EXTREMAL LYAPUNOV EXPONENTS

1. Let $(M, \mathcal{B}, \mu, f, A)$ be such that

(i) $(M, \mathcal{B}, \mu)$ is a probability space, that is, μ is a probability measure defined on a σ-algebra $\mathcal{B}$ of subsets of the space M

(ii) $f : M \to M$ is an invertible $\mathcal{B}$-measurable transformation preserving the probability measure μ

(iii) $A : M \to \mathrm{GL}(d, \mathbb{R})$ is $\mathcal{B}$-measurable and μ-integrable:

$$\log \|A\| \in L^1(\mu) \quad \text{and} \quad \log \|A^{-1}\| \in L^1(\mu).$$

The associated *projective cocycle* is the invertible transformation $F : M \times \mathbb{P}(\mathbb{R}^d)$ defined by $F(x, [v]) = (f(x), [A(x)v])$. Notice that $F^n(x, [v]) = (f^n(x), [A^n(x)v])$ for every $n \in \mathbb{Z}$, where

$$A^n(x) = \begin{cases} A(f^{n-1}(x)) \cdots A(f(x))\, A(x) & \text{if } n > 0 \\ \mathrm{id} & \text{if } n = 0 \\ A(f^n(x))^{-1} \cdots A(f^{-1}(x))^{-1} & \text{if } n < 0 \end{cases}$$

The *extremal Lyapunov exponents* of F are

$$\lambda_+ = \lim_{n\to\infty} \frac{1}{n} \int \log \|A^n\| \, d\mu \quad \text{and} \quad \lambda_- = \lim_{n\to\infty} \frac{1}{n} \int \log \|(A^n)^{-1}\|^{-1} \, d\mu.$$

The limits exist since the sequences $\int \log \|A^{\pm n}\| \, d\mu$ are sub-additive. Observe that $\|A^n(x)\| \|(A^n)^{-1}\| \geq 1$ and so $\lambda_+ \geq \lambda_-$. We are going to study necessary conditions for the equality to occur.

2. Given σ-algebras $\mathcal{A}_1, \mathcal{A}_2 \subset \mathcal{B}$, we say that $\mathcal{A}_1 \subset \mathcal{A}_2 \bmod 0$ if for every $A_1 \in \mathcal{A}_1$ there exists $A_2 \in \mathcal{A}_2$ such that $\mu(A_1 \Delta A_2) = 0$. We say that $\mathcal{A}_1 = \mathcal{A}_2 \bmod 0$ if $\mathcal{A}_1 \subset \mathcal{A}_2 \bmod 0$ and $\mathcal{A}_2 \subset \mathcal{A}_1 \bmod 0$. A σ-algebra is *generating* if the σ-algebra generated by the iterates $f^n(\mathcal{B}_0)$, $n \in \mathbb{Z}$ equals $\mathcal{B} \bmod 0$.

We assume $(M, \mathcal{B}, \mu)$ to be a *Lebesgue space*, that is, a complete separable probability space. Separability means that $\mathcal{B}$ admits a countable subset Γ that separates any two points of M and such that the σ-algebra it generates equals $\mathcal{B} \bmod 0$. Completeness means that every $\cap_{G\in\Gamma} G^*$ consists of exactly one point, where G^* denotes either G or its complement. Every Lebesgue space is isomorphic mod 0 to the union of

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an interval, endowed with Lebesgue measure, and a finite or countable set of atoms. See Rokhlin [8, § 2].

Let m be a probability measure on $M \times \mathbb{P}(\mathbb{R}^d)$ such that $P_* m = \mu$, where $P : M \times \mathbb{P}(\mathbb{R}^d) \to M$ is the canonical projection. Let

$$\{m_x : x \in M\}$$

be the disintegration of m into conditional probability measures along the fibers, that is, the family of probabilities m_x on $M \times \mathbb{P}(\mathbb{R}^d)$, such that $x \mapsto m_x$ is $\mathcal{B}$-measurable, every m_x is supported inside the fiber $\{x\} \times \mathbb{P}(\mathbb{R}^d)$, and

$$m(E) = \int m_x(E)\, d\mu(x)$$

for any measurable set $E \subset M \times \mathbb{P}(\mathbb{R}^d)$. Such a family exists, because $(M, \mathcal{B}, \mu)$ is a Lebesgue space, and it is unique $\bmod 0$. See [8, § 3].

Theorem 1 (Ledrappier [6])**.** *Suppose $\lambda_+ = \lambda_-$. Let $\mathcal{B}_0 \subset \mathcal{B}$ be a generating σ-algebra such that both f and A are $\mathcal{B}_0$-measurable* $\bmod 0$*. Then the disintegration $x \mapsto m_x$ of any F-invariant probability m with $P_* m = \mu$ is $\mathcal{B}_0$-measurable* $\bmod 0$.

We are going to deduce some consequences, following Ledrappier [6]. Then we state a generalization, Theorem 7, whose proof is given elsewhere [1].

3. Given functions $g_\alpha : M \to X_\alpha$ with values in measurable spaces X_α, $\alpha \in I$, we denote by $\operatorname{span}(g_\alpha : \alpha \in I)$ the smallest σ-algebra on M relative to which every g_α is measurable. We call $\{\emptyset, M\}$ the *trivial σ-algebra.*

Theorem 2. *Suppose $\lambda_+ = \lambda_-$ and*

$$\text{(1)} \quad \operatorname{span}(A \circ f^n : n \geq 0) \cap \operatorname{span}(A \circ f^n : n < 0) = \{\emptyset, M\} \quad \bmod 0.$$

Then there exists a probability η on $\mathbb{P}(\mathbb{R}^d)$ such that $A(x)_* \eta = \eta$ for μ-almost every $x \in M$.

For the proof we need the following easy fact:

Lemma 3. *Let $\hat{\mathcal{B}} = \operatorname{span}(A \circ f^n : n \in \mathbb{Z})$ and $\mathcal{B}_0 = \operatorname{span}(A \circ f^n : n \geq 0)$. Then*

1. *the σ-algebra $\hat{\mathcal{B}}$ is separable and complete* $\bmod 0$
2. *the iterates $f^n(\mathcal{B}_0)$, $n \in \mathbb{Z}$ generate $\hat{\mathcal{B}}$*
3. *both f and A are $\mathcal{B}_0$-measurable and, hence, $\hat{\mathcal{B}}$-measurable*

Then $(M, \hat{\mathcal{B}}, \mu)$ is a Lebesgue space. Moreover, both f and A are $\hat{\mathcal{B}}$-measurable. This means that, up to replacing $\mathcal{B}$ by $\hat{\mathcal{B}}$ from the start, we may suppose that the sub-σ-algebra $\mathcal{B}_0$ defined in Lemma 3 is generating.

Thus, applying Theorem 1, we get that $x \mapsto m_x$ is $\mathcal{B}_0$-measurable mod 0, for any F-invariant probability m such that $P_* m = \mu$. Moreover, we may apply the same arguments with f and A replaced by their inverses, and $\mathcal{B}_0$ replaced by $\mathcal{B}_0' = \operatorname{span}(A \circ f^n : n < 0)$. Notice that $x \mapsto A^{-1}(x) = A(f^{-1}(x))^{-1}$ is $\mathcal{B}_0'$-measurable. We conclude that $x \mapsto m_x$ is also $\mathcal{B}_0'$-measurable.

Thus, in view of (1), the disintegration is measurable mod 0 with respect to the trivial σ-algebra. In other words, there exists η such that $m_x = \eta$ for μ-almost every $x \in M$. Finally, note that $A(x)_* m_x = m_{f(x)}$ for μ-almost every x, because m is F-invariant and f is invertible. This completes the proof of Theorem 2 from Theorem 1.

4. As a further consequence we obtain a theorem of Furstenberg on products of random matrices. We call $(M, \mathcal{B}, \mu, f, A)$ an *independent product of random matrices* if there exists a probability ν supported on some $G \subset \mathrm{GL}(d, \mathbb{R})$ such that $M = G^{\mathbb{Z}}$, $\mathcal{B}$ is the product σ-algebra on M, μ is the Bernoulli measure $\nu^{\mathbb{Z}}$, f is the shift map of M, and $A(\underline{g}) = g_0$ for every $\underline{g} = (g_n)_{n \in \mathbb{Z}}$ in M.

Theorem 4 (Furstenberg [2])**.** *Let $(M, \mathcal{B}, \mu, f, A)$ be an independent product of random matrices and suppose $\lambda_+ = \lambda_-$. Then there exists a probability measure η on $\mathbb{P}(\mathbb{R}^d)$ such that $g_* \eta = \eta$ for every $g \in G$.*

Indeed, $\operatorname{span}(A \circ f^n : n \geq 0)$ is the σ-algebra generated by the cylinders $[0; G_0, \ldots, G_l]$, $l \geq 1$, and $\operatorname{span}(A \circ f^n : n < 0)$ is the σ-algebra generated by the cylinders $[-l; G_{-l}, \ldots, G_{-1}]$, $l \geq 1$, and so the hypothesis (1) is satisfied in this case. So, by Theorem 2, there exists η on $\mathbb{P}(\mathbb{R}^d)$ such that $A(\underline{g})_* \eta = \eta$ for μ-almost every $\underline{g} \in M$. In other words, $g_* \eta = \eta$ for ν-almost every $g \in G$. Then this invariance relation must hold for every g in $G = \operatorname{supp} \nu$, as claimed in Theorem 4.

Most projective maps have very few invariant measures: for instance, if all the eigenvalues of $g \in \mathrm{GL}(d, \mathbb{R})$ have distinct norms then the only g-invariant probability measures in $\mathbb{P}(\mathbb{R}^d)$ are the convex combinations of Dirac masses at the eigenspaces. Thus, the conclusion of Theorem 4 is very strong: the theorem implies that $\lambda_+ > \lambda_-$ for most independent products of random matrices.

5. We call $(M, \mathcal{B}, \mu, f, A)$ a *Markov product of random matrices* if there exists $G \subset \mathrm{GL}(d, \mathbb{R})$ such that $M = G^{\mathbb{Z}}$, $\mathcal{B}$ is the product σ-algebra on M, μ is a Markov measure on M, f is the shift map of M, and $A(\underline{g}) = g_0$ for every $\underline{g} = (g_n)_{n \in \mathbb{Z}}$ in M. The condition on μ means that there exists a family of *transition probabilities* $p(g, \cdot)$, $g \in G$ such that

$$\mu([k;G_k,\dots,G_n,G_{n+1}]) = \int_{[k;G_k,\dots,G_n]} p(g_n,G_{n+1})\,d\mu(\underline{g})$$

for any $k \le n$ and $G_k,\dots,G_n,G_{n+1} \subset G$.

Theorem 5 (Virtser [11], Guivarc'h [3], Royer [9])**.** *Let $(M,\mathcal{B},\mu,f,A)$ be a Markov product of random matrices and suppose $\lambda_+ = \lambda_-$. Then there exists a measurable family $(\eta_g)_{g\in G}$ of probability measures on $\mathbb{P}(\mathbb{R}^d)$ such that $g_*\eta_g = \eta_h$ for $p(g,\cdot)$-almost every $h \in G$.*

6. Let $(M,\mathcal{B},\mu,f)$ be as before and $A_E : M \to \mathrm{SL}(2,\mathbb{R})$ be defined by

$$A_E(x) = \begin{pmatrix} V(x)-E & -1 \\ 1 & 0 \end{pmatrix}$$

where the *energy* E is a real parameter and the *potential* $V : M \to \mathbb{R}$ is a measurable function satisfying

$$\int \max\{\log|V(x)|,0\}\,d\mu < \infty.$$

Let $\lambda_\pm(E)$ be the extremal Lyapunov exponents of the corresponding linear cocycle. In this case $\lambda_- + \lambda_+ = 0$, because $d = 2$ and $\det A \equiv 1$.

The potential $V : M \to \mathbb{R}$ is called *deterministic* if

$$\bigcap_{k=1}^{\infty} \mathrm{span}(V\circ f^n : n \ge k) = \mathrm{span}(V\circ f^n : n \in \mathbb{Z}) \mod 0. \tag{2}$$

Observe that $\mathrm{span}(V\circ f^n) = f^{-n}(\mathrm{span}(V))$ decreases when n increases. Thus, (2) may be read: *the past values determine the future values of V*. Typically, quasi-periodic potentials (f is an irrational rotation) are deterministic, whereas Bernoulli potentials (f is a Bernoulli transformation) are not.

Theorem 6 (Kotani [5], Simon [10])**.** *If V is non-deterministic then $\lambda_-(E) < 0 < \lambda_+(E)$ for almost every value of E.*

Ledrappier [6, § VI] shows how this result follows from Theorem 1.

7. Let $(M,\mathcal{B},\mu,f)$ be as before and $P : \mathcal{E} \to M$ be a fiber bundle with fibers $\mathcal{E}_x$ diffeomorphic to some Riemannian manifold N. A *non-linear cocycle* over f is a measurable transformation $F : \mathcal{E} \to \mathcal{E}$ such that $P\circ F = f\circ P$ and every $F_x : \mathcal{E}_x \to \mathcal{E}_{f(x)}$ is a diffeomorphism. We always assume that the norms of the derivative $DF_x(\xi)$ and its inverse are uniformly bounded. Then the functions

$$(x,\xi) \mapsto \log\|DF_x(\xi)\| \quad \text{and} \quad (x,\xi) \mapsto \log\|DF_x(\xi)^{-1}\| \tag{3}$$

are integrable, relative to any probability measure on $\mathcal{E}$. The *extremal Lyapunov exponents* of F at a point $(x,\xi)\in\mathcal{E}$ are

$$\lambda_+(F,x,\xi) = \lim_{n\to\infty}\frac{1}{n}\log\|DF_x^n(\xi)\|\,.$$
$$\lambda_-(F,x,\xi) = \lim_{n\to\infty}\frac{1}{n}\log\|DF_x^n(\xi)^{-1}\|^{-1}\,.$$

The limits exist m-almost everywhere, with respect to any F-invariant probability m on $\mathcal{E}$, by sub-additivity (Kingman [4]). Notice that

$$\lambda_-(F,x,\xi)\le\lambda_+(F,x,\xi),$$

because $\|DF_x^n(\xi)\|\|DF_x^n(\xi)^{-1}\|\ge 1$. Denote

$$\lambda_\pm = \lambda_\pm(F,m) = \int \lambda_\pm(F,x,\xi)\,dm(x,\xi).$$

If (F,m) is ergodic then $\lambda_\pm(F,x,\xi)=\lambda_\pm$ for m-almost every (x,ξ).

We consider probability measures m on $\mathcal{E}$ that project down to μ under P. By [8, § 3], such a measure m admits a family $\{m_x : x\in M\}$ of probabilities such that $x\mapsto m_x$ is $\mathcal{B}$-measurable, every m_x is supported inside the fiber $\mathcal{E}_x$, and

$$m(E) = \int m_x(E)\,d\mu(x)$$

for any measurable set $E\subset\mathcal{E}$. Moreover, such a family is essentially unique. We call it the *disintegration* of m and refer to the m_x as its *conditional probabilities* along the fibers. The following result extends Theorem 1:

Theorem 7. *Suppose either $\lambda_+(x,\xi)\le 0$ for m-almost every (x,ξ) or $\lambda_-(x,\xi)\ge 0$ for m-almost every (x,ξ). Let $\mathcal{B}_0\subset\mathcal{B}$ be a generating σ-algebra such that both f and $x\mapsto F_x$ are $\mathcal{B}_0$-measurable* mod 0. *Then $x\mapsto m_x$ is $\mathcal{B}_0$-measurable* mod 0.

8. Let us check that Theorem 1 follows from Theorem 7. Take $\mathcal{E} = M\times\mathbb{P}(\mathbb{R}^d)$. Given $A: M\to \mathrm{GL}(d,\mathbb{R})$, consider F_x to be the projective diffeomorphism induced by $A(x)$ on the projective space $N=\mathbb{P}(\mathbb{R}^d)$. Locally, the points of $\mathbb{P}(\mathbb{R}^d)$ may be represented by unit vectors ξ. Then

$$F_x^n(\xi) = \frac{A^n(x)\xi}{\|A^n(x)\xi\|}$$

for every x, ξ, and n. It follows that,

$$DF_x^n(\xi)\dot\xi = \frac{\mathrm{proj}_{A^n(x)\xi}(A^n(x)\dot\xi)}{\|A^n(x)\xi\|},$$

where $\mathrm{proj}_u v = v - u(u\cdot v)/(u\cdot u)$ is the projection of v to the orthogonal complement of u. This implies that

$$\|DF^n_x(\xi)\| \le \|A^n(x)\|/\|A^n(x)\xi\| \le \|A^n(x)\|\|A^n(x)^{-1}\|$$

for every x, ξ, and n. Consequently, $\lambda(x,\xi) \le \lambda_+(x) - \lambda_-(x)$, where

$$\lambda_+(x) = \lim_{n\to\infty} \frac{1}{n} \log \|A^n(x)\| \quad \text{and} \quad \lambda_-(x) = \lim_{n\to\infty} \frac{1}{n} \log \|A^n(x)^{-1}\|^{-1}.$$

Oseledets [7] ensures that these two limits exist almost everywhere and

$$\lambda_\pm = \int \lambda_\pm(x)\, d\mu(x).$$

Clearly, $\lambda_+(x) \ge \lambda_-(x)$ at μ-almost every x. Hence, $\lambda_+ = \lambda_-$ implies $\lambda_+(x) = \lambda_-(x)$ for μ-almost every x, and so $\lambda(x,\xi) \le 0$ for m-almost every (x,ξ). Thus, Theorem 1 is indeed a particular case of Theorem 7.

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