Psi_h(\mu_1) - \Psi_h(\mu_2)| \\ &\le C|(f_\xi^i)'(f_\xi^h(z_k(\xi)))||\Psi_h(\omega)| \\ &\le C|(f_\xi^{i-1})'(f_\xi^{h+1}(z_k(\xi)))||f''(x_l)||f_\xi^h(z_k(\xi)) - x_l||\Psi_h(\omega)|.\end{aligned}$$

According to the definition of $p(h)$, we have two possibilities. The first case in (29) gives, for some $\eta \in [f_\xi(x_l), f_\xi^{h+1}(z_k(\xi))]$,

$$\begin{aligned}|f_{\mu_0}^i(x_l) - x_{m(i-1)}|e^{-\tau i} &\ge |f_\xi^{h+i}(z_k(\xi)) - f_\xi^i(x_l)| \\ &= |(f_\xi^{i-1})'(\eta)||f_\xi^{h+1}(z_k(\xi)) - f_\xi(x_l)| \\ &\ge C^{-1}|(f_\xi^{i-1})'(\eta)||f''(x_l)||f_\xi^h(z_k(\xi)) - x_l|^2 \\ &\ge C^{-1}|(f_\xi^{i-1})'(f_\xi^{h+1}(z_k(\xi)))||f''(x_l)||f_\xi^h(z_k(\xi)) - x_l|^2,\end{aligned}$$

where μ_0 is chosen as in Steps 16, 27 and the last equality uses (33). Then, combining Lemma 3.3 with the inequalities we have just obtained,

$$\begin{aligned}\frac{|f'(\Psi_{h+i}(\mu_1)) - f'(\Psi_{h+i}(\mu_2))|}{|f'(\Psi_{h+i}(\mu_2))|} &\le C\frac{|\Psi_{h+i}(\mu_1) - \Psi_{h+i}(\mu_2)|}{|\Psi_{h+i}(\mu_2) - x_{m(i-1)}|} \\ &\le Ce^{-\tau i}\frac{|\Psi_h(\omega)||f_{\mu_0}^i(x_l) - x_{m(i-1)}|}{|f_\xi^h(z_k(\xi)) - x_l||f_{\mu_2}^{h+i}(z_k(\mu_2)) - x_{m(i-1)}|} \\ &\le Ce^{-\tau i}\frac{|\Psi_h(\omega)|}{\Delta_h(\omega)}\frac{1}{1 - e^{-\tau i}} \le Ce^{-\tau i}\frac{|\Psi_h(\omega)|}{\Delta_h(\omega)}.\end{aligned}$$

In the second case of (29) we have $\epsilon^{1+\tau}e^{-\tau i} \geq |f_\xi^{h+i}(z_k(\xi)) - f_\xi^i(x_l)|$ and so the same kind of calculations give

$$\begin{aligned}
\frac{|f'(\Psi_h(\mu_1)) - f'(\Psi_h(\mu_2))|}{|f'(\Psi_h(\mu_2))|} &\leq C|\Psi_{h+i}(\mu_1) - \Psi_{h+i}(\mu_2)| \\
&\leq Ce^{-\tau i}\frac{|\Psi_h(\omega)|\epsilon^{1+\tau}}{|f_\xi^h(z_k(\xi)) - x_l|} \leq \quad Ce^{-\tau i}\frac{|\Psi_h(\omega)|}{\Delta_h(\omega)}.
\end{aligned}$$

This proves that the total contribution of $[h, h+p(h)]$ to (43) is bounded by

$$\sum_{i=0}^{p(h)} Ce^{-\tau i}\frac{|\Psi_h(\omega)|}{\Delta_h(\omega)} \leq C\frac{|\Psi_h(\omega)|}{\Delta_h(\omega)},$$

as we claimed. Analogously, if $p(h) = 0$ then either

$$\frac{|f'(\Psi_h(\mu_1)) - f'(\Psi_h(\mu_2))|}{|f'(\Psi_h(\mu_2))|} \leq C\frac{|\Psi_h(\mu_1) - \Psi_h(\mu_2)|}{|\Psi_h(\mu_2) - x_l|} \leq C\frac{|\Psi_h(\omega)|}{\Delta_h(\omega)},$$

for iterates in $[-\epsilon, \epsilon]$, or

$$\frac{|f'(\Psi_h(\mu_1)) - f'(\Psi_h(\mu_2))|}{|f'(\Psi_h(\mu_2))|} \leq C|\Psi_h(\mu_1) - \Psi_h(\mu_2)| \leq C\frac{|\Psi_h(\omega)|}{\Delta_h(\omega)},$$

for iterates outside $[-\epsilon, \epsilon]$. As a consequence, the overall contribution of the iterates $j_0 \leq h \leq n-1$ in (43) is bounded by

$$C\sum_{h\in S}\frac{|\Psi_h(\omega)|}{\Delta_h(\omega)} \tag{44}$$

where the sum is over the set S of iterates not belonging to any binding period. The final step is to bound the sum in (44). Let $\bar{l}$ and $\bar{s}$ be fixed and denote $S(\bar{l}, \bar{s}) = \{h_0 > h_1 > \ldots > h_i > \ldots\}$ the set of iterates $h \geq j_0$ whose triple (l, s, j) has $l = \bar{l}$ and $s = \bar{s}$. Lemmas 5.3b), and 5.4 imply

$$|\Psi_{h_0}(\omega)| \geq C^{-1}\sigma^{\frac{h_0-h_i}{3}}|\Psi_{h_i}(\omega)|.$$

Moreover, by construction, $\Delta_{h_i}(\omega) \geq C^{-1}|R(\bar{l}, \bar{s})|$ and

$$|\Psi_{h_0}(\omega)| \geq \frac{|R(\bar{l}, \bar{s})|}{(\bar{l}+\bar{s})^3}.$$

It follows that

$$\begin{aligned}
\sum_{h\in S(\bar{l},\bar{s})}\frac{|\Psi_h(\omega)|}{\Delta_h(\omega)} &= \sum_{i\geq 0}\frac{|\Psi_{h_i}(\omega)|}{\Delta_{h_i}(\omega)} \leq C\sum_{i=0}\frac{\sigma^{(h_i-h_0)/3}|\Psi_{h_0}(\omega)|}{|R(\bar{l}, \bar{s})|} \\
&\leq C\sum_{i\geq 0}\frac{\sigma^{-i/3}}{(\bar{l}+\bar{s})^3} \leq \frac{C}{(\bar{l}+\bar{s})^3}.
\end{aligned}$$

We conclude that
$$\sum_{h\in S}\frac{|\Psi_h(\omega)|}{\Delta_h(\omega)}\le C\sum_{\bar l,\bar s}\frac{1}{(\bar l+\bar s)^3}\le C,$$
which completes the proof of Lemma 5.5. □

8 Parameter exclusions

Now we prove the estimates in Lemmas 5.6 and 5.7 concerning the measure of the set of excluded parameters.

The strategy in the proof of Lemma 5.6 is to show that the intervals $\omega'\subset\omega$ in Step 21 corresponding to partition intervals $R(l,s,j)$ with large $|l|+|s|$ must be small. In proving this we use the previous bounded distortion lemmas.

Let L be a connected component of $\widetilde{G}$. We treat first the case when L belongs in $\mathcal{P}_{k,n-1}$, that is, at stage n it did not yet suffer decomposition (Step 21). Let $j_0=j_0(L,k)$. Then, by definition of large component, $m(\Psi_{j_0}(L))\ge\epsilon$. Moreover, recall Lemma 6.1, there are at least $m_0\ge c\,\log(1/\epsilon)$ iterates (beginning at time j_0) during which the critical orbit remains in a region where $|f'_\mu|\ge\sigma_0$. In particular, $n\ge j_0+m_0\ge c\,\log(1/\epsilon)$ and so, using bounded distortion,
$$\frac{m(E'_{k,n}\cap L)}{m(L)}\le C\frac{m(\Psi_n(E'_{k,n}\cap L))}{m(\Psi_n(L))}\le\frac{C\min\{e^{-\rho n},\epsilon\}}{C^{-1}\epsilon\,\sigma_0^{m_0}}.$$
We have used the fact that the measure of the set excluded by (31) is bounded by $C\min\{e^{-\rho n},\epsilon\}$. If $\rho n\le\log(1/\epsilon)$, it follows
$$\frac{m(E'_{k,n}\cap L)}{m(L)}\le C\sigma_0^{-c\log(1/\epsilon)}\le C\sigma_0^{-c\rho n}.$$
Otherwise,
$$\frac{m(E'_{k,n}\cap L)}{m(L)}\le C\frac{e^{-\rho n}}{\epsilon^{1-c\log\sigma_0}}\le Ce^{-c\rho(\log\sigma_0)n}.$$
So, the lemma is proved in this case.

From now on, suppose that L has already been decomposed, and let ω be any of the intervals in $\mathcal{P}_{k,n-1}$ contained in L. Define $n_i=\min\{m:\omega\in\mathcal{P}_{k,m}\}\le n-1$, the iterate when ω was created. Then $\Psi_{n_i}(\omega)$ contains some interval $R(l_i,s_i,j_i)$. Our distortion bounds imply
$$\begin{aligned}|\Psi_n(\omega)| &= \frac{|\Psi'_n(\xi)|}{|\Psi'_{n_i}(\xi)|}|\Psi_{n_i}(\omega)|\ge\frac{1}{C}\frac{|\Psi'_n(\mu_0)|}{|\Psi'_{n_i}(\mu_0)|}|\Psi_{n_i}(\omega)|\\ &\ge \frac{1}{C}|(f^{n-n_i}_{\mu_0})'(f^{n_i}_{\mu_0}(z_k(\mu_0)))||\Psi_{n_i}(\omega)|\\ &\ge \frac{1}{C}|(f^{n-n_i}_{\mu_0})'(f^{n_i}_{\mu_0}(z_k(\mu_0)))|\frac{e^{-(\pi/\beta)(|l_i|+|s_i|)}}{(|l_i|+|s_i|)^3}.\end{aligned}$$

We treat first the case when ω is such that $p_i = p(n_i) = 0$. By Lemma 5.3b),

$$|(f_{\mu_0}^{n-n_i})'(f_{\mu_0}^{n_i}(z_k(\mu_0)))| \geq \frac{1}{C} e^{(\pi/\beta)|l_i|(2-\alpha)} \sigma^{(n-n_i)/3} \geq \frac{1}{C} e^{(\pi/\beta)|l_i|(2-\alpha)} \tag{45}$$

and so

$$\begin{aligned} \frac{m(E'_{k,n} \cap \omega)}{m(\omega)} &\leq C \frac{m(\Psi_n(E'_{k,n} \cap \omega))}{m(\Psi_n(\omega))} \leq \frac{C \min\{e^{-\rho n}, \epsilon\}(|l_i| + |s_i|)^3}{e^{(\pi/\beta)|l_i|(2-\alpha)} e^{-(\pi/\beta)(|l_i|+|s_i|)}} \\ &\leq C \min\{e^{-\rho n}, \epsilon\}(|l_i| + |s_i|)^3 e^{-(\pi/\beta)|l_i|(2-\alpha-1)}. \end{aligned}$$

In the last inequality we use the fact that $p_i = 0$ implies that s_i is (uniformly) bounded. By (31), $|l_i| + |s_i|$ is either equal to $k_0 + 1 \leq C \log(1/\epsilon)$ or smaller than $\rho n_i < \rho n$. If $\rho n \leq \log(1/\epsilon)$, we find

$$\frac{m(E'_{k,n} \cap \omega)}{m(\omega)} \leq C\epsilon \,(C \log(1/\epsilon))^3 \leq C\epsilon^{1/2} \leq e^{-\rho n/2}.$$

Otherwise,

$$\frac{m(E'_{k,n} \cap \omega)}{m(\omega)} \leq Ce^{-\rho n}(C\rho n)^3 \leq Ce^{-\rho n/2}.$$

Now we suppose that ω is so that $p_i > 0$, and write

$$\begin{aligned} |(f_{\mu_0}^{n-n_i})'(f_{\mu_0}^{n_i}(z_k(\mu_0)))| &\geq \tfrac{1}{C}|(f_{\mu_0}^{p_i+1})'(f_{\mu_0}^{n_i}(z_k(\mu_0)))| \cdot \\ &\qquad \cdot |(f_{\mu_0}^{n-n_i-p_i-1})'(f_{\mu_0}^{n_i+p_i+1}(z_k(\mu_0)))|. \end{aligned} \tag{46}$$

The second factor is bounded from below by $C^{-1}\sigma^{\frac{1}{3}(n-n_i-p_i)} \geq C^{-1}$. For the first one we derive two different lower bounds. On the one hand, the estimates in (38), (39) give

$$|(f_{\mu_0}^{p_i+1})'(f_{\mu_0}^{n_i+1}(z_k(\mu_0)))| \geq \frac{1}{C}\sigma^{p_i/3}\epsilon^{-(1-\alpha-\tau)/2} \geq \frac{1}{C}\epsilon^{-(1-\alpha-\tau)/2}. \tag{47}$$

On the other hand, we also have

$$\begin{aligned} &|(f_{\mu_0}^{p_i+1})'(f_{\mu_0}^{n_i}(z_k(\mu_0)))| \geq \\ &\qquad \geq \tfrac{1}{C}|(f_{\mu_0})''(x_{l_i})|\, e^{-(\pi/\beta)(|l_i|+|s_i|)} \cdot |(f_{\mu_0}^{p_i})'(f_{\mu_0}^{n_i+1}(z_k(\mu_0)))| \\ &\qquad \geq \tfrac{1}{C}|(f_{\mu})''(x_{l_i})|\, e^{-(\pi/\beta)(|l_i|+|s_i|)} \cdot \tfrac{|f_{\mu_0}^{n_i+p_i+1}(z_k(\mu_0)) - f_{\mu_0}^{p_i+1}(x_{l_i})|}{|(f_{\mu_0})''(x_{l_i})|\, e^{-2(\pi/\beta)(|l_i|+|s_i|)}} \\ &\qquad \geq \tfrac{1}{C}e^{(\pi/\beta)(|l_i|+|s_i|)}\epsilon^{1+\tau}e^{-(\rho+\tau)p_i}. \end{aligned} \tag{48}$$

In the second inequality we used the mean value theorem, in the same way as in the proof of Lemma 5.3(a). For the third inequality recall the estimates leading to (38) and (39). Now we distinguish two subcases. Suppose first that n is small enough so that $\rho n \le 2\log(1/\epsilon)$, that is, $\epsilon \le e^{-\rho n/2}$. Then we use (47) to get

$$\frac{m(E'_{k,n}\cap\omega)}{m(\omega)} \le C\frac{\min\{e^{-\rho n},\epsilon\}(|l_i|+|s_i|)^3}{\epsilon^{-(1-\alpha-\tau)/2}\,e^{-(\pi/\beta)(|l_i|+|s_i|)}}.$$

Since $p_i > 0$, we must have $\rho n_i \ge \log(1/\epsilon)$ and $e^{-(\pi/\beta)(|l_i|+|s_i|)} \ge e^{-\rho n_i} \ge e^{-\rho n}$, by (31). Thus,

$$\frac{m(E'_{k,n}\cap\omega)}{m(\omega)} \le C\epsilon^{(1-\alpha-\tau/2)}(\rho n_i)^3 \le \epsilon^{(1-\alpha-\tau)/4} \le e^{(1-\alpha)n/10}, \tag{49}$$

as long as τ and ϵ are small enough. Otherwise, if $\rho n > 2\log(1/\epsilon)$, then we use (48), which yields

$$\frac{m(E'_{k,n}\cap\omega)}{m(\omega)} \le C\frac{e^{-\rho n}(|l_i|+|s_i|)^3}{\epsilon^{1+\tau}e^{-(\rho+\tau)p_i}}.$$

Using Lemma 5.3a) and (31),

$$e^{(\rho+\tau)p_i}(|l_i|+|s_i|)^3 \le Ce^{(2\rho(\rho+\tau)/\log\sigma)n_i}(\rho n_i)^3 \le Ce^{\rho n/6}, \tag{50}$$

as long as ρ and τ are small enough. Replacing above and taking τ sufficiently small, we get

$$\frac{m(E'_{k,n}\cap\omega)}{m(\omega)} \le Ce^{-\rho n(1-1/6-(1+\tau)/2)} \le Ce^{-\rho n/6}.$$

We have covered all the possibilities for $\omega \subset L$, and obtained exponential estimates for $m(E'_{k,n}\cap\omega)$ in every case. Summing these estimates over all the $\omega \subset L$, we get

$$m(E'_{k,n}\cap L) \le Ce^{-b_1 n}m(L),$$

where $b_1 = \min\{(1-\alpha)/10, \rho/6\}$. This concludes the proof of the lemma. $\square$

For proving Lemma 5.7, we fix constants $0 < \gamma_1 < \gamma_2 < 1$, with

$$\gamma_1 > \max\{(3+\alpha)/4, 1-\alpha\}.$$

Given any connected component L of $\tilde{G}$ and any $\omega \in \mathcal{P}_{k,n-1}$ contained in L, take $L = \tilde\omega_0 \supset \tilde\omega_1 \supset \ldots \supset \tilde\omega_{n-1} = \omega$, with $\tilde\omega_j \in \mathcal{P}_{k,j}$ for $0 \le j \le n-1$. Let $n_0 = j_0(L,k)$ and $n_i > \cdots > n_1 > n_0$ be the values of $1 \le j \le n-1$ for which $\tilde\omega_j \ne \tilde\omega_{j-1}$. Then let $\omega_r = \tilde\omega_{n_r}$ for $0 \le r \le i$. Note that $\omega_i = \omega$ and $\omega_0 = L$, in particular, $m(\Psi_{n_0}(\omega_0)) \ge \epsilon$. Moreover, by construction, each $\Psi_{n_r}(\omega_r)$ contains some interval $R(l_r, s_r, j_r)$. Then, we claim that

(1) for all $1 < r \le i$ with $(l_r, s_r, j_r) \neq (\pm k_0, 1, 1)$,

$$\frac{|\omega_r|}{|\omega_{r-1}|} \le e^{-(\pi/\beta)(|l_r|+|s_r|)} e^{(\pi/\beta)\gamma_1(|l_{r-1}|+|s_{r-1}|)} \ ;$$

(2) moreover, $|\omega_1|/|\omega_0| \le Ce^{-(\pi/\beta)(|l_1|+|s_1|)}\,\epsilon^{\alpha-1}$.

Starting the proof of claim (1), we note that

$$\frac{|\omega_r|}{|\omega_{r-1}|} \le C\frac{|\Psi_{n_r}(\omega_r)|}{|\Psi_{n_r}(\omega_{r-1})|} \le C\frac{e^{-(\pi/\beta)(|l_r|+|s_r|)}}{|\Psi_{n_r}(\omega_{r-1})|}$$

because of bounded distortion. Observe that in the last inequality we use the assumption $(l_r, s_r, j_r) \neq (\pm k_0, 1, 1)$. Now we split the argument into three subcases, cf. the proof of the previous lemma. If $p_{r-1} = p(n_{r-1}(\omega)) = 0$ then s_{r-1} is uniformly bounded and the claim follows from

$$\begin{aligned} |\Psi_{n_r}(\omega_{r-1})| &\ge \tfrac{1}{C}e^{(\pi/\beta)|l_{r-1}|(2-\alpha)}e^{-(\pi/\beta)(|l_{r-1}|+|s_{r-1}|)}(|l_{r-1}|+|s_{r-1}|)^{-3} \\ &\ge \tfrac{1}{C}e^{(\pi/\beta)(|l_{r-1}|+|s_{r-1}|)(2-\alpha-1)}(|l_{r-1}|+|s_{r-1}|)^{-3} \\ &\ge e^{-(\pi/\beta)\gamma_1(|l_{r-1}|+|s_{r-1}|)}, \end{aligned}$$

where we also use $(2-\alpha-1) > 0 > -\gamma_1$ and $|l_{r-1}| + |s_{r-1}| \ge k_0 >> 1$. Next we assume $p_{r-1} > 0$. Corresponding to (47) and (48), we have

$$|\Psi_{n_r}(\omega_{r-1})| \ge \frac{1}{C}\epsilon^{-(1-\alpha-\tau)/2}e^{-(\pi/\beta)(|l_{r-1}|+|s_{r-1}|)}(|l_{r-1}|+|s_{r-1}|)^{-3} \qquad (51)$$

and also

$$|\Psi_{n_r}(\omega_{r-1})| \ge \frac{1}{C}\epsilon^{1+\tau}e^{-(\rho+\tau)p_{r-1}}(|l_{r-1}|+|s_{r-1}|)^{-3}. \qquad (52)$$

We distinguish the two remaining subcases according to the relation between $\log(1/\epsilon)$ and $|l_{r-1}| + |s_{r-1}|$. If $(\pi/\beta)(|l_{r-1}| + |s_{r-1}|) \le 2\log(1/\epsilon)$ then we use (51) to obtain

$$\begin{aligned} \tfrac{|\omega_r|}{|\omega_{r-1}|} &\le Ce^{-(\pi/\beta)(|l_r|+|s_r|)}\,\epsilon^{(1-\alpha-\tau)/2}\,e^{(\pi/\beta)(|l_{r-1}|+|s_{r-1}|)}(l_{r-1}|+|s_{r-1}|)^3 \\ &\le Ce^{-(\pi/\beta)(|l_r|+|s_r|)}\,e^{(\pi/\beta)(|l_{r-1}|+|s_{r-1}|)(3+\alpha+\tau)/4}\,(l_{r-1}|+|s_{r-1}|)^3 \qquad (53) \\ &\le e^{-(\pi/\beta)(|l_r|+|s_r|)}\,e^{(\pi/\beta)\gamma_1(|l_{r-1}|+|s_{r-1})|} \end{aligned}$$

as long as we take τ small, recall that $1 > \gamma_1 > (3+\alpha)/4$. In the last inequality we also use $|l_{r-1}| + |s_{r-1}| \ge k_0 >> 1$. Finally, if $(\pi/\beta)(|l_{r-1}| + |s_{r-1}|) > 2\log(1/\epsilon)$

then we use (52), which leads to

$$
\begin{aligned}
\frac{|\omega_r|}{|\omega_{r-1}|} &\leq C(l_{r-1}|+|s_{r-1}|)^3 e^{-(\pi/\beta)(|l_r|+|s_r|)}\epsilon^{-(1+\tau)}e^{(\rho+\tau)p_{r-1}} \\
&\leq C(l_{r-1}|+|s_{r-1}|)^3 e^{-(\pi/\beta)(|l_r|+|s_r|)}e^{(\pi/\beta)(|l_{r-1}|+|s_{r-1}|)((1+\tau)/2+2(\rho+\tau)/\log\sigma)} \\
&\leq e^{-(\pi/\beta)(|l_r|+|s_r|)}\, e^{(\pi/\beta)\gamma_1(|l_{r-1}|+|s_{r-1}|)}. \qquad (54)
\end{aligned}
$$

In the second inequality we use $p_{r-1} \leq 2(\pi/\beta)(|l_{r-1}|+|s_{r-1}|)/\log\sigma$, which is given by (41). In the last one we suppose that ρ and τ are small enough. The proof of claim (1) is complete.

Next, the definition of large component means that $|\Psi_{n_0}(\omega_0)| \geq \epsilon$. Since $\Psi_{n_0}(\omega_0) \subset [-C\epsilon^\alpha, C\epsilon^\alpha]$, it must contain at least a fraction $C^{-1}\epsilon^{1-\alpha}$ of some fundamental domain of the fixed point of $\tilde{f}$ at the origin. Therefore, by bounded distortion, $|\Psi_{n_1}(\omega_0)| \geq C^{-1}\epsilon^{1-\alpha}$ and

$$
\frac{|\omega_1|}{|\omega_0|} \leq C\frac{|\Psi_{n_1}(\omega_1)|}{|\Psi_{n_1}(\omega_0)|} \leq Ce^{-(\pi/\beta)(|l_1|+|s_1|)}\epsilon^{\alpha-1}.
$$

This proves claim (2).

Then, set $B_r = \sum_{n=n_r}^{n_{r+1}-1} p(n)$, so that $B(n,\omega,k) = B_0 + \cdots + B_i$. We claim

(3) $B_r \leq C(|l_r|+|s_r|)$ for every r;

(4) $B_r = 0$ whenever $r=0$ or $(l_r, j_r, s_r) = (\pm k_0, 1, 1)$.

In order to prove (3), let $t_0 = n_r$ and $n_r < t_1 < \ldots < t_q < n_{r+1}$ be the values of t for which $p(t) > 0$. We may suppose $q \geq 1$: otherwise, $B_r = p(n_r)$ and (3) is given by (41). Given $0 \leq j \leq q$ we have, by Lemma 5.3(b),

$$
|\Psi_{t_j}(\omega_r)| \geq \sigma^{\frac{t_j-t_0}{3}}|\Psi_{t_0}(\omega_r)|.
$$

We denote by λ_j, σ_j the values of l, s corresponding to t_j, in particular $\lambda_0 = l_r$, $\sigma_0 = s_r$. Then, the previous equation gives

$$
g(e^{-(\pi/\beta)(|\lambda_j|+|\sigma_j|)}) \geq \sigma^{\frac{t_j-t_0}{3}} g(e^{-(\pi/\beta)(|\lambda_0|+|\sigma_0|)}), \qquad (55)
$$

where $g(t) = (\pi/\beta)^3 t/\log^3(1/t)$. Let $\alpha_j = e^{-(\pi/\beta)(|\lambda_j|+|\sigma_j|)}$ and $T = t_j - t_0$. Then, clearly,

$$
g(\sigma^{-T/6}\alpha_j) = g(\alpha_j)\sigma^{-T/6}\left(\frac{\log 1/\alpha_j}{\log 1/\alpha_j + T/6\log\sigma}\right)^3.
$$

If $\epsilon > 0$ is small then α_j must be large, in particular $3/(\log 1/\alpha_j) \leq 1$. As a consequence,

$$
\left(\frac{\log(1/\alpha_j) + T/6\log\sigma}{\log(1/\alpha_j)}\right)^3 = \left(1 + \frac{T}{18}\frac{3\log\sigma}{\log(1/\alpha_j)}\right)^3 \leq \sigma^{T/6}.
$$

It follows that

$$g(\sigma^{-T/6}\alpha_j) \geq g(\alpha_j)\sigma^{-T/3} \geq g(\alpha_0),$$

where the last inequality results from (55). Using the monotonicity of g, we obtain $\sigma^{-T/6}\alpha_j \geq \alpha_0$, that is,

$$e^{-(\pi/\beta)(|\lambda_j|+|\sigma_j|)} \geq \sigma^{1/6(t_j-t_0)}e^{-(\pi/\beta)(|\lambda_0|+|\sigma_0|)}. \tag{56}$$

Combining this with the binding period estimates we get,

$$\begin{aligned} p(t_j) &\leq (|\lambda_j|+|\sigma_j|) \leq C(|\lambda_0|+|\sigma_0|) - C(t_j - t_0) \\ &\leq C(|l_r|+|s_r|) - c\,(p(t_0)+\cdots+p(t_{j-1})), \end{aligned}$$

for positive constants $C \geq 1 \geq c > 0$. It follows that

$$c\,p(t_j) \leq p(t_j) \leq C(|l_r|+|s_r|) - c(p(t_0)+\ldots+p(t_j))$$

and then

$$c\,(\sum_{i=0}^{j} p(t_i)) \leq C(|l_r|+|s_r|).$$

As this holds for $1 \leq j \leq q$, the proof of claim (3) is complete.

We also need to check that $B_0 = 0$. Reasoning by contradiction, suppose that there exists $n_0 \leq t_1 < n_1$ such that $p(t_1) > 0$, and take t_1 minimum with this property. Now,

$$|\Psi_{t_1}(\omega_0)| \leq C\frac{e^{-(\pi/\beta)(|l_1|+|s_1|)}}{(|l_1|+|s_1|)^3} \leq Ce^{-(\pi/\beta)(|l_1|+|s_1|)} \leq C\epsilon.$$

On the other hand, the same argument as in the proof of claim (2), gives

$$|\Psi_{t_1}(\omega_0)| \geq \frac{1}{C}\epsilon^{1-\alpha} >> \epsilon$$

which is the contradiction we were looking for. Moreover, a similar reasoning applies when $(l_r, s_r, j_r) = (\pm k_0, 1, 1)$, with $\epsilon^{1-\alpha}$ replaced by $(\log(1/\epsilon))^{-3}$. Indeed, in this case $\Psi_{n_r}(\omega_r)$ covers a subinterval of $[-\epsilon, \epsilon]$ with length greater than $C^{-1}\epsilon(\log(1/\epsilon))^{-3}$ and thus contains a fraction larger than $C^{-1}(\log(1/\epsilon))^3$ of some fundamental domain. The proof of claim (4) is also complete.

Now we introduce

$$I = \sum_{\omega} e^{4\theta B(n,\omega,k)} m(\omega) = |\omega_0| \sum_{\omega} e^{4\theta B(n,\omega,k)} \frac{|\omega_1|}{|\omega_0|} \cdots \frac{|\omega_i|}{|\omega_{i-1}|}$$

where $\theta > 0$ is to be fixed appropriately small, and the sum runs over all the elements $\omega \in \mathcal{P}_{k,n-1}$ contained in the large component L. If $(l_r, s_r, j_r) \neq (\pm k_0, 1, 1)$

then we use claims (1) or (2) to bound $|\omega_r|/|\omega_{r-1}|$. On the other hand, given any $r_1 < r_2$ such that $(l_r, s_r, j_r) \neq (\pm k_0, 1, 1)$ for all $r_1 \leq r < r_2$ then (take r_2 maximum)

$$\begin{aligned}\frac{|\omega_{r_1}|}{|\omega_{r_1-1}|}\cdots\frac{|\omega_{r_2-1}|}{|\omega_{r_2-2}|}\frac{|\omega_{r_2}|}{|\omega_{r_2-1}|} &\leq 1\cdots 1\cdot e^{-(\pi/\beta)(|l_{r_2}|+|s_{r_2}|)}(\log(1/\epsilon))^3 \\ &\leq e^{-(\pi/\beta)(|l_{r_2}|+|s_{r_2}|)}e^{(\pi/\beta)\gamma_1(|l_{r_1-1}|+|s_{r_1-1}|)}.\end{aligned}$$

The first inequality follows from the same kind of estimates as in the proof of claim (4). In the second one we use $e^{(\pi/\beta)\gamma_1(|l_{r_1-1}|+|s_{r_1-1}|)} \geq C^{-1}\epsilon^{-\gamma_1} >> (\log(1/\epsilon))^3$. Replacing these estimates we find

$$\begin{aligned}\frac{I}{|\omega_0|} &\leq \sum_\omega e^{4C\theta\sum_{r=1,*}^{i}(|l_r|+|s_r|)}C\epsilon^{\alpha-1}\prod_{r=1,*}^{i}e^{-(\pi/\beta)(|l_r|+|s_r|)}\prod_{r=1,*}^{i-1}e^{(\pi/\beta)\gamma_1(|l_r|+|s_r|)} \\ &\leq C\epsilon^{\alpha-1}\sum_\omega e^{-(\pi/\beta)\gamma_1(|l_i|+|s_i|)}e^{4C\theta\sum_{r=1,*}^{i}(|l_r|+|s_r|)}e^{-(\pi/\beta)(1-\gamma_1)\sum_{r=1,*}^{i}(|l_r|+|s_r|)} \\ &\leq C\sum_\omega e^{(4C\theta-(\pi/\beta)(1-\gamma_1))\sum_{r=1,*}^{i}(|l_r|+|s_r|)},\end{aligned}$$

where the star $*$ indicates that all the sums and products are over the values of r for which $(l_r, s_r, j_r) \neq (\pm k_0, 1, 1)$. In the last step we used $e^{-(\pi/\beta)(|l_r|+|s_r|)} \leq \epsilon$ together with $\alpha - 1 + \gamma_1 > 0$. Then, taking θ sufficiently small,

$$\begin{aligned}\frac{I}{|\omega_0|} &\leq C\sum_\omega e^{-(\pi/\beta)(1-\gamma_2)\sum_{r=1,*}^{i}(|l_r|+|s_r|})} \\ &\leq C\sum_M e^{-(\pi/\beta)(1-\gamma_2)M}\#\{\omega\in\mathcal{P}_{k,n-1}:\sum_{r=1,*}^{i}(|l_r|+|s_r|)=M\}.\end{aligned}$$

Now, the number of solutions of $\sum_{r=1,*}^{i}\xi_r = M$ with large (positive) ξ_r is well known to be less than $e^{\gamma_0 M}$, with $\gamma_0 > 0$ small. In particular, $\gamma_0 \leq (\pi/\beta)(1-\gamma_2)/3$ if $\epsilon > 0$ is small enough. Since each sequence (l_r, s_r, j_r), $1 \leq r \leq i$, corresponds to at most one element of $\mathcal{P}_{k,n-1}$, the number of intervals $\omega \in \mathcal{P}_{k,n-1}$ associated to a same solution $\xi_r = |l_r| + |s_r|$ is bounded by

$$4^i\cdot\prod_{r=1,*}^{i}4(|l_r|+|s_r|)^3 \leq e^{\gamma_0 M}4^{2i}$$

if ϵ is small (implying $|l_r| + |s_r| \geq k_0$ large). Observe that we consider these solutions indexed by those $r = 1, \ldots, i$ with $(l_r, s_r, j_r) \neq (\pm k_0, 1, 1)$; the factor 4^i bounds all the possible choices corresponding to the remaining values of r. Note also that $i \leq n/m_0 \leq n/c\log(1/\epsilon) \leq \theta n/(2\log 4)$, for small enough ϵ. Altogether,

$$\frac{I}{|\omega_0|} \leq C\sum_M e^{-(\pi/\beta)(1-\gamma_2)+2\gamma_0}e^{\theta n} \leq C\sum_M e^{-\gamma_0 M}e^{\theta n} \leq Ce^{\theta n}.$$

This implies

$$Ce^{\theta n}|\omega_0| \geq I \geq e^{2n\theta} \sum_{\{\omega\subset\omega_0 : B(n,\omega,k)\geq n/2\}} |\omega| = e^{2n\theta} m(E''_{k,n}\cap\omega_0).$$

from where the lemma follows immediately (recall that $\omega_0 = L$). □

9 Proof of Theorem A

We begin this section with the proof of Lemma 5.1: the critical orbits exhibit exponential growth of the derivative for all the parameters satisfying (30) and (31). To do this we write

$$|(f_\mu^n)'(z_k(\mu))| =$$
$$= \prod_{i=0}^{j_0-1} |f'(f_\mu^i(z_k(\mu))) \prod_{\{i:p(i)=0\}} |f'(f_\mu^i(z_k(\mu)))| \prod_{\{i:p(i)>0\}} |f'(f_\mu^i(z_k(\mu)))|.$$

The first factor is larger than $\epsilon^{-(1-\gamma)j_0}$, by Corollary 4.4, and the second one is larger than $\sigma_0^{\#\{i:\, p(i)=0\}}$, by construction. Moreover, Lemma 5.3 implies that the third factor is larger than 1. Thus,

$$|(f_\mu^n)'(z_k(\mu))| \geq \sigma_0^{(n-\#\{i:\, p(i)>0\})} \geq \sigma^{2(n-B(n,\omega,k))} \geq \sigma^n,$$

completing the proof. □

Similar arguments show that $|(f_\mu^n)'(x)|$ grows exponentially fast as $n\to+\infty$, for Lebesgue almost every point x. Let us explain why, without going into technical details. Suppose first that x does not belong in the pre-orbit of any of the x_l. Then we can define binding periods for x in the same way as we did for critical points, and the second part of Lemma 5.3 remains valid. Then choosing a subsequence $(n_k)_k$ of iterates such that no n_k belongs to a binding period, we get $|(f_\mu^{n_k})'(x)| \geq \sigma^{n_k/3}$ for all k. This is a consequence of Lemma 5.3(b), and the fact that f_μ is expanding away from critical points. In particular, $\limsup(1/n)\log|(f_\mu^n)'(x)| \geq \log\sigma/3$ for every such point x.

Note that the length of the binding period corresponding to an iterate n can be bounded in terms of the distance from $f_\mu^n(x)$ to the nearest critical point, in the same way as before (41). However, now we are imposing no lower bound on this distance, and so binding periods can be arbitrarily long (the sequence n_k can be very sparse).

Yet, refining these arguments and restricting to a full Lebesgue measure subset of points x, one may replace lim sup by lim inf above. The main step is to prove, using the distortion bounds we obtained in the previous sections, that for Lebesgue almost every x there is some $k \geq 0$ such that $y = f_\mu^k(x)$ has $\min\{|f_\mu^j(x) - x_k| : |k| \geq k_0\} \geq e^{-\rho j}$ for all $j \geq 0$. Then the same arguments as we used for critical orbits show that $|(f_\mu^n)'(y)| \geq \sigma^{n/3}$ for all $n \geq 0$. Therefore, $\limsup(1/n)\log|(f_\mu^n)'(x)| = \limsup(1/n)\log|(f_\mu^n)'(y)|$ is bounded from below by $\log\sigma/3$, as we claimed.

Next, we show that f_μ is transitive, for every $\mu \in S$, where $S = \cap_{k,n} S_{k,n}$ is the set of parameters surviving all the exclusions. Let $P \notin [-\epsilon, \epsilon]$ be some fixed point of $\tilde{f}$ and P_μ denote its continuation as a fixed point of f_μ. It is easy to see that the unstable manifold $W^u(P_\mu)$ coincides with the whole S^1: it suffices to note that this holds for $W^u(P)$ and that f_μ is close to f, which differs from $\tilde{f}$ only on the small interval $[-\epsilon, \epsilon]$. For the same reasons, the fact that the negative orbit of P under $\tilde{f}$ is dense in S^1 implies that $f_\mu^{-n}(P_\mu)$ intersects $R(\pm k_0, 1, 1)$ for some $n \geq 0$. Now let $J \subset S^1$ be an arbitrary open interval. We construct sequences

- $J = J_0 \supset J_1 \supset \cdots J_n \supset \cdots$ of intervals,
- $0 < \nu_1 < \cdots < \nu_n < \cdots$ of iterates,
- and $(l_1, s_1, j_1), \ldots, (l_n, s_n, j_n), \ldots$

such that $f_\mu^{\nu_i}(J_i)$ contains $R(l_i, s_i, j_i)$ and $|l_i| + |s_i|$ is strictly decreasing. This is done, by induction, as follows. For each $i \geq 1$, we take ν_i to be the first iterate such that $f_\mu^{\nu_i}(J_{i-1})$ contains some $R(l_i, s_i, j_i)$. Note that ν_i must exist: otherwise, $f_\mu^\nu | J_{i-1}$ would be monotone for every $\nu \geq 1$ and so, due to the expanding behavior of f_μ, the length of $f_\mu^\nu(J_{i-1})$ would be unbounded, which is a contradiction. In fact, this expanding behavior implies that $f_\mu^{\nu_i}(J_{i-1})$ is much larger than $f_\mu^{\nu_{i-1}}(J_{i-1})$: Lemma 5.6 together with (45) and (47) yield

$$|f_\mu^{\nu_i}(J_{i-1})| \geq \frac{1}{C}\epsilon^{-(1-\alpha-\tau)/2}|f_\mu^{\nu_{i-1}}(J_{i-1})|.$$

We fix (l_i, s_i) as above, with $|l_i| + |s_i|$ minimum, and we set $J_i = f^{-\nu_i}(R(l_i, s_i, j_i))$. Suppose that $(l_i, s_i) \neq (\pm k_0, 1)$. Then

$$C\frac{e^{-(\pi/\beta)(|l_i|+|s_i|)}}{(|l_i| + |s_i|)^3} \geq |f_\mu^{\nu_i}(J_{i-1})| >> |f_\mu^{\nu_{i-1}}(J_{i-1})| \geq \frac{1}{C}\frac{e^{-(\pi/\beta)(|l_{i-1}|+|s_{i-1}|)}}{(|l_{i-1}| + |s_{i-1}|)^3},$$

if $\epsilon > 0$ is small. Using the monotonicity of $t \mapsto t/(\log^3(1/t))$ once again we get that $|l_i| + |s_i| < |l_{i-1}| + |s_{i-1}|$. This completes the inductive step. Now, observe that the previous argument breaks down when $(l_i, s_i) = (\pm k_0, 1)$, because in this case the upper bound for $|f_\mu^{\nu_i}(J_{i-1})|$ is not necessarily valid. Therefore, the conclusion is that we must, eventually, reach some $\nu \geq 1$ for which

$f_\mu^\nu(J) \supset f^\nu(J_i) \supset R(\pm k_0, 1, 1)$. It follows that $f_\mu^{\nu+n}(J)$ must contain P_μ and then $\cup_j f_\mu^{\nu+n+j}(J) \supset W^u(P_\mu) = S^1$. This proves that f_μ is transitive.

Finally, we show that $S = \cap_{k,n} S_{k,n}$ does indeed have positive Lebesgue measure. By construction, $S_{k,n+1} = S_{k,n} \setminus E_{k,n}$ and $E_{k,n} = E'_{k,n} \cup E''_{k,n} \cup E'''_{k,n}$. For the reasons given in Section 5, one does not need to take $E'''_{k,i}$ in consideration when estimating the measure of the total excluded set:

$$\bigcap_k S_{k,n+1} = \bigcap_k S_{k,n} \setminus \bigcup_k (E'_{k,n} \cup E''_{k,n})$$

and so

$$L \setminus \bigcap_k S_{k,n+1} = \left(L \setminus \bigcap_k S_{k,n}\right) \bigcup \left(\bigcup_k (E'_{k,n} \cup E''_{k,n}) \cap L\right)$$

for every large component L of G. Therefore,

$$m\left(L \setminus \bigcap_k S_{k,n+1}\right) \leq m\left(L \setminus \bigcap_k S_{k,n}\right) + m\left(\bigcup_k (E'_{k,n} \cup E''_{k,n}) \cap L\right). \tag{57}$$

Now, Lemmas 5.6 and 5.7 imply

$$m\left((E'_{k,n} \cup E''_{k,n}) \cap L\right) \leq B\, e^{-4bn} m(L), \tag{58}$$

for absolute constants $B > 0$, $b > 0$, and every $|k| \geq k_0$.

Now we must bypass the fact that we are dealing with infinitely many critical points. With that in mind, we note that the nth iterate of $z_k(\mu)$ is very close to the nth iterate of $z_{\pm\infty}(\mu) = \pm\mu$ if $|k| \geq k_1(\epsilon, n) = (3\beta/\pi\alpha) n \log(1/\epsilon)$:

$$\begin{aligned} |f_\mu^n(z_k(\mu)) - f_\mu^n(z_\infty(\mu))| &\leq (C\mu^{\alpha-1})^n \, |z_k(\mu) - z_\infty(\mu)| \\ &\leq (C\epsilon^{2(\alpha-1)})^n \, C e^{-(\pi/\beta)\alpha k} \leq \epsilon^n. \end{aligned}$$

Taking ϵ small, ϵ^n is much smaller than $e^{-\rho n}$, and so it is also much smaller than the distance from $f_\mu^n(z_{\pm\infty}(\mu))$ to the nearest critical point. Therefore, cf. comments at the end of Section 5, the conclusions of Lemmas 5.1-5.7 are valid for all the values of l with $k_1 \leq |l| < \infty$ if $z_{\pm\infty}$ satisfy (30) and (31). This means that if $\mu \in S_{l,n}$ for every l with $|l| < k_1$ as well as for $l = \pm\infty$, then the conclusions of Lemmas 5.1-5.7 are valid for all the critical values z_l, $k_0 \leq |l| \leq \infty$. Hence, we need only add over k with $|k| \leq k_1$ or $|k| = \infty$, which gives

$$m\left(\bigcup_k (E'_{k,n} \cup E''_{k,n}) \cap L\right) \leq \sum_{|k| \leq k_1(\epsilon,n),\ |k|=\infty} B e^{-4bn} m(L) \leq C n e^{-4bn} \log\frac{1}{\epsilon}\, m(L).$$

Now observe that we only have to deal with $n \geq m_0 \geq c \log(1/\epsilon)$. For such an n,

$$m\left(\bigcup_k (E'_{k,n} \cup E''_{k,n}) \cap L\right) \leq C n e^{-4bn} \log\frac{1}{\epsilon}\, m(L) \leq \epsilon^{bc} e^{-bn} m(L).$$

Replacing in (57), we get

$$m\left(L \setminus \bigcap_k S_{k,n+1}\right) \leq m\left(L \setminus \bigcap_k S_{k,n}\right) + \epsilon^{bc} e^{-bn} m(L)$$

for each $n \geq m_0$, and so, taking ϵ small,

$$m(L \setminus S) \leq \sum_n \epsilon^{bc} e^{-bn} m(L) \leq C\epsilon^{bc} m(L) < m(L).$$

This completes the proof of the theorem.

References

[ACT] A. Arneodo, P. Coullet, C. Tresser, *Possible new strange attractors with spiral structure*, Comm. Math. Phys. 79 (1981), 573–579.

[BC] M. Benedicks, L. Carleson, *The dynamics of the Hénon map*, Ann. Math. 133 (1991), 73-169.

[CKR] L. Chua, A. Khibnik, D. Roose, *On periodic orbits and homoclinic bifurcations in Chua's circuit with a smooth nonlinearity*, Bifur. and Chaos 3-2 (1993), 363-384.

[Ja] M. Jakobson, *Absolutely continuous invariant measures for one-parameter families of one-dimensional maps*, Comm. Math. Phys. 81 (1981), 39-88.

[PRV] M. J. Pacifico, A. Rovella, M. Viana, *Persistence of global spiral attractors*, in preparation.

[Sh] L.P. Shil'nikov, *A case of the existence of a denumerable set of periodic motions*, Sov. Math. Dokl. 6 (1965), 163–166.

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