Entropy, old and new

Marcelo Viana

IMPA - Rio de Janeiro

Mathwest 2011, UWA, Perth

Statistical mechanics

Consider a 1D lattice $\mathcal{L} = \mathbb{Z}$. Each node $\xi \in \mathcal{L}$ may be in one of finitely many states $\{1, \ldots, d\}$, where each state i has probability $p_i > 0$ of occurring (with $p_1 + \cdots + p_d = 1$).

Entropy (Boltzmann, Gibbs)

$$S = \sum_{i=1}^{d} -p_i \log p_i .$$

Symbolic dynamics (Bernoulli shifts)

Consider $X = \{1, \ldots, d\}$ and a probability vector $(p_1, \ldots, p_d)$. On $X^{\mathbb{Z}}$, consider the shift map

$$\sigma : (\xi_n)_n \mapsto (\xi_{n+1})_n$$

and the σ-invariant probability measure μ characterized by

$$\mu\left(\{(\xi_n)_n : \xi_a = i_a, \ldots, \xi_b = i_b\}\right) = p_{i_a} \cdots p_{i_b}.$$

Entropy (Shannon)

$$h_\mu(\sigma) = \sum_{i=1}^{d} -p_i \log p_i.$$

Smooth dynamics

Consider a continuous transformation $f : M \to M$ and an f-invariant probability μ on M. Given a finite partition $\mathcal{P}$ of M,

$$H_\mu(\mathcal{P}) = \sum_{P \in \mathcal{P}} -\mu(P) \log \mu(P)$$

$$\mathcal{P}^n = \{P_0 \cap f^{-1}(P_1) \cap \cdots \cap f^{-n+1}(P_{n-1}) : P_0, P_1, \ldots, P_{n-1} \in \mathcal{P}\}.$$

Entropy (Kolmogorov-Sinai)

$$h_\mu(f) = \sup_{\mathcal{P}} \left(\lim_n \frac{1}{n} H_\mu(\mathcal{P}^n) \right).$$

I.e., the average amount of information generated by each iteration of the map f.

What is it good for?

Entropy is the first and most important equivalence (conjugacy) invariance. For Bernoulli shifts it is even a complete invariant:

Theorem (Ornstein)

Two Bernoulli shifts are ergodically equivalent if and only if they have the same entropy.

Why is it useful?

Entropy relates well to many other geometric and dynamical invariants, which makes it a very computable object.

For instance, for smooth systems it coincides with the sum of all positive Lyapunov exponents:

Theorem (Pesin)

If $f : M \to M$ is a diffeomorphism and μ is a volume measure, then

$$h_\mu(f) = \int \Big(\sum_i \max\{0, \lambda_i\} \Big) \, d\mu.$$

Topological entropy

Consider a continuous transformation $f : M \to M$ on some metric space. A set $E \subset M$ is (n, ε)-separated if for any $x \neq y$ in E there exists $0 \leq i < n$ such that $d(f^i(x), f^i(y)) > \varepsilon$.

Topological entropy (Adler, Konheim, McAndrew; Bowen)

$$h_{top}(f) = \lim_{\varepsilon \to 0} \left(\lim_{n \to \infty} \frac{1}{n} \log s(n, \varepsilon) \right)$$

where $s(n, \varepsilon)$ is the largest cardinality of an (n, ε)-separated set.

Measure-theoretical *vs* topological entropy

Variational Principle (Dinaburg, Goodman, Goodwin)

If f is continuous and M is compact then

$$h_{top}(f) = \sup_{\mu} h_{\mu}(f),$$

where the supremum is over all f-invariant probabilities.

In many cases the supremum is attained, e.g., if f is expansive: there is $\varepsilon > 0$ such that for any $x \neq y$ in M there exists $n \in \mathbb{Z}$ satisfying $d(f^n(x), f^n(y)) > \varepsilon$.

Entropy conjecture

Let $f : M \to M$ be a diffeomorphism. For each $0 \leq k \leq \dim M$, let $\mathrm{sp}(f_k)$ be the spectral radius of the action $f_k : H_k(M) \to H_k(M)$ of f on k-dimensional homology.

Entropy conjecture (Shub)

$$h_{top}(f) \geq \sup_k \log \mathrm{sp}(f_k).$$

Results

- True for an open and dense subset of homeomorphisms, if $\dim M \neq 4$ (Palis, Pugh, Shub, Sullivan).
- $h_{top}(f) \geq \log \mathrm{sp}(f_1)$ always true for homeomorphisms; hence, the conjecture is true for $\dim M \leq 3$ (Manning).
- True for hyperbolic (Shub, Williams; Ruelle, Sullivan) and certain partially hyperbolic systems (Saghin, Xia).
- Not always true for piecewise affine homeomorphisms (Shub).

The smooth case

Theorem (Yomdin)

The entropy conjecture holds for every C^∞ diffeomorphism.

Key fact: $h_{top}(f)$ coincides with the rate of volume growth under iteration by f. That is usually false in finite differentiability.

Main result

Theorem (Liao Gang, MV, Jiagang Yang)

The entropy conjecture holds for every C^1 diffeomorphism away from homoclinic tangencies.

Best result for finitely differentiable systems. We have to explain the meaning of "away from homoclinic tangencies".

Homoclinic points

Let $p \in M$ be a fixed (or periodic) point of f which is hyperbolic: the spectrum of $Df(p)$ does not intersect the unit circle.

There are smooth submanifolds $W^s(p)$ and $W^u(p)$ that intersect transversely at p and satisfy

- $f^n(q) \to p$ as $n \to \infty$ for every $q \in W^s(p)$
- $f^{-n}(q) \to p$ as $n \to \infty$ for every $q \in W^u(p)$.

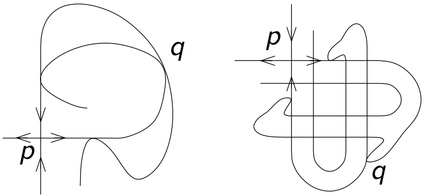
Points in $W^s(p) \cap W^u(p) \setminus \{p\}$ are called homoclinic points.

Fact

Existence of transverse homoclinic points implies $h_{top}(f) > 0$. The converse is true in low dimensions.

Homoclinic tangencies

Non-transverse homoclinic points are even more interesting, as they lie at the heart of most dynamical instability phenomena:



A map is away from tangencies if no map close to it exhibits homoclinic tangencies.

Entropy expansiveness

Theorem

Every C^1 diffeomorphism away from tangencies is entropy expansive.

Recall: f is expansive if there is $\varepsilon > 0$ such that $x \neq y$ implies that $d(f^n(x), f^n(y)) > \varepsilon$ for some n.

Roughly speaking: f is entropy expansive if the exceptions to expansiveness, assuming they exist, carry no entropy.

Semi-continuity of entropy

Corollary

The map $f \mapsto h_{top}(f)$ is upper semi-continuous on the set of diffeomorphisms away from tangencies.

The entropy conjecture follows, using Yomdin's theorem and the fact that C^∞ diffeomorphisms form a dense subset.

Semi-continuity and symbolic extensions

Corollary

If f is away from tangencies then the map $\mu \mapsto h_\mu(f)$ is upper semi-continuous on the space of f-invariant probability measures.

In particular, f admits measures of maximum entropy (variational principle).

Semi-continuity and symbolic extensions

Corollary

If f is away from tangencies then the map $\mu \mapsto h_\mu(f)$ is upper semi-continuous on the space of f-invariant probability measures.

In particular, f admits measures of maximum entropy (variational principle).

Moreover, every C^1 diffeomorphism away from tangencies admits a principal symbolic extension: roughly speaking, it may be realized as a subsystem of a shift. That is not true close to tangencies.