The Zorich – Kontsevich Conjecture

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Translation Surfaces

Compact Riemann surface endowed with a non-vanishing holomorphic closed 1-form (Abelian differential) $\omega = f(z)dz$.

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$$\Downarrow$$

$$g = |\omega|$$

singularities = zeros of ω

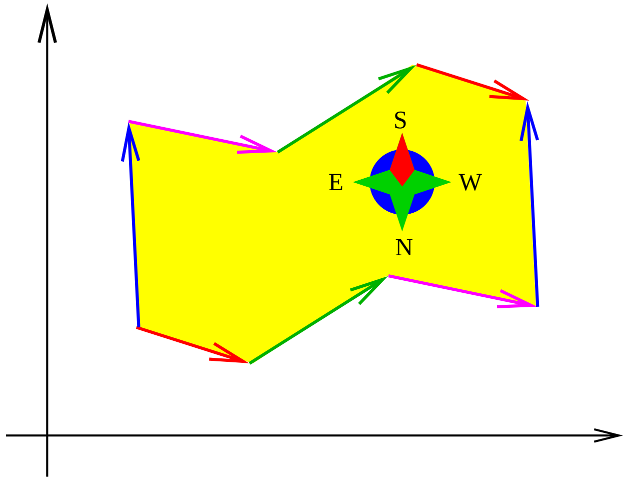
X such that $\omega \cdot X = 1$

$$\Downarrow$$

Compact orientable surface endowed with a flat metric g with finitely many conical singularities and a unit parallel vector field X.

Construction of Translation Surfaces

Consider a planar polygon whose sides can be grouped in pairs of (non-adjacent) segments that are parallel and have the same length.

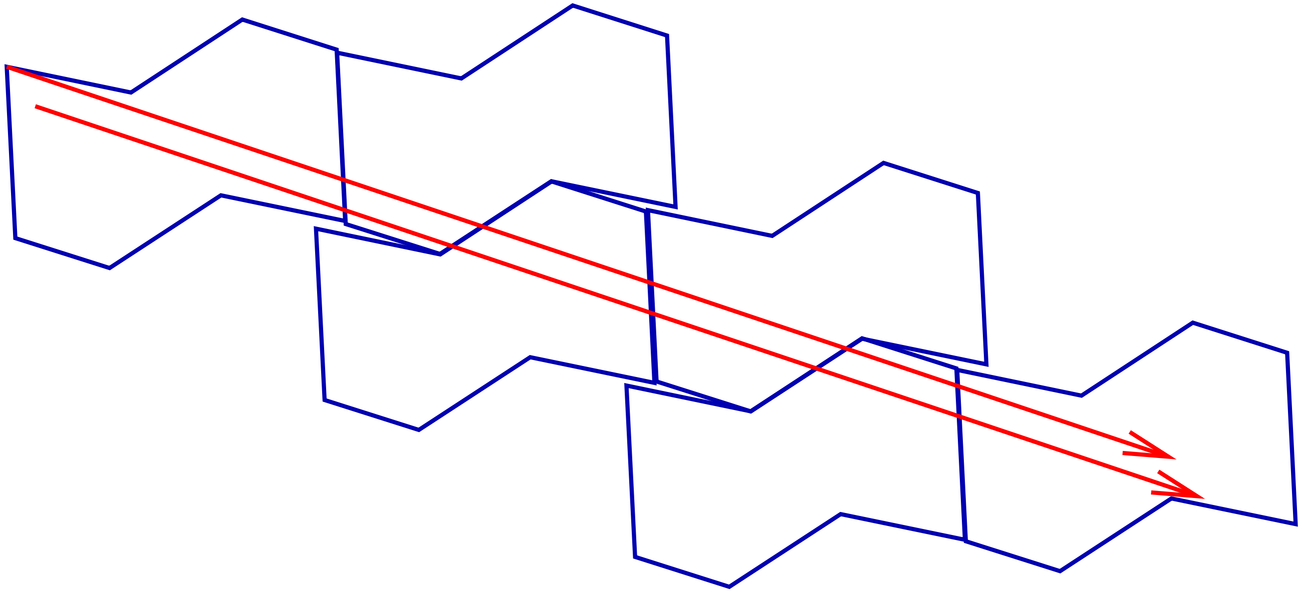


Identifying the two sides in the same pair, by translation, one obtains a translation surface.

$$\omega = dz \qquad g = \text{Euclidean} \qquad X = (0,1)$$

singularities $\subset$ vertices

Geodesic Flows on Translation Surfaces



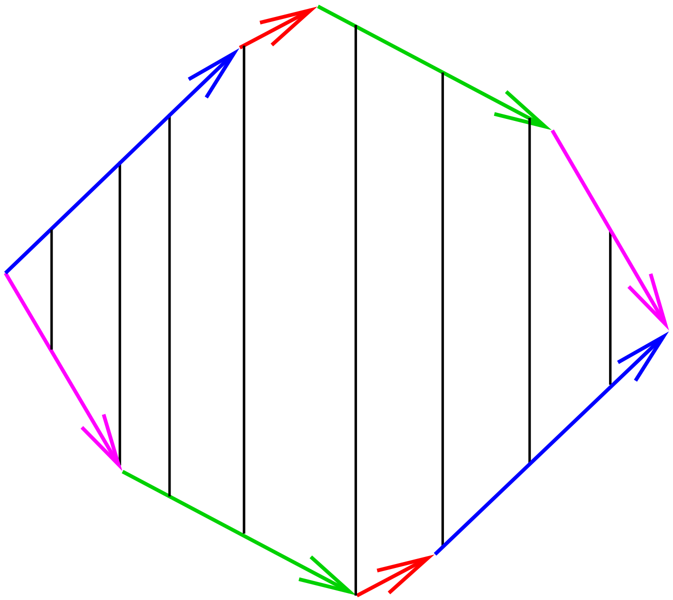
We want to understand the behavior of geodesics with a given direction. In particular,

- When are the geodesics closed ?
- When are they dense in the surface ?

Measured Foliations

A foliation on a surface is measured if it is defined by some 1-form α with isolated zeros: the leaves are tangent to the kernel of α at every non-singular point.

Example: parallel foliations on translation surfaces.



$\alpha = dx$ singularities $\subset$ vertices

Measured Foliations

Maier 43: Given a measured foliation, the ambient splits into

- periodic components: all leaves are closed;
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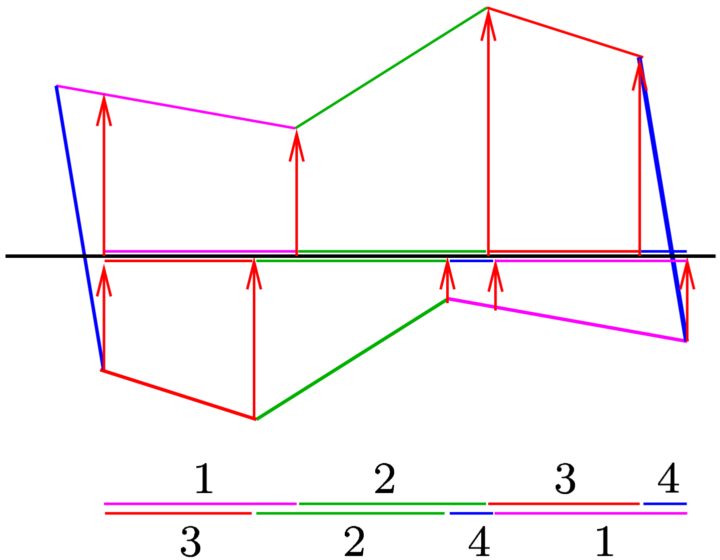
Calabi 69, Katok 73: Every measured foliation $\mathcal{F}$ without saddle-connections is the vertical foliation with respect to some flat translation metric.

Necessary and sufficient: no closed paths homologous to zero formed by positively oriented saddle-connections.

Interval Exchange Transformations

Associated to the vertical foliation on a translation surface there is an interval exchange transformation: the return map to some section transverse to the foliation.



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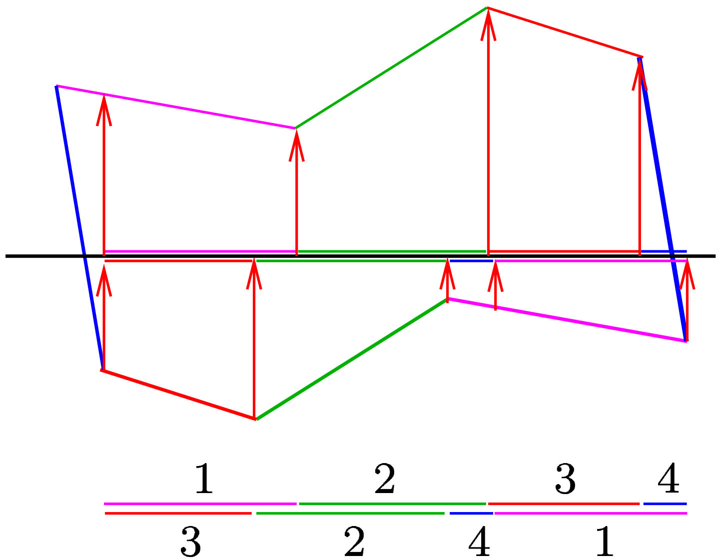
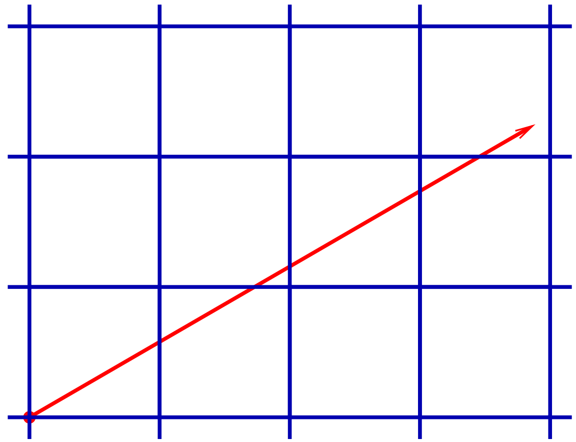


Conversely, every interval exchange transformation may be suspended to the vertical flow on some translation surface (not unique).

Geodesic Flows on the Flat Torus



Let $c_1 = (v_1, v_2)$ define the direction of the geodesics.

- If v_1/v_2 is rational then every geodesic is closed.
- If v_1/v_2 is irrational then the flow is uniquely ergodic.

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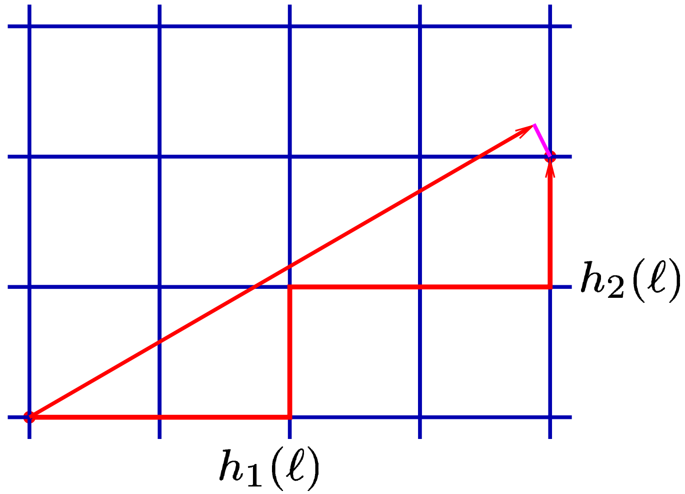

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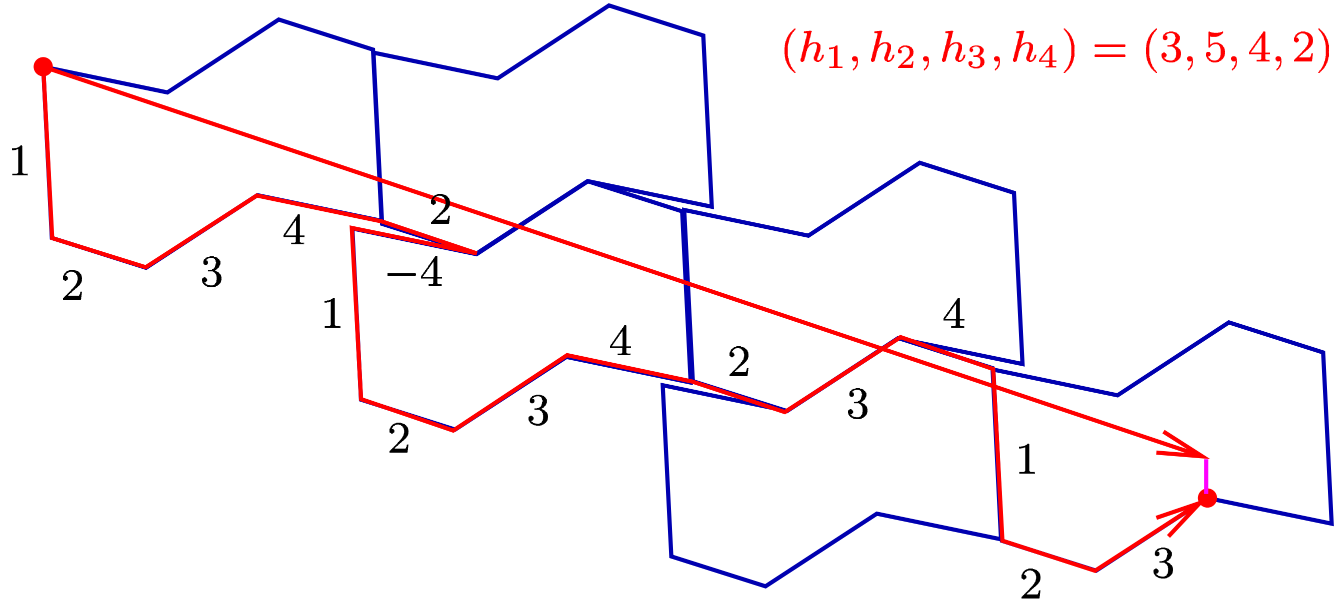
- If v_1/v_2 is rational then every geodesic is closed.
- If v_1/v_2 is irrational then the flow is uniquely ergodic.

Given a geodesic segment of length ℓ, "close" it to get $h(\ell) = (h_1(\ell), h_2(\ell)) \in H_1(\mathbb{T}, \mathbb{Z})$. Then $h(\ell) - c_1\ell$ is bounded.

Geodesic Flows in Higher Genus



Given any geodesic segment of length ℓ, close it to get an element $h(\ell) = (h_1(\ell), \dots, h_d(\ell))$ of $H_1(S, \mathbb{Z})$.

Asymptotic Cycles

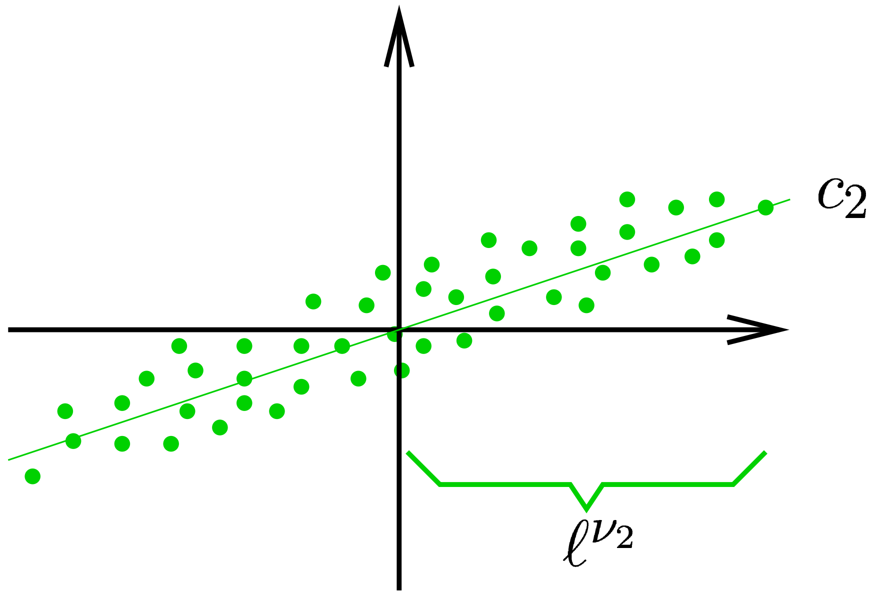
Schwartzmann 57:
The asymptotic cycle of a pair (surface, direction) is the limit

$$c_1 = \lim_{\ell \to \infty} \frac{1}{\ell} h(\ell) \in H_1(S, \mathbb{R}).$$

Kerckhoff, Masur, Smillie 86: For every translation surface and for almost every direction, the geodesic flow is uniquely ergodic. In particular, the asymptotic cycle is well defined, and every geodesic is dense.

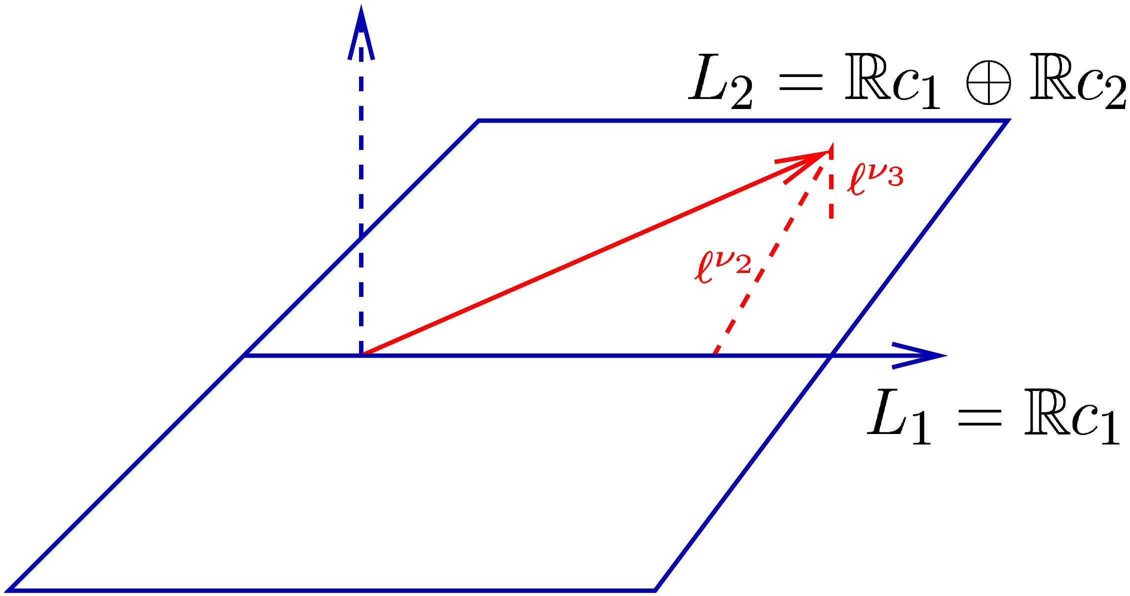
Deviations from the Limit Direction

Zorich discovered that the deviation of the vector $h(\ell)$ from the direction of c_1 distributes itself along a favorite direction c_2, with amplitude ℓ^{ν_2} for some $\nu_2 < 1$:



Deviations from the Limit Direction

The same phenomenon is observed in higher orders: the component of $h(\ell)$ orthogonal to $\mathbb{R}c_1 \oplus \mathbb{R}c_2$ has a favorite direction $c_3 \in \mathbb{R}^d$, and amplitude ℓ^{ν_3} for some $\nu_3 < \nu_2$, and so on.



Conjecture and Main Result

Conjecture (Zorich, Kontsevich). *There are numbers* $1 = \nu_1 > \nu_2 > \cdots > \nu_g > 0$ *and subspaces* $L_1 \subset L_2 \subset \cdots \subset L_g$ *with* $c_1 \in L_1$ *and* $\dim L_i = i$ *for every* i*, such that*

- *the deviation of* $h(\ell)$ *from* L_i *has amplitude* $\ell^{\nu_{i+1}}$ *for all* $i < g$
- *the deviation of* $h(\ell)$ *from* L_g *is bounded.*

(where g *= genus of the flat surface).*

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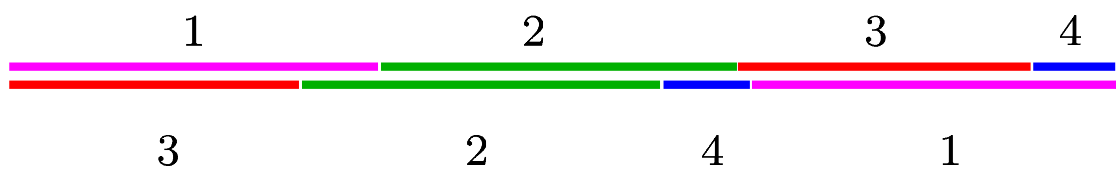
Work of Kontsevich and Zorich translated the claim of the conjecture into a statement in Dynamics:

The Rauzy Algorithm

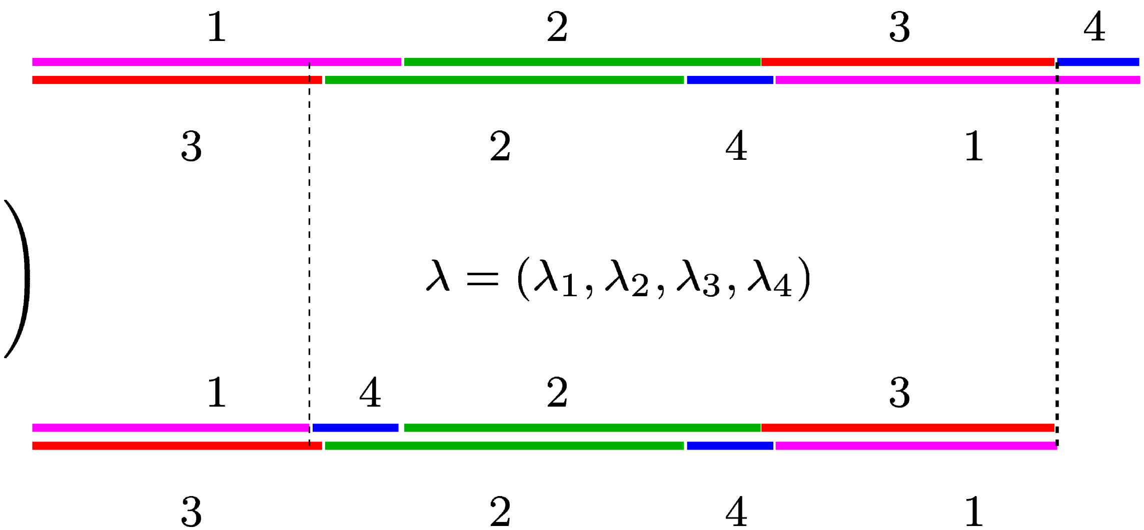


$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix} \qquad \lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$$

To analyze the behavior of longer and longer geodesics, we consider return maps to shorter and shorter cross-sections.

One way to do this is the Rauzy renormalization algorithm in the space of interval exchange transformations. We describe the algorithm through an example.

The Rauzy Algorithm

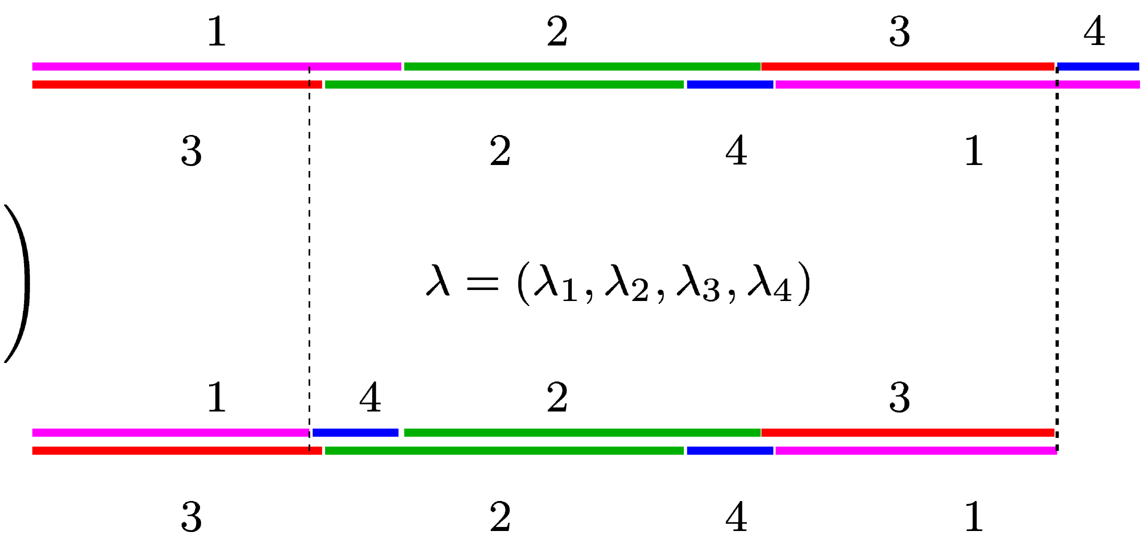


$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix} \qquad \lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$$

$$\pi' = \begin{pmatrix} 1 & 4 & 2 & 3 \\ 3 & 2 & 4 & 1 \end{pmatrix} \qquad \lambda' = (\lambda_1 - \lambda_4, \lambda_2, \lambda_3, \lambda_4)$$

Each interval exchange transformation is replaced by the corresponding return map to a certain subinterval.
Above is a “bottom” case: of the two rightmost intervals, the bottom one is longest.

The Rauzy Algorithm



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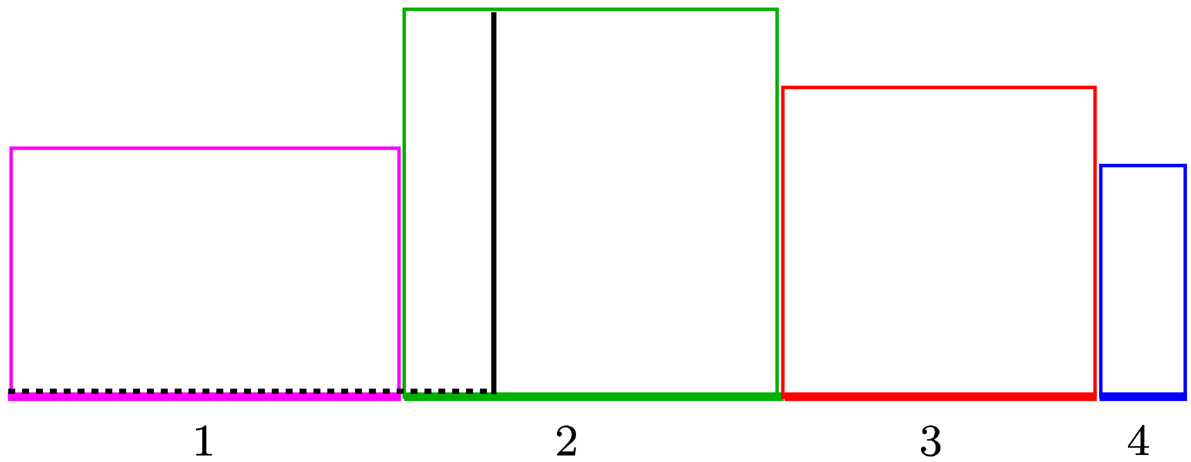
$$\pi' = \begin{pmatrix} 1 & 4 & 2 & 3 \\ 3 & 2 & 4 & 1 \end{pmatrix} \qquad \lambda' = (\lambda_1 - \lambda_4, \lambda_2, \lambda_3, \lambda_4)/(1 - \lambda_4)$$

The Rauzy transformation is defined by $R : (\pi, \lambda) \mapsto (\pi', \lambda')$. It admits an invariant measure ν absolutely continuous with respect to Lebesgue measure in the λ-space.

The Rauzy Cocycle

Now we analyze the effect of this algorithm on the return map (suspension of the interval exchange transformation):



Consider a geodesic segment that leaves from the i'th interval and returns to the cross-section. "Close" it by joining the endpoints to some chosen point.
This defines some $v_i \in H_1(S, \mathbb{Z})$.

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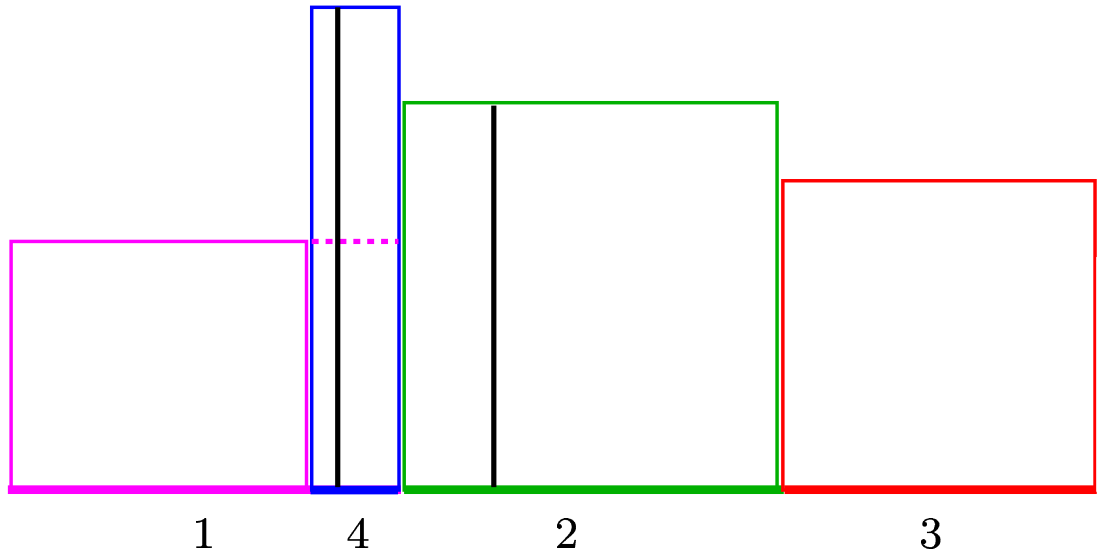

$$v'_1 = v_1 \quad v'_2 = v_2 \quad v'_3 = v_3 \quad v'_4 = v_1 + v_4$$

This corresponds to a linear cocycle $\mathcal{R}(\pi, \lambda, v) = (\pi', \lambda', v')$ over the Rauzy map $R(\pi, \lambda) = (\pi', \lambda')$.

The Zorich Cocycles

The invariant measure ν of R is infinite... Zorich introduced an accelerated algorithm

$$Z(\pi, \lambda) = R^n(\pi, \lambda) \quad \text{and} \quad \mathcal{Z}(\pi, \lambda, v) = \mathcal{R}^n(\pi, \lambda, v),$$

where $n = n(\pi, \lambda)$ is smallest such that the Rauzy iteration changes from “top” to “bottom” or vice-versa.

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- The transformation Z admits an absolutely continuous invariant ergodic probability μ over every Rauzy class = smallest invariant subset of the set of permutations π
- the cocycle $\mathcal{Z}$ acts symplectically on $v \in H_1(S, \mathbb{R}) \approx \mathbb{R}^{2g}$.

This brings us to the setting of the Oseledets theorem:

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Conjecture (Zorich, Kontsevich). *The Lyapunov exponents of every Zorich cocycle are non-zero and distinct:*

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Theorem (Veech 84). $\nu_2 < 1$.

Theorem (Forni 02). $\nu_g > 0$.

Theorem (Avila, Viana 04). *All exponents are non-zero and distinct.*

Main Steps of the Proof

1. A general criterium for multiplicity 1 of the Lyapunov exponents of linear cocycles.
2. Checking that this criterium applies to every Zorich cocycle.

Multiplicity of a Lyapunov exponent = dimension of the corresponding invariant subbundle in the Oseledets decomposition.

Linear Cocycles

Let $f : M \to M$ be a measurable transformation and $A : M \to GL(d, \mathbb{R})$ be a measurable function.

They define a linear cocycle F, through

$$F : M \times \mathbb{R}^d \to M \times \mathbb{R}^d, \quad F(x, v) = \big(f(x), A(x)v\big).$$

Assume:

1. The map f has a finite or countable Markov partition.
2. A is constant on each element of the partition.
3. A bounded distortion condition (inverse branches of iterates of f and their Jacobians are equicontinuous).

The Criterium

We call the cocycle simple if the map f has

1. (pinching) Some periodic point $p \in M$, with period $\kappa \geq 1$, such that all the eigenvalues of $A^\kappa(p)$ have distinct norms.
2. (twisting) Some homoclinic point $z \in \operatorname{supp} \mu$

$$z \in W^u_{loc}(p) \quad \text{and} \quad f^m(z) \in W^s_{loc}(p)$$

   such that $A^m(z)E \oplus F = \mathbb{R}^d$ for any invariant subspaces E and F of $A^\kappa(p)$ with $\dim E + \dim F = d$.

twisting $\Leftrightarrow$ the algebraic minors of the matrix of $A^m(z)$ in an eigenbasis of $A^\kappa(p)$ are all different from zero.

The Criterium

Theorem 0. *If the cocycle is simple then all its Lyapunov exponents have multiplicity* 1.

Previous results were obtained by
Guivarc'h, Raugi and Gol'dsheid, Margulis, for products of independent random matrices, and
Bonatti, Viana, for cocycles over subshifts of finite type.

Checking the Criterium

Theorem 1. *Every Zorich cocycle is simple.*

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The proof is by induction on the number of intervals. We consider combinatorial operations of reduction/extension:

$$\pi = \begin{pmatrix} a_1 & \cdots & a_{i-1} & c & a_{i+1} & \cdots & \cdots & \cdots & a_d \\ b_1 & \cdots & & \cdots\cdots & b_{j-1} & c & b_{j+1} & \cdots & b_d \end{pmatrix}$$

$$\updownarrow$$

$$\pi' = \begin{pmatrix} a_1 & \cdots & a_{i-1} & a_{i+1} & \cdots & \cdots & a_d \\ b_1 & \cdots & \cdots & b_{j-1} & b_{j+1} & \cdots & b_d \end{pmatrix}$$

This has a topological and geometric counterpart for the corresponding surfaces:

Checking the Criterium

Given π with d symbols, there exists π' with $d-1$ symbols such that π is an extension of π'. Then, either $g(\pi) = g(\pi')$ or $g(\pi) = g(\pi') + 1$.

1. If $g(\pi) = g(\pi')$ then the corresponding Zorich actions are (symplectically) conjugate.

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2. If $g(\pi) = g(\pi') + 1$ then $H_1(S(\pi'), \mathbb{R})$ may be seen as a symplectic reduction of $H_1(S(\pi), \mathbb{R})$

$= $ symplectic orthogonal of v_c inside $H_1(S(\pi), \mathbb{R})/v_c$

and the Zorich action on $H_1(S(\pi'), \mathbb{R})$ is conjugate to the natural action on the symplectic reduction.

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In this way one can prove twisting for π from twisting for π'. Pinchi ng also requires a careful combinatorial analysis.