Magna est veritas et .... Yes, when it gets a chance.

There is a law, no doubt – and likewise a law regulates your luck in the throwing of dice.

It is not Justice, the servant of men, but accident, hazard, Fortune – the ally of patient Time – that holds an even and scrupulous balance.

Joseph Conrad, Lord Jim

Dynamics:

A Probabilistic & Geometric

Perspective

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Basic setup

I will consider smooth systems on a compact manifold M:

- maps $\quad f : M \to M$

  $$f^1 = f, \quad f^{n+1} = f \circ f^n, \quad n \in \mathbb{N}$$

  for diffeomorphisms: $f^{-n} = (f^n)^{-1}$

- flows $\quad X^t : M \to M, \quad t \in \mathbb{R}$

  $$\frac{d}{dt} X^t(z) \mid_{t=s} = X(X_s(z))$$

  where X is a vector field on M

General problems

For *most* dynamical systems, describe how *most* orbits $f^n(z)$, $X^t(z)$ behave, specially as time $n,\ t \to \infty$.

An **attractor** is a compact invariant set $A \subset M$ containing dense orbits and whose **basin of attraction** $B(A)$ has positive Lebesgue probability in M.

$B(A)$ is the set of points whose orbits accumulate in A as time $\to +\infty$.

An **SRB** (or **physical**) **measure** is an invariant probability measure μ such that, for every continuous function φ,

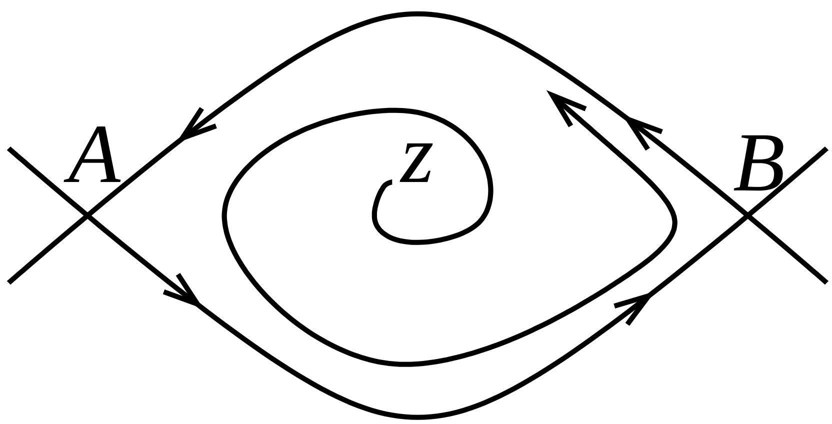
$$\lim_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(z)) = \int \varphi \, d\mu$$

for a set of points $z \in M$ with positive Lebesgue probability in M.

The **basin** $B(\mu)$ is the set of points $z \in M$ for which this holds.

Definitions for flows are similar.

SRB measures *not always* exist, e.g. the following planar flow (Bowen):



$$\frac{1}{T}\int_0^T \varphi(X^t(z))\,dt \quad \text{diverges}$$

as $T \to +\infty$, whenever $\varphi(A) \neq \varphi(B)$.

Is the dynamical behaviour robust (stable) under small modifications of the system ?

Given (f, μ) with μ an SRB measure, consider the iteration scheme

$$x_n \mapsto x_{n+1}$$

where x_{n+1} is picked at random close to $f(x_n)$. (random noise)

For instance, choose maps f_n at random close to f, and take $x_{n+1} = f_n(x_n)$.

(f,μ) is **stochastically stable** if, given any continuous φ, we have

$$\lim \frac{1}{n}\sum_{j=0}^{n-1}\varphi(x_j) \approx \int \varphi\, d\mu$$

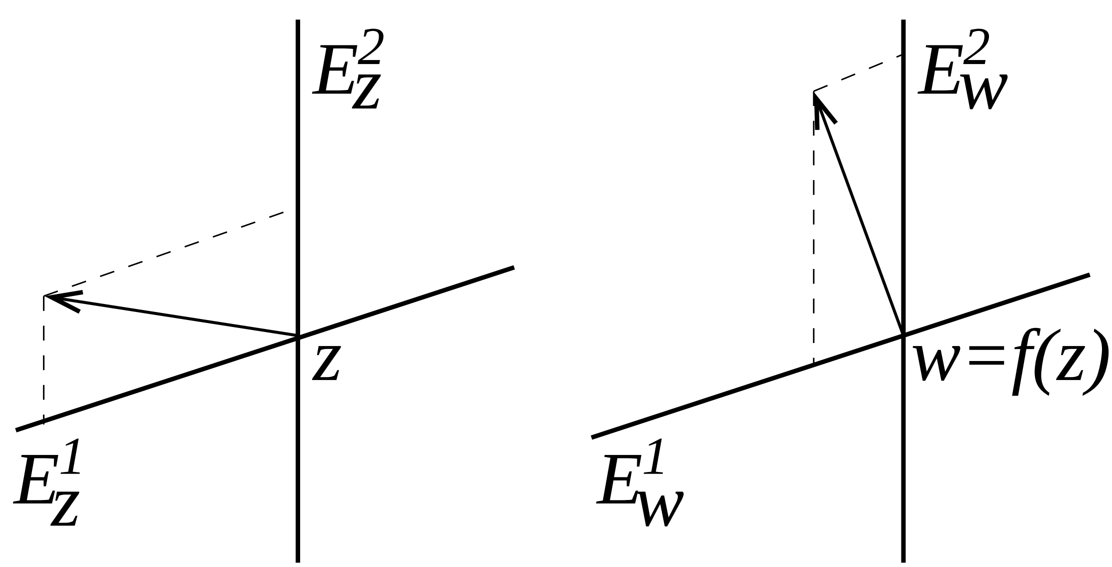
for almost every random orbit $(x_n)_n$, if the random noise is small.

In the continuous time case, random perturbations are described by (diffusion type) stochastic flows.

Uniformly hyperbolic systems

This notion (Axiom A) was introduced by Smale in the early sixties:

uniform contraction and expansion along complementary tangent directions, over the *relevant* part of the dynamics.



The **limit set** L is the smallest closed set containing the accumulation sets of all orbits.

the system is **uniformly hyperbolic**

$$\Updownarrow$$

its limit set is **hyperbolic**

L is **hyperbolic** if there is an invariant splitting of the tangent bundle

$$T_L M = E^1 \oplus E^2$$

such that E^1 is contracted by future iterates and E^2 is contracted by past iterates (at uniform rates).

- Axiom A systems admit a notably precise description, in statistical terms.

Sinai, Ruelle, Bowen: there are finitely many SRB measures, and their basins cover a full Lebesgue probability set.

Kifer, Young proved stochastic stability.

- Axiom A systems are, essentially, the structurally stable ones.

Anosov, Palis, Smale, Robin, de Melo, Robinson $\Rightarrow$

Mañé, Hayashi $\Leftarrow$ (C^1 topology)

Structural stability: nearby systems are all equivalent, up to continuous global change of coordinates.

On the other hand, systems can be persistently non-hyperbolic: e.g.

- Newhouse: maps with infinitely many periodic attractors.
- "chaotic" systems like the Lorenz flows or the Hénon maps.

One-dimensional maps

Main model: the family of quadratic maps of the interval

$$f(x) = a - x^2 \qquad a \in (0, 2)$$

Two main types of behaviour are known, depending on the parameter a:

- **periodic:** Lebesgue almost every orbit converges to a periodic one.
- **stochastic:** there exists an invariant probability μ absolutely continuous with respect to Lebesgue measure.

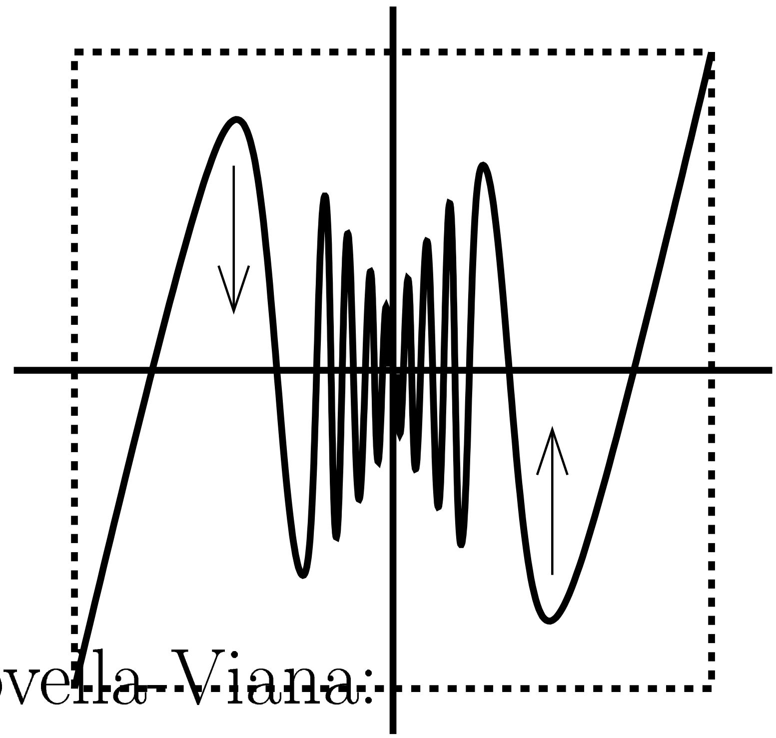
Jakobson: $\{stochastic\}$ has positive Lebesgue probability in parameter space.

Swiatek, Lyubich: $\{periodic\}$ is dense (and open) in parameter space.

Lyubich: $\{periodic\} \cup \{stochastic\}$ has total measure in parameter space.

In particular, almost every quadratic map has a unique SRB measure.

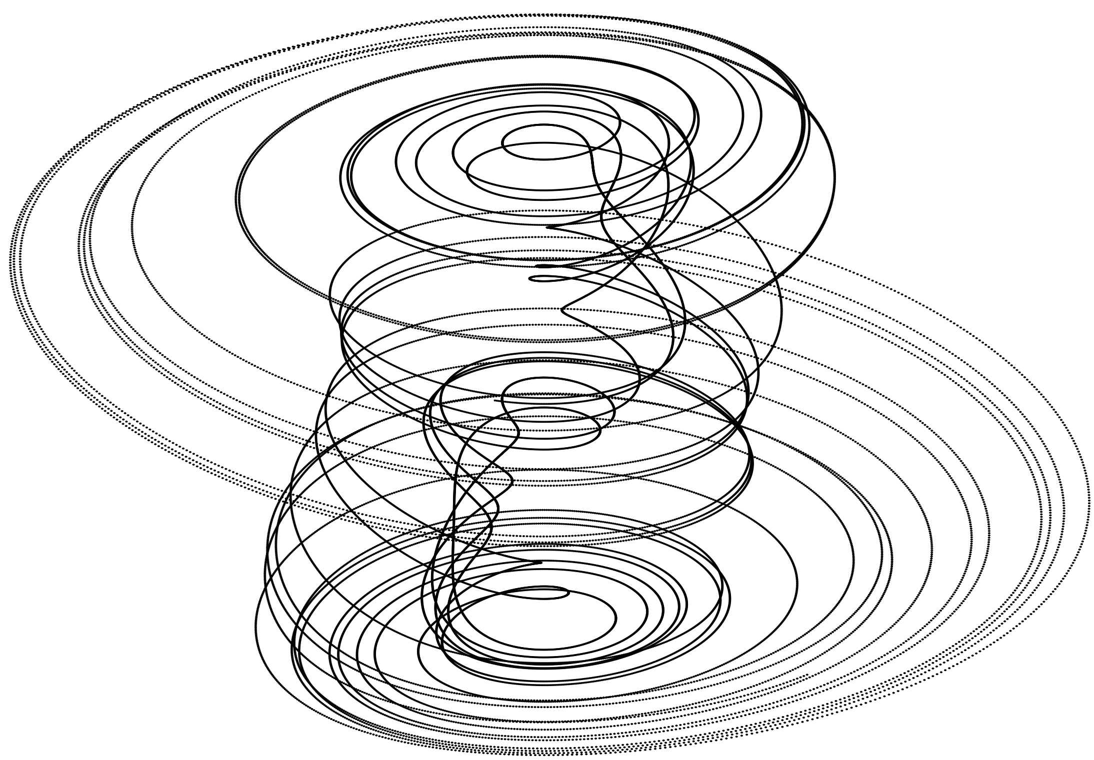
Infinite-modal maps



Pacifico-Rovella-Viana:

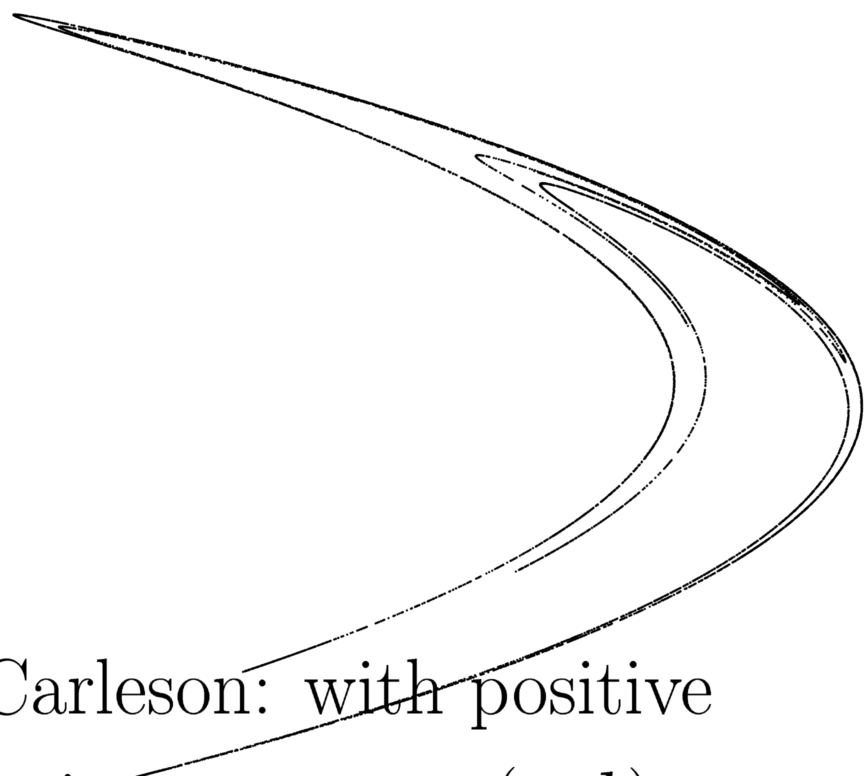
stochastic behaviour has positive Lebesgue probability in parameter space.

This has applications for 3-dim flows: existence of spiral attractors



Hénon-like maps

Model: $f(x,y) = (1 - ax^2 + y, bx)$



Benedicks-Carleson: with positive probability in parameter (a,b) space, f has a non-hyperbolic attractor A.

Then,

Benedicks-Young: the attractor A supports a unique SRB measure μ.

Benedicks-Viana: there are no holes in the basin of the attractor:

$$B(\mu) = B(A)$$

up to zero Lebesgue measure sets.

And (f, μ) is stochastically stable.

Multidimensional attractors

Viana: **open** sets of diffeomorphisms with **multidimensional** attractors of Hénon type (in dimension ≥ 4).

- several expanding directions (positive Lyapunov exponents), at almost every point in the basin of attraction
- topological dimension > 1.

Non-invertible examples exist already in dimension ≥ 2:

For instance, in the cylinder $S^1 \times \mathbb{R}$,

$$\varphi(x, y) = (16\,x, a(x) - y^2)$$

with $a(x)$ a convenient Morse function.

Alves: Every map C^3-close to φ has a unique SRB measure.

This measure is absolutely continuous with respect to Lebesgue measure.

A global scenario

A program towards a global view of dynamical systems was proposed a few years ago by Palis:

Main conjecture

Every system can be approximated by another having only finitely many attractors, which are **nice**, and whose basins cover almost all of the ambient space M.

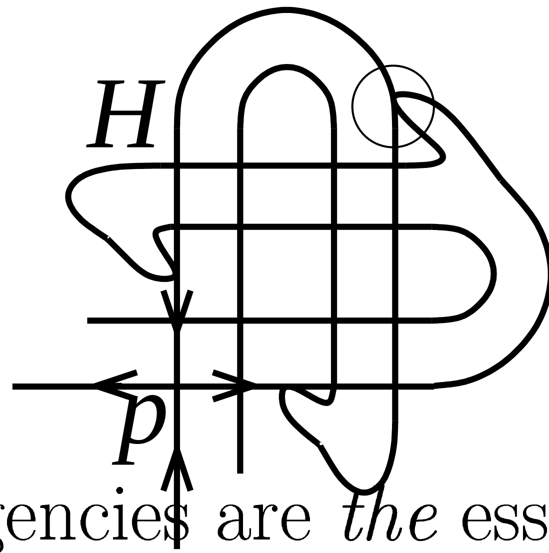
Nice attractors:

- support SRB measures, and have no holes in their basins of attraction;
- the dynamics restricted to each basin of attraction is stochastically stable.

Moreover,

for almost all small perturbations of such systems, 99% of M (in measure) is covered by the basins of a finite number of nice attractors.

Homoclinic tangencies



In dim 2, tangencies are *the* essential obstruction to hyperbolicity (Palis):

Pujals-Sambarino: every diffeomorphism on a surface can be C^1-approximated by another either uniformly hyperbolic or with a homoclinic tangency.

Unfolding the tangency: which dynamics are *typical* in families of maps f_t, $t \in \mathbb{R}$, through $f_0 = f$?

Works by Newhouse, Palis, Takens, and Yoccoz disclosed a link between fractal dimensions of the set H and frequency of hyperbolicity in parameter space.

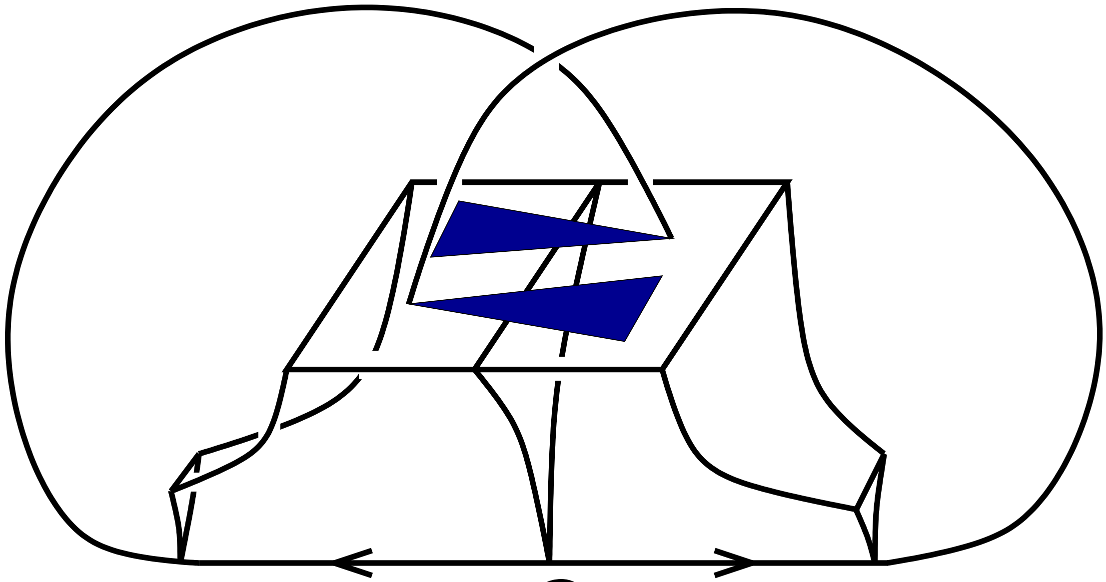
Moreira-Yoccoz: most parameters (full Lebesgue density at $t = 0$) correspond to

- either uniform hyperbolicity
- or persistent homoclinic tangencies.

Flows : Lorenz-like attractors

Lorenz (1963) exhibited a 3-dimensional o.d.e. whose solutions seemed to depend sensitively on the initial point.

Geometric models for this behaviour were proposed in the seventies by Afraimovich-Bykov-Shil'nikov and by Guckenheimer-Williams:



These models have attractors A that are

singular: A contains a singularity (equilibrium) O, besides regular orbits

transitive: A contains dense orbits

robust: every C^1-near flow exhibits a similar attractor

Morales-Pacifico-Pujals: in dimension 3,

A is robustly singular transitive

$$\Downarrow$$

A is **singular hyperbolic**, and either an attractor or a repeller

Singular hyperbolicity: dominated invariant splitting

$$T_\Lambda M = E^1 \oplus E^2$$

E^1 contracting, E^2 volume expanding, either for X^t or for X^{-t}

Tucker: The original Lorenz equations
do exhibit a robust strange attractor
(for the classical parameter values).

Flows in higher dimensions

Are there Lorenz-like attractors with
dimension $W^u(O) > 1$?

Bonatti-Pumariño-Viana: Yes.
In some manifolds of any dim ≥ 4.
They have a unique SRB measure.

Partial hyperbolicity

$f : M \to M$ is **partially hyperbolic** if there exists an invariant splitting

$$TM = E^1 \oplus E^2$$

such that

1. the splitting is **dominated**

$$\|Df \mid E^1_z\| \, \|Df^{-1} \mid E^2_{f(z)}\| < 1$$

2. either E^1 is uniformly contracting or E^2 is uniformly expanding.

Shub, Mañé, Bonatti-Díaz constructed robustly transitive diffeomorphisms in dimension ≥ 3 that are not uniformly hyperbolic (Anosov); all their examples are partially hyperbolic.

Ergodic properties of these maps have been studied by Brin, Pesin, Sinai, Carvalho, Kan, Grayson, Pugh, Shub, Wilkinson

If f is C^1 robustly transitive then

Mañé: if dimension $M = 2$

f is uniformly hyperbolic (Anosov)

Díaz-Pujals-Ures: if dimension $M = 3$

f is partially hyperbolic

Bonatti-Díaz-Pujals: in any case

f admits a dominated splitting

The statements are best possible, even for volume preserving diffeomorphisms.

Towards an ergodic theory of partially hyperbolic systems

So, let me present two general theorems on existence & finitude of SRB measures for partially hyperbolic diffeomorphisms.

The **central direction** E^c is the one invariant subbundle, either E^1 or E^2, that fails to be hyperbolic (contracting or expanding).

Bonatti-Viana:

Suppose E^c is **mostly contracting:** vectors in the central direction are non-uniformly contracted (negative Lyapunov exponents) as $n \to +\infty$, at Lebesgue almost all points.

- Then f has finitely many SRB measures, whose basins cover a full Lebesgue probability subset of M.

Alves-Bonatti-Viana:

Suppose E^c is **mostly expanding:** non-uniform expansion of all central vectors as $n \to +\infty$

$$\lim_{n\to+\infty} \frac{1}{n} \log \prod_{j=1}^{n} \|Df^{-1}|E^c_{f^j(z)}\| < 0$$

Lebesgue almost everywhere.

- Lebesgue almost every point $z \in M$ is in the basin of some SRB measure of the map f.
- If the limit is bounded away from

zero, the SRB's are finitely many.

Final theme

These results suggest that *non-uniform hyperbolicity may suffice* for the system to have good statistical properties.

In this direction, I state

Conjecture: *If a smooth map has only non-zero Lyapunov exponents, Lebesgue almost everywhere, then it admits some SRB measure.*