FLAVORS OF PARTIAL HYPERBOLICITY

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Abstract. The notion of partial hyperbolicity has been used in the literature in various forms, not all equivalent. We discuss some of these variations, to advocate that the "right" definition should be *existence of a pointwise dominated decomposition into at least two invariant subbundles, one of which is hyperbolic (either expanding or contracting).* To support this point of view, we include proofs of Hölder continuity and absolute continuity of the hyperbolic laminations, under the pointwise domination assumption.

1. Introduction

1. There has long been a consensus on the proper definition of uniform hyperbolicity: the tangent space admits a Df-invariant splitting into two subbundles, restricted to one of which the derivative is eventually exponentially contracting while restricted to the other it is eventually exponentially expanding. Introduced by Smale [**?**] some forty years ago, this notion lead to a surprisingly precise theory of a class of dynamical systems whose behavior is often very complex. It was also at the very heart of Anosov's celebrated proof that the geodesic flow on manifolds with negative curvature is ergodic [**?**].

A few years later, Brin, Pesin [**?**] proposed to weaken the hyperbolicity assumption to what they called *partial hyperbolicity*: roughly speaking, one allows for a Df-invariant subbundle which is neither expanding nor contracting, in addition to the hyperbolic ones. In particular, they proved that such systems admit invariant foliations tangent to their hyperbolic subbundles which are absolutely continuous, meaning that their holonomy maps (projections between cross-sections along the leaves of the foliation) preserve the class of zero Lebesgue measure sets. The corresponding statement for hyperbolic systems had been the crucial technical ingredient in Anosov's argument. Shortly afterwards, Hirsch, Pugh, Shub carried out a thorough investigation of the related notion of *normal hyperbolicity*, which culminated in their book [**?**].

2. Much more recently, in the last decade or so, a series of key developments put the notion of partial hyperbolicity back at the forefront of Dynamics. Initially, there were three main projects, which have been gradually merging to one another.

One was the retaking by Pugh, Shub (see [**?**]) of Brin, Pesin's original program: to prove (stable) ergodicity for typical volume preserving systems under weak hyperbolicity assumptions. This benefitted from important contributions from several other mathematicians, especially Wilkinson, Burns, Dolgopyat, Pesin, Nitica, Torok, Tahzibi, F. Rodriguez-Hertz, J. Rodriguez-Hertz, and Ures. Detailed surveys can be found in [**?**, **?**, **?**], as well as [**?**, Chapter 8].

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Another one (see [**?**, Chapters 7 and 9]), was the search for a characterization of robust (stable) transitivity for diffeomorphisms and flows, and their attractors and repellers. Shub and Mañé had shown that robustly transitive diffeomorphisms need not be hyperbolic, and other examples were exhibited more recently by Bonatti, Díaz, Viana. A fundamental recent development was due to Bonatti, Diaz, Morales, Pacifico, Pujals, Ures [**?**, **?**, **?**], who proved that some form of partial hyperbolicity, or just the existence of a dominated invariant splitting, is *necessary* for robust transitivity. Here one deals mostly with dissipative systems, but works of Arbieto, Matheus [**?**] and Horita, Tahzibi [**?**] treat the conservative case as well, enhancing the connection with the issues in the previous paragraph, and the next one.

Yet another active program (see [**?**, Chapter 11]), was the development of an ergodic theory of partially hyperbolic dissipative systems: proofs of existence and finiteness of physical (Sinai-Ruelle-Bowen) measures for some classes of such systems were provided by Alves, Bonatti, Viana [**?**, **?**] and some their results have been much sharpened by Tsujii [**?**] in the case of 2-dimensional maps. Some recent progress suggests there is some significant connection with the study of cocycles over hyperbolic systems (see [**?**, Chapter 12]).

3. It must be noted, however, that these results often use somewhat different definitions of partial hyperbolicity. Let $f : M \to M$ be a diffeomorphism and $\Lambda \subset M$ be a compact f-invariant set. Brin and Pesin's [**?**] original definition was in terms of the push-forward operator

$$f_* : \mathcal{X}(\Lambda) \to \mathcal{X}(\Lambda), \quad f_*X(y) = Df(f^{-1}(y))X(f^{-1}(y))$$

in the space of continuous vector fields on Λ: they called Λ *partially hyperbolic* if the spectrum of the linear operator f_* splits into three subsets contained in disjoint open annuli and *at least two of which are non-empty.* Then there exists an f_*-invariant spectral decomposition $\mathcal{E}^u \oplus \mathcal{E}^c \oplus \mathcal{E}^s$ of $\mathcal{X}(\Lambda)$, and, hence, a Df-invariant splitting

$$T_\Lambda M = E^u \oplus E^c \oplus E^s, \quad E^*_x = \{X_x : X \in \mathcal{E}^*\}$$

of the tangent bundle over the invariant set Λ, where at least two of the subbundles have positive dimension.

4. Then Hirsch, Pugh, Shub [**?**] proposed no less than four different definitions of partial hyperbolicity. In all of them one asks for the existence of a Df-invariant splitting $T_\Lambda M = F \oplus E$ where F dominates E, meaning that $Df \mid F$ is more expanding/less contracting than $Df \mid E$, and one of the two subbundles is uniformly hyperbolic: If E is uniformly contracting, we say that the splitting $E \oplus F$ is of *strong stable type* and write E^{cu} for F and E^s for E. If F is uniformly expanding we call the splitting of *strong unstable type* and write E^u for F and E^{cs} for E. The variations between different definitions reside in the precise formulation of the idea of domination, or normal hyperbolicity: *absolute* versus *relative* (or pointwise) and *eventual* versus *immediate.*

Absolute partial hyperbolicity means that, given any two points in the set, the behavior of the derivative along the subbundle E is uniformly stronger than the behavior along the subbundle F: A compact invariant set Λ is called *(eventually) absolutely partially hyperbolic* if there exists a Df-invariant splitting $F \oplus E$ of $T_\Lambda M$, a Riemmann metric on M, and constants $C > 0$ and $\lambda \in (0, 1)$ such that (ad) the splitting $F \oplus E$ is *absolutely dominated*: for every ξ and η in Λ

$$\|Df^{-n} \mid F_\xi\| \, \|Df^n \mid E_\eta\| \le C\lambda^n,$$

(s∨u) either $Df \mid E$ is uniformly contracting or $Df \mid F$ is uniformly expanding: either $\|Df^n \mid E_\xi\| \le C\,\lambda^n$ for every $\xi \in \Lambda$ and $n \in \mathbb{N}$ or $\|Df^{-n} \mid F_{f^n(\xi)}\| \le C\,\lambda^n$ for every $\xi \in \Lambda$ and $n \in \mathbb{N}$.

Relative partial hyperbolicity, which Michael Herman liked to call *Brazilian partial hyperbolicity*, only requires pointwise domination: a compact invariant set Λ is called *(eventually) pointwise partially hyperbolic* if there exists a Df-invariant splitting $F \oplus E$ of $T_\Lambda M$, a Riemann metric on M, and constants $C > 0$ and $\lambda \in (0,1)$ such that:

(pd) the splitting $F \oplus E$ is *pointwise dominated*: for every $\xi \in \Lambda$ and $n \in \mathbb{N}$ we have

$$\|Df^n \mid E_\xi\| \, \|Df^{-n} \mid F_{f^n(\xi)}\| \le C\,\lambda^n$$

(s∨u) either $Df \mid E$ is uniformly contracting or $Df \mid F$ is uniformly expanding, in the same sense as before.

Notice that, in either case, changing the Riemann metric only amounts to choosing a different constant C, the definition is otherwise unaffected. Then we also have stronger versions of these definitions, called *immediate absolute partial hyperbolicity* and *immediate relative partial hyperbolicity*, where one requires the constant C to be equal to 1, for some choice of the metric.

5. Actually, Hirsch, Pugh, Shub show, in [**?**, Proposition 2.2], that immediate and eventual *absolute* partial hyperbolicity are equivalent conditions, and that they are also equivalent to the spectral definition of Brin, Pesin [**?**]. Surprisingly, the equivalent question for *pointwise* partial hyperbolicity is much more subtle. Some special cases can be done easily (see [**?**, page 5]), but the general problem was solved only very recently: Gourmelon [**?**] shows that any eventually partially hyperbolic set is immediately partially hyperbolic for some choice of the metric. Accordingly, in what follows we drop the distinction immediate/eventual altogether.

On the other hand, it is easy to see that absolute partial hyperbolicity is strictly stronger than its pointwise counterpart. One simple construction goes as follows. Start with a hyperbolic set (a horseshoe) of a surface diffeomorphism and consider two of its periodic points. Deform the diffeomorphism near one of the points so as to make its expanding eigenvalue become 1, while keeping the other eigenvalue bounded from 1. Then perform a dual deformation near the other periodic point, so as to make its contracting eigenvalue equal to 1, while keeping the other eigenvalue bounded from 1. This can be done in such a way that the new diffeomorphism still has an invariant set, topologically conjugate to the original horseshoe, admitting a Df-invariant splitting of its tangent bundle and this splitting is pointwise dominated. However, the presence of eigenvalues equal to 1 along both subbundles prevents the splitting from being absolutely dominated.

Clearly, this kind of example is not robust. That is in the nature of things in low dimensions: according to a theorem of Pujals, Sambarino [**?**], pointwise dominated decompositions of generic surface diffeomorphisms are actually uniformly hyperbolic and, thus, absolutely partially hyperbolic. On the other hand, using ideas from [**?**] for instance, one can easily construct robust examples of transitive diffeomorphisms on $M = T^3$, say, with pointwise partially hyperbolic splitting $E^u \oplus E^{cs}$

with $\dim E^u = 1$ and $\dim E^{cs} = 2$ and such that

$$\|Df(p) \mid E^u_p\| < 2 \quad \text{and} \quad \|Df(q) \mid E^{cs}_q\| > 3$$

for some fixed points p and q. See Figure 1. Then the splitting is not absolutely partially hyperbolic.

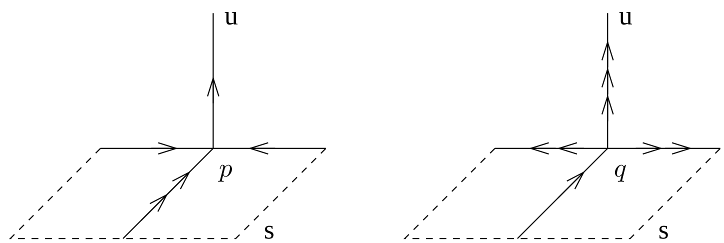

FIGURE 1. Pointwise dominated but not absolutely dominated splitting

6. Both the absolute and the relative/pointwise flavors of the definition have been used in the literature, and the distinction is not always clearly made. Not surprisingly, works where partial hyperbolicity is part of the conclusions tend to adopt the relative point of view. In most cases, this is actually necessary: For instance, the assertions in [**?**, **?**, **?**] that robust transitivity implies partial hyperbolicity or, at least, existence of a dominated splitting of the tangent bundle, would not hold in the absolute setting. Another such example is the work of Bochi, Viana [**?**, **?**] showing that, generically, the Lyapunov exponents of conservative maps and $SL(d)$-cocycles vanish, or else there exists an invariant dominated splitting. Once more, the statement must be in terms of pointwise domination.

In contrast, works where partial hyperbolicity and domination are part of the hypotheses tend to use the stronger absolute definition. This is usually much less justified [1]. Indeed, it should be possible to establish all main results under the, more general, pointwise domination condition. In view of the global picture of the field we have been presenting, it is highly desirable to do so. This point of view is adopted, for instance, in the recent work of Burns, Wilkinson [**?**] where the authors prove Pugh-Shub's ergodicity conjecture, under a center-bunching hypothesis.

As some evidence in favor of this point of view we are advocating, in this paper we give a proof of the absolute continuity theorem in [**?**, **?**] for relatively partially hyperbolic systems. While the statement will not surprise the experts, being well-known in the absolute case, a proof does not seem to be available in the literature, and that may be a reason why authors sometimes shy away from setting their results in greater generality. Let us mention that the statement we give here has already been used in the previously mentioned work of Burns, Wilkinson [**?**]. Under the same pointwise assumption, we also prove two other general features of strong invariant (strong-stable and strong-unstable) foliations, namely that they have Hölder continuous tangent bundles and holonomy maps.

7. Before getting into the statements, let us mention there is yet another variation in the definitions used in the literature. This concerns the number (either two or

[1] Occasionally, one reads: "Partial hyperbolicity as defined here is an absolute concept. Most of what we prove, however, remains valid when the system is relatively partially hyperbolic"...

three) of required invariant subbundles. We say that a compact invariant set Λ is *partially hyperbolic with three subbundles* if there exists a Df-invariant splitting $E^u \oplus E^c \oplus E^s$ of $T_\Lambda M$, a Riemmann metric on M, and constants $C > 0$ and $\lambda \in (0,1)$ such that:

(d) $E^u \oplus E^c \oplus E^s$ is *dominated*: both $(E^u \oplus E^c) \oplus E^s$ and $E^u \oplus (E^c \oplus E^s)$ are (absolutely or pointwise) dominated.

(s∧u) $Df \mid E^s$ is uniformly contracting *and* $Df \mid E^u$ is uniformly expanding.

Thus, Λ is simultaneously of strong-stable type and strong-unstable type:

As we mentioned before, the original definition in Brin, Pesin [**?**] asked for three invariant subbundles, *at least two of which have positive dimension.* That also turned out to be the right choice for the characterization, given by Díaz, Pujals, Ures [**?**], of robustly transitive diffeomorphisms in 3-manifolds. Indeed, robust transitivity may coexist with periodic points exhibiting complex eigenvalues, and the presence of such periodic points is an obstruction to the existence of three invariant subbundles ([**?**] also shows that this is the *sole* obstruction in dimension 3).

On the other hand, the approach to proving ergodicity via accessibility, proposed by Pugh, Shub [**?**], naturally calls for the existence of both a strong-stable subbundle and a strong-unstable subbundle. In view of the remarkable effectiveness of this approach, it would be interesting to devise a counterpart for partially hyperbolic systems with only two invariant subbundles. A natural candidate to replace accessibility is minimality of the strong invariant foliation, that was first addressed from this perspective in [**?**] and has been investigated in [**?**, **?**].

8. Henceforward, *partial hyperbolicity* will always be meant in the *relative (pointwise)* sense. We say that a subbundle E is *Hölder continuous* if, in the neighborhood of any point, there are Hölder continuous linearly independent vector fields spanning E.

Proposition A. *Let Λ be a partially hyperbolic set for a C^2 diffeomorphism f. Then the corresponding invariant subbundles E and F are Hölder continuous. If Λ is partially hyperbolic with three subbundles E^u, E^s, E^c then they are all Hölder continuous.*

Corresponding statements in the absolute case can be found in [**?**, Corollary 2.1] and [**?**, Theorem 6.4]. The next result concerns the transverse regularity of strong-stable and strong-unstable foliations:

Proposition B. *Let Λ be a partially hyperbolic set of strong-unstable (respectively, strong-stable) type for a C^1 diffeomorphism f. Then the local holonomy maps of the strong-unstable (respectively, strong-stable) foliation of Λ are Hölder continuous. If Λ is partially hyperbolic with three subbundles then both holonomies, strong-stable and strong-unstable, are Hölder continuous.*

A similar statement appeared in [**?**, Theorem A']. We include a proof, since it is short and prepares the way for the next result. An example of Wilkinson [**?**] shows that the integral foliation of a Hölder continuous subbundle needs not be transversely Hölder continuous (i.e., the holonomy might not be Hölder). Therefore, Proposition B is not a consequence of Proposition A, even in the C^2 case.

Finally, we state the absolute continuity theorem. This fundamental property was first established by Anosov, Sinai [**?**, **?**] for uniformly hyperbolic systems, as a main step in the proof that the geodesic flow on manifolds with negative curvature is ergodic. Pugh-Shub [**?**] and Brin-Pesin [**?**, Theorem 3.1] extended it to absolutely partially hyperbolic systems. We check that it holds under our weaker (pointwise) partial hyperbolicity condition.

Theorem C. *Suppose f is a C^2 partially hyperbolic diffeomorphism of strong-stable type. Then for any local holonomy map $\pi : \Sigma_1 \to \Sigma_2$ of the strong-stable foliation of f there exists $K > 0$ such that*

$$\frac{1}{K}\, m_{\Sigma_1}(B) \le m_{\Sigma_2}(\pi(B)) \le K m_{\Sigma_1}(B) \quad \text{for any measurable set } B \subset \Sigma_1,$$

where m_{Σ_i} is the Riemannian volume induced on the cross-section Σ_i, $i = 1, 2$.

The assumption means that $\Lambda = M$ is a partially hyperbolic set for f, of strong-stable type. The same arguments hold when Λ is a partially hyperbolic attractor, of strong-stable type. Taking the inverse map, one gets dual results for the strong-unstable foliation of partially hyperbolic diffeomorphisms and repellers of strong-unstable type.

2. Hölder Continuity

We begin by proving Propositions A and B. For both of them, we may suppose that Λ is of strong-unstable type: the strong-stable case follows, replacing f by its inverse, and the situation with three subbundles is easily reduced to the one with two subbundles, by writing

$$E^u \oplus E^c \oplus E^s = E^{cu} \oplus E^s = E^u \oplus E^{cs}$$

with $E^{cu} = E^u \oplus E^c$ and $E^{cs} = E^c \oplus E^s$. Note, in addition, that the statements do not change when one replaces f by some iterate f^N, with N large. Up to doing this right from the start, we may suppose that the conditions in the definition of partial hyperbolicity hold with $C = 1$: there is $\lambda < 1$ such that

$$(1) \qquad \|Df^{-1} \mid E^u_x\| \le \lambda \quad \text{and} \quad \|Df^{-1} \mid E^u_{f(x)}\| \, \|Df \mid E^{cs}_x\| \le \lambda \quad \text{for all } x \in \Lambda.$$

2.1. Invariant Subbundles.

Here we prove Proposition A. Let $\lambda < 1$ be as in (1) and $\varepsilon > 0$ be small enough so that $\lambda^2 + 2\varepsilon < \lambda - \varepsilon$. Extend E^u and E^{cs} continuously to a neighborhood of Λ, then let F^u and F^{cs} be C^1-approximations such that for every $x \in \Lambda$ the derivative $Df(x) : F^{cs}_x \oplus F^u_x \to F^{cs}_{f(x)} \oplus F^u_{f(x)}$ is given by a matrix

$$\begin{pmatrix} A_x & B_x \\ C_x & D_x \end{pmatrix}$$

where A_x and D_x are invertible matrices such that $\|D_x^{-1}\|$ and $\|D_{f(x)}^{-1}\| \, \|A_x\|$ are bounded by $\lambda + \varepsilon$, while $\|B_x\|$ and $\|C_x\|$ are bounded by ε. Let $\mathcal{E}$ be the C^1 bundle over Λ whose fiber at $x \in \Lambda$ is the space $\mathcal{L}(F^u_x, F^{cs}_x)$ of linear maps $L : F^u_x \to F^{cs}_x$, endowed with the usual operator norm. Set $\mathcal{E}(1) = \{(x, L) \in \mathcal{E} : \|L\| \le 1\}$. Consider now the linear graph transform $\Gamma : \mathcal{E}(1) \to \mathcal{E}$ given by

$$\Gamma(x, L) = (f(x), \Gamma_x(L)), \quad \Gamma_x(L) = (B_x + A_x\, L) \circ (L\, C_x + D_x)^{-1}$$

A direct calculation (see [**?**, pp 62–64]) shows that Γ maps $\mathcal{E}(1)$ into $\mathcal{E}(1)$, and every Γ_x contracts the fiber of $\mathcal{E}$ by a uniform factor

$$\kappa = \frac{\lambda^2 + 2\varepsilon}{\lambda - \varepsilon} < 1.$$

Thus, Γ admits a continuous invariant section $\sigma : \Lambda \to \mathcal{E}(1)$. This invariant section yields precisely the unstable direction: $E^u_x = \mathrm{graph}(\sigma(x))$ for all $x \in \Lambda$. We shall show that σ is Hölder continuous (Lemma 1 below).

Via the exponential chart we identify $T_\Lambda M$ locally with $\mathbb{R}^d \times (\mathbb{R}^u \oplus \mathbb{R}^{cs})$, where $\mathbb{R}^d$ corresponds to a neighborhood in M and $\mathbb{R}^u \oplus \mathbb{R}^{cs}$ corresponds to the decomposition of $T_x M$ as the direct sum $F^u \oplus F^{cs}$. We endow $\mathbb{R}^d \times (\mathbb{R}^u \oplus \mathbb{R}^{cs})$ with the product metric. Having made this identification, we endow the bundle $\mathcal{E}$ with the metric

$$d_{\mathcal{E}}\big((x,L),(y,K)\big) = \max\{d(x,y), \|L-K\|\}.$$

Note that Γ is of class C^1, because F^u and F^{cs} are C^1 and the map f is C^2. In particular, Γ is α-Hölder for every $\alpha \in (0,1)$. Set $\mu = \mathrm{Lip}(f^{-1}|_\Lambda)$ and fix $\alpha > 0$ small enough so that

$$\kappa\, \mu^\alpha < 1.$$

Then let $C > 0$ be such that $d_{\mathcal{E}}(\Gamma(x,L),\Gamma(y,K)) \le C\, d_{\mathcal{E}}((x,L),(y,K))^\alpha$ for any (x,L) and (y,K) in $\mathcal{E}$, that is, Γ is (C,α)-Hölder.

Lemma 1. *We have* $d_{\mathcal{E}}(\sigma(x),\sigma(y)) \le \dfrac{C\,\mu^\alpha}{1-\kappa\mu^\alpha}\, d(x,y)^\alpha$ *for all* $x, y \in \Lambda$.

Proof. (The estimates are lifted straight out of [**?**, p. 46]). We are going to show

$$d_{\mathcal{E}}(\sigma(x),\sigma(y)) \le \kappa^n\, d_{\mathcal{E}}\big(\sigma(f^{-n}x),\sigma(f^{-n}y)\big) + C\sum_{j=1}^{n} \mu^{\alpha j}\,\kappa^{j-1} d(x,y)^\alpha. \tag{2}$$

for every $n \ge 1$ and any $x, y \in \Lambda$. The case $n = 1$ is given by

$$\begin{aligned}
d_{\mathcal{E}}(\sigma(x),\sigma(y)) &= d_{\mathcal{E}}\big(\Gamma_{f^{-1}x}(\sigma(f^{-1}x)),\Gamma_{f^{-1}y}(\sigma(f^{-1}y))\big)\\
&\le d_{\mathcal{E}}\big(\Gamma_{f^{-1}x}(\sigma(f^{-1}x)),\Gamma_{f^{-1}x}(\sigma(f^{-1}y))\big)\\
&\qquad + d_{\mathcal{E}}\big(\Gamma_{f^{-1}x}(\sigma(f^{-1}y)),\Gamma_{f^{-1}y}(\sigma(f^{-1}y))\big)\\
&\le \kappa\, d_{\mathcal{E}}\big(\sigma(f^{-1}x),\sigma(f^{-1}y)\big) + C\, d(f^{-1}x, f^{-1}y)^\alpha\\
&\le \kappa\, d_{\mathcal{E}}\big(\sigma(f^{-1}x),\sigma(f^{-1}y)\big) + C\,\mu^\alpha\, d(x,y)^\alpha.
\end{aligned} \tag{3}$$

Now we use induction to show that the claim holds for all n: combining (2) with (3) (with x and y replaced by $f^{-n}(x)$ and $f^{-n}(y)$)

$$\begin{aligned}
d_{\mathcal{E}}(\sigma(x),\sigma(y)) &\le \kappa^n\, d_{\mathcal{E}}\big(\sigma(f^{-n}x),\sigma(f^{-n}y)\big) + C\sum_{j=1}^{n}\mu^{\alpha j}\,\kappa^{j-1}\, d(x,y)^\alpha\\
&\le \kappa^n\Big[\kappa d_{\mathcal{E}}\big(\sigma(f^{-(n+1)}x),\sigma(f^{-(n+1)}y)\big) + C\,\mu^{\alpha(n+1)}\, d(x,y)^\alpha\Big]\\
&\qquad + C\sum_{j=1}^{n}\mu^{\alpha j}\,\kappa^{j-1}\, d(x,y)^\alpha\\
&\le \kappa^{n+1} d_{\mathcal{E}}\big(\sigma(f^{-(n+1)}x),\sigma(f^{-(n+1)}y)\big) + C\sum_{j=1}^{n+1}\mu^{\alpha j}\,\kappa^{j-1}\, d(x,y)^\alpha,
\end{aligned}$$

which means that (2) holds for $n+1$. This completes the proof of our claim. Now, since $\sum_{j=0}^{\infty}(\mu^\alpha\,\kappa)^j = 1/(1-\kappa\,\mu^\alpha)$ and $\kappa < 1$, taking the limit in (2) we obtain that

$$d_{\mathcal{E}}(\sigma(x),\sigma(y)) \le \frac{C\,\mu^\alpha}{1-\kappa\,\mu^\alpha}\, d(x,y)^\alpha,$$

as stated. □

This proves that E^u is Hölder continuous. The proof of Hölder continuity for E^{cs} is analogous, iterating backwards instead of forwards.

2.2. **Holonomy Maps.** Now we prove Proposition B. Let λ be as in (1) and $\varepsilon > 0$ be such that $e^{4\varepsilon} < \lambda^{-1}$. Then take $\delta > 0$ and $a > 0$ small enough so that

$$\|Df^{-1} \mid E^u_x\| \le e^{\varepsilon}\|Df^{-1} \mid E^u_y\| \quad \text{and} \quad \|Df \mid V\| \le e^{\varepsilon}\|Df \mid E^{cs}_y\|, \tag{4}$$

for all $x, y \in \Lambda$ with $d(x,y) \le 2\delta$ and any subspace V of T_xM contained in the *center-stable cone*

$$C_a(E^{cs}, x) = \{v^u + v^{cs} \in E^u_x \oplus E^{cs}_x : \|v^u\| \le a\|v^{cs}\|\}$$

of width a. Extend E^u and E^{cs} continuously to a neighborhood U of Λ, small enough so that the extended center-stable cone field remains invariant. Reducing U and δ if necessary, we may suppose that U contains the 2δ-neighborhood of Λ and conditions (1) and (4) remain true for every $x, y \in U$ with $d(x,y) \le 2\delta$.

We want to prove that any local holonomy map π between cross-sections Σ_1 and Σ_2 satisfies a Hölder condition. We use $d_u(\cdot\,,\cdot)$ to represent distance measured along leaves of $\mathcal{F}^u$. Notice $d_u(\cdot\,,\cdot) \ge d(\cdot\,,\cdot)$. It is no restriction to suppose that Σ_1 and Σ_2 are nearby, and their tangents are close to E^{cs} at each point, in the following sense:

(a) $d_u(x, \pi(x)) \le \delta$ for every $x \in \Sigma_1$ and

(b) $T_x\Sigma_i \subset C_a(E^{cs}, x)$ for every $x \in \Sigma_i$, $i = 1, 2$.

Indeed, properties (a) and (b) can always be enforced by considering backward iterates $f^{-n}(\Sigma_1)$ and $f^{-n}(\Sigma_2)$, with n large, in the place of Σ_1 and Σ_2 . Since the foliation $\mathcal{F}^u$ is invariant under f, the holonomy ρ_n from $f^{-n}(\Sigma_1)$ to $f^{-n}(\Sigma_2)$ is given by $\rho_n = f^{-n} \circ \pi \circ f^n$. Hence, π is Hölder continuous if and only if ρ_n is, so that replacing the initial cross-sections does not affect the validity of the conclusion.

Set $\delta_1 = \delta/L$, where $L > 1$ is a Lipschitz constant for f and its inverse. Let $d_\Sigma(\cdot\,,\cdot)$ represent distance measured along some cross-section to the foliation (it will always be clear from the context which one is meant). Notice $d_\Sigma(\cdot\,,\cdot) \ge d(\cdot\,,\cdot)$.

In what follows we denote $x_2 = \pi(x_1)$ and $y_2 = \pi(y_1)$.

By similar considerations to those made above, we may, taking the cone width $a > 0$ sufficiently small and up to replacing Σ_1 and Σ_2 by sufficiently large pre-iterates by f, assume that the following "triangle inequality" holds for all $n \in \mathbb{N}$:

$$d_{\Sigma_2}(f^{-n}x_2, f^{-n}y_2) \le d_u(f^{-n}x_1, f^{-n}x_2) + d_{\Sigma_1}(f^{-n}x_1, f^{-n}y_1) + d_u(f^{-n}y_1, f^{-n}y_2).$$

Since $\mathcal{F}^u$ is a *continuous* foliation, we may fix $\delta_2 > 0$ small enough so that

$$d_\Sigma(x_1, y_1) < \delta_2 \quad \Rightarrow \quad d_\Sigma(\pi(x_1), \pi(y_1)) < \delta_1, \tag{5}$$

for any $x_1, y_1 \in \Lambda \cap \Sigma_1$ and any cross-sections Σ_1 and Σ_2 satisfying (a)–(b).

Lemma 2. *There exists $C > 0$, depending on δ and δ_2 and, given any cross-sections Σ_1 and Σ_2 satisfying (a)–(b) and any points $x_1, y_1 \in \Lambda \cap \Sigma_1$ with $d_\Sigma(x_1, y_1) \le \delta_1$, there exists $n \le 4\log_\lambda d_\Sigma(x_2, y_2)$ such that*

(i) *$d_\Sigma(f^{-j}(x_2), f^{-j}(y_2)) \le \delta$ for all $0 \le j \le n$, and*

(ii) *$d_\Sigma(f^{-n}(x_2), f^{-n}(y_2)) \le C d_\Sigma(f^{-n}(x_1), f^{-n}(y_1))$.*

Proof. Let us treat first the case when $d_\Sigma(f^{-j}(x_2), f^{-j}(y_2))$ is less than δ_1 for all $0 \le j \le 4\log_\lambda d_\Sigma(x_2, y_2)$. Fix any n between $2\log_\lambda d_\Sigma(x_2, y_2)$ and $4\log_\lambda d_\Sigma(x_2, y_2)$. Then we may use (4) and (b), for the backward iterates of the cross-sections, to conclude that

$$d_u(f^{-n}(x_1), f^{-n}(x_2)) \le \prod_{j=0}^{n-1} \left(\|Df^{-1} \mid E^u_{f^{-j}(x_2)}\| e^{\varepsilon}\right) d_u(x_1, x_2)$$

$$d_\Sigma(f^{-n}(x_2), f^{-n}(y_2)) \ge \prod_{j=1}^{n} \left(\|Df \mid E^{cs}_{f^{-j}(x_2)}\| e^{\varepsilon}\right)^{-1} d_\Sigma(x_2, y_2).$$

Together with (a) and the domination property in (1), this yields

$$\frac{d_u(f^{-n}(x_1), f^{-n}(x_2))}{d_\Sigma(f^{-n}(x_2), f^{-n}(y_2))} \le (\lambda e^{2\varepsilon})^n \frac{d_u(x_1, x_2)}{d_\Sigma(x_2, y_2)} \le \lambda^{n/2} \frac{\delta}{d_\Sigma(x_2, y_2)} \le \delta.$$

and, analogously,

$$d_u(f^{-n}(y_1), f^{-n}(y_2)) \le \delta d_\Sigma(f^{-n}(x_2), f^{-n}(y_2)).$$

Then, by the aforementioned "triangle inequality",

$$d_\Sigma(f^{-n}(x_1), f^{-n}(y_1)) \ge (1 - 2\delta) d_\Sigma(f^{-n}(x_2), f^{-n}(y_2)),$$

and this implies claim (ii), as long as we take $\delta \le 1/4$ and $C \ge 2$. Claim (i) is obvious from the construction. Now suppose that, on the contrary, there exists $n \le 4\log_\lambda d_\Sigma(x_2, y_2)$ such that $d_\Sigma(f^{-n}(x_2), f^{-n}(y_2))$ is larger than δ_1. Take such an n minimum. Then,

$$d_\Sigma(f^{-n}(x_2), f^{-n}(y_2)) \le L d_\Sigma(f^{-n+1}(x_2), f^{-n+1}(y_2)) \le L\delta_1 = \delta.$$

Thus, claim (i) is satisfied. Moreover, by (5),

$$d_\Sigma(f^{-n}(x_1), f^{-n}(y_1)) \ge \delta_2.$$

Together with the previous inequality, this implies claim (ii), as long as we take $C \ge \delta/\delta_2$. The proof of the lemma is complete. □

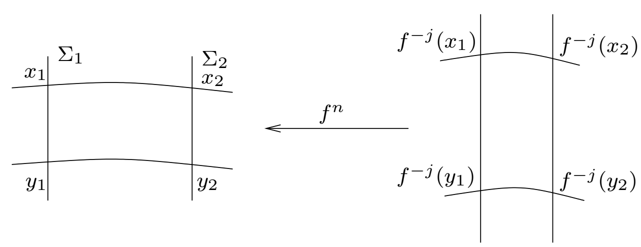

FIGURE 2. Proving holonomies are Hölder

The following corollary contains Proposition B:

Corollary 3. *There are constants $K > 0$ and $\gamma \in (0, 1]$, depending on δ and δ_2, such that*

$$d_\Sigma(x_2, y_2) \le K d_\Sigma(x_1, y_1)^\gamma$$

for any x_1 and y_1 in $\Lambda \cap \Sigma_1$ and any cross-sections Σ_1 and Σ_2 satisfying (a)–(b).

Proof. Clearly, we may restrict ourselves to the case when $d_\Sigma(x_1, y_1) \le \delta_1$. In view of part (i) of Lemma 2, we are in a position to apply (4) and get

$$d_\Sigma(x_2, y_2) \le \prod_{j=1}^{n} \big(\|Df \mid E^{cs}_{f^{-j}(x_2)}\| e^{\varepsilon}\big) d_\Sigma(f^{-n}(x_2), f^{-n}(y_2)).$$

We also need a lower bound for $d_\Sigma(x_1, y_1)$ which is obtained in a similar fashion. We divide the argument into two cases:

Case 1: We have $d_\Sigma(f^j(x_1), f^j(y_1)) < \delta$ for all $j \in [-n, 0]$.

Conditions (1) and (a) imply

$$d_u(f^{-j}(y_1), f^{-j}(y_2)) \le \lambda^j d_u(y_1, y_2) \le \lambda^j \delta \le \delta \tag{6}$$

for every $j \ge 0$, and analogously for $d_u(f^{-j}(x_1), f^{-j}(x_2))$. Then, using (4) and the mean value theorem for f^{-1},

$$d_\Sigma(x_1, y_1) \ge \prod_{j=0}^{n-1} \big(\|Df^{-1} \mid E^{cs}_{f^{-j}(x_2)}\| e^{\varepsilon}\big)^{-1} d_\Sigma(f^{-n}(x_1), f^{-n}(y_1)).$$

(Note that here we use the hypothesis $d_{\Sigma_1}(x_1, y_1) \le L^{-1}\delta$ in order to guarantee that $d(x_2, z) \le 2\delta$ for every z in the geodesic from x_1 to y_1 in Σ_1.)

Let

$$K_1 = e^{2\varepsilon} \sup\{\|Df \mid E^{cs}_\eta\| \, \|Df^{-1} \mid E^{cs}_{f(\eta)}\| \, : \eta \in \Lambda\}.$$

The previous inequalities, combined with (ii), give

$$\frac{d_\Sigma(x_2, y_2)}{d_\Sigma(x_1, y_1)} \le \frac{d_\Sigma(f^{-n}(x_2), f^{-n}(y_2))}{d_\Sigma(f^{-n}(x_1), f^{-n}(y_1))} K_1^n \le C\, K_1^n.$$

Since $n \le 4 \log_\lambda d_\Sigma(x_2, y_2)$, this implies

$$\frac{d_\Sigma(x_2, y_2)}{d_\Sigma(x_1, y_1)} \le C\, d_\Sigma(x_2, y_2)^\theta,$$

where $\theta = 4 \log K_1 / \log \lambda < 0$. The claim then follows with $\gamma' = 1/(1-\theta) < 1$ and $K' = C^{\gamma'}$.

Case 2: there is some $j \in [-n, 0]$ such that $d_\Sigma(f^j(x_1), f^j(y_1)) \ge \delta$.

In this case we have

$$d_{\Sigma_1}(x_1, y_1) \ge \frac{\delta}{L^n}$$

since $L > 1$ is a Lipschitz constant for f.

We also have that $d_\Sigma(x_2, y_2) \le \delta$ and hence

$$\frac{d_\Sigma(x_2, y_2)}{d_\Sigma(x_1, y_1)} \le \frac{L^n\, \delta}{\delta} = L^n$$

Since $n \le 4 \log_\lambda d_\Sigma(x_2, y_2)$, this implies as in Case 1 above that there is $\gamma'' < 1$ such that

$$d_\Sigma(x_2, y_2) \le d_\Sigma(x_1, y_1)^{\gamma''}$$

Setting $\gamma = max\{\gamma', \gamma''\}$ and $K = max\{K', 1\}$ finishes the proof.

□

The proof of Proposition B is complete.

Remark 4. The Hölder constants provided by the proof depend only on the distance between the cross-sections Σ_1 and Σ_2 , and the angle they make with E^{cs} (through the iterate $n \geq 1$ required to obtain properties (a) and (b) in the proof).

Remark 5. If E^{cs} has dimension 1 or, more generally, if Df is conformal in the direction E^{cs}

$$\|Df^{-1} \mid E^{cs}_{f(\xi)}\| = \|Df \mid E^{cs}_{\xi}\|^{-1}$$

then $K_1 = e^{2\varepsilon}$ and so $\theta = 8\varepsilon/\log\lambda$. So, in this case the Hölder constant γ may be taken arbitrarily close to 1. Compare Palis, Viana [**?**], where a similar conclusion is obtained for hyperbolic sets of C^1 diffeomorphisms. This has a number of interesting consequences: for instance, the attractor Λ has well-defined transverse Hausdorff dimension. See Palis, Takens [**?**].

3. Absolute Continuity

Here we prove that invariant strong-stable (or strong-unstable) foliations of partially hyperbolic C^2 diffeomorphisms are absolutely continuous.

Let $f : M \to M$ be the diffeomorphism and $TM = E^{cu} \oplus E^s$ be the Df-invariant splitting (remark: for convenience here we deal with the $E^{cu} \oplus E^s$ case; the $E^u \oplus E^{cs}$ follows from considering f^{-1}, as usual) Let $\mathcal{F}^s$ be the strong-stable foliation, tangent to E^s at every point. We are going to prove

Theorem 6. *For any holonomy map $\pi : \Sigma_1 \to \Sigma_2$ of $\mathcal{F}^s$ there exists a constant $K > 0$ such that*

$$\frac{1}{K} < \frac{m_{\Sigma_1}(D)}{m_{\Sigma_2}(\pi(D))} < K$$

for any disk $D \subset \Sigma_1$.

Theorem C is a direct consequence. Indeed, although the conclusion of Theorem 6 seems weaker, because it refers to disks instead of general measurable sets, it is quite easy to deduce the full statement. Given any measurable set B let $\mathcal{D}$ be any family of disks covering B. By Theorem 6,

$$m_{\Sigma_2}(\pi(B)) \leq \sum_{D\in\mathcal{D}} m_{\Sigma_2}(\pi(D)) \leq K \sum_{D\in\mathcal{D}} m_{\Sigma_1}(D).$$

Since $\mathcal{D}$ may be taken such that $\sum_{D\in\mathcal{D}} m_{\Sigma_1}(D)$ is arbitrarily close to $m_{\Sigma_1}(B)$, it follows that $m_{\Sigma_2}(\pi(B)) \leq K m_{\Sigma_1}(B)$. Similarly, using the inverse holonomy map $\pi^{-1} : \Sigma_2 \to \Sigma_1$, we obtain the left-hand inequality. This shows that Theorem C does follow from Theorem 6.

3.1. **Outline of the proof.** Before starting the proof of Theorem 6, let us explain what are the main steps. Instead of trying to compare the volumes of D and $\pi(D)$ directly, one looks at iterates $f^n(D) \subset f^n(\Sigma_1)$ and $f^n(\pi(D)) \subset f^n(\Sigma_2)$ for some large $n \geq 1$. The point is that $f^n(\Sigma_1)$ and $f^n(\Sigma_2)$ are very close to each other, because Σ_1 and Σ_2 are transverse to the strong-stable foliation $\mathcal{F}^s$, and the leaves of $\mathcal{F}^s$ are exponentially contracted by f^n. This makes it possible to compare the volumes of appropriate subsets of $f^n(\Sigma_1)$ and $f^n(\Sigma_2)$ using the holonomy map $\pi_n = \pi(f^n(\Sigma_1), f^n(\Sigma_2))$. More precisely, we consider balls $\mathcal{B}(n,x) \subset f^n(\Sigma_1)$ around each $f^n(x) \in f^n(\Sigma_1)$, with radius $r(n,x)$ chosen in a judicious way. One important condition is that $r(n,x)$ *should be much larger than the distance between $f^n(x)$ and*

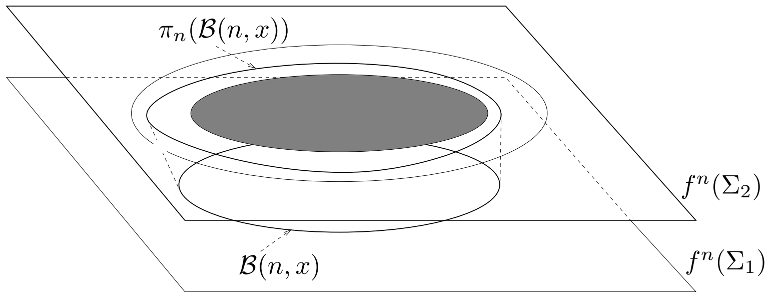

FIGURE 3. Volumes of large balls are almost preserved by the holonomy of nearby cross-sections

$\pi_n(f^n(x))$*:* this ensures that the volume of $\mathcal{B}(n,x)$ is approximately equal to the volume of $\pi_n(\mathcal{B}(n,x))$:

$$\frac{m_{f^n(\Sigma_1)}\big(\mathcal{B}(n,x)\big)}{m_{f^n(\Sigma_2)}\big(\pi_n(B(n,x))\big)}$$

is uniformly close to 1. See Figure 3.

One then considers preimages of the domain $\mathcal{B}(n,x)$ under f^n. The volume of $f^{-n}(\mathcal{B}(n,x))$ is given by the volume of $\mathcal{B}(n,x)$ divided by the Jacobian $J_{\Sigma_1} f^n(\xi)$ of f^n along Σ_1 , at some point ξ in $f^{-n}(\mathcal{B}(n,x))$. Another main condition is that $r(n,x)$ *should be small enough* so that the Jacobians at different points of $f^{-n}(\mathcal{B}(n,x))$ are comparable, up to some factor close to 1. Then, one may take $\xi = x$. Similarly, the volume of

$$f^{-n}(\pi_n(\mathcal{B}(n,x))) = \pi(f^{-n}(\mathcal{B}(n,x)))$$

equals the volume of $\pi_n(\mathcal{B}(n,x))$ divided by the Jacobian $J_{\Sigma_2} f^n(\eta)$ at some point η, and one may take $\eta = \pi(x)$. Now, $J_{\Sigma_1} f^n(\xi)$ and $J_{\Sigma_2} f^n(\eta)$ are comparable up to some factor bounded from zero and infinity. The main reason for this is that $\xi = x$ and $\eta = \pi(x)$ are in the same strong-stable leaf, and so their forward iterates remain forever close. One also needs a Hölder-type estimate for the tangent spaces to iterates of Σ_1 and Σ_2; it is in order to have this Hölder estimate that we require f to be C^2. It follows that the volumes of $f^{-n}(\mathcal{B}(n,x))$ and $\pi(f^{-n}(\mathcal{B}(n,x)))$ are comparable:

$$\frac{m_{\Sigma_1}\big(f^{-n}(\mathcal{B}(n,x))\big)}{m_{\Sigma_2}\big(\pi(f^{-n}(B(n,x)))\big)} \approx \frac{m_{f^n(\Sigma_1)}\big(\mathcal{B}(n,x)\big)}{m_{f^n(\Sigma_2)}\big(\pi(B(n,x))\big)} \frac{J_{\Sigma_2} f^n(\eta)}{J_{\Sigma_1} f^n(\xi)}$$

is uniformly bounded from zero and infinity.

Finally, to prove that the same is true for the volumes of D and $\pi(D)$, it suffices to find an efficient covering of D by sets of the form $f^{-n}(\mathcal{B}(n,x))$: there is a uniform upper bound for the number of elements of the covering that contain a given point. That bound is provided by Besicovich's covering lemma.

3.2. **Preliminaries.** Now, let us get into the details of the proof. According to the definition of partial hyperbolicity, there exist $m \geq 1$ and $\theta < 1$ such that

(a) $\|Df^m \mid E^s_x\| < \theta^2$ and
(b) $\|Df^m \mid E^s_x\| < \theta^2 \, \|Df^{-m} \mid E^{cu}_{f^m(x)}\|^{-1}$

for every $x \in \Lambda$. Up to replacing f by f^m, we may suppose that $m = 1$, and we do so in all that follows. Let us define

$$\tilde{a}(x) \equiv \|Df \mid E^s_x\| \quad \text{and} \quad \tilde{b}(x) \equiv \|Df^{-1} \mid E^{cu}_{f(x)}\|^{-1}.$$

Clearly, $\tilde{a}$ and $\tilde{b}$ are continuous functions, and $\tilde{a}(x) < \theta^2$ and $\tilde{a}(x) < \theta^2\,\tilde{b}(x)$ for every $x \in \Lambda$, by conditions (a) and (b).

Let $\delta > 0$ be some fixed small constant, and set

$$a(x) \equiv \sup\{\tilde{a}(y) : d(x,y) < \delta\} \quad \text{and} \quad b(x) \equiv \sup\{\tilde{b}(y) : d(x,y) < \delta\}.$$

Assuming δ is sufficiently small, we have $a(x) < \theta^2$ and $a(x) < \theta^2 b(x)$, for every $x \in \Lambda$. Now, define

$$\mu(n,x) \equiv \prod_{i=0}^{n-1} a(f^i(x)) \quad \text{and} \quad \sigma(n,x) \equiv \prod_{i=0}^{n-1} b(f^i(x)).$$

Thus, $\mu(x,n)$ is an upper bound for the contraction along the direction of $\mathcal{F}^s$, over any orbit that stays within δ from the orbit of x during the first n iterates. Similarly, $\sigma(x,n)$ is a lower bound for the least expansion along E^{cu} over all such orbits. The previous estimates imply that

$$\mu(n,x) < \theta^{2n} \quad \text{and} \quad \mu(n,x) < \theta^{2n}\,\sigma(n,x) \tag{7}$$

for every x and $n \geq 1$.

Let $c(\cdot)$ be a continuous function such that $\theta^{-1}a(y) \leq c(y) \leq \theta b(y)$ and $c(y) \leq \theta$ for every $y \in \Lambda$. For instance, $c(y) = \theta^{-1}a(y)$. Then define

$$r(n,x) \equiv \prod_{i=0}^{n-1} c(f^i(x)). \tag{8}$$

The following properties are direct consequences of the definition:

$$\text{(R1) } r(n,x) \leq \theta^n, \quad \text{(R2) } \mu(n,x) \leq \theta^n r(n,x), \quad \text{(R3) } r(n,x) \leq \theta^n \sigma(n,x).$$

For each $x \in \Sigma_1$ and $n \geq 1$, we denote by $\mathcal{B}(n,x)$ the ball of radius $r(n,x)$ around $f^n(x)$ inside $f^n(\Sigma_1)$. Conditions (R1) and (R3) are saying that these radii are uniformly small if n is large. Yet, according to (R2), for large n they are much larger than the distance between $f^n(x)$ and $\pi_n(f^n(x))$.

3.3. **Step 1.** The first main step in the proof of Theorem 6 is to show that the volume of the ball $\mathcal{B}(n,x)$ of radius $r(n,x)$ around $f^n(x)$ inside $f^n(\Sigma_1)$ is approximately equal to the volume of its image $\pi_n(\mathcal{B}(n,x))$ under the holonomy map π_n from $f^n(\Sigma_1)$ to $f^n(\Sigma_2)$.

Proposition 7. *There exists a sequence $(\varepsilon_n)_n$ converging to zero such that*

$$\left| \frac{m_{f^n(\Sigma_1)}\,(\mathcal{B}(n,x))}{m_{f^n(\Sigma_2)}\,(\pi_n(\mathcal{B}(n,x)))} - 1 \right| \leq \varepsilon_n$$

for every $n \geq 1$ and $x \in \Sigma_1$.

For the proof of Proposition 7 we need a few auxiliary results. Let $x \in \Sigma_1$ and $n \geq 1$ be fixed throughout. At a few places we assume that δ is small, and n is sufficiently large (all the conditions are independent of the point x). The constants $K_1, \ldots, K_j, \ldots$ that appear in the sequel depend only on Σ_1, Σ_2, and the map f.

Let $1 \le k \le d$, where d is the dimension of the manifold M. For each $x \in M$ and k-dimensional subspaces V^1 and V^2 of $T_x M$, we define

$$\text{angle}(V^1, V^2) = \max_{u_1 \in V^1} \min_{u^2 \in V^2} \angle(u_1, u_2) \tag{9}$$

We are going to use the following elementary fact:

$$\angle(u_1, u_2) \le \frac{\|u_1 - u_2\|}{\|u_2\|} \tag{10}$$

for every nonzero vectors u_1 and u_2 in any Hilbert space.

Our first lemma, which is a consequence of the domination property (b), says that the tangent spaces of $f^n(\Sigma_1)$ and $f^n(\Sigma_2)$ approach the center-unstable bundle E^{cu} exponentially fast as n increases.

Lemma 8. *There exists $K_1 > 0$ such that*

$$\text{angle}\left(T_{f^n(\xi)} f^n(\Sigma_j), E^{cu}(f^n(\xi))\right) \le K_1 \theta^{2n}$$

for any $n \ge 1$, $\xi \in \Sigma_j$, and $j = 1, 2$.

Proof. We consider $j = 1$; the other case is entirely analogous. Every nonzero vector $\tilde{v} \in T_{f^n(\xi)} f^n(\Sigma_1)$ may be written as $Df^n(\xi)v$ for some $v \in T_\xi \Sigma_1$. Let us write $v = v_1 + v_2$ where $v_1 \in E^s_\xi$ and $v_2 \in E^{cu}_\xi$. Using (10) and the fact that Σ_1 is transverse to the direction of E^s, there exists a constant $K_1 > 0$ that depends only on Σ_1 such that $\|v_1\| \le K_1 \|v_2\|$. From

$$\|Df^n(\xi)v_1\| \le \|Df^n \mid E^s_\xi\| \|v_1\| \quad \text{and} \quad \|v_2\| \le \|Df^{-n} \mid E^{cu}_{f^n(\xi)}\| \, \|Df^n(\xi)v_2\|$$

and the domination condition (b), we conclude that

$$\frac{\|Df^n(\xi)v_1\|}{\|Df^n(\xi)v_2\|} \le \theta^{2n} \frac{\|v_1\|}{\|v_2\|} \le K_1 \theta^{2n}. \tag{11}$$

Then, by (10), $\text{angle}(Df^n(\xi)v, Df^n(\xi)v_2) \le K_1 \theta^{2n}$. Since $Df^n(\xi)v_2$ is in $E^{cu}_{f^n(\xi)}$, the definition (9) gives

$$\text{angle}(T_{f^n(\xi)} f^n(\Sigma_1), E^{cu}(f^n(\xi))) \le K_1 \theta^{2n},$$

as we claimed. $\square$

Next, we use (R1) and (R3) to conclude that any point whose nth iterate is in $\mathcal{B}(n, x)$ remains close to the orbit of x during the first n iterates. We denote by d_N the distance induced on a submanifold $N \subset M$ by the Riemannian metric of M. That is, $d_N(p, q)$ is the shortest (infimum) length of a piecewise smooth curve connecting p and q inside N. On the other hand, $d(p, q)$ is the distance between the points p and q in the ambient M.

Lemma 9. *There exists $K_2 > 0$ such that $d(f^j(x), f^j(\xi)) \le K_2 \theta^n$ for any $0 \le j \le n$ and any $\xi \in \Sigma_1$ such that $f^n(\xi) \in \mathcal{B}(n, x)$.*

Proof. Take $K_2 = 3$. The case $j = n$ is clear: by the definition of $\mathcal{B}(n, x)$,

$$d(f^n(x), f^n(\xi)) \le d_{f^n(\Sigma_1)}(f^n(x), f^n(\xi)) < r(n, x) \le \theta^n.$$

Now, given $0 \le j < n$, suppose the statement is known for all $j < i \le n$. More precisely, there exists some piecewise smooth curve γ_n connecting $f^n(x)$ to $f^n(\xi)$ inside $f^n(\Sigma_1)$, whose length is less than $r(n, x)$ and such that the length of $\gamma_i =$

$f^{i-n}(\gamma_n)$ is less than $3\,\theta^n$ for every $j < i \leq n$. We are going to prove that this remains true for $i = j$. Let $\dot\gamma_i$ denote the velocity vector of each γ_i. We decompose

$$\dot\gamma_i = \dot\gamma_i^s + \dot\gamma_i^{cu} \in E^s \oplus E^{cu}.$$

Just as in (11), we have $\|\dot\gamma_i^s\|/\|\dot\gamma_i^{cu}\| \leq K_1\theta^{2j}$ for every $j \leq i \leq n$. Assuming j is large enough, then θ^{2j} is smaller than $1/2$. Then the cases $i = j$ and $i = n$ give

$$\|\dot\gamma_j\| \leq \frac{3}{2}\,\|\dot\gamma_j^{cu}\| \quad \text{and} \quad \|\dot\gamma_n\| \geq \frac{1}{2}\,\|\dot\gamma_n^{cu}\|\,. \tag{12}$$

Now, the induction assumption implies that the length of γ_j is less than

$$\|Df^{-1}\|\ \text{length}(\gamma_{j+1}) \leq \|Df^{-1}\|K_2\theta^n.$$

Assuming n is large enough, this is smaller than δ. So, γ_i is contained in the δ-neighborhood of $f^i(x)$ for all $i \leq j \leq n$. Thus, by the definition of $\sigma(\cdot\,,\cdot)$,

$$\|\dot\gamma_j^{cu}\| = \|Df^{j-n}\cdot\dot\gamma_n^{cu}\| \leq \frac{\|\dot\gamma_n^{cu}\|}{\sigma(n-j,f^j(x))}\,.$$

Together with (12), this gives $\|\dot\gamma_j\| \leq 3\,\|\dot\gamma_n\|\,\sigma(n-j,f^j(x))^{-1}$. Then,

$$\text{length}(\gamma_j) \leq \frac{3\,\text{length}(\gamma_n)}{\sigma(n-j,f^j(x))} \leq \frac{3\,r(n,x)}{\sigma(n-j,f^j(x))}\,.$$

By (R1) and (R3), $r(n,x) = r(j,x)\,r(n-j,f^j(x)) \leq \theta^n\sigma(n-j,f^j(x))$. So, the previous inequality gives $\text{length}(\gamma_j) \leq 3\theta^n = K_2\theta^n$. □

For notational simplicity, given ξ and η in the same strong-stable leaf, we represent by $d_s(\xi,\eta)$ the distance between the two points inside that leaf.

Lemma 10. *There exists $K_3 > 0$ such that $d_s(y,\pi_n(y)) \leq K_3\mu(n,x)$ for every $y \in \mathcal{B}(n,x)$.*

Proof. Recall that $\mu(k,z)$ is an upper bound for the derivative of f^k along the stable direction, for orbits that remain within δ from the orbit of z. Given $y \in \mathcal{B}(n,x)$, let $\xi = f^{-n}(y)$. Since Σ_1 and Σ_2 are compact, there exists a uniform upper bound C_3 for the distance between ξ and $\pi(\xi)$ inside the leaf of $\mathcal{F}^s$ that contains the two points. As f contracts strong-stable leaves, by property (a), it follows that

$$d_s(f^j(\xi),f^j(\pi(\xi))) \leq C_3\sup\|Df^j \mid E^s\| \leq C_3\theta^{2j}$$

for all $j \geq 1$. In particular, fixing $p \geq 1$ so that $C_3\theta^{2p} < \delta/2$, we have

$$d(f^j(\xi),f^j(\pi(\xi))) \leq d_s(f^j(\xi),f^j(\pi(\xi))) < \delta/2$$

for all $j \geq p$. Lemma 9 gives $d(f^j(\xi),f^j(x)) \leq K_2\theta^n$ for all $0 \leq j \leq n$. Assume n is large enough so that $K_2\theta^n < \delta/2$. Then the last two inequalities imply that $f^j(\xi)$ and $f^j(\pi(\xi))$ remain within δ from $f^j(x)$ for, at least, $n-p$ iterates. Therefore,

$$d_s(y,\pi_n(y)) = d_s\big(f^n(\xi),f^n(\pi(\xi))\big) \leq \mu(n-p,f^p(x))\,d_s\big(f^p(\xi),f^p(\pi(\xi))\big).$$

Observe that $\mu(n-p,f^p(x)) = \mu(n,x)/\mu(p,x)$. Moreover, $\mu(p,x)$ admits a uniform lower bound $c_3 > 0$, because f is a diffeomorphism and p has been fixed. It follows that $d_s(y,\pi_n(y)) \leq K_3\mu(n,x)$, where $K_3 = \delta/c_3$. □

3.4. **Local coordinates.** Now we are going to show that $\mathcal{B}(n,x)$ and $\pi_n(\mathcal{B}(n,x))$ may be written as graphs, of C^1-nearby maps, over the center-unstable direction at $f^n(x)$. This will allow us to compare the measures induced by the Riemannian structure of M on $\mathcal{B}(n,x)$ and on $\pi_n(\mathcal{B}(n,x))$, and, thus prove Proposition 7. For the precise statement, it is convenient to introduce local coordinates near $f^n(x)$.

Let $\exp_z : T_zM \to M$ be the exponential map of M at any z. In the tangent space T_zM we consider the inner product defined by the Riemannian metric of M. Let $B^{cu}(z,\rho)$ and $B^s(z,\rho)$ be the balls of radius ρ around the origin inside E^{cu}_z and E^s_z, respectively, and let $B^{TM}(z,\rho) = B^{cu}(z,\rho) \times B^{cu}(z,\rho)$. Assuming ρ is small enough, the exponential map is a diffeomorphism from $B^{TM}(z,\rho)$ onto its image $\exp_z(B^{TM}(z,\rho))$ in M. Assuming δ has been fixed sufficiently small, the inverse map $\phi_z = \exp_z^{-1}$ is well-defined in the δ-neighborhood $B(z,\delta)$ of every $z \in M$, with image contained in $B^{TM}(z,\rho)$. In what follows we often identify points $y \in B(z,\delta)$ with the corresponding images under these local charts ϕ_z. Note that if n is sufficiently large then $r(n,x) \le \theta^n$ is smaller than δ, and so $\mathcal{B}(n,x)$ is contained in $B(f^n(x),\delta)$.

Lemma 11. *There exist disks D_1 and D_2 in $E^{cu}_{f^n(x)}$, and C^1 maps $g_1 : D_1 \to E^s_{f^n(x)}$ and $g_2 : D_2 \to E^s_{f^n(x)}$, such that $\mathcal{B}(n,x) = \operatorname{graph}(g_1)$ and $\pi_n(\mathcal{B}(n,x)) = \operatorname{graph}(g_2)$.*

Proof. For n large enough, $\mathcal{B}(n,x)$ is contained in $B(f^n(x),\delta)$, so via the exponential chart we can think of it as a subset of $T_{f^n(x)}M$; the same applies to $\pi_n(\mathcal{B}(n,x))$, because by Lemma 10 its points are close to their preimages under π_n. Now, by Lemma 8, for n large enough the set $f^n(\Sigma_1)$ is nearly tangent to the center-unstable direction E^{cu}; there is, on the other hand, a uniform lower bound for the angle between the stable direction E^s and the center-unstable direction E^{cu}. Now, since $\mathcal{B}(n,x)$ is a small ball in $f^n(\Sigma_1)$, each point $z \in \mathcal{B}(n,x)$ corresponds uniquely to some pair $(z_{cu}, z_s) \in E^{cu}_{f^n(x)} \times E^s_{f^n(x)}$, where the coordinate z_s ranges over some disk D_1 containing 0 in $E^{cu}_{f^n(x)}$. By the differentiability of local stable manifolds it follows that the map $g_1 : z_{cu} \mapsto z_s$ so defined is C^1. The existence of D_2 and of g_2 follow from identical arguments. □

Lemma 12. *There exist $\alpha \in (0,1]$ and $K_4 > 0$ such that $\|Dg_1(z)\| \le K_4\theta^{\alpha n}$ and $\|Dg_2(w)\| \le K_4\theta^{\alpha n}$ for every $z \in D_1$ and $w \in D_2$.*

Proof. First, we observe that given $y \in \mathcal{B}(n,x)$ then $E^{cu}(y)$ may be written as the graph of a linear map $\xi_y : E^{cu}(f^n(x)) \to E^s(f^n(x))$ with $\|\xi_y\| \le \tilde{C}\theta^{\alpha n}$ for some uniform constants $\tilde{C} > 0$ and $\alpha \in (0,1]$. This is a simple consequence of Lemma 8, together with Proposition A. Indeed, according to the proposition, the subbundle E^{cu} is Hölder continuous. So, there exist constants $C > 0$ and $\alpha \in (0,1]$, depending only on f, such that, for every y in the δ-neighborhood of $f^n(x)$, the subspace $E^{cu}(y)$ may be written as the graph of a linear map $\xi_y : E^{cu}(f^n(x)) \to E^s(f^n(x))$ with $\|\xi_y\| \le Cd(f^n(x),y)^\alpha$. By definition, $d_{f^n(\Sigma_1)}(f^n(x),y) \le r(n,x) \le \theta^n$. So,

$$d(f^n(x),y) \le 2d_{f^n(\Sigma_1)}(f^n(x),y) \le 2\theta^n.$$

(Here the factor 2 accounts for the fact that the local chart ϕ may be slightly expanding.) Setting $\tilde{C} \equiv C2^\alpha$ we conclude that ξ_y is exponentially close to zero:

$$\|\xi_y\| \le \tilde{C}\theta^{n\alpha}.$$

On the other hand, by Lemma 8,

$$\mathrm{angle}(T_y\mathcal{B}(n,x), E^{cu}(y)) \leq K_1\theta^{2n}$$

As long as n is sufficiently large, this implies that $T_y\mathcal{B}(n,x)$ is also a graph over $E^{cu}(f^n(x))$:

$$T_y\mathcal{B}(n,x) = \mathrm{graph}(h_y), \quad h_y : E^{cu}_{f^n(y)} \to E^s_{f^n(y)}.$$

Clearly there is some constant $L > 0$ such that

$$\|h_y - \xi_y\| \leq L\,\mathrm{angle}(T_y\mathcal{B}(n,x), E^{cu}(y))$$

for every y. Then we have that $\|h_y - \xi_y\| \leq L\,\mathrm{angle}(T_y\mathcal{B}(n,x), E^{cu}(y)) \leq LK_1\theta^{2n}$. It follows that $\|h_y\| \leq \|\xi_y\| + \|h_y - \xi_y\| \leq \tilde{C}\theta^{n\alpha} + LK_1\theta^{2n}$. Setting $K_4 \geq \tilde{C} + LK_1$ we are done. □

Lemma 13. *There exist $\alpha' \in (0,1)$ and $K_5 > 0$ such that D_1 and D_2 contain the ball of radius $r(n,x)(1-K_5\theta^{\alpha' n})$ and are contained in the ball of radius $r(n,x)(1+K_5\theta^{\alpha' n})$ around the origin.*

Proof. We deal with the disk D_1 here; the case of D_2 is identical. Note that D_1 is the projection $P(\mathcal{B}(n,x))$ onto $E^{cu}(f^n(x))$ of the disk $\mathcal{B}(n,x) \subset f^n(\Sigma_1)$ along the stable direction E^s. By Lemma 12, $\mathrm{angle}(T_z\mathcal{B}(n,x), E^{cu}_{f^n(x)})$ converges exponentially to 0 with rate θ^α uniformly over all $z \in \mathcal{B}(n,x)$. It is easy to see that given $\alpha' < \alpha$ then the projection $P(\mathcal{B}(n,x)) = D_1$ coincides with the ball of radius $r(n,x)$ around the origin in $E^{cu}_{f^n(x)}$ modulo a factor of order $e^{\theta^{\alpha' n}}$, as desired. □

Now we can prove Proposition 7:

Proof. Now, we start the study of the metrics $m_{f^n(\Sigma_1)}$ and $m_{f^n(\Sigma_2)}$. If we set $\gamma_1(u) = (u, g_1(u))$ and $\gamma_2(u) = (u, g_2(u))$ then these metrics are determined by the first fundamental form $g^\ell_{ij} = \frac{\partial\gamma_\ell}{\partial u_i}\frac{\partial\gamma_\ell}{\partial u_j}$, where $\ell = 1,2$ and $i,j = 1,...,n-k$. That is,

$$g^\ell_{ij} = \delta_{ij} + \frac{\partial g_\ell}{\partial u_i}\frac{\partial g_\ell}{\partial u_j}, \tag{13}$$

where $\ell = 1,2$.

Consider now the ball $A(n,x)$ in $f^n(\Sigma_1)$ of radius $r(n,x)(1-K_5\theta^{\alpha' n})$ and center $f^n(x)$ and the ball $C(n,x)$ in $f^n(\Sigma_1)$ of radius $r(n,x)(1+K_5\theta^{\alpha' n})$ and center $f^n(x)$. By Lemma 13 it follows that D_1 and D_2 contain (the projection onto $E^{cu}(f^n(x))$ of) $A(n,x)$ and are contained in (the projection onto $E^{cu}(f^n(x))$ of) $C(n,x)$. Since, by Lemma 12, $\|Dg_1\|$ and $\|Dg_2\|$ are uniformly bounded, it follows that there exists a sequence $\delta(n)$, converging to zero, such that for every $x \in \Sigma_1$ we have

$$\|\frac{m_{f^n(\Sigma_1)}A(n,x)}{m_{f^n(\Sigma_1)}C(n,x)} - 1\| < \delta(n) \tag{14}$$

and

$$\|\frac{m_{f^n(\Sigma_2)}\pi_n(A(n,x))}{m_{f^n(\Sigma_2)}\pi_n(C(n,x))} - 1\| < \delta(n). \tag{15}$$

We denote by P the projection along the k-plane $E^s(f^n(x))$; that is, $P(u, g_\ell(u)) = u$ for $\ell = 1,2$. By Lemmas 10 and 13 it follows that for large enough n we have

$$P(A(n,x)) \subset P(\pi_n(\mathcal{B}(n,x))) \subset P(C(n,x)) \tag{16}$$

We have, by the definition of $m_{f^n(\Sigma_1)}$ and $m_{f^n(\Sigma_2)}$, that given a disk D then

$$m_{f^n(\Sigma_\ell)}(D) = \iint_{P(D)} \sqrt{det(g_{ij}^\ell)}du_1 du_2 \dots du_{n-k}, \tag{17}$$

where $\ell = 1, 2$ and $D \subset f^n(\Sigma_1)$ or $D \subset f^n(\Sigma_2)$. Then by Lemma 12 and (13), (16), and (17) it follows that given $\varepsilon > 0$ then for large enough n we have

$$\begin{aligned} m_{f^n(\Sigma_1)}(A(n,x)) &= \iint_{P(A(n,x))} \sqrt{\det(g_{ij}^1)}du_1 du_2 \dots du_{n-k} \\ &\leq \iint_{P(\pi_n(\mathcal{B}(n,x)))} \sqrt{\det(g_{ij}^2)}du_1 \dots du_{n-k} + \varepsilon\, m(P(\pi_n(\mathcal{B}(n,x))) \\ &= (1+\varepsilon)\, m(P(\pi_n(\mathcal{B}(n,x))), \end{aligned}$$

where m is an Euclidean measure in $T_{f^n(x)}\mathcal{B}(n,x)$. Now, by (14) we have that for large n the volume $m_{f^n(\Sigma_1)}(A(n,x))$ is a good approximation of $m_{f^n(\Sigma_1)}(\mathcal{B}(n,x))$, while clearly for large n the volume $m(P(\pi_n(\mathcal{B}(n,x)))$ is a good approximation of $m_{f^n(\Sigma_2)}(\pi_n(\mathcal{B}(n,x)))$.

Analogously, we can obtain an upper estimate for $m_{f^n(\Sigma_2)}\pi_n(\mathcal{B}(n,x))$ using $m_{f^n(\Sigma_1)}(C(n,x))$. Combining the two resulting inequalities we obtain Proposition 7, as desired. □

3.5. **Step 2.** The next main step in the proof of Theorem 6 is

Proposition 14. *There exists K_6 such that*

$$\frac{1}{K_6} \leq \frac{m_{\Sigma_1}(f^{-n}(\mathcal{B}(n,x)))}{m_{\Sigma_2}\pi(f^{-n}(\mathcal{B}(n,x)))} \leq K_6$$

This will follow from Proposition 7 and the following distortion lemma for the Jacobian of f along Σ_1 and Σ_2:

Lemma 15. *There exists $K_7 > 0$ such that*

$$\| \log \det Df^{-n}(z_1) \mid T_{z_1}\mathcal{B}(n,x) - \log \det Df^{-n}(z_2) \mid T_{z_2}\mathcal{B}(n,x) \| \leq K_7$$

for all z_1, z_2 in $\mathcal{B}(n,x)$. The same holds with $\pi_n(\mathcal{B}(n,x))$ instead of $\mathcal{B}(n,x)$, for some constant K_8.

Proof. It is easy to see, using the fact that f is a C^2 map, that there exist constants R_1 and R_2 such that, if $z_1, z_2 \in M, d(z_1, z_2) \leq 1$, and A_1, A_2 are subspaces of $\mathbb{R}^n$ with dimension $n-k$, then

$$\| \log \det Df^{-1}(z_1)|A_1 - \log \det Df^{-1}(z_2)|A_2\| \leq R_1 d(z_1,z_2) + R_2 \text{ang}(A_1, A_2). \tag{18}$$

We have by Proposition A that the map $w_2 : x \mapsto E^{cu}(x)$ is (C,α)-Hölder continuous, and hence $\rho : x \mapsto \log \det Df^{-n}(x) \mid E^{cu}$ is Hölder continuous with constants (Q,α). Now, given $z_1, z_2 \in \mathcal{B}(n,x)$, by 18, Lemma 8 (applied to z_1 and

z_2 and to their first n preiterates), and condition (R3) it follows that

$$\begin{aligned}
\|\log\frac{\det Df^{-n}(z_1)\mid T_{z_1}\mathcal{B}(n,x)}{\det Df^{-n}(z_2)\mid T_{z_2}\mathcal{B}(n,x)}\| &\le \|\log\frac{\det Df^{-n}(z_1)\mid T_{z_1}\mathcal{B}(n,x)}{\det Df^{-n}(z_1)\mid E^{cu}(z_1)}\| \\
&+\|\log\frac{\det Df^{-n}(z_2)\mid T_{z_2}\mathcal{B}(n,x)}{\det Df^{-n}(z_2)\mid E^{cu}(z_2)}\| \\
&+\|\log\frac{\det Df^{-n}(z_1)\mid E^{cu}(z_1)}{\det Df^{-n}(z_2)\mid E^{cu}(z_2)}\| \\
&\le R_2\sum_{i=0}^{n-1}\mathrm{angle}(T_{f^{-i}(z_1)}f^{-i}(\mathcal{B}(n,x)),E^{cu}(f^{-i}(z_1)) \\
&+R_2\sum_{i=0}^{n-1}\mathrm{angle}(T_{f^{-i}(z_2)}f^{-i}(\mathcal{B}(n,x)),E^{cu}(f^{-i}(z_2))) \\
&+Q\sum_{i=0}^{n-1}d(f^{-i}(z_1),f^{-i}(z_2))^{\alpha} \\
&\le 2R_2\,K_1\sum_{i=0}^{n-1}\theta^{n-i} \\
&+Q\sum_{i=0}^{n-1}\theta^{\alpha(n-i)}d(z_1,z_2)^{\alpha} \\
&< K_7
\end{aligned}$$

for sufficiently large K_7. In a similar way, we prove that

$$\|\log\frac{\det Df^{-n}(z_1)\mid T_{z_1}\pi_n(\mathcal{B}(n,x))}{\det Df^{-n}(z_2)\mid T_{z_2}\pi_n(\mathcal{B}(n,x))}\|\le K_8$$

for any z_1 and z_2 in $\pi_n(\mathcal{B}(n,x))$. □

Now we can prove Proposition 14:

Proof. Observe that

$$m_{\Sigma_1}f^{-n}(\mathcal{B}(n,x))=\int_{\mathcal{B}(n,x)}\left\|\det Df^{-n}(z)\mid T_z\mathcal{B}(n,x)\right\|\,dm_{f^n(\Sigma_1)}(z) \tag{19}$$

and similarly for $m_{\Sigma_2}\pi(f^{-n}(\mathcal{B}(n,x)))$.

Now, since we know $m_{f^n(\Sigma_1)}\mathcal{B}(n,x)$ and $m_{f^n(\Sigma_2)}\pi_n(\mathcal{B}(n,x))$ are comparable, we just need to compare the expressions inside the integrals.

Setting $K_9=max\{K_8,K_7\}$, we know that if we replace the integrand in (19) by its value at a certain point z_1 then the the resulting number is near $m_{\Sigma_1}f^{-n}(\mathcal{B}(n,x))$ modulo a factor smaller than e^{K_9}. We can estimate the value of $m_{\Sigma_2}\pi(f^{-n}(\mathcal{B}(n,x)))$ similarly.

Now we need only estimate the expression

$$\|\log\det Df^{-n}(z_1)\mid T_{z_1}\mathcal{B}(n,x)-\log\det Df^{-n}(\pi_n(z_1))\mid T_{\pi_n(z_1)}\pi_n(\mathcal{B}(n,x))\|.$$

Following the same arguments as before, we have

$$\begin{aligned} &\|\log\det Df^{-n}(z_1) \mid T_{z_1}B(n,x) - \log\det Df^{-n}(\pi_n(z_1)) \mid T_{\pi_n(z_1)}\pi_n(\mathcal{B}(n,x))\| \\ \leq &\|\log\det Df^{-n}(z_1) \mid T_{z_1}\mathcal{B}(n,x) - \log\det Df^{-n}(z_1) \mid E^{cu}(z_1)\| \\ + &\|\log\det Df^{-n}(\pi_n(z_1)) \mid T_{\pi_n(z_1)}\pi_n(\mathcal{B}(n,x)) - \log\det Df^{-n}(\pi_n(z_1)) \mid E^{cu}(\pi_n(z_1))\| \\ + &\|\log\det Df^{-n}(z_1) \mid E^{cu}(z_1) - \log\det Df^{-n}(\pi_n(z_1)) \mid E^{cu}(\pi_n(z_1))\| \end{aligned}$$

and this is smaller than

$$\begin{aligned} &Q\sum_{i=0}^{n-1} d(f^{-i}(z_1), f^{-i}(\pi_n(z_1)))^{\alpha} + R_2\sum_{i=0}^{n-1}\operatorname{angle}(T_{f^{-i}(z_1)}f^{-i}(\mathcal{B}(n,x)), E^{cu}(f^{-i}(z_1)) \\ &+ R_2\sum_{i=0}^{n-1}\operatorname{angle}(T_{f^{-i}(\pi_n(z_1))}f^{-i}(\pi_n(\mathcal{B}(n,x))), E^{cu}(f^{-i}(\pi_n(z_1)))) \\ &\leq Q\sum_{i=0}^{n-1}\theta^{\alpha(n-i)}d(z_1,\pi_n(z_1))^{\alpha} \\ &+ 2R_2\,K_1\sum_{i=0}^{n-1}\theta^{n-i} < K_{10} \end{aligned}$$

if K_{10} is large enough. The proposition follows. □

At last, we are in a position to complete the proof of the theorem:

Proof of Theorem 6. Consider $\varepsilon > 0$ such that the set $D_\varepsilon = \{y \in D | d_{\Sigma_1}(y, D^c) > \varepsilon\}$ satisfies

$$\|\frac{m_{\Sigma_1}(D_\varepsilon)}{m_{\Sigma_1}(D)} - 1\| < \frac{1}{2} \quad \text{and} \quad \|\frac{m_{\Sigma_2}(\pi(D_\varepsilon))}{m_{\Sigma_2}(\pi(D))} - 1\| < \frac{1}{2}$$

Consider in $f^n(\Sigma_1)$ the set $\mathcal{B}_n = \{\mathcal{B}(n,x) | x \in D_\varepsilon\}$. Note that if n is sufficiently large, then $\mathcal{B}(n,x) \subset f^n(D)$ for all $\mathcal{B}(n,x) \in \mathcal{B}_n$, since by (R3) the radius $r(n,x)$ of $\mathcal{B}(n,x)$ shrinks faster than $d(\partial f^n(D), \partial f^n(D_\varepsilon))$, which is bounded below by (some constant times) $\sigma(n,x)$. Now, using Besicovich's covering theorem (see [**?**], for instance) we can cover D_ε with a countable family of balls $\mathcal{G}_n \subset \mathcal{B}_n$, such that each ball in $\mathcal{G}_n$ intersects, at most, ℓ other balls in $\mathcal{G}_n$, where $\ell \in \mathbb{N}$ depends only the dimension of Σ_1. Then

$$\frac{1}{2\ell}m_{\Sigma_1}D \leq \frac{1}{\ell}m_{\Sigma_1}D_\varepsilon \leq \sum_{\mathcal{B}\in\mathcal{G}_n} m_{\Sigma_1}f^{-n}(\mathcal{B}) \leq \ell m_{\Sigma_1}D$$

Analogously,

$$\frac{1}{2\ell}m_{\Sigma_2}\pi(D) \leq \frac{1}{\ell}m_{\Sigma_2}\pi(D_\varepsilon) \leq \sum_{\mathcal{B}\in\mathcal{G}_n} m_{\Sigma_2}\pi(f^{-n}(\mathcal{B})) \leq \ell m_{\Sigma_2}\pi(D).$$

Then, applying Proposition 14 to the balls $\mathcal{B}$, we have the theorem.

□

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