Chaotic dynamical behaviour

Marcelo Viana *

1 Introduction

A main task in Dynamical Systems is the development of a mathematical theory of *chaotic dynamics*, encompassing in a conceptual structure the manifestations of erratic and, to a large extent, unpredictable asymptotic behaviour exhibited by many natural phenomena. After Ruelle-Takens [RT], such "chaotic" behaviour should be interpreted in terms of the presence of a "strange attractor", a notion which can be sketched as follows. By an *attractor* one means a (compact) region in phase space which is invariant under time evolution and to which converge the future trajectories of a large – positive volume or, even, open – set of initial states (the *basin* of attraction). One also requires some condition of dynamical indivisibility, e.g. existence of dense trajectories in the attractor. Lorenz [Lo] highlighted the crucial role of *sensitive dependence on the initial state* as a source of unpredictability of the dynamics and this is a main ingredient here: we call the attractor *strange* (or *chaotic*) if most trajectories corresponding to nearby initial states in the basin move away from each other as they approach the attractor.

An important model Ruelle-Takens had in mind were Smale's Axiom A (or uniformly hyperbolic) attractors [Sm], a class of systems whose dynamics is now well understood. While exhibiting very rich behaviour – in particular, they contain infinitely many periodic trajectories – they are rather robust (or *persistent*) and even structurally *stable*: the qualitative features of both the atractor and its dynamics remain unchanged under any small perturbation of the system. On the other hand, uniformly hyperbolic attractors occur less frequently in applications arising from the experimental sciences than it was thought at some stage. Instead, a large number of numerical studies have been identifying objects of a nonhyperbolic nature, such as the Lorenz-like attractors [Lo] or the Hénon-like attractors [He], as more appropriate models for the chaotic behaviour displayed by many such systems.

The discovery of these and a few other surprising phenomena (including e.g. Feigenbaum-Coullet-Tresser's cascades of period doubling bifurcations) strongly influenced the current trends of mathematical research in Dynamical Systems.

*This work was partially supported by a J. S. Guggenheim Foundation Fellowship.

Indeed, largely in response to these discoveries, a number of results has been obtained in recent years which already provide some deep understanding of chaotic behaviour and the properties of (non-Axiom A) strange attractors. Much in contrast with the unstability of their dynamics – it may be modified by arbitrarily small perturbations of the system – these attractors are, themselves, quite persistent. Indeed, any flow near one with a Lorenz-like attractor also has such an attractor, close to the initial one [GW]. The situation is more subtle in the case of attractors of Hénon type: slightly perturbing the system may cause the chaotic attractor to be, for instance, replaced by periodic attractors, see [Ur]. Yet, as shown by Benedicks-Carleson [BC], Hénon attractors do have a remarkable, probabilistic, form of persistence: they are preserved by many (positive probability) small perturbations of the initial system. Another important result with a similar flavour had been obtained before by Jakobson [Ja], for real quadratic transformations.

In the present work we address the problem of describing the dynamical mechanisms underlying the formation of strange attractors and/or their persistence under perturbations of the dynamical system. The unstable character of most interesting chaotic attractors also means that this problem is closely related to the study of the processes of global or semi-global bifurcation – i.e. modification of the dynamics – and this is the approach we adopt here. It is also motivated by a program recently proposed by Palis, aiming at a description of complicated dynamics in the general setting of smooth dynamical systems. The basic strategy is to depart from some convenient set $\mathcal{B}$ formed by systems with well defined types of bifurcations. One wants $\mathcal{B}$ to approximate every (non-Axiom A) situation of interesting dynamics, while being sufficiently small so as to be analysable. Then, a global understanding of the space of all dynamical systems may be attained by analysing the forms of dynamics occurring in a persistent way in generic parametrized families passing through the elements of $\mathcal{B}$. Note that, in view of the previous discussion, *persistence* is to be understood mostly in a measure-theoretical sense: positive Lebesgue measure set of parameter values. See [PT2] for an extended discussion.

In Sections 2 and 3 we focus on discrete time dynamical systems and discuss two main mechanisms of bifurcation, resp. *homoclinic tangencies* and *critical saddle-node cycles*, which are bound to be important ingredients in the construction of a set $\mathcal{B}$ as above. These bifurcations are accompanied by a wide range of complex dynamical phenomena, including persistent strange attractors, and indeed they provide some unifying setting for the analysis of such phenomena.

Then, in Section 4 we turn into discussing chaotic dynamics in the context of smooth flows. While versions of the preceding results can be derived for this context, just by reducing to Poincaré return maps, we concentrate on some new, interesting phenomena associated to global and semi-global bifurcations involving equilibria of the flow. This includes the study of global strange attractors arising in the unfolding of *homoclinic connections*.

We close this Introduction by observing that, in all the instances of chaotic behaviour we have considered so far, sensitive dependence on the initial state has an essentially one-dimensional character: the chaotic attractor always has topological dimension 1 and this also means that it exhibits only one direction of expansion around each trajectory (in more precise terms, there is one single positive Lyapounov exponent). The study of *multidimensional strange attractors*, having several expanding directions, is mostly open and we refer the reader to [V2] for some recent results on this topic.

The author is most grateful to the hospitality and support of the FIM/ETH-Zurich, the University of Michigan and the IHES-Paris during the preparation of this paper.

2 Homoclinic Bifurcations

Here we consider parametrized families $\varphi_\mu : M \longrightarrow M$, $\mu \in \mathbb{R}$, of diffeomorphisms on a surface M. We also comment on the general case $\dim M \geq 2$ near the end of the section. By *homoclinic bifurcation* we mean the formation of homoclinic trajectories – i.e. trajectories that accumulate on a same saddle point under both positive and negative iteration – as the parameter μ varies. A main way this process may take place is through the appearance of homoclinic tangencies. Let us explain this in more detail. We suppose that for some $\bar{\mu} \in \mathbb{R}$ the stable and the unstable manifolds of a saddle $p = p_{\bar{\mu}}$ of $\varphi_{\bar{\mu}}$ have a point q (and therefore a whole trajectory) of nontransverse intersection, see Figure 1. Usually, one takes such a *homoclinic tangency* to be quadratic – the two invari-

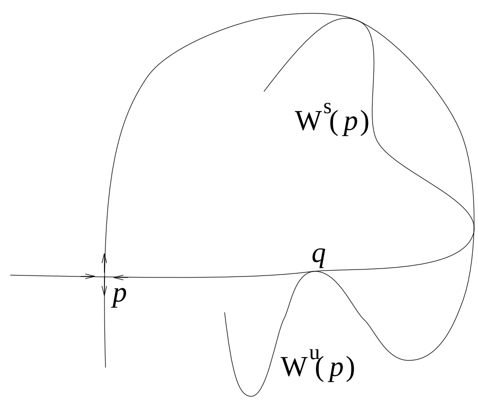

Figure 1: A homoclinic tangency

ant manifolds have different curvatures at q – and to be generically unfolded by $(\varphi_\mu)_\mu$. This last condition means the following: for μ close to $\bar{\mu}$ one considers the continuation p_μ of p – i.e. the unique saddle of φ_μ near p and having the same period – and one requires the stable and the unstable manifolds of p_μ to move with respect to each other near q, with nonzero relative speed, as μ varies. Then, for $\mu > \bar{\mu}$ say, the two invariant manifolds have transverse intersections close to q.

A particularly simple model is the Hénon family of diffeomorphisms of the 2-dimensional plane:

$$h_{\mu,\nu}\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \quad h_{\mu,\nu}(x,y) = (1 - \mu x^2 + \nu y, x). \tag{1}$$

In fact, it is not difficult to check that for each fixed $\nu = \bar{\nu} \neq 0$ there are (infinitely many) values $\mu = \bar{\mu}$ such that $h_{\bar{\mu},\bar{\nu}}$ exhibits quadratic homoclinic tangencies which, moreover, are generically unfolded by $(h_{\mu,\bar{\nu}})_\mu$. Note also that $|\det Dh_{\mu,\nu}| = |\nu|$.

In the sequel we focus on homoclinic tangencies associated to area-dissipative saddles, i.e. $\left|\det D\varphi^l_{\bar{\mu}}(p)\right| < 1$, where l is the period of p. Of course, the case $\left|\det D\varphi^l_{\bar{\mu}}(p)\right| > 1$ can be reduced to the previous one just by reversing time (then one should replace *attractor* by *repeller* in what follows). See also [Du] for interesting recent results in the area-conservative context.

Homoclinic bifurcations are a very common feature occurring in many relevant applications and also a main mechanism for the development of complicated behaviour in a dynamical system. Indeed, the transition of experimental models into complex forms of dynamics – e.g. under variation of physical parameters – very frequently involves the formation of transverse homoclinic trajectories. Moreover, this last process is always accompanied by many other profound dynamical modifications, including

creation of horseshoes: [Sm] For every $\mu > \bar{\mu}$ close to $\bar{\mu}$, φ_μ has invariant hyperbolic sets containing infinitely many periodic saddle points.

cascades of period doubling bifurcations: [YA] For parameters $(\mu_k)_k$ arbitrarily close to $\bar{\mu}$, φ_{μ_k} has periodic attractors of period const 2^k.

creation of saddle-node cycles: For values of μ arbitrarily close to $\bar{\mu}$, φ_μ has critical saddle-node cycles, see next section;

coexistence of infinitely many periodic attractors: [Ne], [R1] There exist intervals I arbitrarily close to $\bar{\mu}$ and generic (Baire second category) $G \subset I$ such that every φ_μ, $\mu \in G$, has infinitely many periodic attractors.

Moreover, extending the methods of Benedicks-Carleson [BC], it was proven in [MV] that the presence of strange attractors is always a persistent phenomenon in this setting of homoclinic bifurcations.

Theorem 1 [MV] *For any generic family $(\varphi_\mu)_\mu$ of surface diffeomorphisms going through a homoclinic tangency, there exists $S \subset \mathbb{R}$ such that φ_μ has Hénon-like strange attractors for every $\mu \in S$ and $S \cap [\bar{\mu} - \varepsilon, \bar{\mu} + \varepsilon]$ has positive Lebesgue measure for all $\varepsilon > 0$.*

By a *Hénon-like strange attractor* of a transformation $\varphi: M \longrightarrow M$ we mean a compact set $\Lambda \subset M$ such that $\varphi(\Lambda) = \Lambda$ and

1. Λ coincides with the closure of the unstable manifold of some saddle of φ;
2. the basin $W^s(\Lambda) = \{z \in M: \lim_{n\to+\infty} \operatorname{dist}(\varphi^n(z), \Lambda) = 0\}$ contains a full neighbourhood of Λ;
3. there is some $\bar{z} \in \Lambda$ whose trajectory $\{\varphi^n(\bar{z}): n \geq 1\}$ is dense in Λ and
4. such dense trajectory may be taken exhibiting exponential expansion: $\|D\varphi^n(\bar{z})\| \geq c\sigma^n$ for every $n \geq 1$ and some $c > 0$, $\sigma > 1$;
5. there are *critical* points $z \in \Lambda$, admiting nonzero tangent vectors v such that $\|D\varphi^n(z)v\| \to 0$ as both $n \to \pm\infty$ (hence, Λ is not a hyperbolic set).

A stronger form of the crucial sensitivity property 4 (positive Lyapounov exponent) is given in [BY], where *physical* (or Sinai-Ruelle-Bowen) invariant measures are constructed for strange attractors as above: with respect to such a measure, almost every $z \in \Lambda$ satisfies the inequality in 4, for some $c > 0$ depending on z. As a consequence, the trajectories of a large (at least positive volume) subset of points $z \in W^s(\Lambda)$ also exhibit such an exponential expansion.

Now we give a short account of homoclinic bifurcations in high-dimensional manifolds. As before, we consider generic families $\varphi_\mu: M \longrightarrow M$, $\mu \in \mathbb{R}$, such that some $\varphi_{\bar{\mu}}$ has a homoclinic tangency, but now we let $m = \dim M \geq 2$. In this general setting the dynamics of φ_μ close to the tangency, $\mu \approx \bar{\mu}$, frequently involves periodic saddles or sources and also "strange saddles," [Rm]. In order to have attractors one takes the saddle p to be *sectionally dissipative*: the product of any pair of eigenvalues of $D\varphi^l_{\bar{\mu}}(p)$ has norm less than 1 (in particular there is at most one expanding eigenvalue). Then, under this assumption Theorem 1 generalizes to arbitrary dimension (even to arbitrary Hilbert manifolds).

Theorem 2 [V1] *Let $(\varphi_\mu)_\mu$ be a family of diffeomorphisms on a manifold M, $\dim M \geq 2$, unfolding a homoclinic tangency associated to a sectionally dissipative saddle p of $\varphi_{\bar{\mu}}$. Then, there exists $S \subset \mathbb{R}$ such that φ_μ has Hénon-like strange attractors for every $\mu \in S$ and $S \cap [\bar{\mu} - \varepsilon, \bar{\mu} + \varepsilon]$ has positive Lebesgue measure for all $\varepsilon > 0$.*

Newhouse's theorem on coexistence of infinitely many periodic attractors also extends to the general sectionally dissipative setting [PV]. A reformulation in the absence of sectional dissipativeness is proven in [Rm]. See [YA], resp. [MR], for extensions of the above result on cascades of period doubling to the sectionally dissipative case, resp. the general higher-dimensional context.

We close this section with a brief discussion of the following converse question, which plays an important role in Palis' scenario mentioned in the Introduction: *Can any diffeomorphism exhibiting some of the dynamical phenomena* *above be approximated by diffeomorphisms with homoclinic bifurcations ?* A positive answer will mean that homoclinic bifurcations may, in some sense, be viewed as a main unifying mechanism of complicated dynamical behaviour. Although this problem is mostly open, there is presently a certain amount of favourable evidence. On the one hand, all the situations one can actually exhibit of coexistence of infinitely many periodic attractors, resp. occurrence of Hénon-like strange attractors, are indeed approximated by homoclinic tangencies, see [TY], [Ur]. Also, some positive results are being provided for the case of cascades of period doubling bifurcations, in a work in preparation by E. Catsigeras. Finally, there is a close, and fairly well understood, interplay between homoclinic tangencies and yet another, *a priori* quite distinct, bifurcation process: creation of critical saddle-node cycles. This is one of the topics in the forthcoming section.

3 Saddle-Node Cycles

A natural question raised by Theorems 1 and 2 concerns the relative frequency of strange attractors near $\mu = \bar{\mu}$. More precisely, one would like to know if and when Hénon-like attractors are a *prevalent phenomenon*, meaning that

$$\frac{m(S \cap [\bar{\mu} - \varepsilon, \bar{\mu} + \varepsilon])}{2\varepsilon}, \quad m = \text{Lebesgue measure},$$

has a positive limit as $\varepsilon \to 0$. Examples of such a situation have not yet been found (it also follows from [PT1] that the limit above is zero in a large class of cases) but the general answer remains unknown. A relevant variation consists in considering the unfolding of homoclinic tangencies by families with a larger number of parameters, $l > 1$ say, and then asking whether Hénon-like attractors can be a prevalent phenomenon in such families (with respect to the l-dimensional Lebesgue measure). An interesting situation for the study of this question is provided by homoclinic tangencies occurring in the Hénon family at parameter values $(\bar{\mu}, \bar{\nu})$ close to $(2, 0)$.

On the other hand, the limit above is always positive in a closely related context of bifurcations: the creation of critical saddle-node cycles. A diffeomorphism $\varphi: M \longrightarrow M$, $m = \dim M \geq 2$, has a *saddle-node k-cycle* if it has fixed (or periodic) points $p_1, \ldots, p_k$ such that

- p_1 is a saddle-node – $D\varphi(p_1)$ has eigenvalues $\lambda_1 = 1, \lambda_2, \ldots, \lambda_m$ with $|\lambda_i| < 1$ for all $2 \leq i \leq m$ – and $p_2, \ldots, p_k$ are hyperbolic saddles;
- $W^u(p_i)$ intersects $W^s(p_{i+1})$ transversally for all $1 \leq i < k$ and $W^u(p_k)$ also intersects $W^s(p_1)$; if $k = 1$ we just require $W^u(p_1) \subset \text{interior}\,(W^s(p_1))$.

Note that we define $W^s(p_i) = \{z \in M: \varphi^n(z) \to p_i \text{ as } n \to +\infty\}$ and $W^u(p_i) = \{z \in M: \varphi^n(z) \to p_i \text{ as } n \to -\infty\}$ for every $1 \leq i \leq k$. The *strong-stable manifold* $W^{ss}(p_1)$ consists of the points $z \in M$ for which $\varphi^n(z)$ converges exponentially fast to p_1 as $n \to +\infty$. By [NPT], $W^s(p_1)$ admits a unique φ-invariant foliation by codimension-1 submanifolds which are exponentially contracted under positive iteration and such that $W^{ss}(p_1)$ is one of the leaves. We call the cycle *critical* if $W^u(p_k)$ has a nontransverse intersection with some leaf of this *strong-stable foliation* of p_1. See Figure 2 for an example, where the cycle contains a unique periodic point, the saddle-node $p = p_1$.

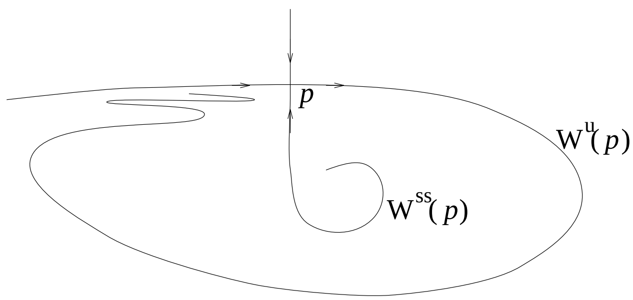

Figure 2: A critical saddle-node 1-cycle

Now, we consider generic families of diffeomorphisms $\varphi_\mu : M \longrightarrow M$, $\mu \in \mathbb{R}$, unfolding such a cycle: for some $\bar{\mu} \in \mathbb{R}$, $\varphi_{\bar{\mu}}$ has a critical saddle-node cycle, satisfying mild conditions of nondegeneracy. The relation between the present situation and the setting of homoclinic bifurcations is twofold: *the generic unfolding of a critical saddle-node cycle always involves the occurrence of (sectionally dissipative) homoclinic tangencies*, [NPT], *and the converse is also true*, as observed by L. Mora. In particular, the same dynamical phenomena are present in both bifurcations, which also means that these two processes are essentially equivalent for what concerns defining a bifurcation set $\mathcal{B}$ as in the Introduction.

On the other hand, families of diffeomorphisms going through a saddle-node cycle allow for a more global description of the dynamics related to the bifurcation than has been attained so far in the previous context of homoclinic tangencies. In particular, one is able to prove in this way that strange attractors always occur for a definite positive fraction of the parameter values near any critical saddle-node cycle.

Theorem 3 [DRV] *Let $(\varphi_\mu)_\mu$ be a generic family of diffeomorphisms on a manifold M, $\dim M \geq 2$, unfolding a critical saddle-node cycle of $\varphi_{\bar{\mu}}$. Then*

$$\liminf_{\varepsilon \to 0} \frac{m(S \cap [\bar{\mu} - \varepsilon, \bar{\mu} + \varepsilon])}{2\varepsilon} > 0.$$

where S is the set of values of μ for which φ_μ has Hénon-like strange attractors.

Now we want to focus on the special case of $1-$cycles, recall Figure 2. An important feature of such cycles is the existence of a *trapping region*: it is not difficult to find a compact domain $V \subset M$ containing the closure of $W^u(p_1)$ and such that $\varphi_{\bar{\mu}}(V) \subset \text{interior}\,(V)$. For instance, one may take V homeomorphic to the solid $m-$torus $S^1 \times B^{m-1}$ if M is orientable; see e.g. [DRV]. Then, for parameter values μ close to $\bar{\mu}$ all the asymptotic dynamics is concentrated in the maximal invariant set

$$\Lambda_\mu = \bigcap_{n>0} \varphi_\mu^n(V),$$

and the previous theorem implies that, quite often, Λ_μ contains Hénon-like attractors. However, a much stronger statement can be obtained in the present context: frequent occurrence of a *unique, global strange attractor* inside V.

Theorem 4 [DRV] *For an open set of families $(\varphi_\mu)_\mu$ unfolding a critical saddle-node 1-cycle there is a set $G \subset \mathbb{R}$ of values of μ satisfying*

$$\liminf_{\varepsilon \to 0} \frac{m(G \cap [\bar{\mu} - \varepsilon, \bar{\mu} + \varepsilon])}{2\varepsilon} > 0$$

and such that Λ_μ is a Hénon-like strange attractor for every $\mu \in G$.

Recall that, according to our definition, this includes the existence of dense trajectories in Λ_μ, $\mu \in G$. A simple, and yet important, remark is that the basin of Λ_μ contains the domain V, which depends only on the initial system $\varphi_{\bar{\mu}}$.

4 Bifurcations of vector fields

As already mentioned, the previous results may be rephrased in the setting of bifurcations of vector fields away from singular (i.e. equilibrium) points, just by considering appropriate Poincaré return maps. On the other hand, some new important phenomena of chaotic dynamics come associated with the presence of equilibria and here we discuss a few recent developments in this direction, focussing on the topic of chaotic attractors.

For simplicity, we deal with vector fields X in 3-dimensional euclidean space $M = \mathbb{R}^3$, even if the results to be discussed below have a broader scope. As a general assumption, we take X to have a hyperbolic saddle equilibrium, at the origin say. Up to reversing time if necessary, we may suppose that the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ of $DX(0)$ satisfy $\lambda_1 > 0 > \mathcal{R}e\,\lambda_2 \geq \mathcal{R}e\,\lambda_3$. We are particularly interested in studying homoclinic connections and the development of chaotic behaviour as they are unfolded by parametrized families of vector fields. By *homoclinic connection* we mean a regular trajectory $\{X^t(z): t \in \mathbb{R}\}$ of the flow $(X^t)_t$ of X, such that $X^t(z) \to 0$ as both $t \to \pm\infty$, see Figure 3. The dynamics of (perturbations of) X close to the homoclinic connection may then be described by the first-return map π associated to some transversal section Σ as in the figure. We point out that π is not a smooth map, in general, due to the presence of the singular point at the origin. Its behaviour depends in an important way on the sign of $\lambda_1 + \mathcal{R}e\lambda_2$, as we explain below.

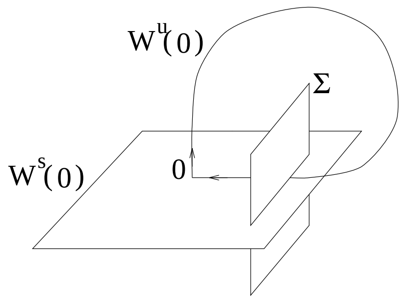

Figure 3: A homoclinic connection

For the time being we take λ_2 and λ_3 to be real (and distinct). The so-called geometric Lorenz attractors, [GW], [ABS], may be related to the present situation in the following way. One departs from a vector field X having a homoclinic connection and such that $\lambda_1 + \lambda_2 > 0$. Actually, in the present situation one takes X symmetric with respect to the z-axis, i.e. invariant under $(x, y, z) \mapsto (-x, -y, z)$, and so there are even two homoclinic trajectories. Then, by unfolding these connections – it is convenient to do it in such a way that the symmetry and the equilibrium point at the origin be preserved – one can find *large* perturbations Y of X for which the two unstable separatrices of 0 accumulate on a compact invariant set Λ containing the singular point together with regular trajectories (some of which are dense in Λ). Moreover, the trajectories of all the points in a neighbourhood of Λ are attracted to it, while exhibiting sensitive dependence on the initial state: for (Lebesgue) typical nearby points z_1, z_2 in this neighbourhood $\operatorname{dist}(Y^t(z_1), Y^t(z_2))$ grows exponentially fast with time $t > 0$ (up to attaining the order of magnitude of $\operatorname{diam}(\Lambda)$). Quite important is the robustness of these features under perturbations: the previous statements remain valid for any smooth vector field close enough to Y.

Let us note that these properties of Lorenz-like attractors strongly rely on the expansion condition $\lambda_1 + \lambda_2 > 0$. The contracting version of geometric Lorenz flows was recently studied by [Rv]: he considers $\lambda_1 + \lambda_2 < 0$ and, as before, takes a convenient large perturbation Y of a (symmetric) vector field X having homoclinic connections associated to the origin, for which the accumulation set of $W^u(0)$ is a strange attractor. Although the geometry of contracting and expanding Lorenz-like attractors is fairly similar, these two kinds of attractors have substantially distinct dynamical properties, in particular contracting Lorenz-like attractors are only measure-theoretically persistent: they occur for a positive measure set of parameter values in families of vector fields $(Y_\mu)_\mu$ passing through Y. In fact, [Rv] also proves that Axiom A flows having only periodic attractors are generic (open and dense set of parameters) in such families.

A crucial point in both the constructions in the two previous paragraphs is to take the perturbed vector field Y in such a way that (any vector field close to) it admits an invariant foliation by curves which are uniformly contracted under the flow. This permits to simplify the analysis of the flow considerably, by reducing it to a 1-dimensional setting. We note that existence of such a foliation for the original equations of Lorenz

$$\begin{cases} \dot{x} &= -\sigma x + \sigma y \\ \dot{y} &= rx - y - xz \\ \dot{z} &= -bz + xy \end{cases} \tag{2}$$

is not yet known and it remains a (slightly) embarassing open question whether strange attractors with the above properties really occur in (2), despite a few conclusive results for similar equations, [R2], [Ry].

On the other hand, the modeling of (2) by these geometric Lorenz flows is restricted to a narrow parameter range near Lorenz' original values $r = 28$, $\sigma = 10$, $b = 8/3$. Indeed, detailed numerical observations by [Sp], see also [HP], suggest that for only slightly larger values of r such an invariant foliation must cease to exist. In an ongoing joint work with S. Luzzatto we are developing an extension of the geometric model, reflecting the behaviour of Lorenz' equations in this broader range. Using methods inspired by [BC] we are proving that the strange attractor survives the breakdown of the invariant foliation, even if in this process it acquires subtler properties, namely a milder form of persistence under perturbations: positive measure set of parameter values.

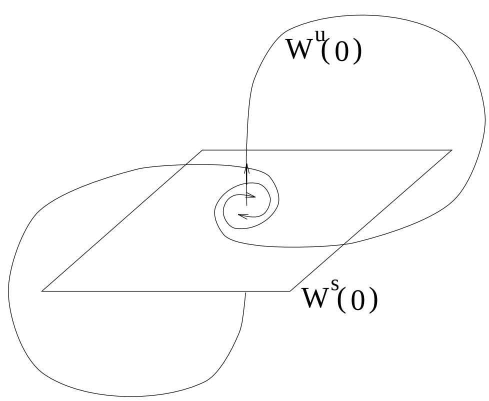

Figure 4: Symmetric saddle-focus connections

Another interesting, possibly even richer, situation corresponds to the equilibrium being a *saddle-focus*, i.e. λ_2, λ_3 being complex conjugate numbers. Figure 4 describes homoclinic connections associated to a saddle-focus. We take the vector field X to be symmetric with respect to the origin, i.e. invariant under $(x, y, z) \mapsto (-x, -y, -z)$ (but see comments below). Also, we consider the, relatively more interesting, expanding case $\lambda_1 + \mathcal{R}e\lambda_2 > 0$, asymptotic behaviour near the homoclinic trajectory being mostly of periodic type in the contracting case. An example of saddle-focus homoclinic connections was considered by [ACT], who found numerical evidence for the existence of a spiraling strange attractor. The spiral structure stems from the behaviour of the flow near the singular point and a rough picture of the attractor may be obtained as follows. Consider a transversal section Σ to both homoclinic trajectories and the first-return map π associated to it. Up to mild assumptions on the global behaviour of the flow (satisfied by an open set of vector fields X) one may take Σ satisfying $\pi(\Sigma) \subset \text{interior}\,(\Sigma)$. Moreover, a simple analysis of the behaviour of trajectories near the origin shows that the image $\pi(\Sigma)$ looks like in Figure 5: the return map is discontinuous along the line $\Sigma \cap W^s(0)$ and the images of the two connected components of $\Sigma \backslash W^s(0)$ spiral around the intersections of Σ with each of the two homoclinic connections.

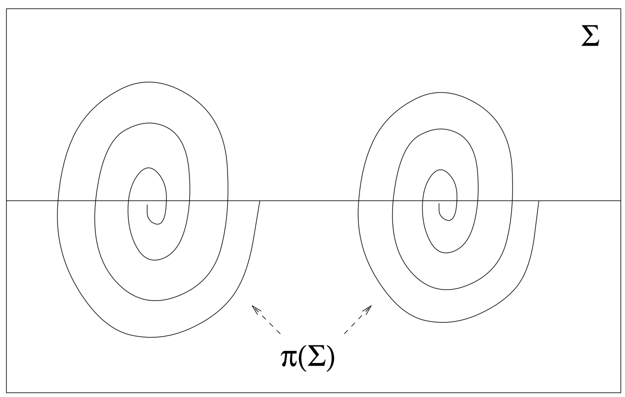

Figure 5: A spiraling strange attractor

Then we set $\Gamma = \cap_{n \geq 0} \pi^n(\Sigma)$ and $\Lambda = \text{closure}\,(\{X^t(x): x \in \Gamma, t \in \mathbb{R}\})$. In general, this "attractor" Λ contains several different forms of dynamics such as, periodic attractors, invariant hyperbolic sets of saddle type containing countably many periodic trajectories, [Si], and strange attractors obtained by suspension along the flow of Hénon-like attractors of π, [Pu]. On the other hand, in a joint work with Pacifico and Rovella we are proving that a persistent global spiral attractor does occur in this setting. More precisely, we consider symmetric vector fields X with saddle-focus connections as above and parametrized families of vector fields $(X_\mu)_\mu$ with $X_0 = X$. For each μ close to zero we consider maximal invariant sets Γ_μ, Λ_μ, defined in the same way as before, in terms of the flow $(X_\mu^t)_t$ and the return map π_μ to Σ associated to X_μ. Then we are proving that Λ_μ is a strange attractor for a positive Lebesgue measure set of values of the parameter near $\mu = 0$. Our approach combines an extension of the techniques of [BC] – extension is required in order to deal with the presence of infinitely many *folding* regions – together with a careful analysis of early iterates of such regions. We remark that this approach applies also in the absence of symmetry – Šil'nikov bifurcation – except that in this case the attractor occurs only for large values of μ.

References

[ABS] V. S. Afraimovich, V. V. Bykov, L. P. Šil'nikov, *On attracting structurally unstable limit sets of Lorenz type attractors*, Trans. Moscow Math. Soc. 44 (1977), 153-216.

[ACT] A. Arneodo, P. Coullet, C. Tresser, *Possible new strange attractors with spiral structure*, Comm. Math. Phys. 79 (1981), 573-579.

[BC] M. Benedicks, L. Carleson, *The dynamics of the Hénon map*, Ann. Math. 133 (1991), 73-169.

[BY] M. Benedicks, L.-S. Young, *SBR-measures for certain Hénon maps*, Invent. Math. 112-3 (1993), 541-576.

[DRV] L. J. Díaz, J. Rocha, M. Viana, *Strange attractors in saddle-node cycles: prevalence and globality*, preprint IMPA 1993 and to appear.

[Du] P. Duarte, *Plenty of elliptic islands for the standard family of area preserving maps*, Ann. Inst. Henri Poincaré, Analyse non linéaire, Vol. 11, n° 4 (1994), 359-409.

[He] M. Hénon, *A two dimensional mapping with a strange attractor*, Comm. Math. Phys. 50 (1976), 69-77.

[HP] M. Hénon, Y. Pomeau, *Two strange attractors with a simple structure*, Lect. Notes Math. 565 (1976), 29-68.

[GW] J. Guckenheimer, R. Williams, *Structural stability of Lorenz attractors*, Publ. Math. IHES 50 (1979), 59-72.

[Ja] M. Jakobson, *Absolutely continuous invariant measures for one-parameter families of one-dimensional maps*, Comm. Math. Phys. 81 (1981), 39-88.

[Lo] E. N. Lorenz, *Deterministic non-periodic flow*, J. Atmos. Sci. 20 (1963), 130-141.

[MR] J. C. Martín-Rivas, *Homoclinic bifurcations and cascades of period-doubling in higher dimensions*, thesis IMPA 1992 and to appear.

[MV] L. Mora, M. Viana, *Abundance of strange attractors*, Acta Math. 171 (1993), 1-71.

[Ne] S. Newhouse, *The abundance of wild hyperbolic sets and nonsmooth stable sets for diffeomorphisms*, Publ. Math. I.H.E.S. 50 (1979), 101-151.

[NPT] S. Newhouse, J. Palis, F. Takens, *Bifurcations and stability of families of diffeomorphisms*, Publ. Math. I.H.E.S. 57 (1983), 7-71.

[PT1] J. Palis, F. Takens, *Hyperbolicity and the creation of homoclinic orbits*, Annals of Math. 125 (1987), 337-374.

[PT2] J. Palis, F. Takens, *Hyperbolicity and Sensitive Chaotic Dynamics at Homoclinic Bifurcations*, Cambridge University Press, 1993.

[PV] J. Palis, M. Viana, *High dimension diffeomorphisms displaying infinitely many periodic attractors*, Ann. of Math. 1994.

[Pu] A. Pumariño, *Coexistence with positive probability of strange attractors in homoclinic connections of saddle-focus type*, thesis Department of Mathematics, University of Oviedo, Spain.

[R1] C. Robinson, *Bifurcation to infinitely many sinks*, Comm. Math. Phys. 90 (1983), 433-459.

[R2] C. Robinson, *Homoclinic bifurcation to a transitive attractor of Lorenz type*, Nonlinearity 2 (1989), 495-518.

[Rm] N. Romero, *Persistence of homoclinic tangencies in higher dimensions*, thesis IMPA 1992 and to appear Ergod. Th. & Dynam. Sys..

[Rv] A. Rovella, *The dynamics of perturbations of the contracting Lorenz attractor*, Bull. Braz. Math. Soc. 24 (1993), 233-259.

[RT] D. Ruelle, F. Takens, *On the nature of turbulence*, Commun. Math. Phys. 20 (1971), 167-192.

[Ry] M. Rychlik, *Lorenz' attractors through Šil'nikov-type bifurcation*, Ergod. Th. & Dynam. Sys. 10 (1990), 793-821.

[Si] L. P. Šil'nikov, *A case of the existence of a denumerable set of periodic motions*, Sov. Math. Dokl. 6 (1965), 163-166.

[Sm] S. Smale, *Differentiable dynamical systems*, Bull. Am. Math. Soc. 73 (1967), 747-817.

[Sp] C. Sparrow, *The Lorenz equations: bifurcations, chaos and strange attractors*, Appli. Math. Sci. 41, Springer Verlag, 1982.

[TY] L. Tedeschini-Lalli, J.A. Yorke, *How often do simple dynamical processes have infinitely many coexisting sinks ?* Comm. Math. Phys. 106 (1986), 635-657.

[Ur] R. Ures, *On the approximation of Hénon-like attractors by homoclinic tangencies*, thesis IMPA 1993 and to appear Ergod. Th. & Dynam. Sys..

[V1] M. Viana, *Strange attractors in higher dimensions*, Bull. Braz. Math. Soc. 24 (1993), 13-62.

[V2] M. Viana, *Multidimensional nonhyperbolic attractors*, preprint IMPA 1995 and to appear.

[YA] J. A. Yorke, K. T. Alligood, *Cascades of period doubling bifurcations: a prerequisite for horseshoes*, Bull. A.M.S. 9 (1983), 319-322.

Marcelo Viana
IMPA, Est. D. Castorina 110, Jardim Botânico, 22460 Rio de Janeiro, Brazil
Tel: +55(21)294-9032 Fax: +55(21)512-4115 E-mail: viana@impa.br