Homoclinic bifurcations and persistence of nonuniformly hyperbolic attractors

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1 Introduction

Let $\varphi: M \longrightarrow M$ be a general smooth transformation on a riemannian manifold. A main object of study in Dynamics is the asymptotic behaviour of the orbits $\varphi^n(z) = \varphi \circ \cdots \circ \varphi(z)$, $z \in M$, as time n goes to infinity. Typical forms of behaviour – occurring for "many" $z \in M$ – are, of course, of particular relevance and this leads us to the notion of attractor. By an *attractor* we mean a (compact) φ-invariant set $\Lambda \subset M$ which is dynamically indivisible and whose basin – the set of points $z \in M$ for which $\varphi^n(z) \to \Lambda$ as $n \to +\infty$ – is a large set. Dynamical indivisibility can be expressed by the existence of a dense orbit in Λ (if Λ supports a "natural" φ-invariant measure, one may also require that φ be ergodic with respect to such measure). As for the basin, it must have positive Lebesgue volume or, even, nonempty interior; in all the cases we will consider here the basin actually contains a full neighbourhood of the attractor.

In addition, we want to focus on forms of asymptotic behaviour which are typical also from the point of view of the dynamical system: we call an attractor *persistent* if it occurs for a large set of maps near φ. "Large" is to be understood in this context in a measure-theoretical sense: positive Lebesgue measure set of parameter values in every generic family of transformations containing φ. On the other hand, stronger forms of persistence – e.g. with "large set" meaning a full neighbourhood of φ – hold in some important situations to be described below.

In the simplest case, Λ reduces to a single periodic orbit of φ. While the presence of a large or, even more so, an infinite number of these periodic attractors – possibly with high periods and strongly intertwined basins – may render the behaviour of individual orbits rather unpredictable, rich asymptotic dynamics comes more often associated with the presence of nonperiodic attractors (having, in many cases, an intrincate geometric structure). Indeed, there is a large amount of numerical evidence for the occurrence of such nontrivial attractors in a wide range of situations in Dynamics, from mathematical models of complex natural phenomena to even the simplest abstract nonlinear systems. A striking feature of many of these systems is the phenomenon of *exponential sensitivity with respect to initial conditions*: typical (pairs of) orbits of nearby points move away from

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each other exponentially fast as they approach the attractor. Note the profound consequences: measurement imprecisions and round-off errors tend to be amplified under iteration and so, in practice, the long-term behaviour of trajectories in the basin of the attractor is unpredictable (or "chaotic").

A conceptual framework for the understanding of such *chaotic dynamics* is currently under active development. Two main general problems in this context are, to describe the (dynamical, geometric, ergodic) structure of chaotic attractors and, to identify the mechanisms responsible for their formation and persistence. A fairly complete solution to these problems is known in the special case of uniformly hyperbolic (or Axiom A) attractors, see e.g. [Sh2], [Bo], and this is a basic ingredient here. On the other hand, uniform hyperbolicity *per se* is seldom observed in dynamical systems arising from actual phenomena in the experimental sciences, where sensitivity with respect to initial conditions is quite more often related to nonuniformly hyperbolic behaviour. This last notion can be defined as follows. We say that φ has *Lyapounov exponents* $\lambda_1, \dots, \lambda_l$ at $z \in M$ if the tangent space may be split $T_z M = E_1 \oplus \cdots \oplus E_l$ in such a way that

$$\lim_{n \to +\infty} \frac{1}{n} \log \|D\varphi^n(z)v\| = \lambda_j \quad \text{for every } v \in E_j \backslash \{0\} \text{ and } 1 \leq j \leq l.$$

By Oseledec's theorem such a splitting exists at almost every point, relative to any finite φ-invariant measure. Then we call the system *nonuniformly hyperbolic* [Pe], if $\lambda_j \neq 0$ for all j and for almost all points (with respect to the relevant measure under consideration); see [Pe]. Note that occurrence of some positive Lyapounov exponent corresponds precisely to (infinitesimal) exponential sensitivity around the trajectory of z. Also, in the situations to be considered here, existence of positive Lyapounov exponents is the key ingredient for nonuniform hyperbolicity, the fact that all the remaining exponents are strictly negative then following from elementary considerations.

The dynamics of nonuniformly hyperbolic attractors is, in general, rather unstable under perturbations of the system and this means that more subtle mechanisms of dynamical persistence occur in this general context than in the Axiom A case (where persistence comes along with structural stability and is, ultimately, an instance of transversality theory). The comprehension of such mechanisms is then directly related to the general study of bifurcations of dynamical systems. This is, in fact, the departing point of the program towards a theory of sensitive dynamics recently proposed by Palis and underlying Section 2 below. The basic strategy is to focus on a convenient set of well-defined bifurcation processes – this set should be dense among all (non-Axiom A) systems exhibiting interesting dynamical behaviour – and to determine which are the persistent forms of dynamics in generic parametrized families unfolding such bifurcations (once more, persistence is meant in the sense of positive Lebesgue measure of parameter values). See e.g. [PT] for precise formulations and an extended discussion.

A central role is played here by the processes of *homoclinic bifurcation* – that is, creation and/or destruction of transverse intersections between the stable and the unstable manifolds of a same hyperbolic saddle, see Figure 1 – which, by themselves, encompass all presently known forms of interesting behaviour in this setting of discrete dynamical systems. Study of homoclinic bifurcations and of their interplay with other main processes of dynamical modification provides a most promising scenario for the understanding of complicated asymptotic behaviour, specially in low-dimensions, and in Section 2 we discuss some of the results already substantiating this scenario.

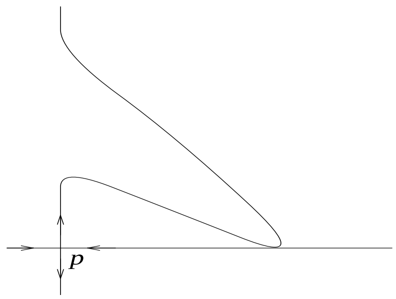

Figure 1: A homoclinic tangency

On the other hand, several of these results actually extend to manifolds of arbitrary dimension and this is an area of considerable ongoing progress. A very interesting topic is the construction and analysis of the properties of *multidimensional* nonuniformly hyperbolic attractors. By multidimensionality we mean existence of several directions of stretching, i.e. several positive Lyapounov exponents (this also implies that the attractor has topological dimension larger than 1). A discussion of recent developments and open problems on this topic occupies most of Section 3.

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2 Bifurcations and attractors

Jakobson's theorem [Ja] provided the first rigorous situations of persistence of chaotic dynamics in a strictly nonuniformly hyperbolic setting: *for a positive measure set of values of* $a \in (1,2)$ *the quadratic real map* $q_a(x) = 1 - ax^2$ *admits an invariant probability measure* μ_a *which is absolutely continuous with respect to the Lebesgue measure*. Moreover, μ_a *is ergodic and has positive Lyapounov exponent:*

$$\lim_{n\to+\infty} \frac{1}{n} \log |Dq_a^n| = \int \log |Dq_a| \, d\mu_a > 0, \quad \mu_a - \textit{almost everywhere}.$$

On the other hand, Benedicks-Carleson [BC] proved that complicated behaviour is also abundant in another important nonlinear model, the Hénon family of diffeomorphisms of the plane $H_{a,b}(x,y) = (1 - ax^2 + by, x)$: *for a positive measure set of parameter values* $H_{a,b}$ *exhibits a compact invariant set* $\Lambda_{a,b} \subset \mathbb{R}^2$ *(the closure of the unstable manifold of a fixed saddle-point) satisfying*

(i) The basin $W^s(\Lambda_{a,b}) = \{z \in \mathbb{R}^2 \colon H_{a,b}^n(z) \to \Lambda_{a,b} \text{ as } n \to +\infty\}$ contains a neighbourhood of $\Lambda_{a,b}$;

(ii) There exists $\hat{z} \in \Lambda_{a,b}$ whose orbit $\{H^n_{a,b}(\hat{z}) : n \geq 0\}$ is dense in $\Lambda_{a,b}$.

Moreover, *this dense orbit may be taken exhibiting a positive Lyapounov exponent:*

(iii) $\left\| DH^n_{a,b}(\hat{z})u \right\| \geq c\sigma^n$ for some $c > 0$, $\sigma > 1$ and $u \in \mathbb{R}^2$ and all $n \geq 0$;

(iv) $\left\| DH^n_{a,b}(\hat{z})v \right\| \to 0$ as $|n| \to \infty$ for some $v \in \mathbb{R}^2$, $v \neq 0$ (and so $\Lambda_{a,b}$ is not uniformly hyperbolic).

A stronger formulation of the sensitivity property (iii) is contained in the construction by [BY] of an SBR-measure $\mu_{a,b}$ supported on the "strange" attractor $\Lambda_{a,b}$: $H_{a,b}$ has a positive Lyapounov exponent $\mu_{a,b}$-almost everywhere (and at every point in a positive Lebesgue volume subset of the basin). An alternative construction of these SBR-measures also giving new information on the geometry of the attractor is being provided in [JN].

Let us outline the mechanism yielding positive Lyapounov exponents in these two situations. A common feature to these and other important models is the combination of fairly hyperbolic behaviour, in most of the dynamical space, with the presence of *critical regions* where hyperbolicity breaks down. In the case of q_a the critical region is just the vicinity of the critical point $x = 0$, where the map is strongly contracting. For Hénon maps, criticality corresponds to the "folding" occurring near $x = 0$, which obstructs the existence of invariant cone fields. Then the proofs of the previous results require a delicate control on the recurrence of the critical region, in order to prevent nonhyperbolic effects from accumulating too strongly. In the 1-dimensional case, for instance, one must impose a convenient lower bound on $|q^n_a(0)|$ for each $n > 0$. This translates into a sequence of conditions on the parameter, which are part of the definition of the positive measure set in the statement. The argument is rather more complex in the Hénon case but it still follows the same basic strategy of *control of the recurrence through exclusion of parameter values*. The dynamical persistence displayed by the maps one gets after these exclusions is all the more remarkable in view of their instability: while an arbitrarily small perturbation of the parameter may destroy the chaotic attractor (e.g. creating periodic attractors, see [Ur]), it is a likely event (positive probability) that the attractor will actually remain after the perturbation.

Departing from these models, we now discuss a number of results and open problems leading to a quantitative and qualitative description of the occurrence of attractors in the general setting of homoclinic bifurcations. Let us begin by defining this setting in a more precise way than we did before. We consider generic smooth families of diffeomorphisms $\varphi_\mu : M \longrightarrow M$, $\mu \in \mathbb{R}$, such that φ_0 exhibits some nontransverse intersection between the stable and the unstable manifolds of a hyperbolic saddle-point p, recall Figure 1. In this section we take M to be a surface. Genericity means that this *homoclinic tangency* is nondegenerate – quadratic – and unfolds generically with the parameter: the two invariant manifolds move with respect to each other with nonzero relative speed, near the tangency. We also suppose $|\det D\varphi_0(p)| \neq 1$ and in what follows we consider $|\det D\varphi_0(p)| < 1$ (in the opposite case just replace φ_μ by φ_μ^{-1}). Then, see e.g. [TY], return-maps to a neighbourhood of the tangency contain small perturbations of the family of singular maps $(x,y) \mapsto (1-ax^2, 0)$. Combining this fact with an extension of the methods in [BC] one can prove that *Hénon-like attractors* – i.e. satisfying (i)-(iv) above – occur in a persistent way whenever a homoclinic tangency is unfolded:

Theorem 1 [MV] *There exists a positive Lebesgue measure set of values of μ, accumulating at $\mu = 0$, for which φ_μ has Hénon-like attractors close to (in a* const $|\mu|$*-neighbourhood of) the orbit of tangency.*

This should also be compared with the well-known theorem of Newhouse on abundance of periodic attractors:

Theorem 2 [Ne] *There exist intervals $I \subset \mathbb{R}$ accumulating at $\mu = 0$ and residual (Baire second category) subsets $B \subset I$ such that for every $\mu \in B$ the diffeomorphism φ_μ has infinitely many periodic attractors close to the orbit of tangency.*

These two contrasting forms of asymptotic behaviour are, actually, strongly interspersed: the values of μ one gets in both the proofs of these results are accumulated by other parameters corresponding to new homoclinic tangencies, [Ur].

Problem 1: (Palis) Can any diffeomorphism exhibiting a Hénon-like attractor, resp. infinitely many periodic attractors, be approximated by another one having a homoclinic tangency ?

Problem 2: Can Newhouse's phenomenon occur for a set S of parameter values with positive Lebesgue measure ?

The answer to Problem 2 is usually conjectured to be negative but it is as yet unknown. Note that the sets B constructed in the proof of Theorem 2 have zero measure, [TY]. An interesting related question is formulated replacing above "positive Lebesgue measure" by "positive Lebesgue density at $\mu = 0$", that is

$$\lim_{\varepsilon \to 0} \frac{m(S \cap [-\varepsilon, \varepsilon])}{2\varepsilon} > 0, \qquad m = \text{Lebesgue measure.}$$

Ongoing progress seems to indicate that the answer to this last question is negative, even if one replaces S by the set of parameter values corresponding to existence of *some* periodic attractor near the tangency. A similar problem can be posed for nonuniformly hyperbolic attractors as in Theorem 1:

Problem 3: Can Hénon-like attractors occur for a set of parameter values having positive density at $\mu = 0$?

While this last problem remains open in the context of homoclinic tangencies, it admits a complete, positive answer in a closely related setting of bifurcations: the unfolding of critical saddle-node cycles. By a *saddle-node k-cycle*, $k \geq 1$, of a diffeomorphism φ we mean a finite set of periodic points $p_1, p_2, \ldots, p_k$ such that

- p_1 is a saddle-node (eigenvalues 1 and λ, with $|\lambda| < 1$) and p_i is a hyperbolic saddle for each $2 \leq i \leq k$;
- $W^u(p_{i-1})$ and $W^s(p_i)$ have points of transverse intersection, for all $2 \leq i \leq k$, and $W^u(p_k)$ intersects the interior of $W^s(p_1)$.

Following [NPT], we call the saddle-node cycle *critical* if $W^u(p_k)$ has a nontransverse intersection with some leaf F of the strong stable foliation of $W^s(p_1)$. Figure 2 describes such a cycle in the case $k = 1$ (in this case we actually require $W^u(p_1)$ to be contained in the interior of $W^s(p_1)$).

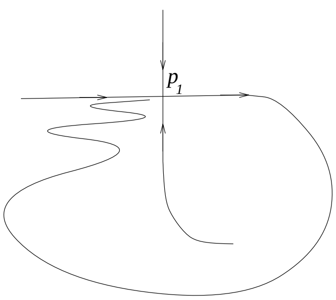

Figure 2: A critical saddle-node cycle

Now we consider the unfolding of such cycles by generic families of diffeomorphisms $\varphi_\mu: M \longrightarrow M$, $\mu \in \mathbb{R}$. More precisely, we suppose that φ_0 has some critical saddle-node cycle satisfying a few mild assumptions: the saddle-node is nondegenerate and unfolds generically with the parameter μ and the criticality – i.e. the nontransverse intersection between $W^u(p_k)$ and F – is quadratic. A theorem of [NPT] asserts that such families always go through homoclinic tangencies, at parameter values arbitrarily close to zero. A converse is also true (Mora): critical saddle-node cycles are formed whenever a homoclinic tangency is unfolded. On the other hand, the present setting is special in that Hénon-like attractors always occur for a *positive fraction* of the parameter values near the one corresponding to the cycle. This is the only bifurcation mechanism known to exhibit such a strong accumulation by chaotic attractors.

Theorem 3 [DRV1] *Let $(\varphi_\mu)_\mu$ be a generic family of diffeomorphisms unfolding a critical saddle-node cycle as above. Then the set of parameter values for which φ_μ exhibits Hénon-like attractors has positive Lebesgue density at $\mu = 0$.*

The proof of Theorem 3 is based on a combination of Theorem 1 with a careful analysis of the distribution of homoclinic tangencies in parameter space, cf. previous remarks. This construction yields Hénon-like attractors which are related to orbits of homoclinic tangency and so have a semi-local nature. While this is unavoidable in the generality of the statement above, attractors of a much more global type can be found in some relevant cases, by using a more direct approach. We mention the case of 1-cycles, recall Figure 2. If φ_0 has a critical 1-cycle then it is not difficult to find a compact domain R containing $W^u(p_1)$ and such that $\varphi_0(R) \subset \text{interior}(R)$. Then, for a sizable portion of the parameter values near zero the asymptotic behaviour of all the points in the domain R (which depends only on the bifurcating diffeomorphism φ_0) is driven by a *unique, global,* nonuniformly hyperbolic attractor:

Theorem 4 [DRV2] *For an open class of families $(\varphi_\mu)_\mu$ unfolding a critical saddle-node 1-cycle, there is a set of values of μ with positive Lebesgue density at $\mu = 0$ for which $\Lambda_\mu = \bigcap_{n\geq 0} \varphi_\mu^n(R)$ is a Hénon-like attractor.*

3 Multidimensional expansion

The unfolding of homoclinic tangencies or saddle-node cycles in higher dimensions leads, more often, to the formation of periodic points with positive unstable index (some expanding eigenvalue) and/or of "strange saddles", see [Ro]. In order to have attractors one makes an assumption of (local) *sectional dissipativeness*: the product of any two of the eigenvalues associated to the saddle p exhibiting the tangency, resp. to the saddle-node p_1 involved in the cycle, has norm less than 1. On the other hand, under this assumption Theorems 1–4 do generalize to manifolds of arbitrary dimension, see [PV], [V1]. In particular, persistent Hénon-like attractors may occur in any ambient manifold.

Now, the attractors one finds in such a sectionally dissipative setting are special in that they exhibit at most one direction of stretching (one single positive Lyapounov exponent). This is also related to the fact that the Hénon-like attractors in the previous paragraph always have topological dimension 1. Our goal in this section is to present a construction of persistent nonuniformly hyperbolic attractors with multidimensional character: *typical orbits in their basin exhibit several stretching directions*. In more precise terms, at Lebesgue almost every point z in the basin there is a splitting $T_zM = E^+ \oplus E^-$ such that

$$\liminf \frac{1}{n} \log \left\| D\varphi^n(z)v^+ \right\| > 0 > \limsup \frac{1}{n} \log \left\| D\varphi^n(z)v^- \right\| \text{ for } v^\pm \in E^\pm \backslash \{0\}$$

and $\dim E^+ > 1$. Previously known examples restricted to rather structured situations, such as Axiom A diffeomorphisms or the persistently transitive examples in [Sh1] or [Ma]. In these last examples, obstruction to uniform hyperbolicity comes from the presence of saddles with different stable indices but the dynamics is actually fairly uniform (in particular, they admit everywhere-defined continuous invariant cone fields).

These examples of *multidimensional attractors* we now describe are the first ones in the presence of *critical* behaviour (in the sense of Section 2). In fact, the basic idea here is to couple nonuniform models such as Hénon maps, with convenient uniformly hyperbolic systems. On the other hand, the attractors we obtain in this way are considerably more robust than the low-dimensional Hénon-like ones: *they persist in a whole open set of diffeomorphisms*. Let us sketch this construction in a simple situation, details being provided in [V2]. We start by considering diffeomorphisms of the form

$$\varphi\colon \mathrm{T}_3 \times \mathbb{R}^2 \longrightarrow \mathrm{T}_3 \times \mathbb{R}^2, \quad \varphi(\Theta, x, y) = (g(\Theta), f(\Theta, x, y))$$

where g is a solenoid map on the solid torus $\mathrm{T}_3 = \mathrm{S}^1 \times \mathrm{B}^2$, see e.g. [Sh2], and $f(\Theta, x, y) = (a(\Theta) - x^2 + by, -bx)$. Here b is a small positive number, a is some nondegenerate function (e.g. a Morse function) with $1 < a(\Theta) < 2$ and we take the solenoid to be sufficiently expanding along the S^1-direction. Then, for an appropriate choice of these objects, φ is contained in an open set of diffeomorphisms exhibiting a multidimensional nonuniformly hyperbolic attractor:

Theorem 5 [V2] *There is a compact domain $K \subset \mathbb{R}^2$ such that for every diffeomorphism $\psi: T_3 \times \mathbb{R}^2 \longrightarrow T_3 \times \mathbb{R}^2$ sufficiently close to φ (in the C^3-sense) $\psi(T_3 \times K) \subset \text{interior}(T_3 \times K)$ and ψ has two stretching directions at Lebesgue almost every point of $T_3 \times K$.*

A crucial fact distinguishing these examples from the quadratic models in Section 2 is that their critical regions are too large for the same kind of recurrence control as we described there to be possible in the present situation. In order to motivate this remark we observe that in the (singular) limit $b = 0$ the critical set of φ coincides with $\{\det D\varphi = 0\}$, a codimension-1 submanifold, and, therefore, is bound to have robust intersections with (some of) its iterates. In other words, close *returns* of the critical region back to itself cannot be avoided by any sort of parameter exclusions, which means that we are forced to deal with the accumulation of contracting/nonhyperbolic effects associated to such returns. This is done through a statistical type of argument which we can (very roughly) sketch as follows. Given $z \in T_3 \times \mathbb{R}^2$, the nonhyperbolic effect introduced at each time $\nu \geq 1$ for which $\varphi^\nu(z)$ is close to the critical region is estimated in terms of an appropriate integrable function $\Delta_\nu(z)$. The definition of $\Delta_\nu(z)$ in the actual situation of Theorem 5 – with $b > 0$ – is fairly complicated and we just mention that in the (much simpler) limit case $b = 0$ one may take $\Delta_\nu(z) = -\log|x_\nu|$, where x_ν is the x-coordinate of $\varphi^\nu(z)$. Then we derive two crucial stochastic properties of these Δ_ν:

1. the expected (i.e. average) value of Δ_ν is small for each $\nu \geq 1$;
2. the probability distributions of Δ_μ and Δ_ν are (fairly) independent from each other if $|\mu - \nu|$ is large enough.

This allows us to use probabilistic arguments (of large deviations type) to conclude that, for most trajectories, the overall nonhyperbolic effect corresponding to iterates near the critical region is smaller than (i.e. dominated by) the hyperbolic contribution coming from the iterates taking place away from that region.

The proof of 2 above is based on the fast decay of correlations exhibited by uniformly hyperbolic systems such as solenoids and, in fact, this seems to be the key property of the map g for what concerns our construction (in its present form the proof makes use of a few other properties of solenoid maps, in an apparently less important way). This suggests that a similar type of argument should apply if the solenoid is replaced in the construction above by more general (not necessarily uniformly hyperbolic) maps having such fastly mixing character. As a first step in this direction we pose

Problem 4: Prove that $\varphi(x, y) = (g(x), a(x) - y^2)$ has two positive Lyapounov exponents for a large set of choices of $a(x)$, where g is some convenient – possibly multimodal – smooth transformation of the real line exhibiting chaotic behaviour in the sense of Jakobson's theorem.

Finally, in the view of the discussion in the Introduction, one should try and relate the present topic with the general study of bifurcations of higher-dimensional smooth systems, in the spirit of Section 2. Again, a first step may be

Problem 5: Describe generic bifurcation mechanisms leading to the formation of multidimensional nonuniformly hyperbolic attractors.

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