EQUILIBRIUM STATES FOR HYPERBOLIC POTENTIALS

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Abstract. We prove existence of finitely many ergodic equilibrium states associated to local homeomorphisms and hyperbolic potentials. In addition, if the dynamics is transitive we obtain the uniqueness of equilibrium state. Under the assumptions of uniform contraction on the fiber and non-uniformly expansion on some region of the base we also prove existence of equilibrium states for partially hyperbolic skew-products.

1. Introduction

The theory of equilibrium states, developed by Sinai, Ruelle and Bowen in the seventies and eighties, came into existence through the application of techniques and results from statistical mechanics to smooth dynamics.

In the classical setting, given a continuous map $f : M \to M$ on a compact metric space M and a continuous potential $\phi : M \to \mathbb{R}$, we say that μ_ϕ is an equilibrium state associated to (f, ϕ), if μ_ϕ is an f-invariant probability measure characterized by the following variational principle:

$$P_f(\phi) = h_{\mu_\phi}(f) + \int \phi \, d\mu_\phi = \sup_{\mu \in \mathcal{M}_f(M)} \left\{ h_\mu(f) + \int \phi \, d\mu \right\}$$

where $P_f(\phi)$ denotes the topological pressure, $h_\mu(f)$ denotes the metric entropy and the supremum is taken over all f-invariant probabilities measures.

In the uniformly hyperbolic context, which includes uniformly expanding maps, equilibrium states always exist and they are unique if the potential ϕ is Hölder continuous and the dynamics f is transitive. However, the scenario beyond hyperbolic systems is pretty much incomplete.

Recently, several advances were obtained outside the uniformly hyperbolic setting in the works of Bruin and Keller [5] and Denker and Urbanski [8], for interval maps and rational functions on the Riemann sphere; Leplaideur, Oliveira, Rios [15] on partially hyperbolic horseshoes; and Buzzi and Sarig [7] and Yuri [20], for countable Markov shifts and for piecewise expanding maps in one and higher dimensions, among others.

Before proceeding a few words are in order on the nature of the two problems: existence and uniqueness of equilibrium states. Existence is relatively soft property that can often be established via compacteness arguments. Uniqueness is usually more subtle, and requires a better understanding of the dynamics. Examples of transitive shifts with equilibrium states having nontrivial supports go back to Krieger [12]. So, some conditions on the potential is certainly necessary.

For local diffeomorphisms which present expansion in a non-uniform way we point out the results of Arbieto, Matheus and Oliveira [2]; Oliveira and Viana [16]; Varandas and Viana [19]. They obtained uniqueness of equilibrium states for potentials with small oscillation. In this context, they were able to construct expanding equilibrium states absolutely continuous with respect to the conformal measure; moreover, the equilibrium state is unique. The hypothesis on the potential is used to ensure that most of the pressure emanates from regions of the ambient where the dynamics is actually hyperbolic.

These findings motivate, to some extent, the approach that we follow in the present paper. Indeed, we consider a certain inequality between the pressure of the system (f, ϕ) on the hyperbolic and the nonhyperbolic regions of M, and we prove that, for local homeomorphisms, this hyperbolic hypotheses on the potential is suffices for existence and finiteness of equilibrium states. The results mentioned in the previous paragraph fit in this framework.

Hyperbolicity of the potential is the essential condition in our results. Closely related condition have been considered by Hofbauer and Keller [11], Denker, Keller and Urbanski [9] for piecewise monotonic maps and by Buzzi, Paccaut, Schmitt [6], in the context of piecewise expanding multidimensional maps.

In the case of local diffeomorphisms, hyperbolicity condition of the potential ϕ in our first results imply positive Lyapunov exponents for the equilibrium measure. In this way, a natural objective is to obtain existence and finiteness of equilibrium states for systems which present some contraction direction. In this paper, we also study partially hyperbolic skew-products of fiber contraction maps over non-uniformly expanding dynamics and we prove existence of equilibrium states for this class of transformations.

2. Setting and Statements

Let (M, d) be a compact metric space of dimension m and d be the distance on M. Let $f : M \to M$ be a local homeomorphism. Suppose that every inverse branches f^{-1} are locally Lipschitz continuous, that means, there is a bounded function $x \mapsto \sigma(x)$ such that, for each $x \in M$ there exists a neighborhood V_x of x so that $f_x : V_x \to f_x(V_x)$ is invertible and for every $y, z \in f_x(V_x)$ holds

$$d(f_x^{-1}(y), f_x^{-1}(z)) \leq \sigma(x) d(y, z)$$

Given a positive constant $\sigma > 0$, we denote by $\Sigma_\sigma = \Sigma(\sigma)$ the set whose points satisfy:

$$\Sigma_\sigma := \left\{ x \in M; \ \limsup_{n \to +\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \sigma(f^j(x)) \leq \log \sigma \right\}$$

We say that a real continuous function $\phi : M \to \mathbb{R}$ is a *hyperbolic potential* if the topological pressure of ϕ is located on Σ_σ for some $\sigma \in (0, 1)$, i.e.,

$$P_f(\phi, \Sigma_\sigma^c) < P_f(\phi, \Sigma_\sigma) = P_f(\phi).$$

Note that, for $\sigma \in (0, 1)$ we have f^{-1} is locally a contraction. In this way, Σ_σ is a non-uniformly expanding region on M. A probability measure μ (not necessarily invariant) is *expanding* if $\mu(\Sigma_\sigma) = 1$.

Let $C(M)$ be the complete metric space of real continuous functions $\psi : M \to \mathbb{R}$ equipped with the uniform convergence norm. The *Ruelle-Perron-Fröbenius transfer operator* $\mathcal{L}_\phi : C(M) \to C(M)$ associated to dynamics $f : M \to M$ and a real continuous function $\phi : M \to \mathbb{R}$ is the linear operator defined on $C(M)$ by

$$\mathcal{L}_\phi(\psi)(x) = \sum_{y \in f^{-1}(x)} e^{\phi(y)} \psi(y)$$

The dual operator $\mathcal{L}_\phi^*$ of $\mathcal{L}_\phi$ acts on the space $\mathcal{M}(M)$ of Borel measures in M as follow:

$$\int \psi \, d(\mathcal{L}_\phi^* \rho) = \int \mathcal{L}_\phi(\psi) \, d\rho$$

for every real continuous function ψ.

We say that a Borel measure (not necessarily invariant) ν is a conformal measure associated to (f, ϕ) if ν is an eigenmeasure for $\mathcal{L}_\phi^*$.

Our first result states the existence of a conformal expanding measure for (f, ϕ) associated to the eingenvalue $\lambda = e^{P_f(\phi)}$ as well as the existence of a finite number of ergodic f-invariant probabilities absolutely continuous with respect to this reference measure. In particular, we get uniqueness of such ergodic measure when the dynamics f is transitive.

Theorem A. *Let $f : M \to M$ be a local homeomorphism and let $\phi : M \to \mathbb{R}$ be a hyperbolic Hölder continuous potential. Then, there exist a conformal measure ν associated to the eingenvalue $\lambda = e^{P_f(\phi)}$. Moreover, ν is expanding and there is a finite number of f-invariant ergodic measures absolutely continuous with respect to it.*

Under the same hyperbolicity condition on the potential ϕ, the existence of an expanding conformal measure ν associated to the eingenvalue $\lambda = e^{P_f(\phi)}$ implies that every equilibrium state of (f, ϕ) has to be absolutely continuous with respect to ν. Thus, we use the semicontinuity property of the entropy map to ensure the existence of some equilibrium measure and we combine this with the Theorem A to obtain

Theorem B. *Let $f : M \to M$ be a local homeomorphism and let $\phi : M \to \mathbb{R}$ be a hyperbolic Hölder continuous potential. Then, there exist finitely many ergodic equilibrium states associated to (f, ϕ). In addition, the equilibrium state is unique if f is transitive.*

Since each equilibrium state μ_ϕ is an expanding measure, by a consequence of this result we obtain that all the Lyapunov exponents of f are positive at μ_ϕ-almost every point, if f is a local diffeomorphism.

We would like to point out that our results remains true if the potential ϕ satisfies a weaker condition than Hölder continuity, that of having summable variation on hyperbolic dynamical balls, which means that there exist some positive constant K such that if for some $n \in \mathbb{N}$ holds

$$d(f^{n-j}(y), f^{n-j}(z)) \leq \sigma^j d(f^n(y), f^n(z))$$

for every $1 \leq j \leq n$ and every y, z in the dynamical ball $B_\delta(x, n)$ then

$$K^{-1} \leq \sum_{j=1}^{n-1} \phi(f^j(y)) - \phi(f^j(z)) \leq K$$

A class of examples which satisfies our hyperbolicity condition on the potential was studied by Arbieto, Matheus and Oliveira [2]; Oliveira and Viana [16]; Varandas and Viana [19]. We describe this example below.

Example 2.1. Let $f : M \to M$ be a C^1-local diffeomorphism on a compact manifold M. Assume that there exist positive constants σ_1

close to one and $\sigma_2 > 1$, and a covering $R = \{R_1, ..., R_q, R_{q+1}, ..., R_s\}$ of M by domains of injectivity for f that satisfies

(1) For some open region $A \subset M$, $\|Df^{-1}(x)\| \leq \sigma_1$ for every $x \in A$ and $\|Df^{-1}(x)\| \leq \sigma_2^{-1}$ for all $x \in M \setminus A$.
(2) The region A above can be covered by q elements of the partition R with $\log q < \log \deg(f)$.

Suppose $\phi : M \to \mathbb{R}$ be a Hölder continuous potential with small oscillation, i.e., $\sup \phi - \inf \phi < \log \deg(f) - \log q$.

The authors show that there exists some non-uniformily expanding region $\Sigma \subset M$ such that $P_f(\phi, \Sigma^c) < P_f(\phi, \Sigma) = P_f(\phi)$. Thus, there is a finite number of ergodic equilibrium states for the system.

Besides this class described above, we can also apply our result on the following class of examples.

Example 2.2. Let $f : M \to M$ be the dynamics of the last example and $\Sigma \subset M$ be the non-uniformly expanding set. Since we have the inequality $h_{top}(f, \Sigma^c) < h_{top}(f, \Sigma)$ then any Hölder continuous function $\phi : M \to \mathbb{R}$ that satisfies

$$\sup \{\phi(x);\, x \in \Sigma^c\} \leq \inf \{\phi(x);\, x \in \Sigma\}$$

is a hyperbolic potential. In fact,

$$P_f(\phi, \Sigma^c) \leq h_{top}(f, \Sigma^c) + \sup_{x \in \Sigma^c} \phi(x) < h_{top}(f, \Sigma) + \inf_{x \in \Sigma} \phi(x) \leq P_f(\phi, \Sigma)$$

Thus, by Theorem B, there exists finitely many ergodic equilibrium states for (f, ϕ). If f is transitive, the equilibrium is unique. Note that, in this class it is not necessary ϕ has small oscillation. For example, any positive potential that vanishes on Σ^c satisfies the hypoteses of our theorem.

Example 2.3. Let $f_0 : M \to M$ be an expanding map and fix some cover $\mathcal{P} = \{P_1, P_2, \ldots, P_n\}$ of f_0 by injectivity domain. On each P_i consider a fixed (or periodic) point p_i. Deforming f_0 on a neighborhood B_i of p_i by a Hopf bifurcation or Pitchfork bifurcation, the result map $f : M \to M$ coincides with the expanding map f_0 outside $B_1 \cup \cdots \cup B_n$. Thus, if the deformation satisfies

(1) $f(B_1 \cup \cdots \cup B_n) \subset B_1 \cup \cdots \cup B_n$
(2) $h_{top}(f, B_1 \cup \cdots \cup B_n) < h_{top}(f, (B_1 \cup \cdots \cup B_n)^c)$

then the f-invariant Cantor set

$$\Sigma_\sigma = \bigcap_{j \geq 0} f^{-j}((B_1 \cup \cdots \cup B_n)^c)$$

is a non-uniformly expanding and holds the inequality

$$h_{top}(f,\Sigma_\sigma^c) < h_{top}(f,\Sigma_\sigma)$$

Thus, there exist finitely many maximal entropy measure for f. Morevover, any Hölder continuous function $\phi : M \to \mathbb{R}$ such that

$$\sup\{\phi(x);\, x \in B_1 \cup \cdots \cup B_n\} \leq \inf\{\phi(x);\, x \in (B_1 \cup \cdots \cup B_n)^c\}$$

is a hyperbolic potential and, by Theorem B, there is a finite number of ergodic equilibrium states for (f,ϕ).

The next example shows that the hyperbolicity property is essential for the uniqueness of equilibrium measure.

Example 2.4. Fix some positive constant $\alpha \in (0,1)$ and define on S^1 the local homeomorphism

$$f_\alpha(x) = \begin{cases} x(1+2^\alpha x^\alpha), & \text{if } 0 \leq x \leq \frac{1}{2} \\ x - 2^\alpha(1-x)^{1+\alpha}, & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases}$$

Then, f is a transitive non-uniformily expanding map. Consider the potential $\phi : S^1 \to \mathbb{R}$ defined by $\phi(x) = -\log|f^{'}(x)|$.

Note that, by Pesin formula the SRB measure μ_ϕ satisfies the equality $h_{\mu_\phi}(f) - \int \log|f^{'}(x)|\, d\mu_\phi = 0$ and by Ruelle inequality it is an equilibrium state for (f_α,ϕ). On the other hand, for the Dirac measure δ_0 supported at the fixed point 0 we also have $h_{\delta_0}(f) - \int \log|f^{'}(x)|\, d\delta_0 = 0$. Thus, μ_ϕ and δ_0 are two distincts ergodic equilibrium states. In this case, uniqueness fails because the pressure on hyperbolic region is equal to the pressure on nonhyperbolic region.

We also prove existence of equilibrium states for partially hyperbolic skew-products whose base dynamics is a non-uniformly expanding local diffeomorphism and the fiber dynamics is uniformly contracting. Let us state the precise setting.

Let M be a compact manifold with dimension m and N be a compact metric space. Consider $f : M \to M$ be a C^1 local diffeomorphism and $g : M \times N \to N$ be a fiber contraction that varies β-Hölder continuously on M, i.e., there are constants $C > 0$, $\lambda \in (0,1)$ and $\beta \leq 1$ such that

$$d_N(g_x(y), g_x(z)) \leq \lambda d_N(y,z) \text{ for all } y,z \in N \text{ and } x \in M$$

and

$$d_N(g_x(y), g_z(y) \leq C d_M(x,z)^\beta \text{ for all } x,z \in M \text{ and } y \in N$$

Define on $M \times N$ the skew-product:

$$F : M \times N \to M \times N, \quad F(x,y) = (\,f(x),\, g_x(y)\,)$$

Let $\gamma > 0$ be a constant such that $\gamma \leq \|Df(x)\|$ for all $x \in M$. Fix $\delta < 1$ some positive constant such that $\delta^{\beta}\gamma^{-1} < 1$. Thus there exists $\tilde{\sigma} \in (0,1)$ such that $\lambda\gamma^{-1} \leq \lambda\delta^{-\beta} \leq \tilde{\sigma} < 1$ for all $x \in M$ and we call F a partially hyperbolic skew-product.

Remark. If N is a compact n-dimensional manifold, f is partially hyperbolic of type $E^c \oplus E^u$ and g is differentiable with $\|D_y g\| \leq \lambda$ for all $y \in N$ then F is a partially hyperbolic in the usual sense. But in our work, it is not necessary N to be finite dimensional and f to be partially hyperbolic.

Let $\phi : M \times N \to \mathbb{R}$ be a Hölder continuous potential. When we add to ϕ our hyperbolicity hypothesis on the base M, which means that there exists some non-uniformly expanding region Σ_σ on M such that

$$P_F(\phi, \Sigma_\sigma^c \times N) < P_F(\phi, \Sigma_\sigma \times N) = P_F(\phi)$$

we obtain:

Theorem C. *Let $F : M \times N \to M \times N$ be a partially hyperbolic skew-product and $\phi : M \times N \to \mathbb{R}$ be a Hölder continuous potential hyperbolic on M. Then:*
(i) There exist some equilibrium state μ_ϕ associated to (F, ϕ).
(ii) If the potential ϕ does not depends on the fiber N then there exist a finitely many ergodic equilibrium states for (f, ϕ). In addition, there exists only one equilibrium measure, if f is transitive.

On the item (ii) of the Theorem C, the condition ϕ does not depends on the fiber, it has the meaning $\phi(x, \cdot) : N \to \mathbb{R}$ is a constant function for each $x \in M$ fixed. In particular, Theorem C says that there exist finitely many maximal entropy measures for the partially hyperbolic skew-product, if the topological entropy of the base dynamics f is located on the non-uniformly expanding region. And there exist only one such measure, if f is transitive.

Example 2.5. Let $f : \mathbb{T}^n \to \mathbb{T}^n$ be a local diffeomorphism like in the Examples 2.1, 2.2 or 2.3 with $M = \mathbb{T}^n$. Consider any Hölder continuous map $h : \mathbb{T}^n \to \mathbb{T}^n$ and a positive constant $\lambda < 1$ such that $\lambda\|Df(x)\|^{-1} < 1$ for every $x \in M$. Then the skew-product

$$F : \mathbb{T}^n \times \mathbb{T}^n \to \mathbb{T}^n \times \mathbb{T}^n, \quad F(x,y) = (f(x), h(x) + \lambda y)$$

satisfies the hypotheses of the Theorem C. Hence, it has finitely many ergodic equilibrium states associated to any Hölder continuous potential $\phi : \mathbb{T}^n \times \mathbb{T}^n \to \mathbb{R}$ hyperbolic on M and constant on the fiber. In particular, if f is transitive then there exists a unique maximal entropy measure.

3. Preliminary Results

In this section, we state some facts and notations which will be used throughout this work. This content may be omitted in a first reading and the reader can return here whenever necessary.

3.1. **Equilibrium States.** Let $f : M \to M$ be a continuous transformation defined on a compact metric space (M, d). Let $\phi : M \to \mathbb{R}$ be a real continuous function that we call by *potential*.

Given an open cover α for M we define the pressure $P_f(\phi, \alpha)$ of ϕ with respect to α by

$$P_f(\phi, \alpha) := \lim_{n\to+\infty} \frac{1}{n} \log \inf_{\mathcal{U}\subset\alpha^n} \{\sum_{U\in\mathcal{U}} e^{\phi_n(U)}\}$$

where the infimum is taken over all subcover $\mathcal{U}$ of $\alpha^n = \bigvee_{n\geq 0} f^n\alpha$ and $\phi_n(U)$ indicates $\sup_{x\in U} \sum_{j=0}^{n-1} \phi \circ f^j(x)$.

Definition 3.1. The *topological pressure* $P_f(\phi)$ of the potential ϕ with respect to the dynamics f is defined by

$$P_f(\phi) := \lim_{\delta\to 0} \big\{ \sup_{|\alpha|\leq\delta} P_f(\phi, \alpha)\big\}$$

where $|\alpha|$ denotes the diameter of the open cover α.

An alternatively way to define topological pressure is through the notion of dynamical balls. This approach is from dimension theory and it is very useful to calculate the topological pressure of non-compact sets.

Fix $\delta > 0$. Given $n \in \mathbb{N}$ and $x \in M$ let $B_\delta(x, n)$ the dynamical ball:

$$B_\delta(x, n) := \{y \in M \;/\; d(f^j(x), f^j(y)) < \delta\,, \text{ for } \; 0 \leq j \leq n\}$$

Denote by $\mathcal{F}_N$ the collection of dynamical balls

$$\mathcal{F}_N = \{B_\delta(x, n) \;/\; x \in M \text{ and } n \geq N\}$$

Given a f-invariant subset Λ of M, non necessarily compact, let $\mathcal{U}$ be a finite or countable family of $\mathcal{F}_N$ which cover Λ.

For every $\gamma \in \mathbb{R}$ define

$$m_f(\phi, \Lambda, \delta, N, \gamma) = \inf_{\mathcal{U}\subset\mathcal{F}_N} \{ \sum_{B_\delta(x,n)\in\mathcal{U}} e^{-\gamma n + S_n\phi(B_\delta(x,n))}\}$$

As N goes to infinity we define

$$m_f(\phi, \Lambda, \delta, \gamma) = \lim_{N\to+\infty} m_f(\phi, \Lambda, \delta, N, \gamma)$$

Taking the infimum over γ we obtain

$$P_f(\phi,\Lambda,\delta)=\inf\{\gamma\,/\,m_f(\phi,\Lambda,\delta,\gamma)=0\}$$

The *relative pressure* $P_f(\phi,\Lambda)$ of a subset Λ of M is given by

$$P_f(\phi,\Lambda)=\lim_{\delta\to 0}P_f(\phi,\Lambda,\delta)$$

and holds the inequality

$$P_f(\phi)=\sup\{P_f(\phi,\Lambda),\ P_f(\phi,\Lambda^c)\}$$

We refer the reader to [17] for more details and properties of the relative pressure.

The relationship between the topological pressure $P_f(\phi)$ and the metric entropy $h_\mu(f)$ of the dynamics is given by

Theorem 3.2 (Variational Principle)**.** *Let $\mathcal{M}_f(M)$ be the set of probability measures invariants by a continuous transformation $f:M\to M$ defined on a compact metric space M and let $\phi:M\to\mathbb{R}$ be a continuous potential. Then,*

$$P_f(\phi)=\sup_{\mu\in\mathcal{M}_f(M)}\left\{h_\mu(f)+\int\phi\,d\mu\right\}$$

When Λ is a non-compact f-invariant set for the relative pressure holds the inequality $P_f(\phi,\Lambda)\geq\sup\left\{h_\mu(f)+\int\phi\,d\mu\right\}$ where the supremum is taken over all invariant measures μ such that $\mu(\Lambda)=1$.

The variational principle gives a natural way of choosing important measures of $\mathcal{M}_f(M)$.

Definition 3.3. A measure μ_ϕ of $\mathcal{M}_f(M)$ is called an *equilibrium state* for (f,ϕ) if μ_ϕ is characterized by the variational principle:

$$P_f(\phi)=h_{\mu_\phi}(f)+\int\phi\,d\mu_\phi.$$

3.2. **Hyperbolic Potentials.** Let $f:M\to M$ be a local homeomorphism defined on a compact metric space (M,d) and $\phi:M\to\mathbb{R}$ be a hyperbolic potential, that means, there exists some $\sigma\in(0,1)$ such that

$$P_f(\phi,\Sigma_\sigma^c)<P_f(\phi,\Sigma_\sigma)=P_f(\phi).$$

where Σ_σ denotes the set

$$\Sigma_\sigma:=\left\{x\in M;\ \limsup_{n\to+\infty}\frac{1}{n}\sum_{i=0}^{n-1}\log\sigma(f^j(x))\leq\log\sigma\right\}$$

The above inequality allows us to explore the non-uniform expansion property on Σ. In this direction, the key ingredient are the so called hyperbolic times.

Definition 3.4 (Hyperbolic Times)**.** Given $\sigma \in (0,1)$ we say that n is a *hyperbolic time* for x if for each $1 \leq k \leq n-1$ we have

$$\prod_{j=n-k}^{n-1} \sigma(f^j(x)) \leq \sigma^j$$

The next lemma shows that every point in Σ_σ has infinitely many hyperbolic times. In fact, these times appear with a positive frequence $\theta > 0$. For details see [1] and replacing $\|Df(\cdot)^{-1}\|$ by $\sigma(\cdot)$.

Lemma 3.5. *If $x \in M$ satisfies*

$$\limsup_{n\to+\infty} \frac{1}{n}\sum_{j=1}^{n-1} \log \sigma(f^j(x)) \leq \log \sigma < 0.$$

Then, there exists $\theta > 0$, that depends only on f and σ, and a sequence of hyperbolic times $1 \leq n_1(x) \leq \cdots \leq n_l(x) \leq n$ for x with $l \geq \theta n$.

Moreover, if η is a probability measure for which the limit above holds in almost everywhere then for every borelean $A \subset M$ with positive η-measure we have

$$\liminf_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n-1} \frac{\eta(A\cap\Sigma_j)}{\eta(A)} \geq \frac{\theta}{2}$$

where Σ_j denotes the set whose points have j as hyperbolic time.

The next result point out that on hyperbolic times the dynamics f locally behaves as if it were an expanding map: we see the existence of inverse branches with uniform backward contraction on some small neighborhood.

Lemma 3.6 (Distortion Control)**.** *There exists $\delta > 0$ such that for every $n = n(x)$ hyperbolic time for x, the dynamical ball $B_\delta(x,n)$ is mapped homeomorphically by f^n onto the ball $B(f^n(x),\delta)$ with*

$$d(f^{n-j}(y), f^{n-j}(z)) \leq \sigma^j d(f^n(y), f^n(z)) \tag{3.1}$$

for every $1 \leq j \leq n$ and every $y, z \in B_\delta(x,n)$.

Moreover, given a Hölder continuous potential ψ there exists a positive constant K such that

$$K^{-1} \leq e^{-S_n\psi(y)+S_n\psi(z)} \leq K \tag{3.2}$$

for every $y, z \in B_\delta(x,n)$.

Note that, since the potential ψ is Hölder continuous, the inequality (3.2) above is a direct consequence of the local contraction until hyperbolic time n given by the first property (3.1) of the Lemma. For a proof of the property (3.1) see [1] and replacing $\|Df(\cdot)^{-1}\|$ by $\sigma(\cdot)$.

In order to finish this subsection, we observe that every f-invariant set contains a topological disk up to a set of zero measure.

Lemma 3.7. *Let ν be an expanding measure. If A is an f-invariant set with positive ν-measure then there exists a topological disk Δ of radius $\delta/4$ so that $\nu(\Delta \setminus A) = 0$.*

For a proof of this result, see [19].

3.3. **Natural Extension.** Let (M, d) be a compact metric space and let $f : M \to M$ be a continuous non-invertible transformation. Define the space

$$\hat{M} = \{\hat{x} := (\ldots, x_2, x_1, x_0) \in M^{\mathbb{N}}\,;\ f(x_{i+1}) = x_i \ \text{ for all } \ i \geq 0\}.$$

Fixed $\hat{\delta} \in (0,1)$ we can define the following metric on $\hat{M}$:

$$d_{\hat{M}}(\hat{x}, \hat{y}) := \sum_{j \geq 0} \hat{\delta}^j d_M(x_j, y_j).$$

The *natural extension* of f is the homeomorphism

$$\hat{f} : \hat{M} \to \hat{M}, \quad \hat{f}(\hat{x}) = \hat{f}(..., x_2, x_1, x_0) = (..., x_2, x_1, x_0, f(x_0))$$

Let $\hat{\pi} : \hat{M} \to M$ be the *natural projection*, which is,

$$\hat{\pi}(\hat{x}) = \hat{\pi}(\ldots, x_2, x_1, x_0) = x_0$$

Note that, $\hat{\pi}$ is continuous, surjective and (semi)conjugates f and $\hat{f}$.

On ergodic point of view, given an ergodic measure μ defined on Borel subsets of M there exists a unique measure $\hat{\mu}$ defined on Borel subsets of $\hat{M}$ such that $\hat{\pi}_*\hat{\mu} = \mu$, that mean,

$$\mu(A) = \hat{\mu}(\hat{\pi}^{-1}(A)), \text{ for every measurable set } A \subset M.$$

Moreover, since

$$\hat{\pi}^{-1}(x) = \{(\ldots, x_2, x_1, x_0) \in \hat{M}\,;\, x_0 = x\}$$

we observe that $h_{top}(f, \hat{\pi}^{-1}(x)) = 0$ for every $x \in M$ because we can choose a subset of $\hat{\pi}^{-1}(x)$ with finite cardinality as n-generator for every $n \in \mathbb{N}$.

Thus, we apply the Ledrappier-Walter's formula, and we conclude that if $\hat{\mu}$ projects on μ then $h_{\hat{\mu}}(\hat{f}) = h_{\mu}(f)$.

Theorem 3.8 (Ledrappier-Walter's formula)**.** *Let $\tilde{M}$, M be compact metric spaces and let $\tilde{f} : \tilde{M} \to \tilde{M}$, $f : M \to M$, $\pi : \tilde{M} \to M$ be continuous maps such that π is surjective and $\pi \circ \tilde{f} = f \circ \pi$ then*

$$\sup_{\tilde{\mu};\pi_*\tilde{\mu}=\mu} h_{\tilde{\mu}}(\tilde{f}) = h_\mu(f) + \int h_{top}(\tilde{f}, \pi^{-1}(y))\, d\mu(y)$$

See [14] for a proof of this theorem and [18] for more details about natural extension.

4. Proof of Theorem A

Here, we will show the existence of an expanding conformal measure ν for (f, ϕ) associated to the eingenvalue $\lambda = e^{P_f(\phi)}$. Next, we will construct invariant ergodic measures absolutely continuous with respect to ν. In addition, we will get the uniqueness of such ergodic measure when the dynamics f is transitive.

Let $\mathcal{L}_\phi : C(M) \to C(M)$ be the transfer operator of (f, ϕ) :

$$\mathcal{L}_\phi(\psi)(x) = \sum_{y\in f^{-1}(x)} e^{\phi(y)}\psi(y)$$

Since f is a local homeomorphism defined on a compact metric space and ϕ is a continuous function follows that $\mathcal{L}_\phi$ is a linear continuous operator with $\|\mathcal{L}_\phi\| \leq \deg(f)e^{\|\phi\|_\infty}$

Note that, by induction, we have for all $n \geq 1$:

$$\mathcal{L}^n_\phi(\psi)(x) = \sum_{y\in f^{-n}(x)} e^{S_n\phi(y)}\psi(y)$$

where $S_n\phi$ denotes the Birkhoff sum $S_n\phi = \displaystyle\sum_{k=0}^{n-1} \phi \circ f^k$.

Let $(C(M))^*$ be the dual space of $C(M)$. Since M is compact we can identify $(C(M))^*$ with the space $\mathcal{M}(M)$ of finite signed measure defined on M, i.e., $(C(M))^* \simeq \mathcal{M}(M)$. Therefore, the dual operator $\mathcal{L}^*_\phi$ of $\mathcal{L}_\phi$ acts on $(C(X))^*$ as follow:

$$\begin{aligned} \mathcal{L}^*_\phi : (C(M))^* &\longrightarrow (C(M))^* \\ \mu &\longmapsto \mathcal{L}^*_\phi(\mu) : C(M) \longrightarrow \mathbb{R} \\ &\qquad\qquad \psi \longmapsto \mathcal{L}^*_\phi(\mu)(\psi) = \int_M \mathcal{L}_\phi(\psi)d\mu \end{aligned}$$

Let $\Phi : (C(M))^* \to (C(M))^*$ be the operator: $\Phi(\mu) = \dfrac{\mathcal{L}^*_\phi(\mu)}{\int \mathcal{L}_\phi(1)d\mu}$

This operator is continuous on $\mathcal{M}(M)$ because $\mathcal{L}_\phi^*$ is continuous and

$$\mathcal{L}_\phi(1) = \sum_{y \in f^{-1}(x)} e^{\phi(y)} \geq \deg(f) e^{-||\phi||_\infty} = c > 0$$

implies

$$\int_M \mathcal{L}_\phi(1) d\mu \geq c\mu(M) > 0, \quad \forall \mu \in \mathcal{M}(M).$$

In particular, Φ gives invariant the convex compact subset

$$\mathcal{M}_1(\Sigma_\sigma) = \{\eta \in \mathcal{M}_1(M)\,;\ \eta(\Sigma_\sigma) = 1\}$$

of all probabilities measures defined on M whose full weight is located on the non-uniformly expanding region Σ_σ. So, we can apply the Schauder-Tychonov theorem to conclude that there is some fixed point $\nu \in \mathcal{M}_1(\Sigma_\sigma)$ for Φ.

Theorem 4.1 (Schauder-Tychonov)**.** *Let K be a compact convex subset of a locally convex topological vectorial space and let $\Phi : K \to K$ be a continuous application. Then, Φ has some fixed point.*

This fixed point ν is an eigenmeasure for $\mathcal{L}_\phi^*$:

$$\Phi(\nu) = \nu \implies \mathcal{L}_\phi^*(\nu) = \lambda\nu \text{ where } \lambda = \int_M \mathcal{L}_\phi(1) d\nu > 0$$

Next we derive some importants properties of this eigenmeasure.

The *Jacobian* of a measure μ with respect to f is a measurable function $J_\mu f$ satisfying

$$\mu(f(A)) = \int_A J_\mu f \, d\mu$$

for any measurable set A such that $f|_A$ is injective.

In general, a jacobian may fail to exists, in the meantime it is a standard result that a λ-eigenmeasure has jacobian and it is equal to $\lambda e^{-\phi}$. Indeed, let A be a measurable set such that $f|_A$ is injective. Pick a limited sequence $\{\psi_n\} \in C(M)$ such that $\psi_n \to \chi_A$ for ν-qtp. Then,

$$\lambda\nu(e^{-\phi}\psi_n) = \mathcal{L}_\phi^*\nu(e^{-\phi}\psi_n) = \int_M \mathcal{L}_\phi(e^{-\phi}\psi_n) d\nu \longrightarrow \nu(f(A))$$

Since the first member converges to $\int_A \lambda e^{-\phi} d\nu$, we conclude that

$$\int_A J_\nu f \, d\nu = \nu(f(A)) = \int_A \lambda e^{-\phi} d\nu.$$

From now on, we will consider hyperbolic potentials $\phi : M \to \mathbb{R}$, that mean, there exists some constant $\sigma \in (0,1)$ such that

$$P_f(\phi, \Sigma_\sigma^c) < P_f(\phi, \Sigma_\sigma) = P_f(\phi).$$

The above inequality allows us to explore the non-uniform expansion property on Σ_σ. In this direction, we will use frequently the hyperbolic times. For remenber this property see the preliminaries.

Fix $\delta > 0$ such that the dynamical ball $B_\delta(x, n)$ is mapped homeomorphically by f^n onto the ball $B(f^n(x), \delta)$ whenever n is a hyperbolic time for x.

Proposition 4.2. *Let $f : M \to M$ be a local homeomorphism and let $\phi : M \to \mathbb{R}$ be a hyperbolic Hölder continuous potential. A conformal measure ν ($\mathcal{L}_\phi^*(\nu) = \lambda\nu$) associated to (f, ϕ) has the following properties:*
(i) Fixed $\varepsilon \leq \delta$ there exists $K_\varepsilon > 0$ such that if $x \in supp(\nu)$ and n is a hyperbolic time for x then

$$K_\varepsilon^{-1} \leq \frac{\nu(B_\varepsilon(x,n))}{e^{S_n\phi(y) - n\log\lambda}} \leq K_\varepsilon$$

for all y in the dynamical ball $B_\varepsilon(x, n)$.
(ii) The pressure of (f, ϕ), $P_f(\phi)$, is equal to $\log \lambda$.
(iii) Any two conformal measures associated to λ are equivalents.
(iv) If f is transitive then ν is an open measure, i.e. $\nu(U) > 0$ for every open subset $U \subset M$.

Proof. (i) Let $x \in \operatorname{supp}(\nu)$ and n be a hyperbolic time for x. Since f^n maps homeomorphically $B_\varepsilon(x, n)$ into the ball $B(f^n(x), \varepsilon)$ and the jacobian of ν is bounded away from zero and infinity follows that there exists a uniform constant γ_ε depends on the radius ε of the ball such that

$$\gamma_\varepsilon \leq \nu(f^n(B_\varepsilon(x,n))) = \int_{B_\varepsilon(x,n)} \lambda^n e^{-S_n\phi(z)} d\nu \leq 1$$

By distortion control on hyperbolic times (Lemma 3.6), we have

$$\begin{aligned}\gamma_\varepsilon \leq \int_{B_\varepsilon(x,n)} \lambda^n e^{-S_n\phi(z)} d\nu &= \int_{B_\varepsilon(x,n)} \lambda^n e^{-S_n\phi(y)} \left(\frac{\lambda^n e^{-S_n\phi(z)}}{\lambda^n e^{-S_n\phi(y)}}\right) d\nu \\ &\leq \; K e^{-S_n\phi(y) + n\log\lambda} \nu(B_\varepsilon(x,n))\end{aligned}$$

and

$$e^{-S_n\phi(y)+n\log\lambda}\nu(B_\varepsilon(x,n)) \leq K\int_{B_\varepsilon(x,n)} \lambda^n e^{-S_n\phi(y)}\left(\frac{\lambda^n e^{-S_n\phi(z)}}{\lambda^n e^{-S_n\phi(y)}}\right)d\nu$$

Thus,

$$\gamma_\varepsilon K^{-1} \leq \frac{\nu(B_\varepsilon(x,n))}{e^{S_n\phi(y)-n\log\lambda}} \leq K$$

for all $y \in B_\varepsilon(x,n)$.

(ii) To see the equality $P_f(\phi) = \log\lambda$, let α be a cover of M with diameter less than δ. Given $n \in \mathbb{N}$ denote by $\mathcal{U}$ a subcover of α^n. As ν is an eigenmeasure we have

$$\begin{aligned}\lambda^n = \lambda^n\nu(M) = \int_M \mathcal{L}_\phi^n(1)d\nu &\leq \sum_{U\subset\mathcal{U}}\int_U e^{S_n\phi(x)}d\nu \\ &\leq \sum_{U\subset\mathcal{U}} e^{S_n\phi(U)}\nu(U) \\ &\leq \sum_{U\subset\mathcal{U}} e^{S_n\phi(U)}\end{aligned}$$

Taking the infimum of all subcover $\mathcal{U}$ of α^n we obtain the inequality

$$\log\lambda \leq \frac{1}{n}\log\inf_{\mathcal{U}\subset\alpha^n}\left\{\sum_{U\subset\mathcal{U}} e^{S_n\phi(U)}\right\} \overset{n\to+\infty}{\longrightarrow} P_f(\phi,\alpha) \leq P_f(\phi)$$

The converse inequality will be consequence of the property of the conformal measure ν on hyperbolic times. Since every point x of Σ_σ has infinitely many hyperbolic times, given $\varepsilon < \delta$ small we can fix $N > 1$ suficiently large such that

$$\Sigma_\sigma \subset \bigcup_{n\geq N}\bigcup_{x\in\Sigma_n} B_\varepsilon(x,n)$$

where Σ_n denotes the set whose points have n as hyperbolic time.

By Besicovitch's covering lemma, see [17], there exist a countable family $F_n \subset \Sigma_n$ such that every point $x \in \Sigma_n$ is covering by at most d (depending only the dimension m of the space M) dynamical balls $B_\varepsilon(x,n)$ with $x \in F_n$, for each $n \geq N$.

Let $\mathcal{F}_N = \{B_\varepsilon(x,n);\, x \in F_n \text{ and } n \geq N\}$ be the family of dynamical balls centered on the points $x \in F_n$ with diameter less than ε. In this way, $\mathcal{F}_N$ is a countable open cover for Σ_σ.

Now, given γ, $\gamma > \log\lambda$, using the definition of relative pressure and the property (i) above we calculate

$$\begin{aligned}
\inf_{\mathcal{U}\subset\mathcal{F}_N}\sum_{B_\varepsilon(x,n)\in\mathcal{U}} e^{-\gamma n+S_n\phi(B_\varepsilon(x,n))} &\leq \sum_{n\geq N} Ke^{-(\gamma-\log\lambda)n}\big\{\sum_{x\in G_n}\nu(B_\varepsilon(x,n))\big\} \\
&\leq dK\sum_{n\geq N}e^{-(\gamma-\log\lambda)n} \\
&\leq \frac{dK}{1-e^{-(\gamma-\log\lambda)}}e^{-(\gamma-\log\lambda)N}
\end{aligned}$$

Taking the limite on N we obtain

$$\begin{aligned}
m_f(\phi,\Sigma_\sigma,\varepsilon,\gamma) &= \lim_{N\to+\infty}\inf_{\mathcal{U}}\sum_{n>N}e^{-\gamma n+S_n\phi(B_\varepsilon(x,n))} \\
&\leq \lim_{N\to+\infty}\frac{dK}{1-e^{-(\gamma-\log\lambda)}}e^{-(\gamma-\log\lambda)N} \\
&= 0
\end{aligned}$$

Since this inequality is true for every ε and every $\gamma > \log\lambda$ we have

$$P_f(\phi,\Sigma_\sigma)\leq\log\lambda$$

Since by hypothesis, $P_f(\phi,\Sigma_\sigma^c) < P_f(\phi,\Sigma_\sigma) = P_f(\phi)$ we conclude

$$P_f(\phi) = P_f(\phi,\Sigma_\sigma)\leq\log\lambda$$

(iii) Let ν_1 and ν_2 two conformal measures. By inequality that we obtained in (i), if n is a hyperbolic time for x then there exist some positive constant $\tilde{K}$ such that

$$\tilde{K}^{-1}\nu_2(B_\varepsilon(x,n))\leq\nu_1(B_\varepsilon(x,n))\leq\tilde{K}\nu_2(B_\varepsilon(x,n)).$$

Since the family of hyperbolic dynamical balls has small enough diamenter and it covers the full measure set Σ_σ we get this inequality is true for every Borel set. Therefore ν_1 and ν_2 are equivalent measures.

(iv) Suppose that f is transitive, given $U\subset M$ an open set, we have $M = \bigcup_{s\in\mathbb{N}} f^s(U)$.

Since for each s, f^s is also a local homeomorphism we can decompose U into subsets $V_i(s)\subset U$ such that $f^s|_{V_i(s)}$ is injective. Hence,

$$\begin{aligned}
1=\nu(M)\leq\sum_s\nu(f^s(U)) &\leq \sum_s\sum_i\int_{V_i(s)}\lambda^s e^{-S_s\phi(x)}d\nu \\
&\leq \sum_s\lambda^s\sum_i\sup_{x\in V_i(s)}(e^{S_s\phi(x)})\nu(V_i(s))
\end{aligned}$$

Thus, there exists some $V_i(s) \subset U$ such that $\nu(U) \geq \nu(V_i(s)) > 0$.
□

Until now, we have shown the existence of an expanding conformal measure ν associated to the eingenvalue $\lambda = e^{P_f(\phi)}$. In the following, to conclude the proof of Theorem A, we will construct f-invariant ergodic measures absolutely continuous with respect to ν and, by a compactness argument, we will get the finiteness.

Let $(\mu_n)_n$ be the sequence of the averages of the positive iterates of the measure ν restricted to the expanding set Σ_σ:

$$\mu_n := \frac{1}{n}\sum_{j=0}^{n-1} f_*^j(\nu|_{\Sigma_\sigma})$$

Denoting by Σ_j the set of points in Σ_σ that have $j \geq 1$ as hyperbolic time, we consider the sequence

$$\eta_n := \frac{1}{n}\sum_{j=0}^{n-1} f_*^j(\nu|_{\Sigma_j})$$

Applying Lemma 3.5 there are some $\theta > 0$ and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ holds:

$$\eta_n(M) \;\geq\; \frac{1}{n}\sum_{j=0}^{n-1}\nu(\Sigma_j) \;\geq\; \frac{1}{n}\sum_{j=0}^{n-1}\nu(\Sigma_\sigma \cap \Sigma_j) \;\geq\; \theta\nu(\Sigma_\sigma).$$

Since the jacobian of the measure ν is a positive function bounded away from zero and infinity follows that each η_n is absolutely continuous with respect to ν.

Let (n_k) be a subsequence $n_k \to \infty$ such that μ_{n_k} and η_{n_k} converge in the weak* topology to measures μ and η respectively. By construction, μ is an f-invariant measure and η is a component of μ absolutely continuous with respect to ν. Therefore, if we decompose $\mu = \mu_{ac} + \mu_s$ with μ_{ac} absolutely continuous with respect to ν and μ_s singular to ν we get $\mu_{ac}(\Sigma_\sigma) \geq \eta(\Sigma_\sigma) > 0$. Normalizing μ_{ac}, we show the existence of some f-invariant probability measure absolutely continuous to ν.

Suppose, by contradiction, that there is an infinite number of ergodic f-invariant probabilities $\mu_1, \mu_2, \mu_3...$ absolutely continuous with respect to ν. Let $B(\mu_i)$ be the attraction basin of the measure μ_i, for each $i = 1, 2, ...$ Since the basins $B(\mu_i)$ are disjoint invariant sets of positive ν measure, we can apply Lemma 3.7 to conclude that there exists a topological disk Δ_i of radius $\delta/4$ such that $\nu(\Delta_i \setminus B(\mu_i)) = 0$, for each $i = 1, 2, ...$ But due to the compactness of the space M, it is just possible to the existence of a finite number of such topological disks, which is a contradiction.

Remark. If the dynamic f is transitive then we get uniqueness of such ergodic measure because in this case, given two distinct ergodic measures μ_1 and μ_2 absolutely continuous with respect to ν, applying the same argument on the attraction basins $B(\mu_1)$ and $B(\mu_2)$, there are topological disks Δ_1, Δ_2 such that $\nu(\Delta_i \setminus B(\mu_i)) = 0$, for $i = 1, 2$. Thus, by the transitivity property of f we have $f^n(\Delta_1) \cap \Delta_2 \neq \emptyset$ for some positive iterate. Since $B(\mu_1)$ and $B(\mu_2)$ are invariant sets and ν is an open measure whose jacobian is uniformly bounded we obtain $\nu(B(\mu_1) \cap B(\mu_2)) > 0$, that is a contradiction.

5. Proof of Theorem B

In this section, we are interested on the existence and finiteness of equilibrium states for local homeomorphisms. In our setting it is enough to require hyperbolicity condition on the potentials.

Lemma 5.1. *The map $\eta \mapsto h_\eta(f) + \int \phi d\eta$ is upper semicontinuous over the measures that gives full weight to the expanding set Σ_σ.*

Proof. Let $\mathcal{P}$ be a finite patition with diameter less than δ and η be an expanding measure. For each $n \geq 1$, define the partition $\mathcal{P}_n$ by

$$\mathcal{P}_n = \left\{P_n = P_{i_0} \cap \cdots \cap f^{-(n-1)}(P_{i_{n-1}})\, ;\, P_{i_j} \in \mathcal{P}, 0 \leq j \leq n-1\right\}$$

Let $\mathcal{P}_n(x)$ be the atom of $\mathcal{P}_n$ that contains the point x. Note that, by Lemma 3.6, if k is a hyperbolic time for x then $\text{diam}\mathcal{P}_k(x) \leq \sigma^k \delta$. Since η-almost every point $x \in M$ has infinitely many hyperbolic times and the sequence $\text{diam}\mathcal{P}_n(x)$ is non-increasing, we conclude that the diameter of $\mathcal{P}_n(x)$ goes to zero when n goes to infinity for η-almost every $x \in M$.

Thus, $\mathcal{P} \prec \mathcal{P}_1 \prec \ldots \prec \mathcal{P}_n \prec \cdots$ is a non-increasing sequence of partitions with $\text{diam}\mathcal{P}_n(x) \to 0$ for η-almost every point $x \in M$. By Kolmogorov-Sinai Theorem, holds that $h_\eta(f) = h_\eta(f, \mathcal{P})$ for any expanding measure η.

On the other hand, if $\eta(\partial \mathcal{P}) = 0$ then the map $\beta \mapsto h_\beta(f, \mathcal{P})$ is upper semicontinuous on η.

This shows that if $\eta(\partial \mathcal{P}) = 0$ then the map $\beta \mapsto h_\beta(f)$ is upper semicontiuous on η when restricted to the expanding measure. In fact, given $\varepsilon > 0$ we have

$$h_\beta(f) = h_\beta(f, \mathcal{P}) \leq h_\eta(f, \mathcal{P}) + \varepsilon = h_\eta(f) + \varepsilon$$

Thus, combining this fact with the continuity of the integral we conclude that the map $\eta \mapsto h_\eta(f) + \int \phi d\eta$ is upper semicontinuous when restricted to expanding measures.

□

Let $\{\mu_k\}_k$ be a sequence that aproximate the pressure, that is,

$$\limsup_{k\to+\infty} \left\{ h_{\mu_k}(f) + \int \phi d\mu_k \right\} = P_f(\phi)$$

Since the potential ϕ is hyperbolic, by the Variational Principle, we have that μ_k is expanding for k big enough because

$$P_f(\phi, \Sigma_\sigma^c) < P_f(\phi, \Sigma) = P_f(\phi) = \limsup_{k\to+\infty} \left\{ h_{\mu_k}(f) + \int \phi d\mu_k \right\}$$

By compacteness of $\mathcal{M}_1(M)$, there exist some acumulation point μ of the sequence $\{\mu_k\}_k$. Using the upper semicontinuity of the map $\eta \mapsto h_\eta(f) + \int \phi d\eta$, we obtain

$$P_f(\phi) = \limsup_{k\to+\infty} \left\{ h_{\mu_k}(f) + \int \phi d\mu_k \right\} \le h_\mu(f) + \int \phi d\mu \le P_f(\phi)$$

This show that μ is an equilibrium state associated to (f, ϕ).

To get the finiteness of this measures, we need to evoque the following abstract result done by Ledrappier [13] for SRB measures and extended by Varandas and Viana [19] for conformal measures.

Theorem 5.2. *Let $f : M \to M$ be a local homeomorphism, $\phi : M \to \mathbb{R}$ be a Hölder continuous potential and ν be a conformal measure such that $J_\nu f = \lambda e^{-\phi}$, where $\lambda = e^{P_f(\phi)}$. Assume that η is an equilibrium state for (f, ϕ) gives full weight to supp(ν). If η is expanding then η is absolutely continuous with respect to ν.*

Now we are in position to conclude the Theorem B. Let η be an equilibrium state for (f, ϕ). By The Ergodic Decomposition Theorem, we may assume that η is an ergodic measure. Observing that

$$P_f(\phi) = P_f(\phi, \Sigma_\sigma) > P_f(\phi, \Sigma_\sigma^c) \ge \sup\{h_\mu(f) + \int \phi d\mu\}$$

where the supremum is taken over all measures such that $\mu(\Sigma_\sigma^c) = 1$, we conclude that if η is an ergodic equilibrium state then we must have $\eta(\Sigma_\sigma) = 1$, i.e. η is an expanding measure.

Applying the Theorem 5.2. we have that η is absolutey continuous with respect the reference measure ν. By the Theorem A, there is only finitely many ergodic measures with this property. In particular, if f

is transitive there is only one such a measure. This ends the proof of Theorem B.

6. Proof of Theorem C

Finally, we are going to prove the existence of equilibrium states for partially hyperbolic skew-products $F : M \times N \to M \times N$ and Hölder continuous potentials $\phi : M \times N \to \mathbb{R}$. In this scenario, it is enough to require uniform contraction on the fibers and hyperbolicity condition of the potential on the base, i.e., there exists a non-uniformly expanding region Σ_σ on M such that

$$P_F(\phi, \Sigma_\sigma^c \times N) < P_F(\phi, \Sigma_\sigma \times N) = P_F(\phi).$$

Let us start to proving the item (ii) of the Theorem C.

Consider $\phi : M \times N \to \mathbb{R}$ be a Hölder continuous potential hyperbolic on M that does not depends on the fiber. This means that the function $\phi(x, \cdot) : N \to \mathbb{R}$ is constant, for each $x \in M$ fixed.

In this way, fixed a point $y_0 \in N$, the potential ϕ induces a Hölder continuous potential $\psi : M \to \mathbb{R}$ defined by $\psi(x) = \phi(x, y_0)$.

Let $\pi : M \times N \to M$ be the projection on M, i.e., $\pi(x, y) = x$, for every $y \in N$. Observe that the application π is a continuous semi-conjugation between F and f because $\pi(F(x, y)) = f(x) = f(\pi(x, y))$. The next lemma states that there exist a bijection between equilibrium measures of (F, ϕ) and (f, ψ).

Lemma 6.1. *Let $\mu_\psi \in \mathcal{M}_1(M)$ be an ergodic measure. There exists only one ergodic measure $\mu_\phi \in \mathcal{M}_1(M \times N)$ such that $\mu_\psi = \pi_* \mu_\phi$. Moreover, if μ_ψ is an ergodic equilibrium state of (f, ψ) then μ_ϕ is an equilibrium state for (F, ϕ).*

Proof. Suppose the existence of two ergodic measures μ_ϕ^1 and μ_ϕ^2 such that $\pi_* \mu_\phi^1 = \mu_\psi = \pi_* \mu_\phi^2$.

Denote by $B_{\mu_\phi^1}(F)$ and $B_{\mu_\phi^2}(F)$ the attraction basins of μ_ϕ^1 and μ_ϕ^2 respectively. Since g is a fiber contraction and μ_ϕ^1, μ_ϕ^2 are ergodic measures follows that

$$B_{\mu_\phi^1}(F) = A_1 \times N, \;\; B_{\mu_\phi^2}(F) = A_2 \times N \;\text{ with }\; A_1 \cap A_2 = \emptyset$$

From the ergodicity of the measure μ_ψ and the f-invariance of the sets $\pi(B_{\mu_\phi^1}(F))$ and $\pi\big(B_{\mu_\phi^2}(F)\big)$ we conclude that

$$\mu_\psi\big(\pi(B_{\mu_\phi^1}(F))\big) = \mu_\psi\big(\pi(B_{\mu_\phi^1}(F))\big) = 1$$

Thus,

$$\pi\big(B_{\mu_\phi^1}(F)\big) \cap \pi\big(B_{\mu_\phi^2}(F)\big) \neq \emptyset$$

i.e. $A_1 \cap A_2 \neq \emptyset$. Hence, by ergodicity, $\mu_\phi^1 = \mu_\phi^2$.

For the second statement of the lemma, observe that the uniform contraction of g on the fiber $\pi^{-1}(\{x\})$ gives us $h_{top}(F, \pi^{-1}(x)) = 0$ for every $x \in M$. Applying Ledrappier-Walters formula, we have

$$\begin{aligned} P_f(\psi) \leq P_F(\phi) &= \sup\left\{h_{\tilde{\mu}}(F) + \int \phi d\tilde{\mu}\right\} \\ &= \sup\left\{h_\mu(f) + \int h_{top}(F, \pi^{-1}(x))\, d\mu(x) + \int \psi d\mu\right\} \\ &= P_f(\psi) \end{aligned}$$

This inequality shows that $P_f(\psi) = P_F(\phi)$. Thus, if μ_ψ is an ergodic equilibrium state of (f, ψ) then μ_ϕ is an equilibrium state for (F, ϕ). Conversely, note that any equilibrium for (F, ϕ) projects on an equilibrium for (f, ψ).

□

To complete the proof of item (ii) of Theorem C, it is just observe that $\psi : M \to \mathbb{R}$ is a hyperbolic potential. In fact,

$$P_f(\psi, \Sigma_\sigma^c) \leq P_F(\phi, \Sigma_\sigma^c \times N) < P_F(\phi, \Sigma_\sigma \times N) = P_F(\phi) = P_f(\psi)$$

Thus, by the Theorem B jointly with the above lemma, we conclude that there exists finitely many ergodic equilibrium states for (F, ϕ). In addition, the equilibrium measure is unique if the dynamics f is transitive.

For the proof of item (i), we will understand the dynamics of the skew-product

$$F : M \times N \to M \times N,\ \ F(x, y) = (f(x),\, g_x(y)\,)$$

from the natural extension of the base dynamics.

Let $\hat{f}$ be the natural extension of f:

$$\hat{f} : \hat{M} \to \hat{M}, \quad \hat{f}(\hat{x}) = \hat{f}(..., x_2, x_1, x_0) = (..., x_2, x_1, x_0, f(x_0))$$

Since $\delta < 1$ is fixed, we will consider $\hat{M}$ provided with the metric

$$d_{\hat{M}}(\hat{x}, \hat{y}) := \sum_{j \geq 0} \delta^j d_M(x_j, y_j).$$

Define the new skew-product on $\hat{M} \times N$ by

$$\hat{F} : \hat{M} \times N \to \hat{M} \times N; \quad \hat{F}(\hat{x}, y) = (\hat{f}(\hat{x}),\, g_{\hat{\pi}(\hat{x})}(y)\,)$$

In order to not load the notation, we will denote $g_{\hat{\pi}(\hat{x})}(y)$ by $g(\hat{x}, y)$. However, it is clear that g depends only on x and not of his past, that means, if $\hat{\pi}(\hat{x}) = \hat{\pi}(\hat{z})$ then $g(\hat{x}, y) = g(\hat{z}, y)$ for all $y \in N$. Moreover, we will denote by g^n the iterates of g:

$$g^n(\hat{x}, y) := g(\hat{f}^n(\hat{x}), g^{n-1}(\hat{x}, y)) \text{ with } g^1(\hat{x}, y) = g(\hat{x}, y)$$

for all $(\hat{x}, y) \in \hat{M} \times N$.

Note that, $\hat{f}$ is a homeomorphism dominated by g, because for all $n \in \mathbb{N}$ holds

$$d_{\hat{M}}(\hat{f}^{-n}(\hat{x}), \hat{f}^{-n}(\hat{z})) \leq \delta^{-n} d_{\hat{M}}(\hat{x}, \hat{z}) \quad \text{and} \quad \lambda\delta^{-1} \leq \tilde{\sigma} < 1 \tag{6.1}$$

Furthermore, g is a fiber contraction which varies β-Hölder continuously on $\hat{M}$:

$$d_N(g(\hat{x}, y), g(\hat{z}, y)) \leq C d_M(x, z)^{\beta} \leq C d_{\hat{M}}(\hat{x}, \hat{z})^{\beta} \tag{6.2}$$

for all $\hat{x}, \hat{z} \in \hat{M}$ and $y \in N$.

Thus, we can apply the following proposition due to M. Hirsch, C. Pugh and M. Shub [10] which ensures that $\hat{F}$ has an attractor $\hat{\Lambda}$ given by a graph of a Hölder continuous function.

Proposition 6.2. *Let $\hat{M}$ be a compact metric space and N be a complete metric space. Consider $\hat{F} : \hat{M} \times N \to \hat{M} \times N$ be the skew-product $\hat{F}(\hat{x}, y) = (\hat{f}(\hat{x}),\, g_{\hat{\pi}(\hat{x})}(y)\,)$. Suppose that $\hat{f} : \hat{M} \to \hat{M}$ is a homemorphism satisfies (6.1) and $g : \hat{M} \times N \to N$ is continuous satisfies (6.2). Then, there exists a Hölder continuous function $\xi : \hat{M} \to N$ such that the graph of ξ is $\hat{F}$-invariant and attracting for all $(\hat{x}, y) \in \hat{M} \times N$.*

This result allows us to ensure that the behavior of the skew-product $\hat{F}$ is essentially equal to the behavior of the base dynamics $\hat{f}$. In fact, given a potential $\hat{\phi} : \hat{M} \times N \to \mathbb{R}$, since graph$(\xi)$ is an attracting invariant set for $\hat{F}$ we have, by variational principle, $P_{\hat{F}}(\hat{\phi}) = P_{\hat{F}}\big(\hat{\phi}\big|_{\mathrm{graph}(\xi)}\big)$.

Let $F : M \times N \to M \times N$ be the partially hyperbolic skew-product $F(x, y) = (f(x),\, g_x(y)\,)$ and $\phi : M \times N \to \mathbb{R}$ be a Hölder continuous potential, hyperbolic on M.

Define the potential $\hat{\phi} : \hat{M} \times N \to \mathbb{R}$ by $\hat{\phi}(\hat{x}, y) = \phi\,(\,\hat{\pi}(\hat{x}), y)$ where $\hat{\pi} : \hat{M} \to M$ is the natural projection. Notice that $\hat{\phi}$ is Hölder continuous on $\hat{M} \times N$:

$$\begin{aligned} d_{\mathbb{R}}(\hat{\phi}(\hat{x}, y), \hat{\phi}(\hat{z}, w)) &= d_{\mathbb{R}}(\phi(\hat{\pi}(\hat{x}), y), \phi(\hat{\pi}(\hat{z}), w)) \\ &\leq k\, d_{M \times N}((\hat{\pi}(\hat{x}), y), (\hat{\pi}(\hat{z}), w)) \\ &\leq k\, d_{\hat{M} \times N}((\hat{x}, y), (\hat{z}, w)) \end{aligned}$$

for all $(\hat{x}, y), (\hat{z}, w) \in \hat{M} \times N$.

Lemma 6.3. *There exist some equilibrium state $\mu_{\hat{\phi}}$ for $(\hat{F}, \hat{\phi})$.*

Proof. First we observe that since f is a C^1 local diffeomorphism then the natural extension $\hat{f}^{-1}$ is locally Lipschitz continuous, i.e., given $\hat{x} \in \hat{M}$ there exists a neighborhood $V_{\hat{x}}$ such that for every $\hat{y}, \hat{z} \in \hat{f}(V_{\hat{x}})$ we have

$$d_{\hat{M}}(\hat{f}^{-1}(\hat{y}), \hat{f}^{-1}(\hat{z})) \leq \hat{\sigma}(\hat{x})\, d_{\hat{M}}(\hat{y}, \hat{z})$$

where $\hat{\sigma}(\hat{x}) = \|Df^{-1}\| \circ \hat{\pi}(\hat{x})$.

In particular, if Σ_σ is a non-uniformly expanding set on M then for every $\hat{x} \in \hat{\pi}^{-1}(\Sigma_\sigma)$ holds

$$\limsup_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \hat{\sigma}(\hat{f}^j(\hat{x})) \leq \log\sigma < 0$$

Hence, $\hat{f}$ is a non-uniformly expanding map on $\hat{\Sigma}_\sigma := \hat{\pi}^{-1}(\Sigma_\sigma)$.

Furthermore, since $\mathrm{Rec}(\hat{F}) = \mathrm{graph}(\xi)$ and $\xi : \hat{M} \to N$ is a Hölder continuous function follows that $\hat{\phi}$ induces a Hölder continuous potential $\hat{\psi}$ for $\hat{f}$ defined by

$$\hat{\psi} : \hat{M} \to \mathbb{R},\ \hat{\psi}(\hat{x}) := \hat{\phi}(\hat{x}, \xi(\hat{x})) \text{ with } P_{\hat{f}}(\hat{\psi}) = P_{\hat{F}}(\hat{\phi}\big|_{\mathrm{graph}(\xi)}) = P_{\hat{F}}(\hat{\phi})$$

Note also that fixed $\varepsilon > 0$ and $n_0 \in \mathbb{N}$, if $\hat{\mathcal{F}}_{n_0}$ is a collection of dynamical balls

$$\hat{\mathcal{F}}_{n_0} = \{B_\varepsilon(\hat{x}, n) \,/\, \hat{x} \in \hat{M} \text{ and } n \geq n_0\}$$

that cover $\hat{\Sigma}^c_\sigma \subset \hat{M}$ then the collection $\mathcal{F}_{n_0}$ of dynamical balls

$$\mathcal{F}_{n_0} = (\hat{\pi}, \xi)(\hat{\mathcal{F}}_{n_0}) = \{(\hat{\pi}, \xi)(B_\varepsilon(\hat{x}, n)) / \hat{\pi}(\hat{x}) \in M,\ \xi(\hat{x}) \in N \,\text{and}\, n \geq n_0\}$$

cover $\Sigma^c_\sigma \times \xi(\hat{\Sigma}^c_\sigma) \subset M \times N$ and for every $\alpha \in \mathbb{R}$ holds

$$\inf_{B_\varepsilon(\hat{x},n)} \sum_{n\geq n_0} e^{-\alpha n + S_n\hat{\psi}(B_\varepsilon(\hat{x},n))} \quad\leq\quad \inf_{B_\varepsilon((x,\xi(\hat{x})),n)} \sum_{n\geq n_0} e^{-\alpha n + S_n\phi((\hat{\pi},\xi)(B_\varepsilon(\hat{x},n)))}$$

Thus for every ε holds

$$P_{\hat{f}}(\hat{\psi}, \hat{\Sigma}^c_\sigma, \varepsilon) \leq P_F(\phi, \Sigma^c_\sigma \times N, \varepsilon)$$

So,

$$\begin{aligned} P_{\hat{f}}(\hat{\psi}, \hat{\Sigma}^c_\sigma) \leq P_F(\phi, \Sigma^c_\sigma \times N) &< P_F(\phi, \Sigma_\sigma \times N) \\ &= P_F(\phi) \\ &\leq P_{\hat{F}}(\hat{\phi}) = P_{\hat{f}}(\hat{\psi}) \end{aligned}$$

In this way, $\hat{\psi}$ is a hyperbolic Hölder continuous potential for $\hat{f}$. Hence, by the semicontinuity of the map $\eta \mapsto h_\eta(\hat{f}) + \int \hat{\psi} d\eta$, there exists some equilibrium state $\mu_{\hat{\psi}}$ for $(\hat{f}, \hat{\psi})$.

Let $\pi : \hat{M} \times N \to \hat{M}$ be the canonical projection. Since g is an uniform contraction on the fiber $\pi^{-1}(\hat{x})$ follows that $h_{\text{top}}(\hat{F}, \pi^{-1}(\hat{x})) = 0$ for all $\hat{x} \in \hat{M}$. Thus, we can apply the Ledrappier-Walter's formula to conclude that the measure $\mu_{\hat{\phi}}$ such that $\pi_* \mu_{\hat{\phi}} = \mu_{\hat{\psi}}$ is an equilibrium state for $(\hat{F}, \hat{\phi})$:

$$\begin{aligned} P_{\hat{F}}(\hat{\phi}) = P_{\hat{f}}(\hat{\psi}) &= h_{\pi_*\mu_{\hat{\phi}}}(\hat{f}) + \int \hat{\psi} d\mu_{\pi_*\mu_{\hat{\phi}}} \\ &= h_{\mu_{\hat{\phi}}}(\hat{F}) - \int h_{top}(\hat{F}, \hat{\pi}^{-1}(x))\, d\mu_{\pi_*\mu_{\hat{\phi}}} + \int \hat{\phi}\big|_{\text{graph}(\xi)} d\mu_{\hat{\phi}} \\ &= h_{\mu_{\hat{\phi}}}(\hat{F}) + \int \hat{\phi} d\mu_{\hat{\phi}} \end{aligned}$$

□

To complete the proof of item (i) of the Theorem C, we consider $\mu_\phi = \big(\hat{\pi}, id\big)_* \mu_{\hat{\phi}}$ the projection of $\mu_{\hat{\phi}}$ by $\big(\hat{\pi}, id\big)$. In this way, μ_ϕ is an equilibrium state for (F, ϕ) :

$$\begin{aligned} P_{\hat{F}}(\hat{\phi}) \geq P_F(\phi) &\geq h_{\mu_\phi}(F) + \int \phi d\mu_\phi \\ &= h_{\mu_{\hat{\phi}}}(\hat{F}) - \int h_{top}(\hat{F}, (\hat{\pi}, id)^{-1}(z))\, d\mu_\phi(z) + \int \hat{\phi} d\mu_{\hat{\phi}} \\ &= P_{\hat{F}}(\hat{\phi}) \end{aligned}$$

On the equality, we used the fact $h_{top}(\hat{F}, (\hat{\pi}, id)^{-1}(z)) = 0$ for all $z \in \hat{M} \times N$ because $\hat{f}$ is the natural extension of f and g is a fiber contraction.

Acknowledgements

This work is part of the first author's Ph.D. thesis at IMPA. We would like to thank P. Varandas and V. Pinheiro for very useful conversations. We also thank CNPq-Brazil and PRONEX-Dynamical Systems for their financial support.

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