

Dynamics in the moduli space of Abelian differentials

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Adapted coordinates form a **translation atlas**: coordinate changes near any regular point have the form

$$\zeta' = \zeta + \text{const.}$$

Translation surfaces

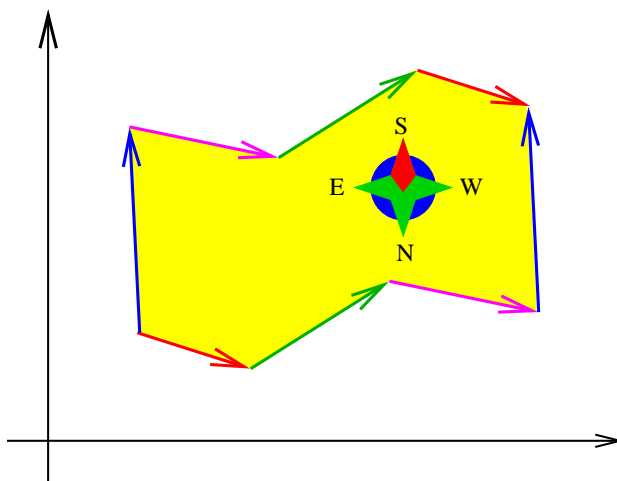
The translation atlas defines

- a **flat metric** with a finite number of conical singularities;
- a parallel unit vector field (the “upward” direction) defined on the complement of the singularities.

Conversely, the flat metric and the upward vector field characterize the translation structure completely.

Geometric representation

Consider any planar polygon with even number of sides, organized in pairs of parallel sides with the same length.

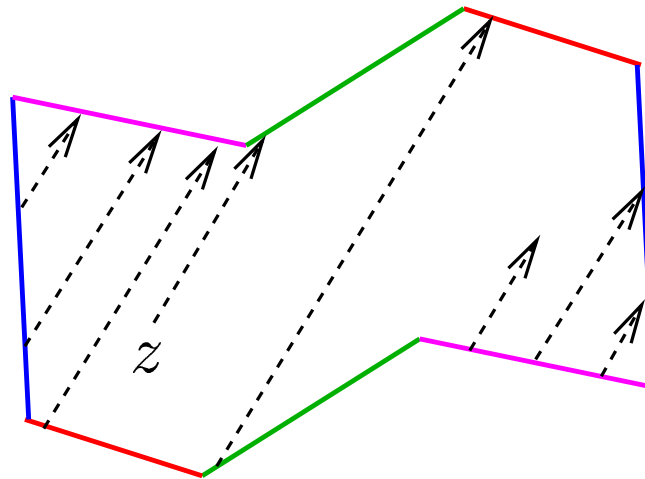


Identifying the two sides in the same pair, by translation, one obtains a translation surface.

Every translation surface can be represented in this way, but not uniquely.

Geodesic flows

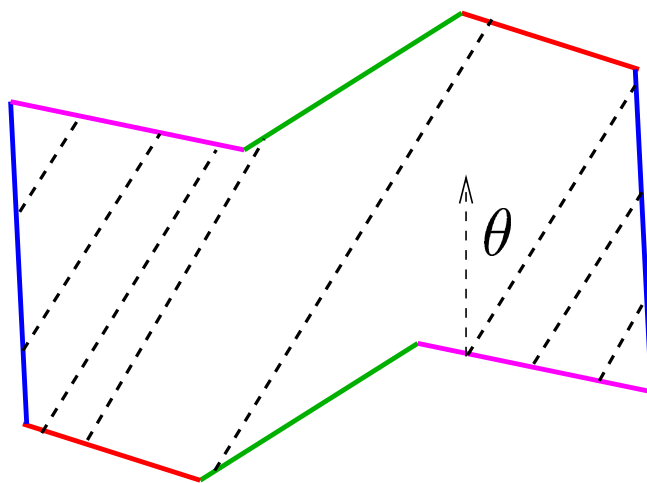
The **trajectories** of the Abelian differential are the geodesics on the corresponding translation surface.



When are geodesics closed ? When are they dense ? How do geodesics distribute themselves on the surface ?

Measured foliations

Geodesics in a given direction define a foliation of the surface which is a special case of a **measured foliation**: it is given by the kernel of a real closed 1-form $\Re(e^{i\theta}\omega)$.



Calabi, Katok, Hubbard-Masur, Kontsevich-Zorich: Every measured foliation with no saddle connections (leaves that connect singularities) is of this form.

Moduli spaces

\mathcal{M}_g = moduli space of Riemann surfaces of genus $g \geq 2$

\mathcal{A}_g = moduli space of Abelian differentials on Riemann surfaces of genus $g \geq 2$

$$\dim_{\mathbb{C}} \mathcal{M}_g = 3g - 3 \quad \dim_{\mathbb{C}} \mathcal{A}_g = 4g - 3$$

\mathcal{A}_g is a fiber bundle (“cotangent bundle”) over \mathcal{M}_g .

Strata of \mathcal{A}_g

Consider any $m_1, \dots, m_\sigma \geq 1$ with $\sum_{i=1}^{\sigma} m_i = 2g - 2$.

$\mathcal{A}_g(m_1, \dots, m_\sigma)$ = subset of Abelian differentials having σ zeroes, with multiplicities m_1, \dots, m_σ .

$$\dim_{\mathbb{C}} \mathcal{A}_g(m_1, \dots, m_\sigma) = 2g + \sigma - 1$$

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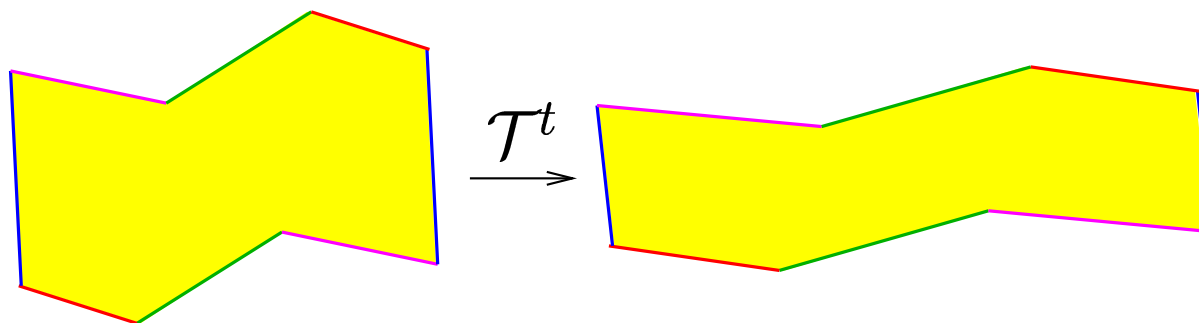
Each stratum may have up to 3 connected components. Kontsevich-Zorich classified all connected components.

Teichmüller flow

The **Teichmüller flow** is the natural action \mathcal{T}^t on the fiber bundle \mathcal{A}_g by the diagonal subgroup of $\mathrm{SL}(2, \mathbb{R})$:

$$\mathcal{T}^t(\omega)_z = [e^t \Re \omega_z] + i [e^{-t} \Im \omega_z]$$

Geometrically:

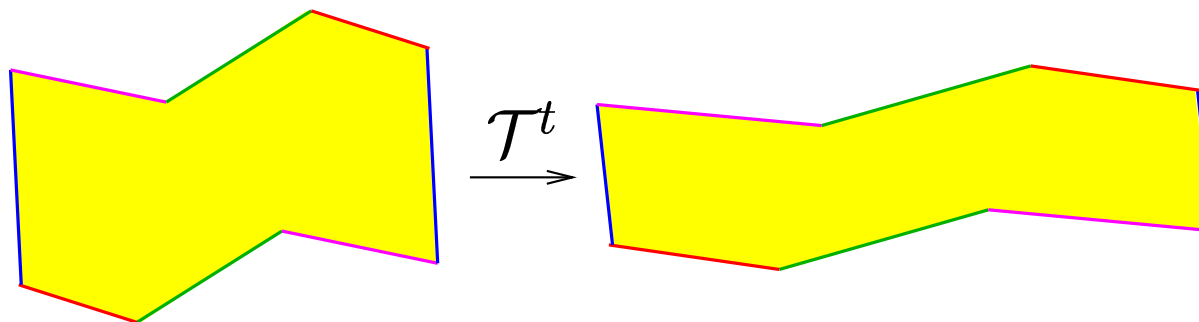


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This flow leaves invariant the volume on every stratum and also preserves the area of the translation surface S .

General Principle

Properties of the Teichmüller flow reflect upon dynamical properties of almost all Abelian differentials.

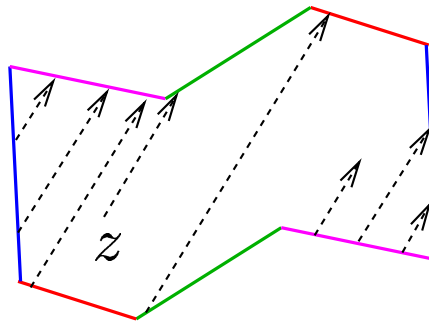
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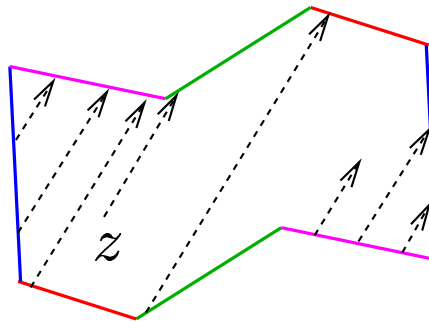
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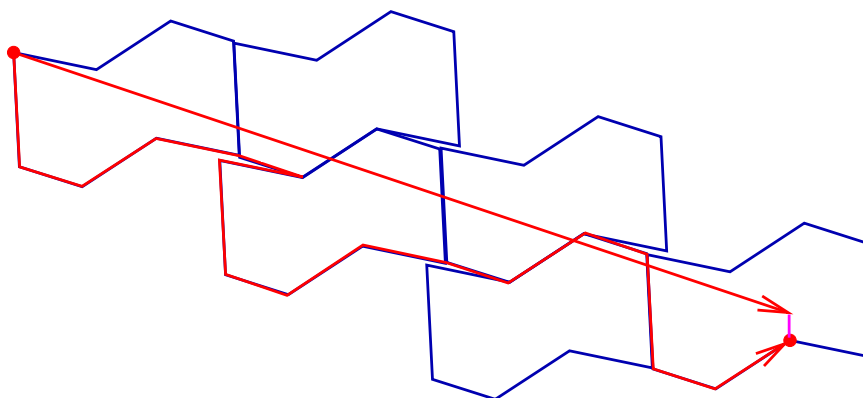
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The result was much improved by **Kerchhoff, Masur, Smillie:** unique ergodicity holds for **every** Abelian differential and almost every direction.

Asymptotic cycles

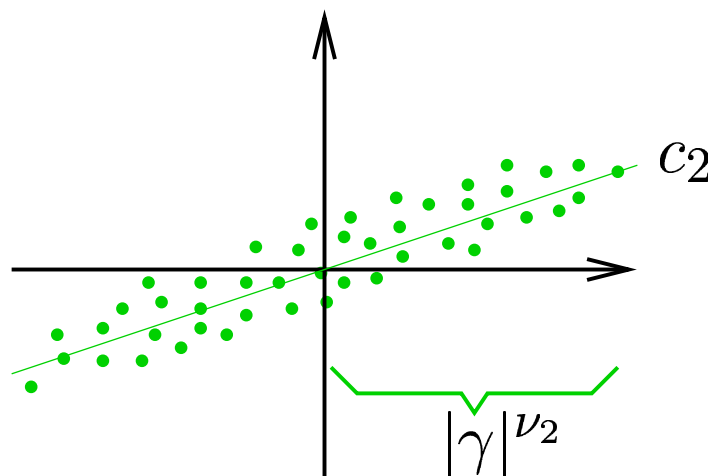
Any geodesic segment γ may be “closed” to get an element $h(\gamma)$ of $H_1(S, \mathbb{Z})$:



Unique ergodicity implies $h(\gamma)/|\gamma|$ converges uniformly to some $c_1 \in H_1(S, \mathbb{R})$ when the length $|\gamma|$ goes to infinity, and the **asymptotic cycle** c_1 does not depend on the initial point, only the surface and the direction.

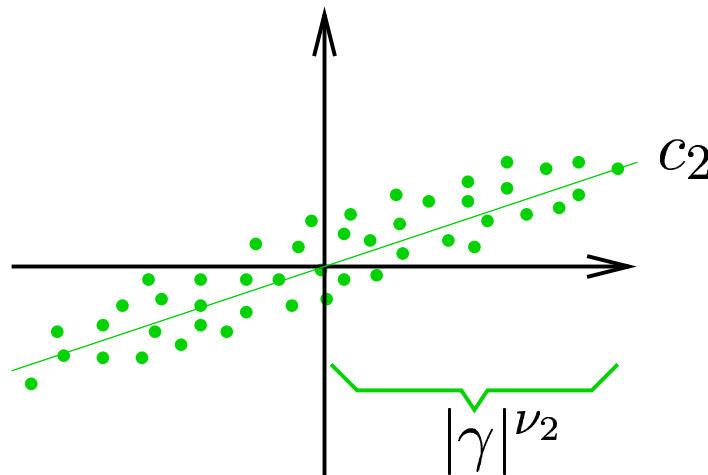
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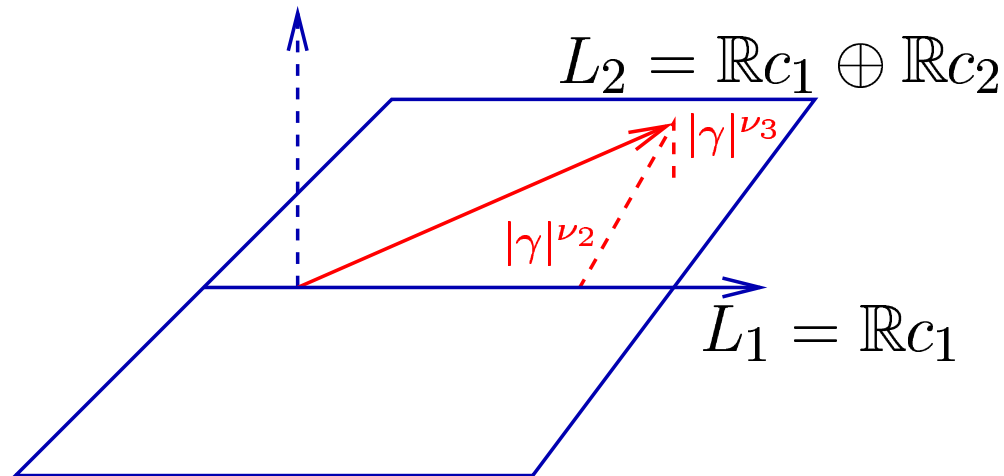


Similarly in higher order: the component of $h(\gamma)$ orthogonal to $\mathbb{R}c_1 \oplus \mathbb{R}c_2$ has a favorite direction c_3 , and amplitude $|\gamma|^{\nu_3}$ for some $\nu_3 < \nu_2$, and so on **up to order $g = \text{genus}$** .

Asymptotic flag conjecture

Conjecture (Zorich, Kontsevich). *There are $1 > \nu_2 > \dots > \nu_g > 0$ and subspaces $L_1 \subset L_2 \subset \dots \subset L_g$ of $H_1(S, \mathbb{R})$ with $\dim L_i = i$ for every i , such that*

- *the deviation of $h(\gamma)$ from L_i has amplitude $|\gamma|^{\nu_{i+1}}$ for all $i < g$*
- *the deviation of $h(\gamma)$ from L_g is bounded ($g = \text{genus}$).*



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The Lyapunov exponents of the Teichmüller flow are

$$\begin{aligned} 2 &\geq 1 + \nu_2 \geq \cdots \geq 1 + \nu_g \geq 1 = \cdots = 1 \geq 1 - \nu_g \geq \cdots \geq 1 - \nu_2 \geq 0 \\ &\geq -1 + \nu_g \geq \cdots \geq -1 + \nu_g \geq -1 = \cdots = -1 \geq -1 - \nu_g \geq \cdots \geq -1 - \nu_2 \geq -2. \end{aligned}$$

Avila, Viana prove that all inequalities above are strict.
The Z-K conjecture follows.