Dynamics in the moduli space of Abelian differentials

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Adapted coordinates form a translation atlas: coordinate changes near any regular point have the form

$$\zeta' = \zeta + \text{const.}$$

Translation surfaces

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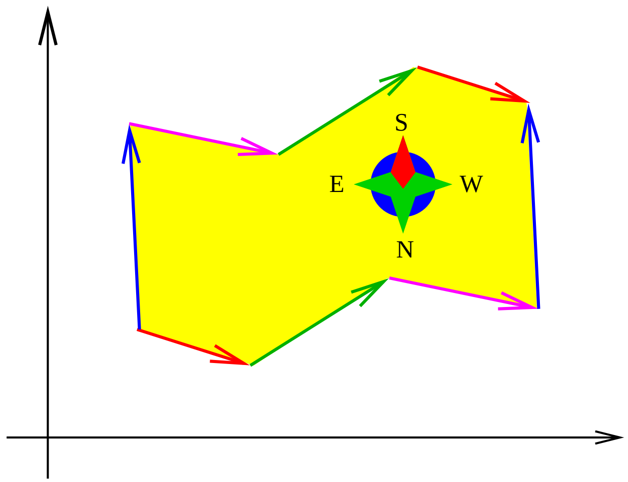
Conversely, the flat metric and the upward vector field characterize the translation structure completely.

Geometric representation

Consider any planar polygon with even number of sides, organized in pairs of parallel sides with the same length.



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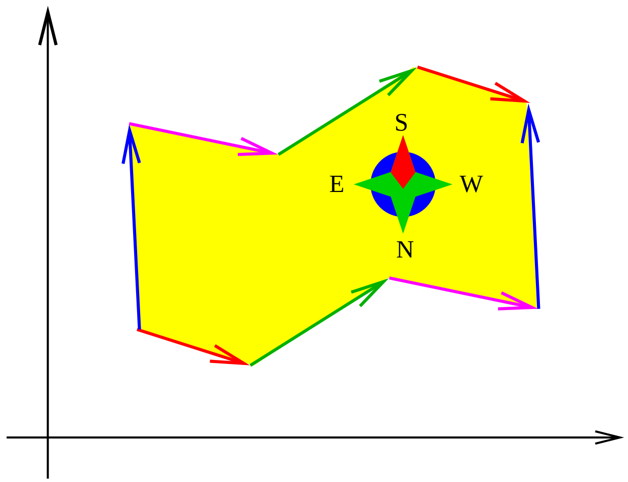

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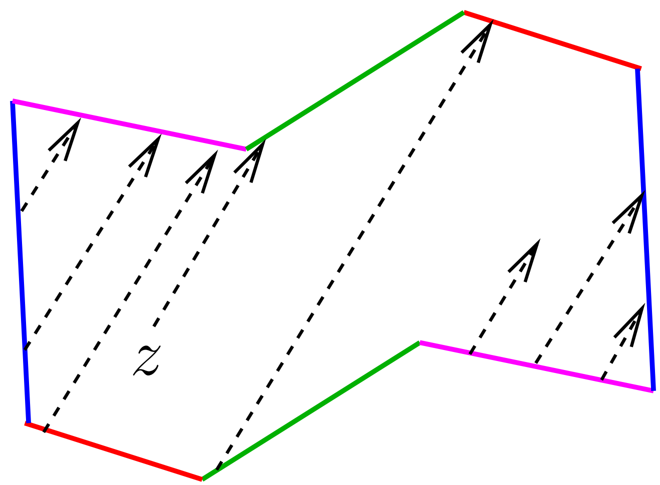
Every translation surface can be represented in this way, but not uniquely.

Geodesic flows

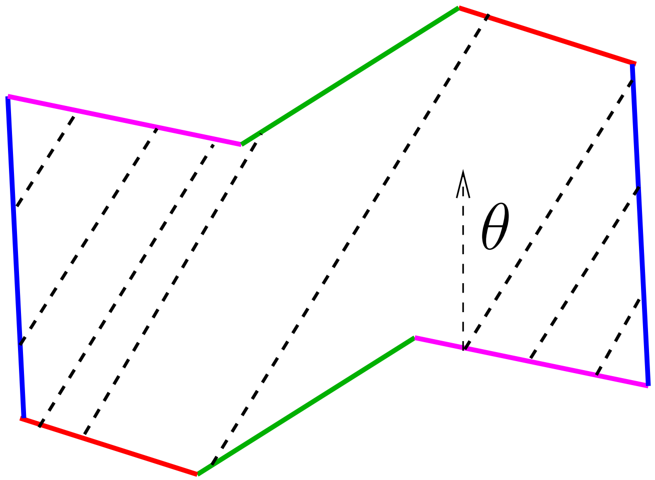
The trajectories of the Abelian differential are the geodesics on the corresponding translation surface.



When are geodesics closed ? When are they dense ? How do geodesics distribute themselves on the surface ?

Measured foliations

Geodesics in a given direction define a foliation of the surface which is a special case of a measured foliation: it is given by the kernel of a real closed 1-form $\Re(e^{i\theta}\omega)$.



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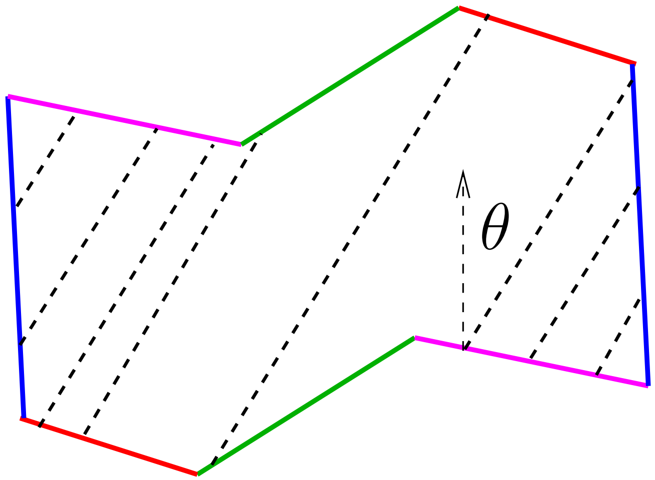

Calabi, Katok, Hubbard-Masur, Kontsevich-Zorich: Every measured foliation with no saddle connections (leaves that connect singularities) is of this form.

Moduli spaces

$\mathcal{M}_g =$ moduli space of Riemann surfaces of genus $g \geq 2$

$\mathcal{A}_g =$ moduli space of Abelian differentials on Rieman surfaces of genus $g \geq 2$

$$\dim_{\mathbb{C}} \mathcal{M}_g = 3g - 3 \qquad \dim_{\mathbb{C}} \mathcal{A}_g = 4g - 3$$

$\mathcal{A}_g$ is an orbifold and a fiber (“cotangent”) bundle over $\mathcal{M}_g$.

Strata of $\mathcal{A}_g$

Consider any $m_1, \ldots, m_\kappa \geq 1$ with $\sum_{i=1}^{\kappa} m_i = 2g-2$.

$\mathcal{A}_g(m_1, \ldots, m_\kappa) =$ subset of Abelian differentials having κ zeroes, with multiplicities $m_1, \ldots, m_\kappa$.

$$\dim_{\mathbb{C}} \mathcal{A}_g(m_1, \ldots, m_\kappa) = 2g + \kappa - 1$$

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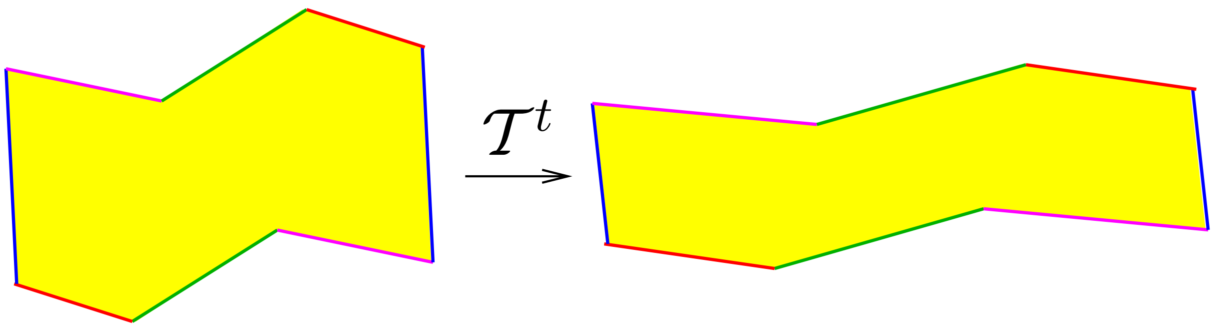
Each stratum may have up to 3 connected components. Kontsevich, Zorich classified all connected components.

Teichmüller flow

The Teichmüller flow is the natural action $\mathcal{T}^t$ on the fiber bundle $\mathcal{A}_g$ by the diagonal subgroup of $\mathrm{SL}(2,\mathbb{R})$:

$$\mathcal{T}^t(\omega)_z = \left[e^t \Re \omega_z\right] + i\left[e^{-t} \Im \omega_z\right]$$

Geometrically:



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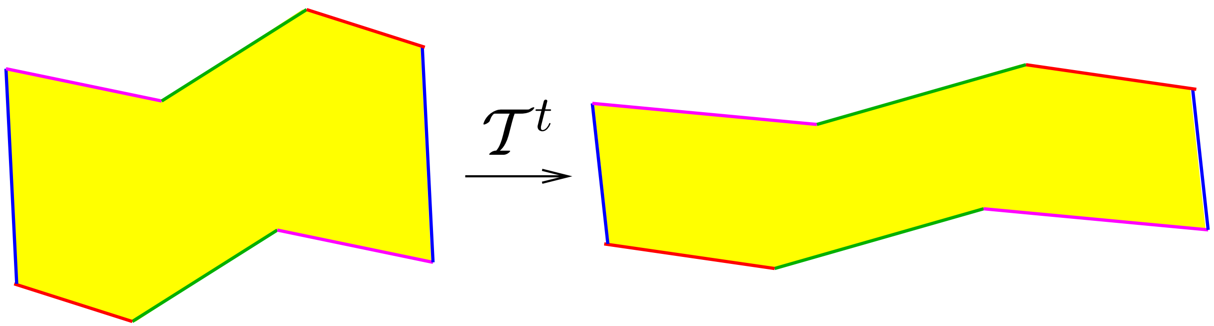

This flow leaves invariant the volume on every stratum and also preserves the area of the translation surface S.

General Principle

Properties of the Teichmüller flow reflect upon dynamical properties of almost all Abelian differentials.

The orbits of the Teichmüller flow “know” the properties of the translation surfaces contained in them.

Ergodicity

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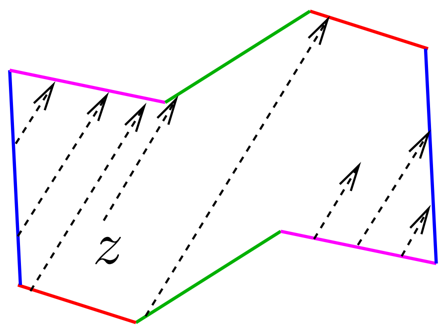

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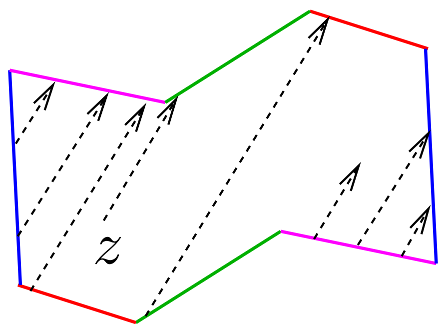
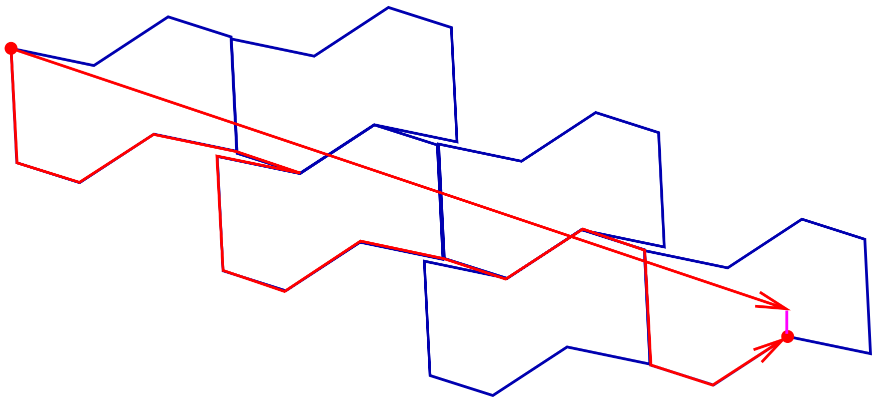


Kerckhoff, Masur, Smillie: unique ergodicity holds for every Abelian differential and almost every direction.

Asymptotic cycles

Any geodesic segment γ may be “closed” to get an element $h(\gamma)$ of $H_1(S,\mathbb{Z})$:

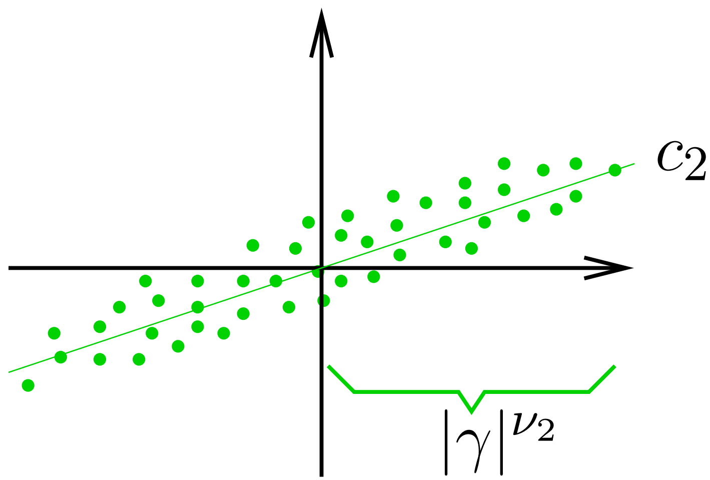


Unique ergodicity implies $h(\gamma)/|\gamma|$ converges uniformly to some $c_1 \in H_1(S,\mathbb{R})$ when the length $|\gamma|$ goes to infinity,

and the asymptotic cycle c_1 does not depend on the initial point, only the surface and the direction.

Zorich phenomenon

The deviation of $h(\gamma)$ from the direction of the asymptotic cycle c_1 distributes itself along a favorite direction c_2, with amplitude $|\gamma|^{\nu_2}$ for some $\nu_2 < 1$:



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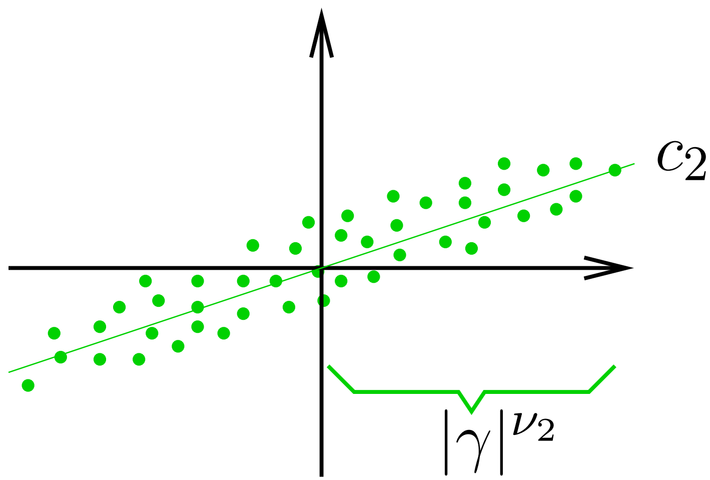
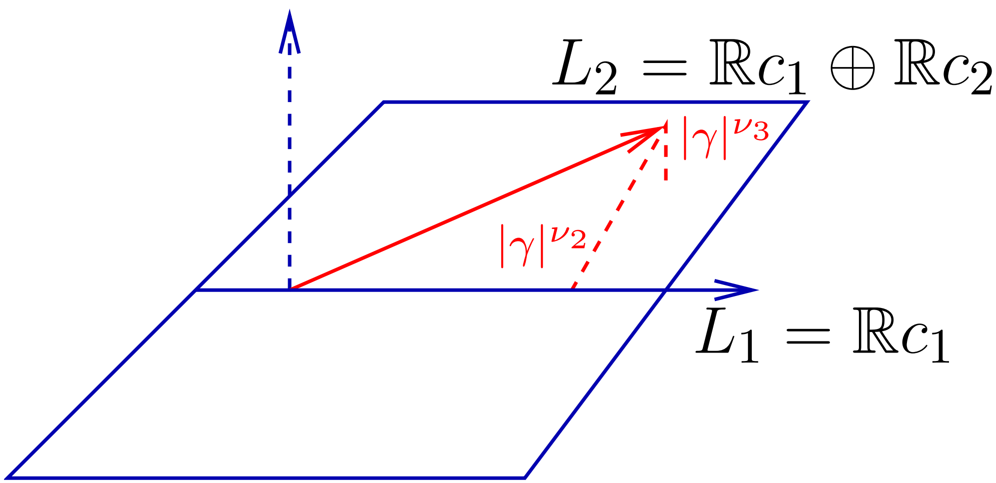


Similarly in higher order: the component of $h(\gamma)$ orthogonal to $\mathbb{R}c_1 \oplus \mathbb{R}c_2$ has a favorite direction c_3, and amplitude $|\gamma|^{\nu_3}$ for some $\nu_3 < \nu_2$, and so on up to order $g =$ genus.

Asymptotic flag conjecture

Conjecture (Zorich, Kontsevich)**.** *There are* $1 > \nu_2 > \cdots > \nu_g > 0$ *and subspaces* $L_1 \subset L_2 \subset \cdots \subset L_g$ *of* $H_1(S, \mathbb{R})$ *with* $\dim L_i = i$ *for every* i*, such that*

- *the deviation of* $h(\gamma)$ *from* L_i *has amplitude* $|\gamma|^{\nu_{i+1}}$ *for all* $i < g$
- *the deviation of* $h(\gamma)$ *from* L_g *is bounded* (g = *genus*).



Main result

Theorem (Avila, Viana). *The Zorich-Kontsevich conjecture is true.*

Previous results

Kontsevich, Zorich translated the conjecture to a statement on the Teichmüller flow.

The Lyapunov exponents of the Teichmüller flow are

$$2 > 1+\nu_2 \geq \cdots \geq 1+\nu_g \geq 1 = \cdots = 1 \geq 1-\nu_g \geq \cdots \geq 1-\nu_2 \geq 0$$
$$\geq -1+\nu_g \geq \cdots \geq -1+\nu_g \geq -1 = \cdots = -1 \geq -1-\nu_g \geq \cdots \geq -1-\nu_2 > -2.$$

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Avila, Viana prove that all inequalities above are strict (including $\nu_g > 0$). The Z-K conjecture follows.

Linear Cocycles

A linear cocycle over a flow $f^t : M \to M$, $t \in \mathbb{R}$ is a flow extension

$$F^t : M \times \mathbb{R}^d \to M \times \mathbb{R}^d, \quad F^t(x,v) = \big(f^t(x), A^t(x)v\big)$$

where $A^t : M \to GL(d,\mathbb{R})$.

Lyapunov exponents

Oseledets: Let μ be an ergodic invariant probability such that $\log\|A^{\pm 1}\|$ are integrable. Then there exist numbers $\lambda_1 > \lambda_2 > \cdots > \lambda_k$, and for μ-almost every $x \in M$ there exists a decomposition $\mathbb{R}^d = E_x^1 \oplus E_x^2 \oplus \cdots \oplus E_x^k$ such that

$$\lambda_i(x) = \lim_{|t|\to\infty} \frac{1}{t} \log\|A^t(x)v\|$$

for every non-zero $v \in E_x^i$ and $1 \le i \le k$.

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for every non-zero $v \in E_x^i$ and $1 \leq i \leq k$.

The dimension of the subspace E^i is called the multiplicity of the Lyapunov exponent λ_i of the linear cocycle.

Main steps in the proof

(1) A sufficient condition for the Lyapunov exponents of a general linear cocycle to to have multiplicity 1.

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(2) This criterium is met by the Kontsevich-Zorich cocycle

$$F^t(x,v) = (f^t(x), A^t(x)) \quad \text{on} \quad \mathcal{A}_g(m_1,\ldots,m_\kappa) \times \mathbb{R}^{2g}$$

F^t where f^t is the Teichüller flow and A^t describes the action of this flow on the homology group $H_1(M) = \mathbb{R}^{2g}$.

Conclusion

The Lyapunov exponents of this cocycle (with multiplicity) are related to those of the Teichmüller flow: they are

$$1 \geq \nu_2 \geq \cdots \geq \nu_g \geq 0 \geq -\nu_g \geq \cdots \geq -\nu_2 \geq -1.$$

The criterium implies that all inequalities are strict, and so the conjecture follows.