Geometry of Flat Surfaces

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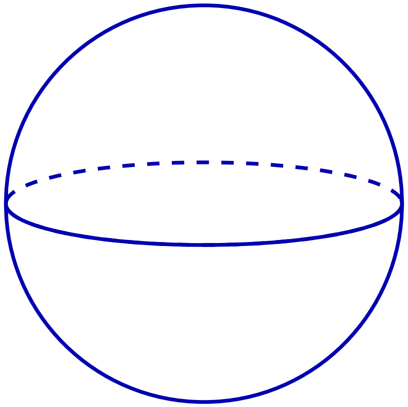
IMPA - Rio de Janeiro

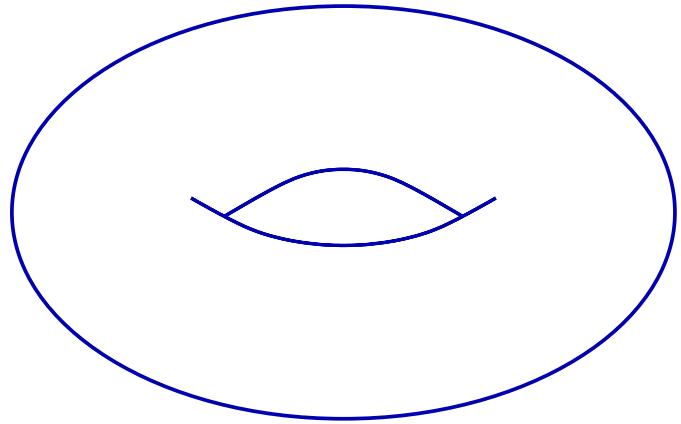
Xi'an Jiaotong University 2005

Some (non-flat) surfaces

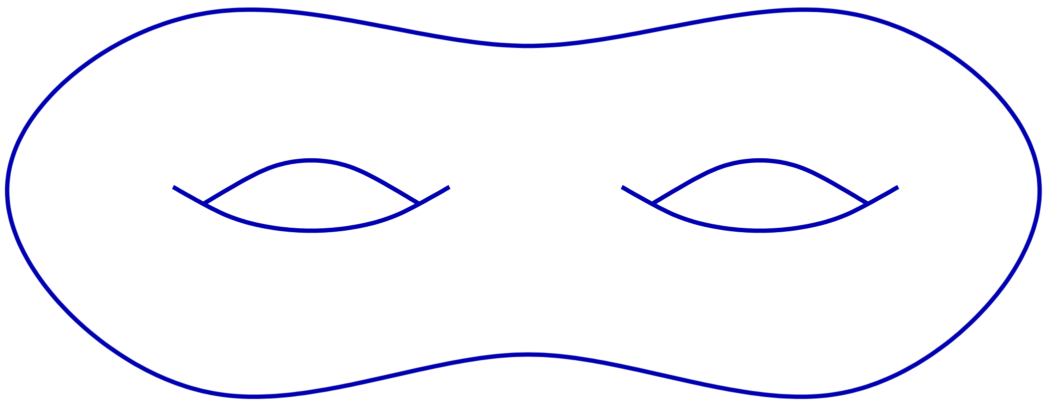
Sphere ($g = 0$)



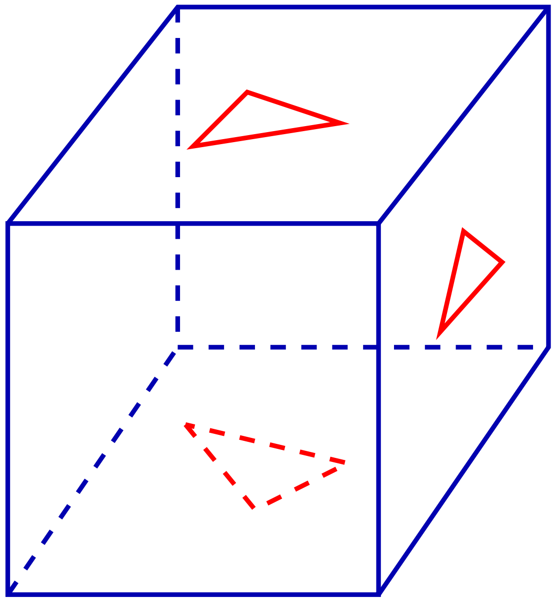
Torus ($g = 1$)



Bitorus ($g = 2$)



One flat "sphere": the cube

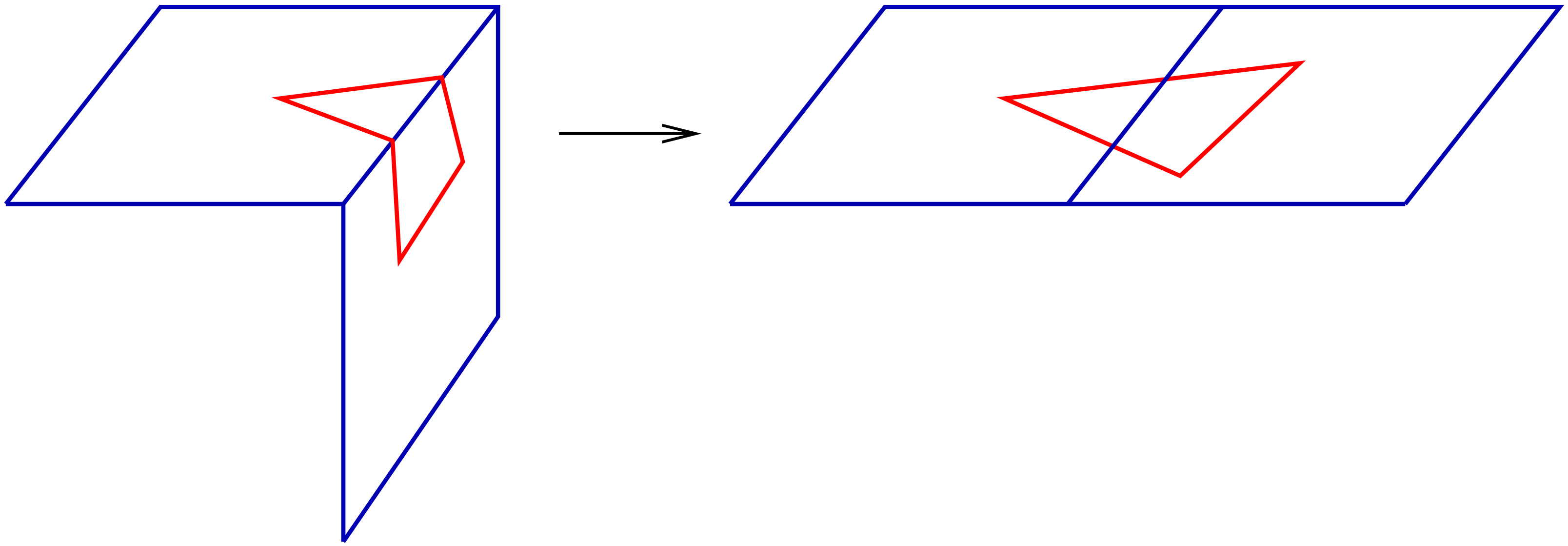


Flat surface: the sum of the internal angles of any triangle on the surface is equal to 180 degrees.

Any triangle ?...

What about the edges ?

Every edge can be "flattened" without deforming the surface:

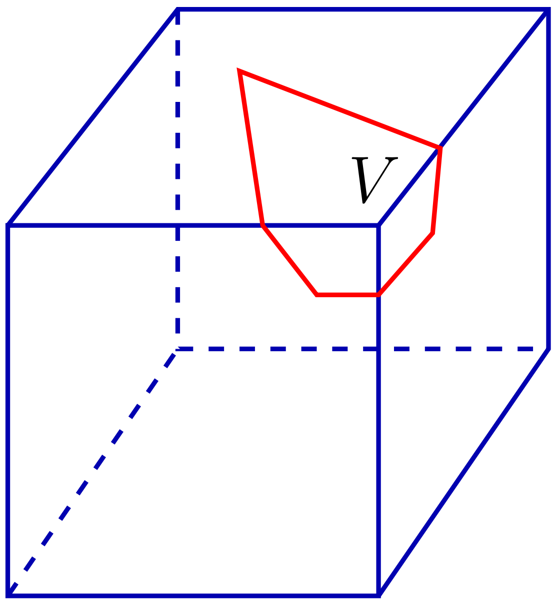


The geodesics ("shortest paths") correspond to straight line segments after flattening.

The sum of the internal angles of a triangle is 180 degrees.

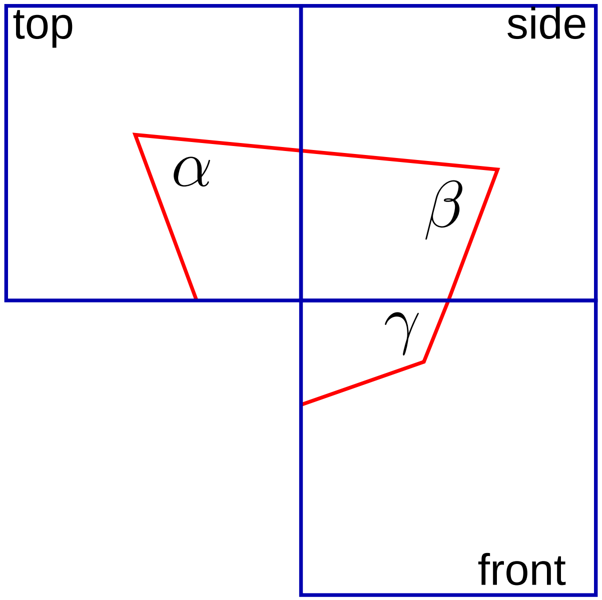
What about the vertices ?

Define $\text{ang}(V)$ = sum of the angles of the faces of the surface adjacent to a given vertex V. In the case of the cube $\text{ang}(V) = 3\pi/2$.



Whenever $\text{ang}(V) \neq 2\pi$, the vertex can not be “flattened" without deforming or tearing the surface.

Triangles on a vertex



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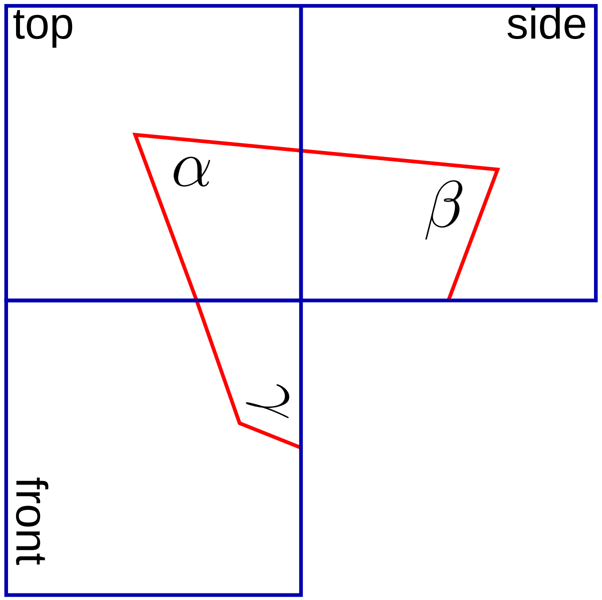
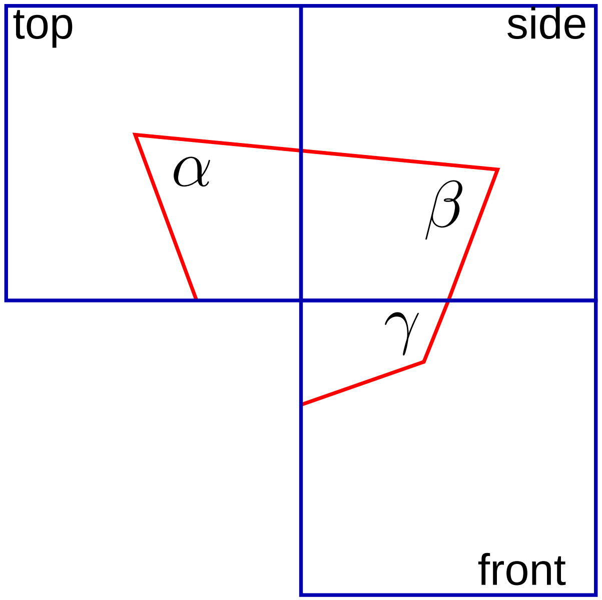
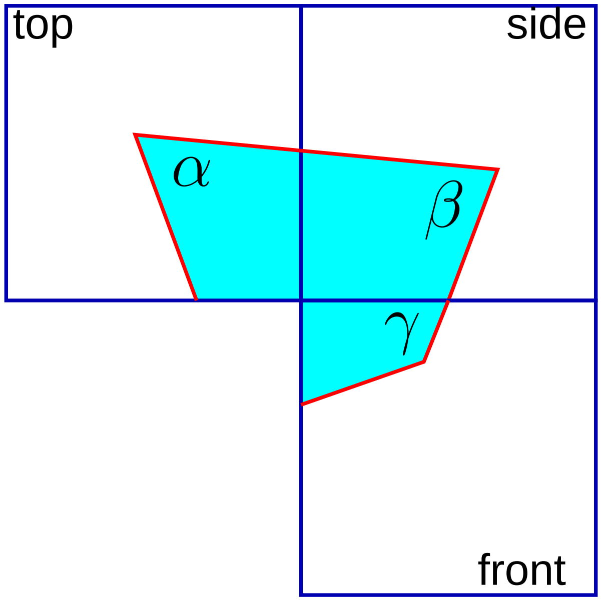


Triangles on a vertex



Sum of the internal angles



The sum of the internal angles of this hexagon is

$$\alpha + \beta + \gamma + \operatorname{ang}(V) + \pi = 4\pi,$$

so the sum of the angles of the triangle on the cube is
$\alpha + \beta + \gamma = 3\pi - \operatorname{ang}(V) = 3\pi/2$. General rule ?

Theorem of Gauss-Bonnet

On a smooth surface the integral of the Gaussian curvature is equal to $2\pi\mathcal{X}$,
where $\mathcal{X} = 2 - 2g$ is the Euler characteristic of the surface.

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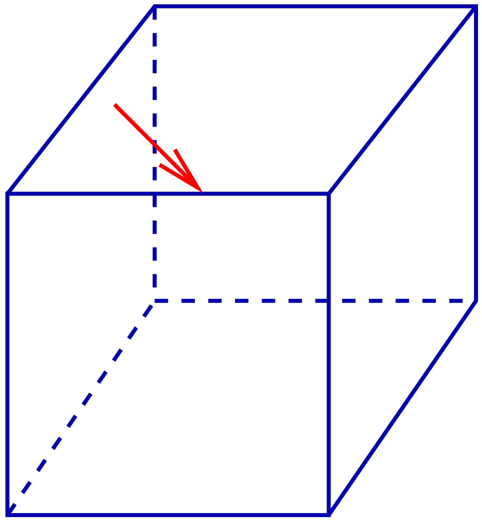
where $\mathcal{X} = 2 - 2g$ is the Euler characteristic of the surface.

Version for flat surfaces:

The sum $\sum_{i=1}^{N} \left(2\pi - \mathrm{ang}(V_i)\right)$ is equal to $2\pi\mathcal{X}$, where $V_1, \ldots, V_N$ are the vertices of the surface.

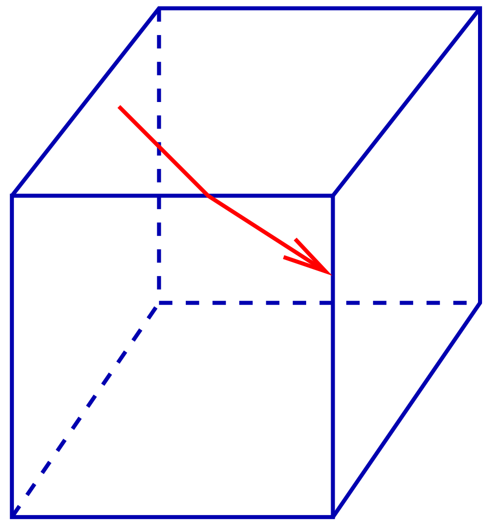
Flat surface: the sum of the internal angles of any triangle is equal to 180 degrees,
except at a finite number of points, the vertices, where is concentrated all the curvature of the surface.

Geodesic walks



We consider "straight lines" (geodesics) in a given direction, from different points on the surface.

Geodesic walks



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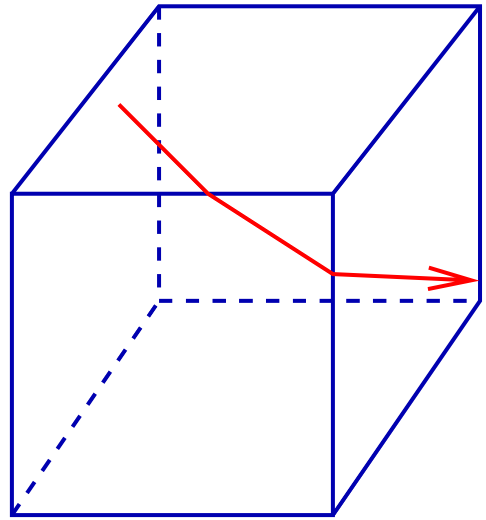


We want to understand the behavior of these geodesics, the way they "wrap" around the surface:

- When are the geodesics closed curves ?
- When are they dense on the surface ?
- What is their quantitative behavior ?

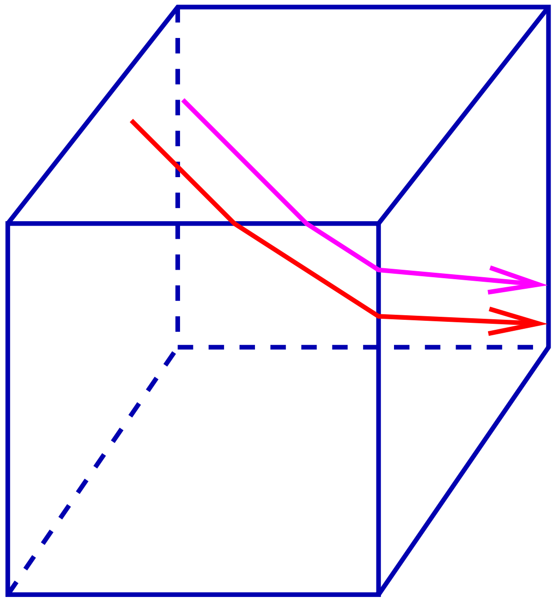
Motivation

The geodesic flow on flat surfaces is related to:

- Interval exchange transformations
- Dinamics of measured foliations
- Lyapunov exponents of linear cocycles
- Teichmüller spaces and flows
- Moduli spaces of Riemann surfaces
- Quadratic differentials
- Continued fraction expansions
- Billiards on polygonal tables
- Renormalization operators
- ...

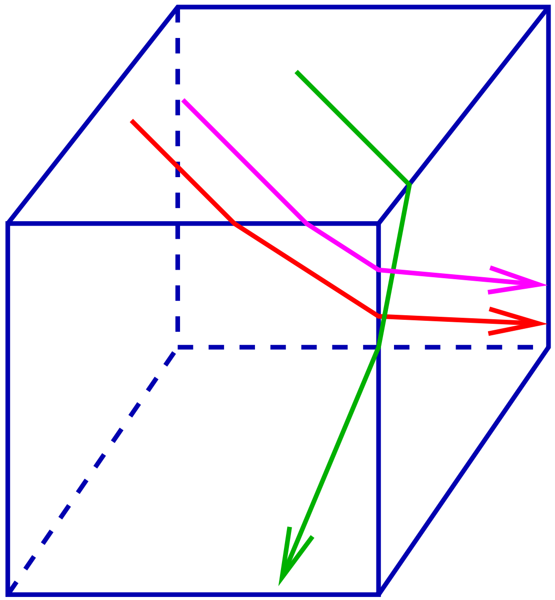
$\Rightarrow$

Geodesic walks



At first sight, the behavior does not depend much on the initial points: geodesics starting in the same direction remain parallel.

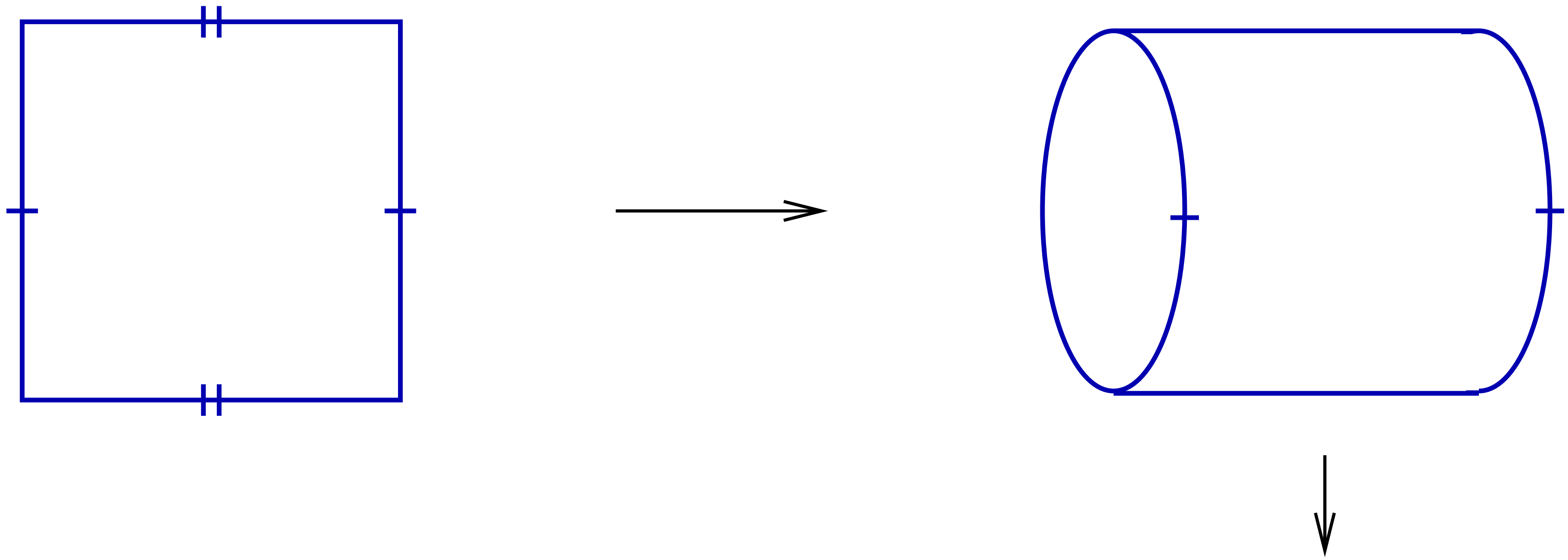
Geodesic walks



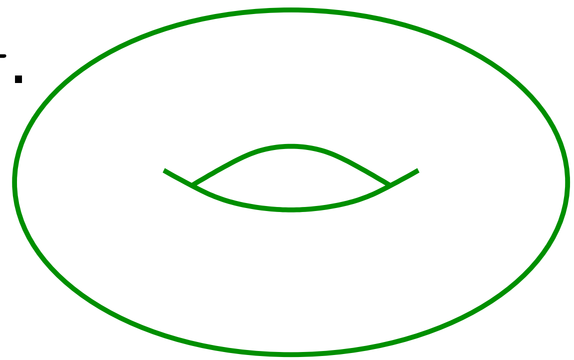
At first sight, the behavior does not depend much on the initial points: geodesics starting in the same direction remain parallel.

But the presence of the vertices may render the situation much more complicated.

The flat torus

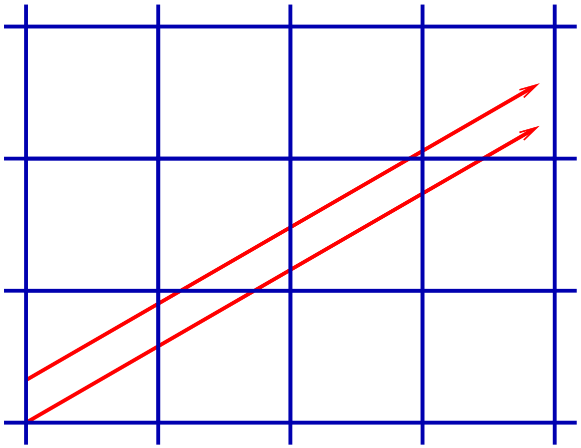


One single vertex V, with $\mathrm{ang}(V) = 2\pi$.



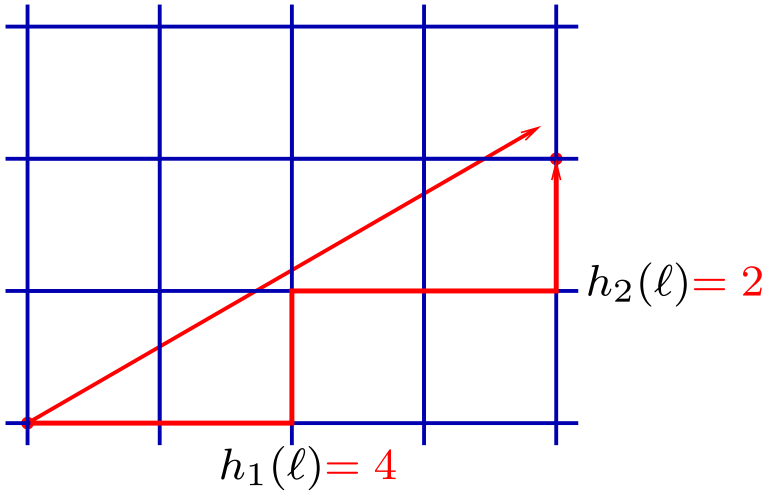
The flat torus does not embed in $\mathbb{R}^3$.

Geodesic walks on the torus



Geodesics in a given direction remain parallel.

Geodesic walks on the torus

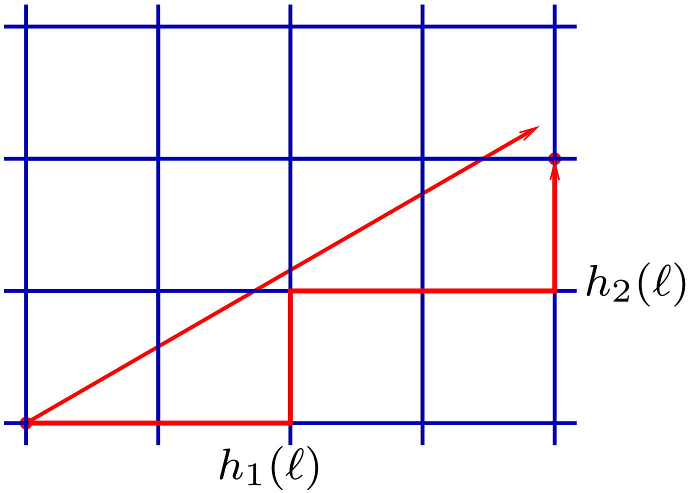


Their behavior may be described using the vector

$$(v_1, v_2) = \lim_{\ell \to \infty} \frac{1}{\ell}(h_1(\ell), h_2(\ell)),$$

where $h_1(\ell)$, $h_2(\ell) =$ “number of turns" of a geodesic segment of length ℓ makes around the torus, in the horizontal and the vertical direction.

Geodesic walks on the torus

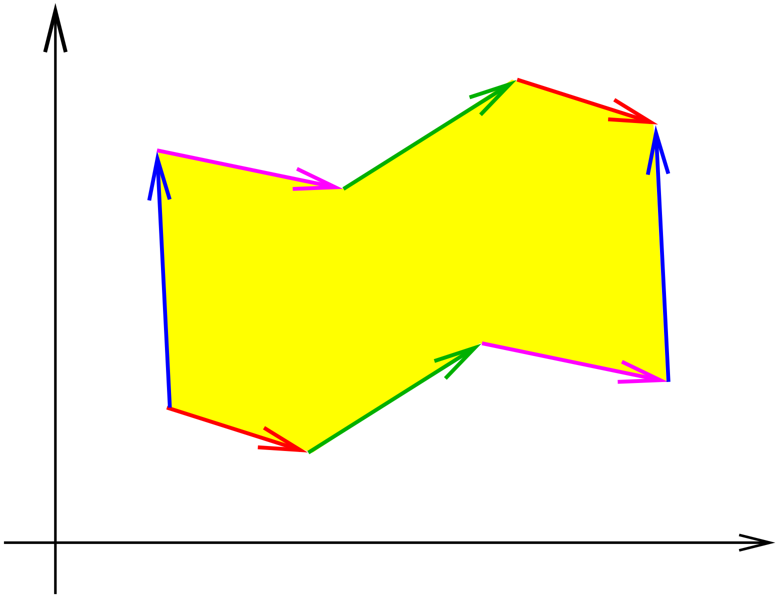


Theorem.

1. *If* v_1/v_2 *is rational then every geodesic is closed.*
2. *If* v_1/v_2 *is irrational then every geodesic is dense and even uniformly distributed (the flow is uniquely ergodic).*

A more general construction

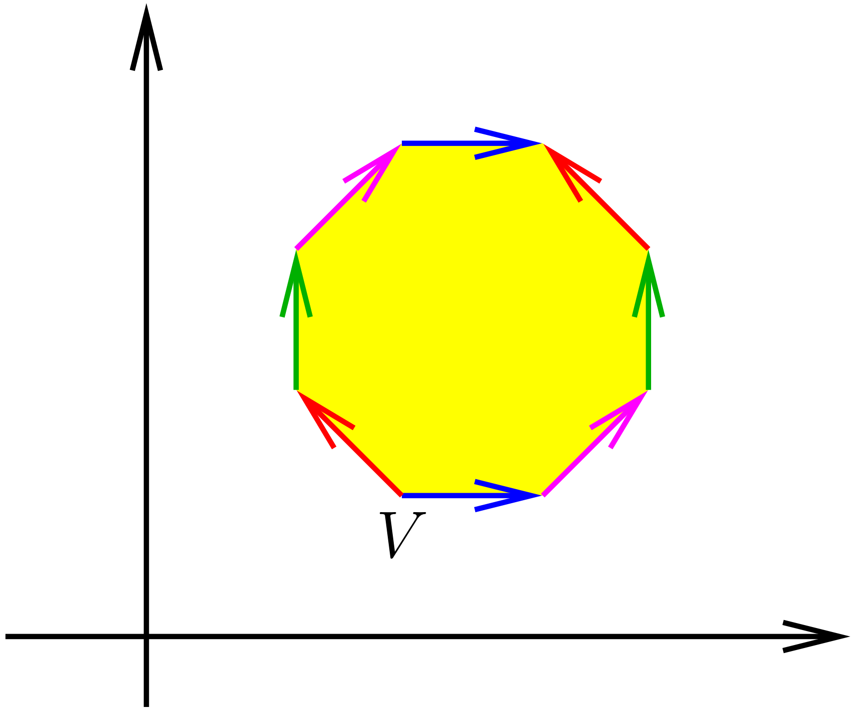
Let us consider any planar polygon bounded by an even number of pairs of (non-adjacent) line segments which are parallel and have the same length.



Identifying the segments in each pair we get a flat surface.

An example

Let us consider the regular octagon:



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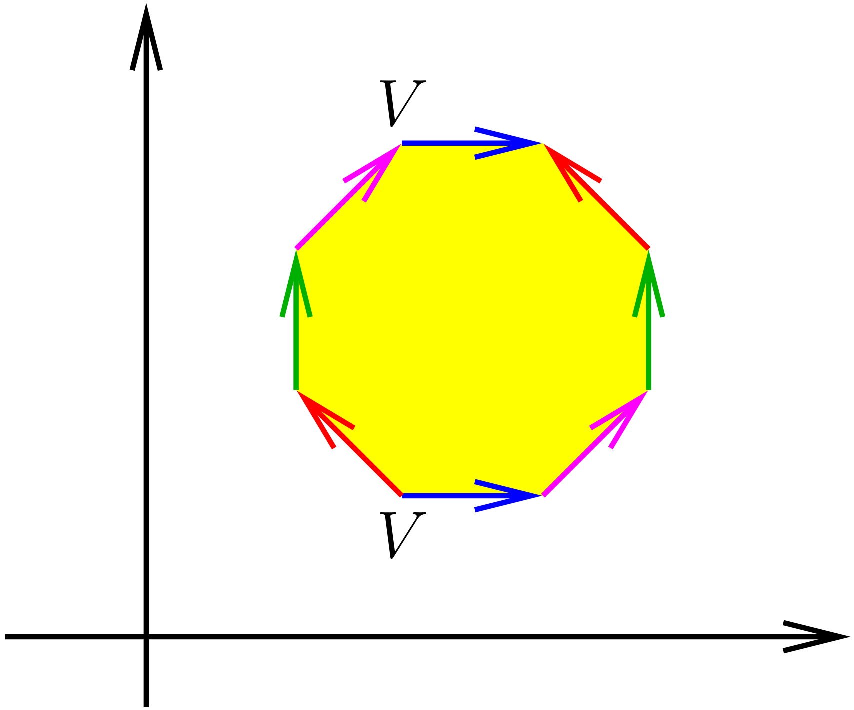

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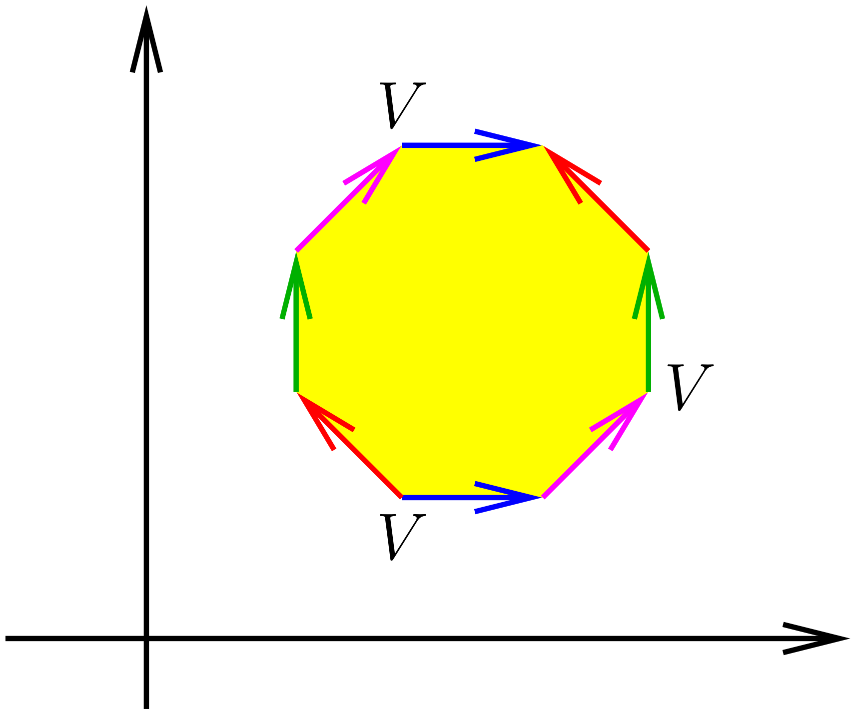

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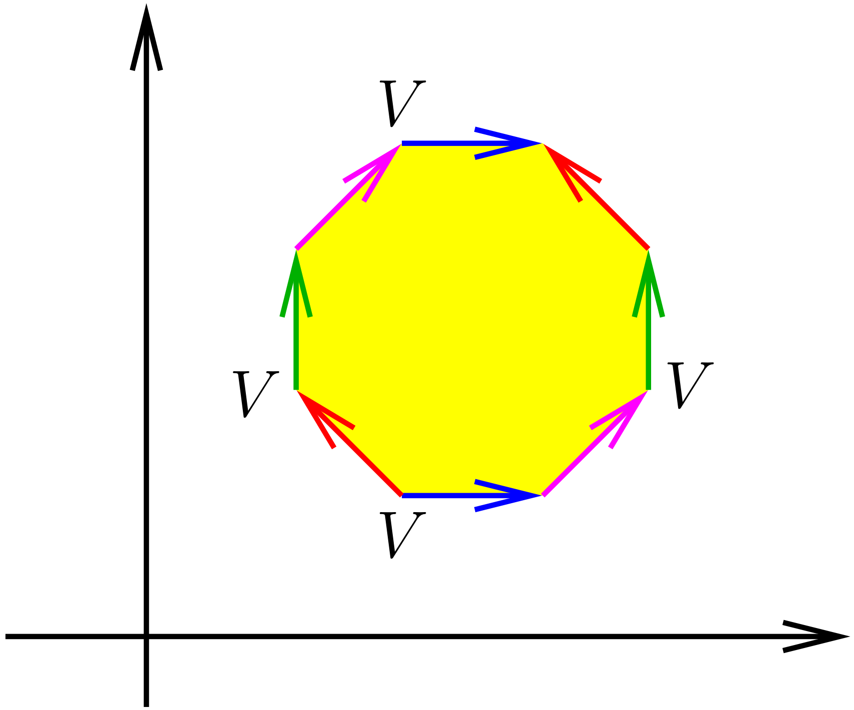

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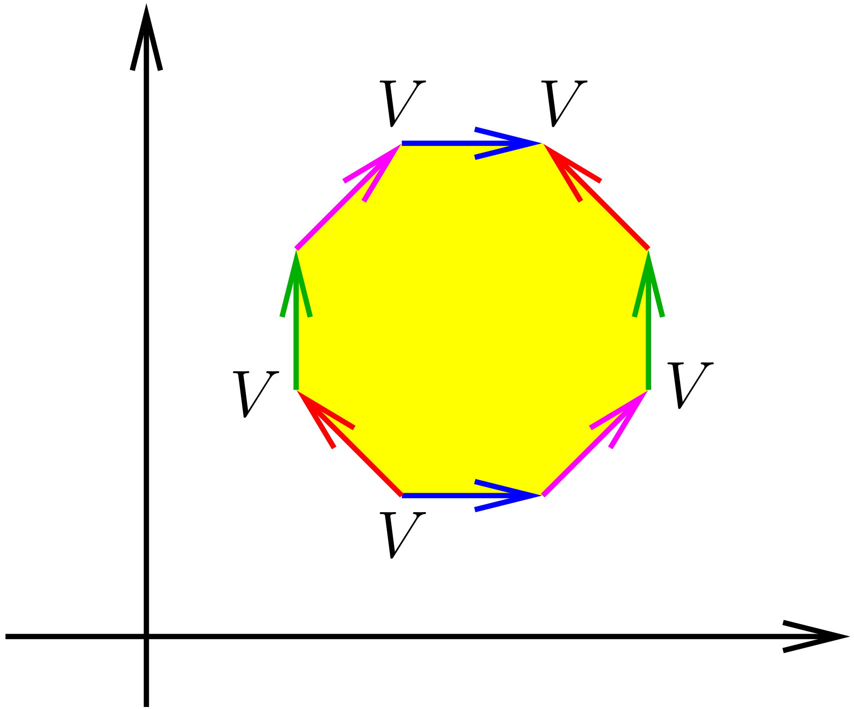

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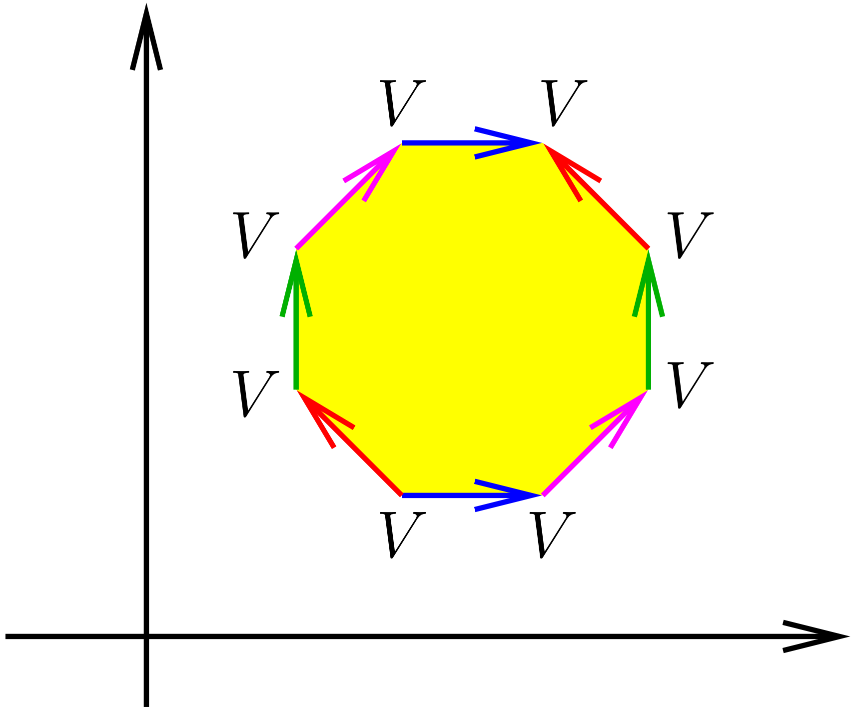

The surface has a unique vertex V, with $\mathrm{ang}(V) = 6\pi$.

How can the angle be bigger than 2π ?

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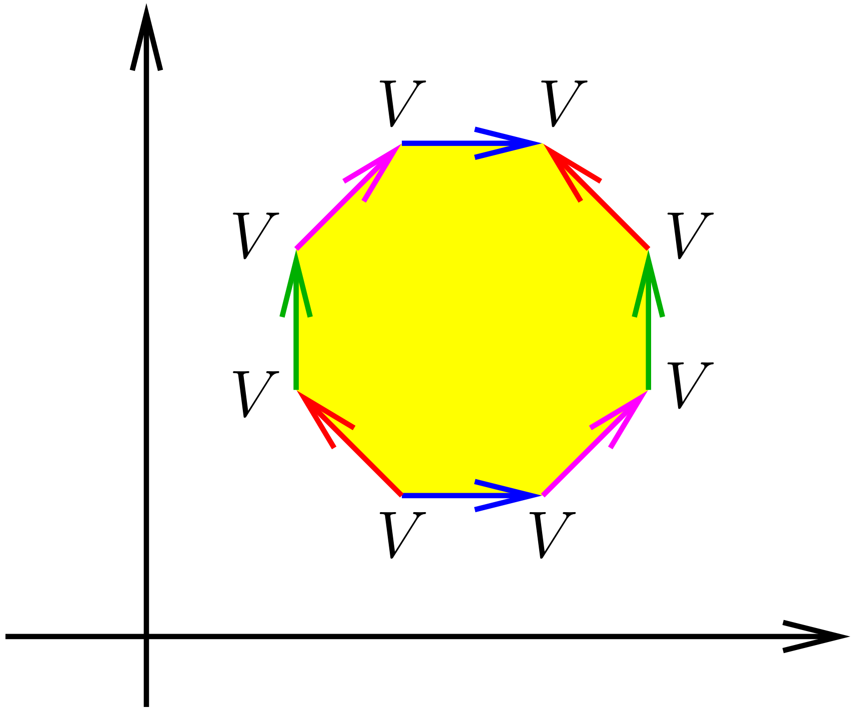

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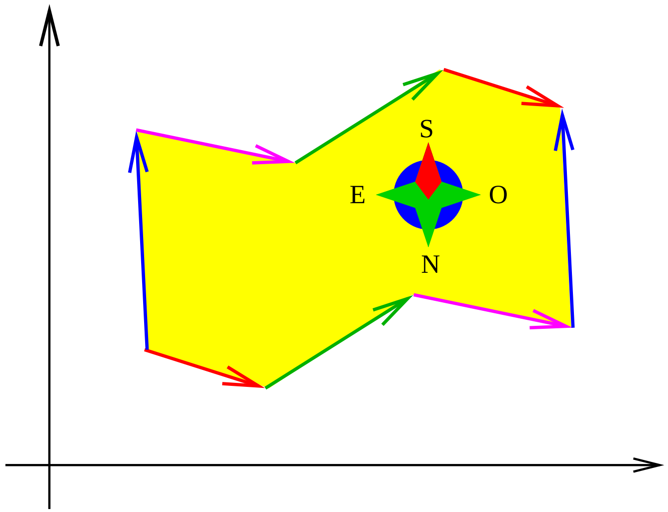
How can the angle be bigger than 2π ?

So, by Gauss-Bonnet, it has genus $g = 2$: flat bitorus.

Translation surfaces

The flat surfaces obtained from planar polygons have some additional structure: a globally defined "compass".



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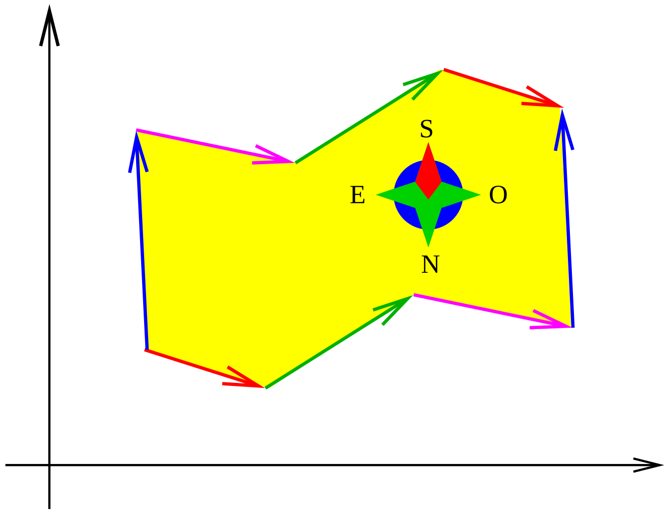
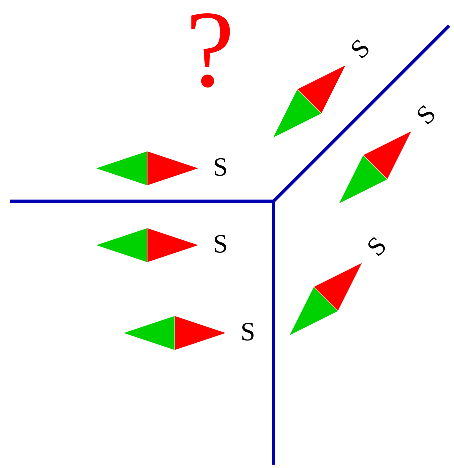
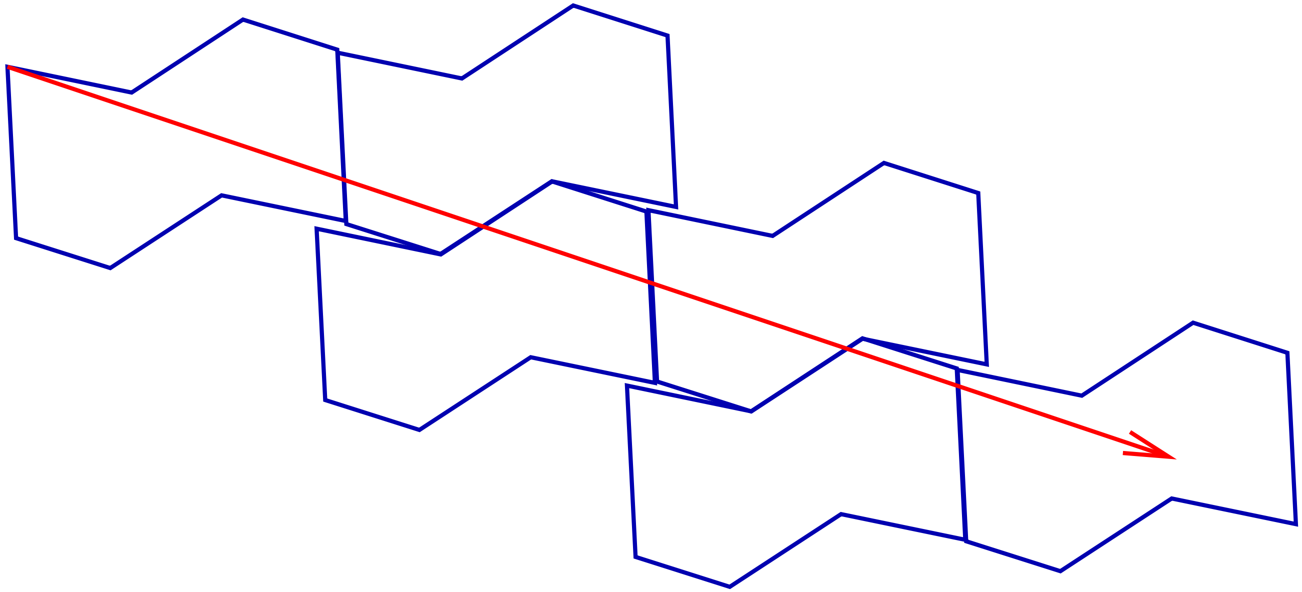


That is not the case of the cube:

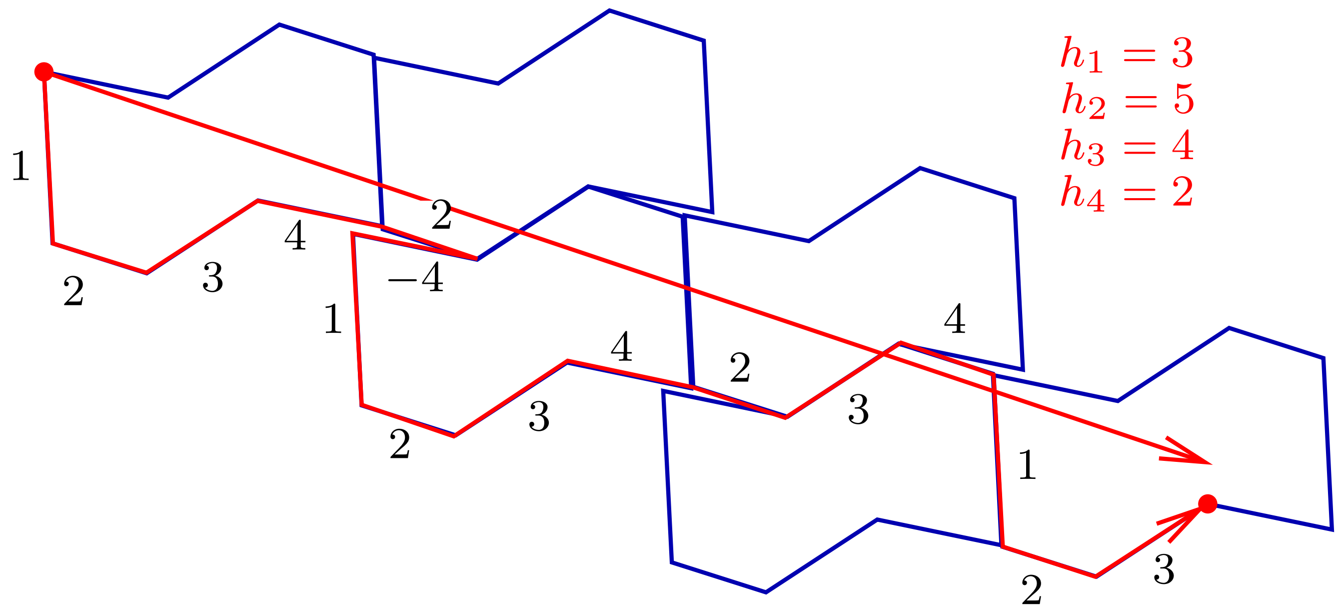


Translation flows



Just as we did for the torus, let us consider geodesics with a given direction starting from points on the surface.

Translation flows



To each geodesic segment of length ℓ we can associate an integer vector $H(\ell) = (h_1(\ell), \ldots, h_d(\ell))$

where $h_i(\ell) =$ “number of turns" in the direction of the i’th side of the polygon.

Asymptotic cycles

S. Schwartzmann (1957):
the *asymptotic cycle* of a pair (surface, direction) is the limit

$$c_1 = \lim_{\ell \to \infty} \frac{1}{\ell} H(\ell)$$

This vector $c_1 \in \mathbb{R}^d$ describes the “average number of turns" of geodesics around the various sides of the polygon, per unit of length.

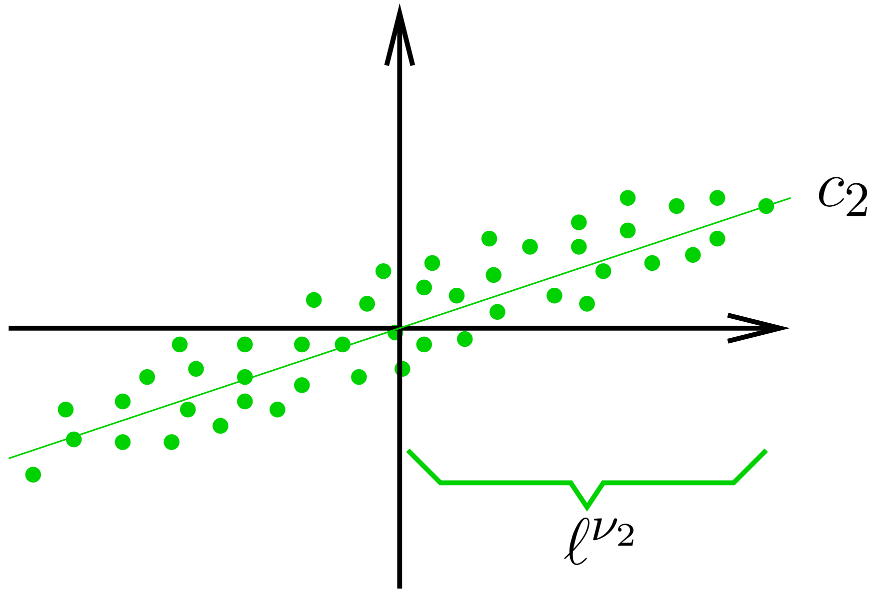
Theorem (Kerckhoff, Masur, Smillie 1986)**.** *For any translation surface and almost any direction, the geodesic flow is uniquely ergodic. In particular, the asymptotic cycle exists and every geodesic is dense.*

Deviations from the limit

Numerical experiments by Anton Zorich suggest that the differences

$$H(\ell) - \ell c_1$$

are distributed along some direction $c_2 \in \mathbb{R}^d$



and their order of magnitude is ℓ^{ν_2} for some $\nu_2 < 1$.

Deviations from the limit

Refining these experiments, he observed that second order deviations

$$\text{“}H(\ell) - \ell c_1 - \ell^{\nu_2} c_2\text{”}$$

are also distributed along some direction $c_3 \in \mathbb{R}^d$ and their order of magnitude is ℓ^{ν_3} for some $\nu_3 < \nu_2$.

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The same type of behavior is observed for higher order deviations:

Conjecture (Zorich-Kontsevich ∼1995). *There exist* c_1, c_2, *...*, c_g *in* $\mathbb{R}^d$ *and numbers* $1 > \nu_2 > \cdots > \nu_g > 0$ *such that*

$$“\ H(\ell) = c_1\ell + c_2\ell^{\nu_2} + c_3\ell^{\nu_3} + \cdots + c_g\ell^{\nu_g} + R(\ell)\ ”$$

where $R(\ell)$ *is a bounded function.*

Zorich – Kontsevich conjecture

Kontsevich, Zorich gave a dynamical interpretation of the vectors c_i and the numbers ν_i (Lyapunov exponents). The main point was to prove

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Theorem (Veech 1984)**.** $\nu_2 < 1$.

Theorem (Forni 2002)**.** $\nu_g > 0$.

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$$1 > \nu_2 > \cdots > \nu_g > 0$$

Theorem (Veech 1984)**.** $\nu_2 < 1$.

Theorem (Forni 2002)**.** $\nu_g > 0$.

Theorem (Avila, Viana 2004)**.** *The ZK conjecture is true.*

The End

That's not all, folks!

...

Let $2d$ be the number of sides of the polygon. Apparently,

- for $d = 2$ we have $\nu_2 = 0$
- for $d = 3$ we have $\nu_2 = 0$
- for $d = 4$ we have $\nu_2 = 1/3$
- for $d = 5$ we have $\nu_2 = 1/2$
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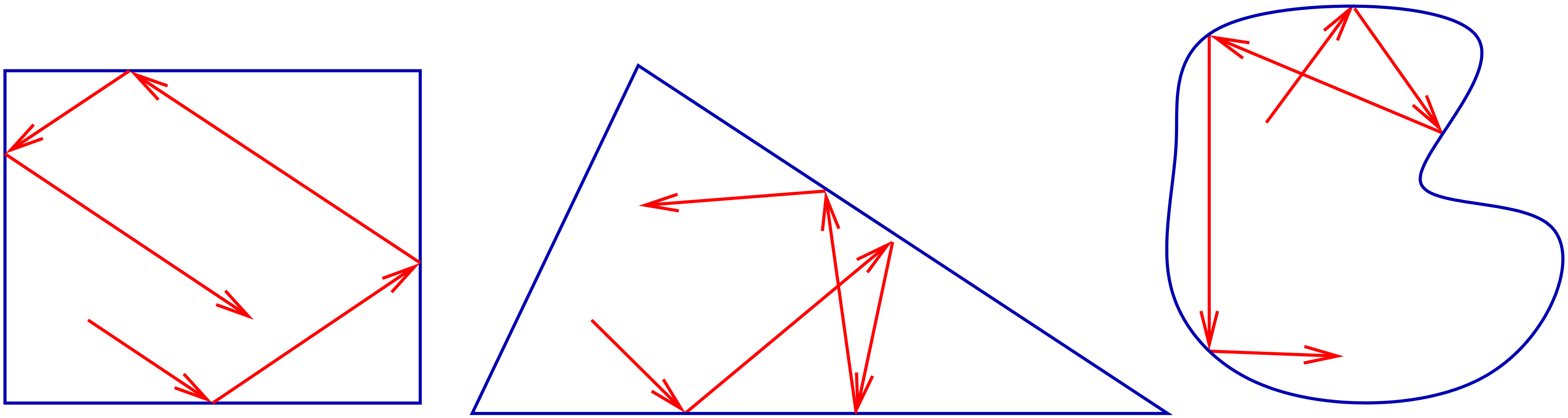
  (probably irrational...)

Conjecture (Kontsevich-Zorich). *The sum $\nu_1 + \nu_2 + \cdots + \nu_g$ is a rational number for all $g \geq 3$.*

Billiards

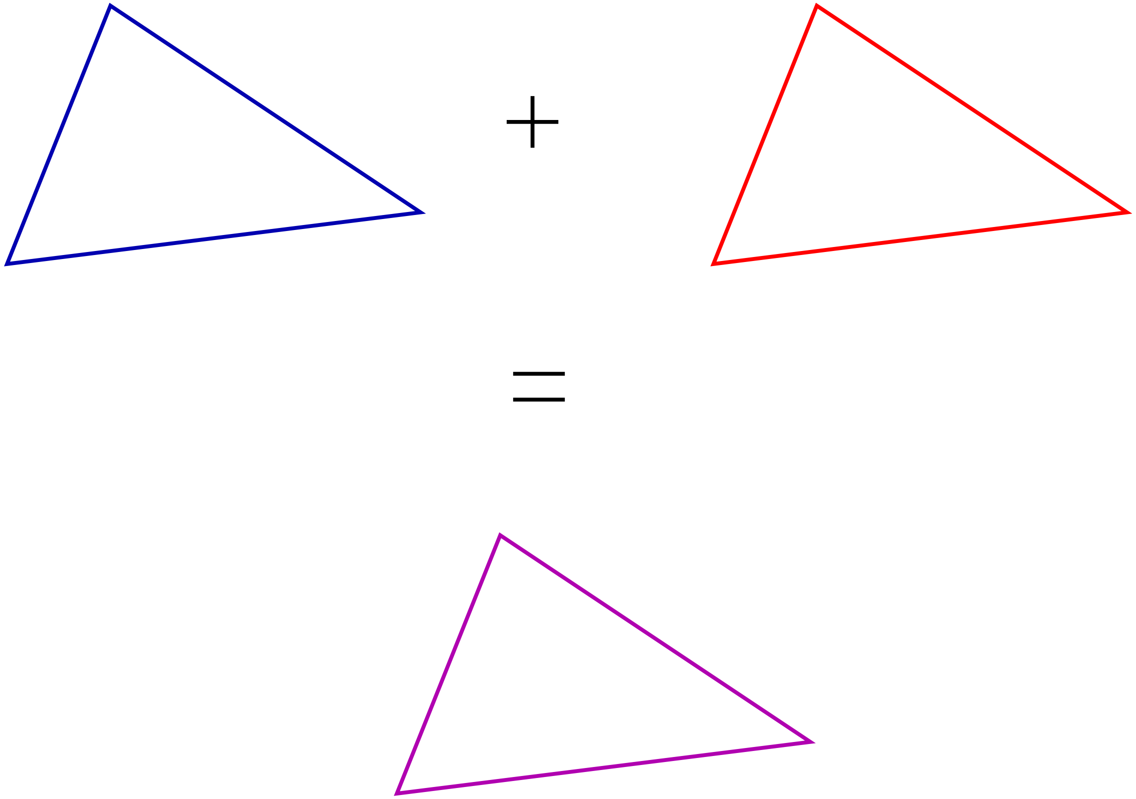
Billiards model the motion of point particles inside bounded regions in the plane, with constant speed and elastic reflections on the boundary:



Let us focus on polygonal table billiards, that are more directly related to geodesics flows on flat surfaces.

Flat spheres

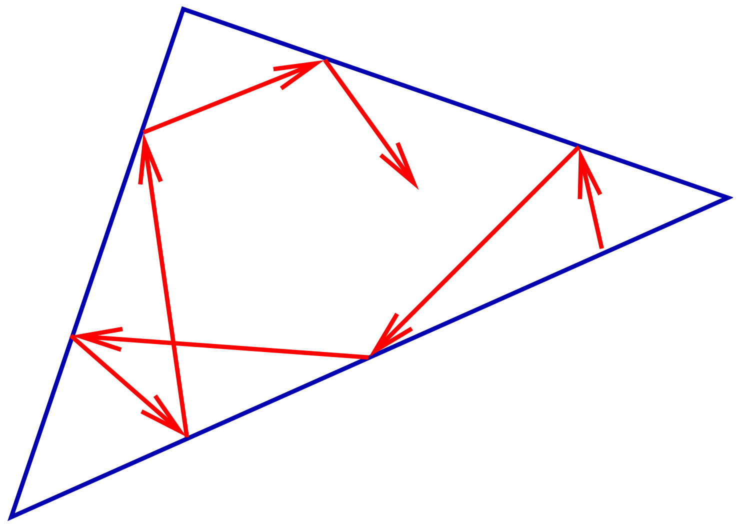
Gluing two identical triangles along their boundaries we obtain a flat sphere with 3 vertices:



Triangular tables

Billiard in a triangular table $\Leftrightarrow$

$\Leftrightarrow$ geodesic flow on a flat sphere with three vertices.



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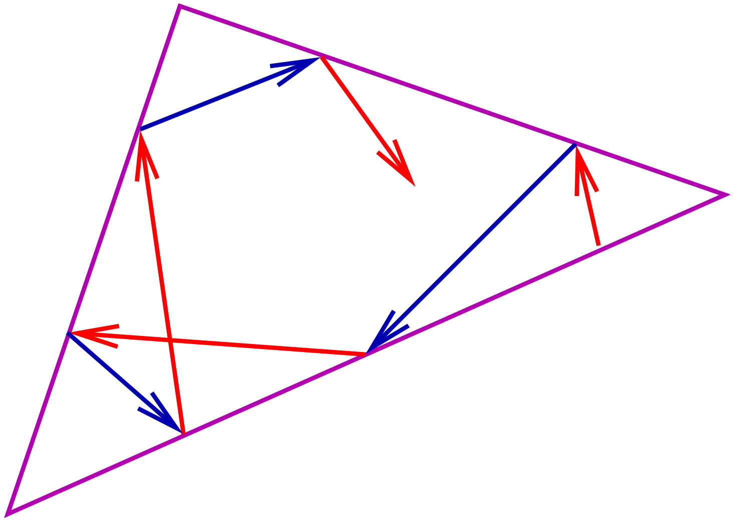
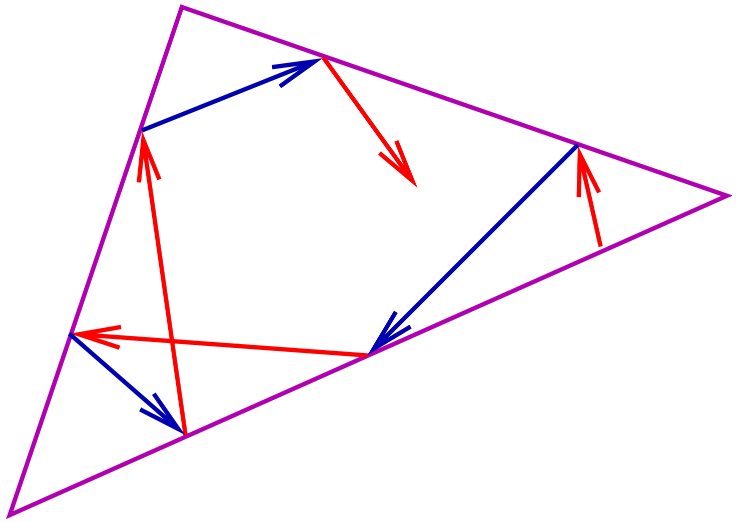


An open problem

Billiard in a triangular table $\Leftrightarrow$

$\Leftrightarrow$ geodesic flow on a flat sphere with three vertices.



Does every flat sphere with three vertices have some closed geodesic ?

Does every billiard on a triangular table have some closed trajectory ?

When the angles are ≤ 90 degrees, the answer is Yes.

Smooth spheres

For smooth spheres with positive curvature there always exist at least 3 closed geodesics: