LYAPUNOV EXPONENTS, STRANGE ATTRACTORS

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Lyapunov Exponents

The *Lyapunov exponents* of a sequence $\{A^n, n \geq 1\}$ of square matrices of dimension $d \geq 1$, are the values of

$$\lambda(v) = \limsup_{n\to\infty} \frac{1}{n} \log \|A^n \cdot v\| \tag{1}$$

over all non-zero vectors $v \in \mathbb{R}^d$. For completeness, set $\lambda(0) = -\infty$. It is easy to see that $\lambda(cv) = \lambda(v)$ and $\lambda(v + v') \leq \max\{\lambda(v), \lambda(v')\}$ for any non-zero scalar c and any vectors v, v'. It follows that, given any constant a, the set of vectors satisfying $\lambda(v) \leq a$ is a vector subspace. Consequently, there are at most d Lyapunov exponents, henceforth denoted by $\lambda_1 < \cdots < \lambda_{k-1} < \lambda_k$, and there exists a filtration $F^1 < \cdots < F^{k-1} < F^k = \mathbb{R}^d$ into vector subspaces, such that

$$\lambda(v) = \lambda_i \text{ for all } v \in F_i \setminus F_{i-1}$$

and every $i = 1, \ldots, k$ (write $F_0 = \{0\}$). In particular, the largest exponent

$$\lambda_k = \limsup_{n\to\infty} \frac{1}{n} \log \|A^n\|. \tag{2}$$

One calls $\dim F_i - \dim F_{i-1}$ the *multiplicity* of each Lyapunov exponent λ_i.

There are corresponding notions for continuous families of matrices A^t, $t \in (0, \infty)$, taking the limit as t goes to ∞ in the relations (1) and (2). The theories for the two types of families, discrete and continuous, are analogous and so at each point of what follows we refer to either one or the other.

Lyapunov stability

Consider the linear differential equation

$$\dot{v}(t) = B(t) \cdot v(t) \tag{3}$$

where $B(t)$ is a bounded function with values in the space of $d \times d$ matrices, defined for all $t \in \mathbb{R}$. The theory of differential equations ensures that there exists a *fundamental matrix* A^t, $t \in \mathbb{R}$ such that

$$v(t) = A^t \cdot v_0$$

is the unique solution of (3) with initial condition $v(0) = v_0$.

If the Lyapunov exponents of the family A^t, $t > 0$ are all negative then the trivial solution $v(t) \equiv 0$ is asymptotically stable, and even exponentially stable. The stability theorem of A. M. Lyapunov asserts that, under an additional regularity condition, stability is still valid for non-linear perturbations

$$w(t) = B(t) \cdot w + F(t, w) \tag{4}$$

with $\|F(t, w)\| \leq \text{const}\, \|w\|^{1+c}$, $c > 0$. That is, the trivial solution $w(t) \equiv 0$ is still exponentially asymptotically stable.

The regularity condition means, essentially, that the limit in (1) does exist, even if one replaces vectors v by elements $v_1 \wedge \cdots \wedge v_l$ of any lth exterior power of $\mathbb{R}^d$, $1 \leq l \leq d$. By definition, the norm of an l-vector $v_1 \wedge \cdots \wedge v_l$ is the volume of the parallelepiped determined by the vectors $v_1, \ldots, v_k$. This condition is usually tricky to check in specific situations. However, the multiplicative ergodic theorem of V. I. Oseledets asserts that, for very general matrix-valued stationary random processes, regularity is an almost sure property. This result sets the foundation for the modern theory of Lyapunov exponents. We are going to discuss the precise statement of the theorem in the slightly broader setting of linear cocycles, or vector bundle morphisms.

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Linear cocycles

Let μ be a probability measure on some space M and $f : M \to M$ be a measurable transformation that preserves μ. Let $\pi : \mathcal{E} \to M$ be a finite-dimensional vector bundle, endowed with a Riemannian metric $\|\cdot\|_x$ on each fiber $\mathcal{E}_x = \pi^{-1}(x)$. Let $\mathcal{A} : \mathcal{E} \to \mathcal{E}$ be a *linear cocycle* over f. What we mean by this is that

$$\pi \circ \mathcal{A} = f \circ \pi$$

and the action $A(x) : \mathcal{E}_x \to \mathcal{E}_{f(x)}$ of $\mathcal{A}$ on each fiber is a linear isomorphism. Notice that the action of the nth iterate $\mathcal{A}^n$ is given by

$$A^n(x) = A(f^{n-1}(x)) \cdots A(f(x)) \cdot A(x)$$

for every $n \geq 1$.

Assume the function $\log^+ \|A(x)\|_x$ is μ-integrable:

$$\log^+ \|A(x)\|_x \in L^1(\mu) \tag{5}$$

(we write $\log^+ \phi = \log \max\{\phi, 1\}$, for any $\phi > 0$). It is clear that the sequence of functions $a_n(x) = \log \|A^n(x)\|_x$ satisfies

$$a_{m+n}(x) \leq a_m(x) + a_n(f^m(x))$$

for every m, n, and x. It follows from J. Kingman's sub-additive ergodic theorem that the limit

$$\lim_{n\to\infty} \frac{1}{n} a_n(x)$$

exists for μ-almost all x. In view of (2), this means that the largest Lyapunov exponent $\lambda_k(x)$ of the sequence $A^n(x)$, $n \geq 1$ is a limit, and not just a lim sup, at almost every point.

Multiplicative ergodic theorem

The Oseledets theorem states that the same holds for all Lyapunov exponents. Namely, for μ-almost every $x \in M$ there exists $k = k(x) \in \{1, \ldots, d\}$, a filtration

$$F_x^1 < \cdots < F_x^{k-1} < F_x^k = \mathcal{E}_x,$$

and numbers $\lambda_1(x) < \cdots < \lambda_k(x)$ such that

$$\lim_{n\to\infty} \frac{1}{n} \log \|A^n(x)\|_x = \lambda_i(x) \tag{6}$$

for all $v \in F_x^i \setminus F_x^{i-1}$ and $i \in \{1, \ldots, k\}$.

The Lyapunov exponents $\lambda_i(x)$, and their number $k(x)$, are measurable functions of x and they are constant on orbits of the transformation f. In particular, if the measure μ is ergodic then k and the λ_i are constant on a full μ-measure set of points. The subspaces F_x^i also depend measurably on the point x and are invariant under the linear cocycle:

$$A(x) \cdot F_x^i = F_{f(x)}^i.$$

It is in the nature of things that, usually, these objects are *not* defined everywhere and they depend discontinuously on the base point x.

When the transformation f is invertible one obtains a stronger conclusion, by applying the previous kind of result also to the inverse of the cocycle. Namely, assuming that $\log^+ \|A^{-1}\|$ is also in $L^1(\mu)$, one gets that there exists a decomposition

$$\mathcal{E}_x = E_x^1 \oplus \cdots \oplus E_x^k,$$

defined at almost every point and such that $A(x) \cdot E_x^i = E_{f(x)}^i$ and

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \|A^n(x)\|_x = \lambda_i(x) \tag{7}$$

for all $v \in E_x^i$ different from zero and all $i \in \{1, \ldots, k\}$. These *Oseledets subspaces* E_x^i are related to the subspaces F_x^i through

$$F_x^j = \oplus_{i=1}^j E_x^i.$$

Hence, $\dim E_x^i = \dim F_x^i - \dim F_x^{i-1}$ is the multiplicity of the Lyapunov exponent $\lambda_i(x)$.

The angles between any two Oseledets subspaces decay sub-exponentially along orbits of f:

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \operatorname{angle}(E_{f^n(x)}^i, E_{f^n(x)}^j) = 0$$

for every $i \neq j$ and almost every point. These facts imply the regularity condition mentioned previously and, in particular,

(8)
$$\lim_{n\to\pm\infty} \frac{1}{n} \log|\det A^n(x)| = \\ = \sum_{i=1}^{k} \lambda_i(x) \dim E_x^i$$

Consequently, for cocycles with values in $\mathrm{SL}(d,\mathbb{R})$ the sum of all Lyapunov exponents, counted with multiplicity, is identically zero.

As we are dealing with almost certain properties, we may generally restrict the vector bundle to some full measure subset over which it is trivial. Then each fiber $\mathcal{E}_x$ is identified with the space $\mathbb{R}^d$, and we may think of $A(x)$ as a $d \times d$ matrix. Then $A_n(x) = A(f^n(x))$ is a stationary random process relative to (f, μ). Thus, in this context it is no serious restriction to view a linear cocycle as a stationary random process with values in the linear group $\mathrm{GL}(d,\mathbb{R})$ of invertible $d \times d$ matrices.

Furthermore, given any such random process A_n, $n \geq 0$, one may consider its normalization $B_n = A_n/|\det A_n|$. The Lyapunov exponents of the two random processes A_n, $n \geq 0$ and B_n, $n \geq 0$ differ by the time average

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log|\det A_j(x)|$$

of the determinant. The Birkhoff ergodic theorem ensures that the time average is well-defined almost everywhere, as long as the function $\log|\det A|$ is in $L^1(\mu)$; that is the case, for instance, if both $\log^+\|A^{\pm 1}\|$ are integrable. This relates the general case to random processes with values in the special linear group $\mathrm{SL}(d,\mathbb{R})$ of $d \times d$ matrices with determinant ± 1.

The Oseledets theorem was extended by D. Ruelle to certain linear cocycles in infinite dimension. He assumes that the $A(x)$ are compact operators on a Hilbert space H and $\log^+\|A\|$ is in $L^1(\mu)$. The conclusion is the same as in finite dimension, except that the filtration

$$\cdots < F_x^i < \cdots < F_x^1 = H$$

may involve infinitely many subspaces, and the Lyapunov exponents may be $-\infty$. There is also a version for cocycles over invertible transformations, where one assumes each $A(x)$ to be invertible and the sum of a unitary operator with a compact operator, such that both $\log\|A^{\pm}\|$ are integrable. The conclusion is that there exists an Oseledets decomposition $H = E_x^1 \oplus \cdots \oplus E_x^i \oplus \cdots$ at almost every point, with finitely or countably many factors.

Random matrices

Relation (8) implies that, for $\mathrm{SL}(d,\mathbb{R})$ cocycles, if there is only one Lyapunov exponent (with full multiplicity) then it must be zero. When this happens the theory contains no information on the behavior of the iterates $A^n(x) \cdot v$, apart from the fact that there is no exponential growth nor decay of their norms. Thus, the question naturally arises under which conditions is there more than one Lyapunov exponent or, equivalently, under which conditions is the largest Lyapunov exponent strictly positive.

This problem was first addressed by H. Furstenberg for products of independent random variables, corresponding to the following class of linear cocycles. Let ν be a probability measure on the group $G = \mathrm{GL}(d,\mathbb{R})$. Consider $M = G^{\mathbb{N}}$ and $\mu = \nu^{\mathbb{N}}$ (or $M = G^{\mathbb{Z}}$ and $\mu = \nu^{\mathbb{Z}}$), and let $f : M \to M$ be the shift map

$$f\big((\alpha_j)_j\big) = (\alpha_{j+1})_j.$$

It is clear that μ is invariant and also ergodic for the transformation f. Consider the cocycle $\mathcal{A} : \mathcal{E} \to \mathcal{E}$ defined by $\mathcal{E} = M \times \mathbb{R}^d$ and

$$A\big((\alpha_j)_j\big) \cdot v = \alpha_0 \cdot v.$$

Clearly,

$$A^n\left((\alpha_j)_j\right)\cdot v = \alpha_{n-1}\cdots\alpha_1\alpha_0\cdot v.$$

Corresponding to the hypothesis of the multiplicative ergodic theorem, assume that $\log^+\|\alpha\|$ (and $\log^+\|\alpha^{-1}\|$) are ν-integrable functions of the matrix α.

Furstenberg's theorem states that if the closed group $G(\nu)$ generated by the support of ν is non-compact and strongly irreducible in $\mathbb{R}^d$ then the largest Lyapunov exponent of the cocycle $\mathcal{A}$ is strictly positive. *Strong irreducibility* means that there exists no finite union of subspaces of $\mathbb{R}^d$ that is invariant under all elements of the group. Improvements, extensions, and alternative proofs have been obtained by several authors since then.

Especially, Y. Guivarc'h, A. Raugi provided conditions under which there are exactly d distinct Lyapunov exponents or, in other words, the multiplicity of every Lyapunov exponent is equal to 1. A matrix semigroup has the *contraction property* if there exists a sequence of elements h_n and a probability measure on the projective space of $\mathbb{R}^d$ that gives zero weight to any projective subspace, such that the images $(h_n)_*m$ of m under the h_n converge to a Dirac mass in the projective space. They proved that if the closed semigroup $H(\nu)$ generated by the support of the probability ν is strongly irreducible and has the contraction property then the largest Lyapunov exponent has multiplicity 1. Applying this to the exterior powers of the cocycle, one obtains sufficient conditions for simplicity of the other Lyapunov exponents as well.

This statement has been improved by I. Ya. Gol'dsheid, G. A. Margulis, who formulated the hypotheses in terms of the algebraic closure $\tilde{G}(\nu)$ of the semigroup $H(\nu)$. They assumed that $\tilde{G}(\nu)$ has the contraction property and the connected component of the identity inside $\tilde{G}(\nu)$ is *irreducible* in $\mathbb{R}^d$, meaning that its elements do not have any common invariant subspace. Then the largest Lyapunov exponent is simple.

Schrödinger cocycles

The 1-dimensional discrete Schrödinger equation is the second order difference equation

$$-(u_{n+1}+u_{n-1})+V_n u_n = E u_n \tag{9}$$

derived from the stationary Schrödinger equation in dimension 1 by space discretization. Here the *energy* E is a constant and $V_n = V(f^n(\theta))$, where the *potential* $V(\cdot)$ is a bounded scalar function and $f: M\to M$ is a transformation preserving some probability measure μ on M. In what follows we take μ to be ergodic. The equation (9) may be rewritten as a first order relation,

$$\begin{pmatrix} u_{n+1} \\ v_{n+1}\end{pmatrix} = \begin{pmatrix} V_n - E & -1 \\ 1 & 0\end{pmatrix}\begin{pmatrix} u_n \\ v_n\end{pmatrix}.$$

Hence, it may also be interpreted as a linear cocycle $\mathcal{A}$ over f, where the vector bundle is $\mathcal{E} = M\times\mathbb{R}^2$ and

$$A(\theta) = \begin{pmatrix} V(\theta) - E & -1 \\ 1 & 0\end{pmatrix} \tag{10}$$

takes values in $\mathrm{SL}(\mathbb{R},2)$. By ergodicity, the Lyapunov exponents are essentially independent of the base point θ. Let $\lambda(E)$ denote the largest exponent: by the relation (8), the other one is $-\lambda(E)$.

The Lyapunov exponent $\lambda(E)$ is related to the spectral theory of the linear operators $\mathcal{L}_\theta$

$$(\mathcal{L}_\theta u)_n = -(u_{n+1}+u_{n-1}) + V_n u_n.$$

on the space $\ell^2(\mathbb{Z})$ of complex square-integrable sequences u_n, $n\in\mathbb{Z}$. These are bounded Hermitian operators and so the spectra are compact subsets of $\mathbb{R}$. Using the assumption that μ is ergodic one can prove that the spectrum $\mathrm{spec}(\mathcal{L}_\theta)$ is constant almost everywhere. If the transformation f is minimal, the spectrum is even independent of the point θ.

Moreover, for all energies,

$$\lambda(E) \geq \text{const dist}(E, \text{spec}(\mathcal{L}_\theta)).$$

In particular, $\lambda(E)$ is always positive on the complement of the spectrum.

A fundamental problem (Anderson *localization*) is to decide when the spectrum is pure-point. This is reasonably well understood for a few classes of base dynamics only. One of them are very chaotic systems such as Bernoulli and Markov processes (*random potentials*) or uniformly hyperbolic maps and flows. Another one are the irrational rotations on the d-dimensional torus (*quasi-periodic potentials*). In the latter case, the results are more complete when there is only one frequency ($d = 1$). It was shown by K. Ishii and by L. Pastur that if $\lambda(E)$ is positive for almost all values of E in some Borel set then the absolutely continuous part of the spectrum is essentially disjoint from that set. The converse is also true, and due to S. Kotani. Thus, checking that $\lambda(E)$ is positive is an important step towards proving localization.

A very general criterion for positivity of the Lyapunov exponent was obtained by Kotani. Namely, he proved that if the potential is not deterministic then $\lambda(E)$ is positive for almost all E. In particular, for non-deterministic potentials the absolutely continuous spectrum is empty, almost surely. In simple terms, the hypothesis means that from the values of the potential for negative n one can not determine the values for positive n. More formally, one calls the potential *deterministic* if every V_n, $n \geq 0$ is almost everywhere a measurable function of $\{V_n : n \leq 0\}$. For instance, quasi-periodic potentials are deterministic, whereas Bernoulli potentials are not.

Subharmonicity method

Let $\mathbb{D}^m$ be the set of complex vectors $(z_1, \ldots, x_m) \in \mathbb{C}^m$ such that $|z_j| \leq 1$ for all j and let $\mathbb{T}^m$ be the subset defined by $|z_j| = 1$ for all j. Let $f : \mathbb{T}^m \to \mathbb{T}^m$ and $A : \mathbb{T}^m \to \text{SL}(d, \mathbb{R})$ be continuous maps that admit holomorphic extensions to the interior of $\mathbb{D}^m$ with $f(0) = 0$. Assume that f preserves the natural (Haar) measure μ on $\mathbb{T}^m$. Let

$$\lambda(A, \mu) = \int_{\mathbb{T}^m} \lambda(z)\, d\mu,$$

where $\lambda(z)$ denotes the largest Lyapunov exponent for the cocycle defined by A over f. It also follows from the subadditive ergodic theorem that

$$\lambda(A, \mu) = \lim \frac{1}{n} \int_{\mathbb{T}^m} \log \|A^n(z)\|\, d\mu.$$

M. Herman observed that, since the function $\log \|A^n(z)\|$ is plurisubharmonic on $\mathbb{D}^m$, one may use the maximum principle to conclude that

$$\frac{1}{n} \int_{\mathbb{T}^m} \log \|A^n(z)\|\, d\mu \geq \frac{1}{n} \log \|A^n(0)\|.$$

Then, taking the limit when $n \to \infty$ one obtains that

$$\lambda(A, \mu) \geq \rho(A) \tag{11}$$

where $\rho(A)$ denotes the spectral radius of the matrix $A(0)$. Starting from this observation, he developed a very effective method for bounding Lyapunov exponents from below, that received several applications and extensions, in particular, to the theory of Schrödinger cocycles with quasi-periodic potentials.

The best known application is the following bound for integrated Lyapunov exponents of 2-dimensional cocycles. Let $f : M \to M$ be a continuous transformation on a compact metric space, preserving some probability measure μ, and $A : M \to \text{SL}(2, \mathbb{R})$ be a continuous map. For each fixed θ, let AR_θ be the cocycle obtained by multiplying $A(x)$, at every point x, by the rotation of angle θ. Herman proved that

$$\frac{1}{2\pi} \int \lambda(AR_\theta, \mu)\, d\theta \geq \int_M N(x)\, d\mu,$$

(A. Avila, J. Bochi later showed that the equality holds) where

$$N(x) = \log \frac{\|A(x)\| + \|A(x)^{-1}\|}{2}.$$

Apart from the exceptional case when A acts by rotation at every point in the support of μ, the right hand side of the inequality is positive, and so the Lyapunov exponent of the cocycle AR_θ is positive for many values of θ.

Non-uniform hyperbolicity

The prototypical example of a linear cocycle is the derivative of a smooth transformation on a manifold. More precisely, let M be a finite-dimensional manifold and $f : M \to M$ be a diffeomorphism, that is, a bijective smooth map whose derivative $Df(x)$ depends continuously on x and is an isomorphism at every point. Let $\mathcal{E} = TM$ be the tangent bundle to the manifold and $\mathcal{A} = Df$ be the derivative. If M is compact or, more generally, if the norms of both Df and its inverse are bounded, then the hypothesis in Oseledets theorem is automatically satisfied for any f-invariant probability μ. Lyapunov exponents yield deep geometric information on the dynamics of the diffeomorphism, especially when they do not vanish. For most results that we mention in the sequel, one needs the derivative Df to be Hölder continuous:

$$\|Df(x) - Df(y)\| \le \text{const}\, d(x,y)^c.$$

Let E^s_x be the sum of the Oseledets subspaces corresponding to negative Lyapunov exponents. Pesin's stable manifold theorem states that there exists a family of embedded disks $W^s_{loc}(x)$ tangent to E^s_x at almost every point and such that the orbit of every $y \in W^s_{loc}(x)$ is exponentially asymptotic to the orbit of x. This lamination $\{W^s(x)\}$ is invariant, in the sense that

$$f(W^s(x)) \subset W^s(f(x))$$

and has an "absolute continuity" property. There are analogous results for the sum E^u_x of the Oseledets subspaces corresponding to positive Lyapunov exponents

The entropy of a partition $\mathcal{P}$ of M is defined by

$$h_\mu(f,\mathcal{P}) = \lim_{n\to\infty} \frac{1}{n} H_\mu(\mathcal{P}^n),$$

where $\mathcal{P}^n$ is the partition into sets of the form $P = P_0 \cap f^{-1}(P_1) \cap \cdots \cap f^{-n}(P_n)$ with $P_j \in \mathcal{P}$ and

$$H_\mu(\mathcal{P}^n) = \sum_{P\in\mathcal{P}^n} -\mu(P)\log\mu(P).$$

The *Kolmogorov-Sinai entropy* $h_\mu(f)$ of the system is the supremum of $h_\mu(f,\mathcal{P})$ over all partitions $\mathcal{P}$ with finite entropy. The Ruelle-Margulis inequality says that $h_\mu(f)$ is bounded above by the average sum of the positive Lyapunov exponents. A major result of the theory, Pesin's entropy formula, asserts that if the invariant measure μ is smooth (e.g. a volume element) then the two invariants coincide:

$$h_\mu(f) = \int \Big(\sum_{j=1}^k \lambda_j^+\Big) d\mu.$$

A complete characterization of the invariant measures for which the entropy formula is true was given by F. Ledrappier and L. S. Young.

The invariant measure μ is called *hyperbolic* if all Lyapunov exponents are non-zero at almost every point. Hyperbolic measures are *exact dimensional*: the pointwise dimension

$$d(x) = \lim_{r\to 0} \frac{\log \mu(B_r(x))}{\log r}$$

exists at almost every point, where $B_r(x)$ is the neighborhood of radius r around x. This fact was proved by L. Barreira, Ya. Pesin, and J. Schmeling. Note that it means that the measure $\mu(B_r(x))$ of neighborhoods scales as $r^{d(x)}$ when the radius r is small.

Another remarkable feature of hyperbolic measures, proved by A. Katok, is that periodic motions are dense in their supports. More than that, assuming the measure is non-atomic, there exist Smale horseshoes H_n with topological entropy arbitrarily close to the entropy $h_\mu(f)$ of the system. In this context, the *topological entropy* $h(f, H_n)$ may be defined as the exponential rate of growth

$$\lim_{k\to\infty} \frac{1}{k} \log \#\{x \in H_n : f^k(x) = x\}.$$

of the number of periodic points on H_n.

Generic systems

Given any area preserving diffeomorphism on any surface M, one may find another whose first derivative is arbitrarily close to the initial one and which has Lyapunov exponents identically zero at almost every point, or else is globally uniformly hyperbolic (Anosov). This surprising fact was discovered by R. Mañé, and a complete proof was given by J. Bochi. Uniform hyperbolicity means that the tangent bundle admits a Df-invariant splitting

$$TM = E^s \oplus E^u$$

such that the line bundle E^s is uniformly contracted and E^u is uniformly expanded by the derivative. It is well-known that Anosov diffeomorphisms can only occur if the surface is the torus $\mathbb{T}^2$.

In fact, the theorem of Mañé-Bochi is stronger: for a residual subset (a countable intersection of open dense sets) of all once-differentiable area preserving diffeomorphisms on any surface, either the Lyapunov exponents vanish almost everywhere or the diffeomorphism is Anosov. This shows that zero Lyapunov exponents are actually quite common for surface diffeomorphisms that are only once-differentiable. Moreover, this theorem has been extended to diffeomorphisms on manifolds with arbitrary dimension, in a suitable formulation, by J. Bochi and M. Viana.

However, this phenomenon should be specific to systems with low differentiability. Indeed, already for Hölder continuous linear cocycles over chaotic transformations it is known that vanishing Lyapunov exponents can only occur with infinite codimension. That is, unless the cocycle satisfies an infinite number of independent constraints, there exists some positive exponent. By chaotic we mean here that the invariant probability μ of the base transformation is assumed to be hyperbolic and have local product structure: it is locally equivalent to a product of two measures, respectively, along stable and unstable sets.

Under additional assumptions one can even prove that all Lyapunov exponents have multiplicity 1 outside an infinite codimension subset. This follows from extensions of the Guivarc'h-Raugi criterion for certain linear cocycles over chaotic transformations, obtained by A. Avila, C. Bonatti, and M. Viana.

Strange attractors

This expression was coined by D. Ruelle and F. Takens in their celebrated study on the nature of fluid turbulence. E. Hopf and also L. D. Landau and E. M. Lifshitz had suggested that turbulent motion arises from the existence in the phase space of invariant tori carrying quasi-periodic flows with large number of frequencies. Ruelle and Takens observed that dissipative systems such as viscous fluids do not generally have such quasi-periodic tori, and concluded that turbulence must be credited to a different mechanism: the presence of some "strange" attractor.

While they did not propose a precise definition, two main features were mentioned: *Complex geometry:* a strange attractor is not reduced to an equilibrium point or a periodic solution of the system and, generally, should have a fractal structure. *Chaotic dynamics:* solutions accumulating on the attractor should be sensitive to their initial states. As more examples were found, it became apparent that these two features do not always come together. This led to two types of definition in the literature, depending on whether one emphasizes the geometry or the dynamics. We adopt the second point of view, and propose to define *strange attractor* as one carrying an invariant ergodic physical measure which has some positive Lyapunov exponent. The notion of physical measure will be defined near the end. The condition on the Lyapunov exponent ensures that the dynamics near the attractor is (exponentially) sensitive to the initial states.

Lorenz-like attractors

The uniformly hyperbolic attractors introduced by S. Smale provided an interesting class of examples of strange attractors, both chaotic and fractal. Perhaps more striking, given that they originated from a concrete problem in fluid dynamics, were the strange attractors introduced by E. N. Lorenz. The Lorenz system of differential equations

$$\begin{aligned} \dot{x} &= -\sigma x + \sigma y & \sigma &= 10 \\ \dot{y} &= rx - y - xz & r &= 28 \\ \dot{z} &= xy - bz & b &= 8/3 \end{aligned} \tag{12}$$

was derived from Lord Rayleigh's model for thermal convection, by Fourier expansion of the stream function and temperature and truncation of all but three modes. Lorenz observed that its solutions depend sensitively on their initial states. Consequently, predictions based on the numerical integration of the equations may turn out to be very inaccurate, given that the initial data obtained from experimental measurements is never completely precise. This remarkable observation brought the issue of predictability in deterministic systems to a whole new light and motivated intense investigation of this and many other chaotic systems.

The dynamical behavior of the equations (12) was first interpreted through certain geometric models where the presence of strange attractors, both chaotic and fractal, could be proved rigorously. It was much harder to prove that the original equations (12) themselves have such an attractor. This was achieved just a few years ago, by W. Tucker, by means of a computer assisted rigorous argument. At about the same time, a mathematical theory of Lorenz-like attractors in 3-dimensional space was developed by C. Morales, M. J. Pacifico, and E. Pujals. In particular, this theory shows that uniformly hyperbolic attractors and Lorenz-like attractors are the only ones which are robust under all small modifications of the vector field.

Hénon-like attractors

Starting from the work of Lorenz, many models of strange attractors have been found and described to some extent, often related to concrete problems. From a mathematical point of view, it is usually hard to give even a rough description of the dynamics in the chaotic regime. However, this was especially successful for the family of strange attractors introduced by M. Hénon. He considered a very simple non-linear system, particularly suited for numerical experimentation: the transformation

$$f(x, y) = (1 - ax^2 + by, x) \tag{13}$$

where a and b are constant parameters. In a breakthrough, M. Benedicks and L. Carleson were able to prove that, for a set of parameter values with positive probability, this transformation has some non-hyperbolic attractor such that the orbits accumulating on it are sensitive to the starting point. The system (13) is also a model for many other situations, including the phenomenon of creation of homoclinic motions as parameters unfold, and the conclusions of Benedicks and Carleson have been extended to such situations, starting from the work of L. Mora and M. Viana.

Moreover, a detailed theory of Hénon-like attractors has been developed by M. Benedicks, M. Viana, D. Wang, L. S. Young, and other authors. It follows from this theory that these attractors carry an invariant ergodic probability measure μ which describes the statistical behavior of almost all trajectories $f^j(x)$, $j \geq 1$ that accumulate the attractor:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \varphi(f^j(x)) = \int \varphi \, d\mu$$

for any continuous function φ. This property implies that, despite the fact that it is supported on a zero volume set, the measure μ is, in some sense, physically observable. For this reason one calls it a *physical measure*. In other words, time averages along typical orbits in the domain of attraction coincide with the space averages determined by the probability μ. Another property with physical relevance is that μ is the zero-noise limit of the stationary measures associated to the Markov chains obtained by adding random noise to f. One says that the system (f, μ) is *stochastically stable*.

See also

Chaos and attractors. Ergodic theory. Fractal dimensions in dynamics. Generic properties of dynamical systems. Homoclinic phenomena. Hyperbolic dynamical systems. Random dynamical systems.

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