Dynamics and Geometry of Flat Surfaces

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Outline

Abelian differentials

Abelian differential = holomorphic 1-form $\omega_z = \varphi(z)dz$ on a (compact) Riemann surface.

Adapted local coordinates: $\zeta = \int_p^z \varphi(w)dw$ then $\omega_\zeta = d\zeta$

near a zero with multiplicity m:

$$\zeta = \left(\int_p^z \varphi(w)dw\right)^{\frac{1}{m+1}} \text{ then } \omega_\zeta = d\left(\zeta^{m+1}\right)$$

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Translation structures

Adapted coordinates form a translation atlas: coordinate changes near any regular point have the form

$$\zeta' = \zeta + \text{const} \quad (\text{because } d\zeta = d\zeta')$$

This translation atlas defines

- a flat metric with a finite number of conical singularities;
- a parallel unit vector field (the "upward" direction) on the complement of the singularities.

Conversely, the flat metric and the upward vector field characterize the translation structure completely.

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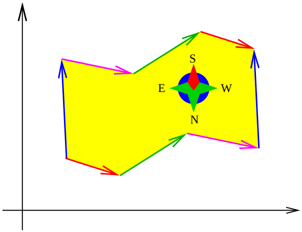
Geometric representation

Consider any planar polygon with even number of sides, organized in pairs of parallel sides with the same length.



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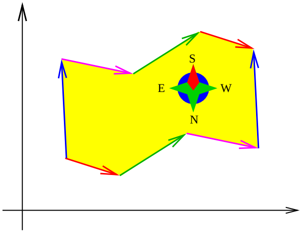

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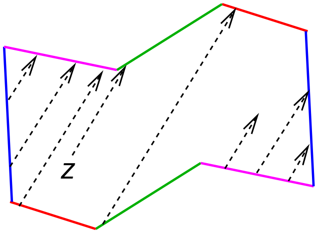
Geodesic flows

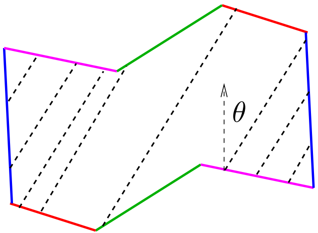
The trajectories of the Abelian differential are the geodesics on the corresponding translation surface.



When are geodesics closed ? When are they dense ? How do geodesics distribute themselves on the surface ?

Measured foliations

Geodesics in a given direction define a foliation of the surface which is a special case of a measured foliation: it is tangent to the kernel of a certain real closed 1-form, namely $\Re(e^{i\theta}\omega)$.



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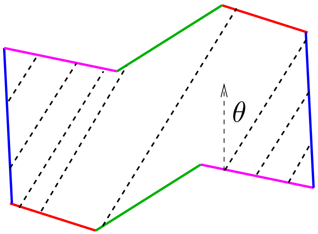

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Moduli spaces

$\mathcal{M}_g$ = moduli space of Riemann surfaces of genus g

$\mathcal{A}_g$ = moduli space of Abelian differentials on Riemman surfaces of genus g

$$\dim_{\mathbb{C}} \mathcal{M}_g = 3g - 3 \qquad \dim_{\mathbb{C}} \mathcal{A}_g = 4g - 3 \qquad (\text{for } g \geq 2)$$

$\mathcal{A}_g$ is an orbifold and a fiber bundle over $\mathcal{M}_g$: the fiber is the first cohomology of the surface.

Strata of $\mathcal{A}_g$

Consider any $m_1, \ldots, m_\kappa \geq 1$ with $\sum_{i=1}^{\kappa} m_i = 2g - 2$.

$\mathcal{A}_g(m_1, \ldots, m_\kappa)$ = subset of Abelian differentials having κ zeroes, with multiplicities $m_1, \ldots, m_\kappa$.

$$\dim_{\mathbb{C}} \mathcal{A}_g(m_1, \ldots, m_\kappa) = 2g + \kappa - 1$$

Each stratum carries a canonical volume measure. These volumes are all finite (Masur, Veech) and they have been computed by Eskin, Okounkov, Pandharipande.

Each stratum may have up to 3 connected components. Kontsevich, Zorich catalogued all connected components.

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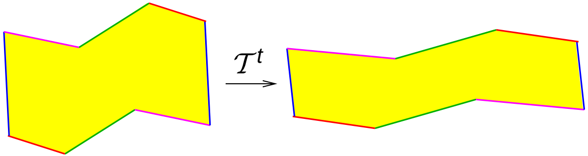
Teichmüller flow

The Teichmüller flow is the natural action $\mathcal{T}^t$ on the fiber bundle $\mathcal{A}_g$ by the diagonal subgroup of $\mathrm{SL}(2,\mathbb{R})$:

$$\mathcal{T}^t(\omega)_z = \left[e^t \Re\omega_z\right] + i\left[e^{-t}\Im\omega_z\right]$$

Geometrically:



This flow leaves invariant the strata and their canonical volumes, and also preserves the area of the surface S.

General Principle

Properties of the Teichmüller flow in the space of all translation surfaces reflect upon the properties of the geodesic flows on typical individual surfaces.

The orbits of the Teichmüller flow “know” the properties of the translation surfaces contained in them.

Ergodicity

Masur, Veech: The (normalized) Teichmüller flow is ergodic on every connected component of every stratum. That is, almost all orbits of the flow are uniformly distributed (according to the volume measure) in the connected component.

Consequence: The geodesic flow of almost every Abelian differential in almost every direction is uniquely ergodic: all geodesics are uniformly distributed (according to the area measure) on the surface.

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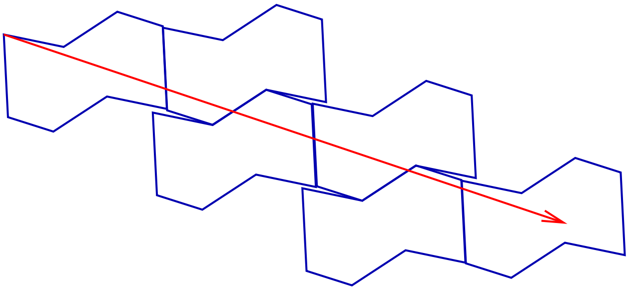
Asymptotics of geodesics

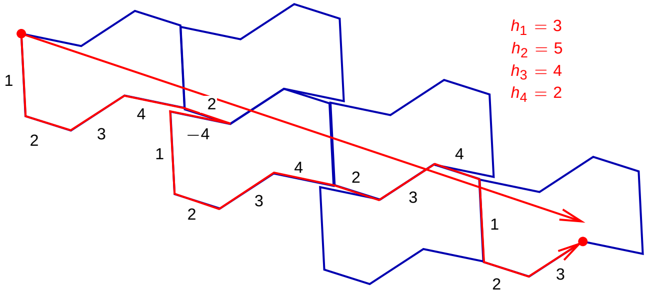
Consider any long geodesic segment γ in a given direction, starting from any point on the surface.



Form a closed curve by concatenating γ with a segment with bounded length.

Representation in homology

There is an associated integer vector $h(\gamma) = (h_1(\gamma), \ldots, h_d(\gamma))$,



where $h_i(\gamma) =$ "number of turns" of γ in the direction of the i'th side of the polygon (counted with sign).

Asymptotic cycles

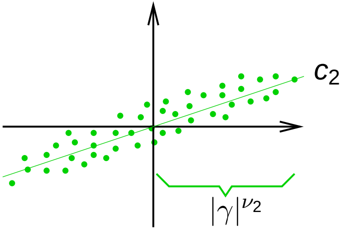
Unique ergodicity implies that

$$\frac{1}{|\gamma|} h(\gamma)$$

converges uniformly to some $c_1 \in H_1(S, \mathbb{R})$ when the length $|\gamma|$ goes to infinity, and the asymptotic cycle c_1 does not depend on the initial point, only the surface and the direction.

Zorich phenomenon

Numerical experiments suggested that the deviation of $h(\gamma)$ from the direction of c_1 distributes itself along a preferred direction c_2, with amplitude $|\gamma|^{\nu_2}$ for some $\nu_2 < 1$:



Zorich phenomenon

Similarly in higher order: the component of $h(\gamma)$ orthogonal to $\mathbb{R}c_1 \oplus \mathbb{R}c_2$ has a favorite direction c_3, and amplitude $|\gamma|^{\nu_3}$ for some $\nu_3 < \nu_2$, and so on up to order $g =$ genus.

Informally: There are $c_1, c_2, \ldots, c_g$ in $\mathbb{R}^d$ and numbers $1 > \nu_2 > \cdots > \nu_g > 0$ such that

$$h(\gamma) = c_1|\gamma| + c_2|\gamma|^{\nu_2} + c_3|\gamma|^{\nu_3} + \cdots + c_g|\gamma|^{\nu_g} + R(|\gamma|)$$

where $R(\cdot)$ is bounded.

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Asymptotic flag theorem

Conjecture (Zorich, Kontsevich)

There are $1 > \nu_2 > \cdots > \nu_g > 0$ and subspaces $L_1 \subset L_2 \subset \cdots \subset L_g$ of $H_1(S, \mathbb{R})$ with $\dim L_i = i$ for every i, such that

- *the deviation of $h(\gamma)$ from L_i has amplitude $|\gamma|^{\nu_{i+1}}$ for all $i < g$*
- *the deviation of $h(\gamma)$ from L_g is bounded $\quad$ (g = genus).*

Theorem (Avila, Viana)

The Zorich-Kontsevich conjecture is true (for almost all translation surfaces).

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The Zorich-Kontsevich conjecture is true (for almost all translation surfaces).

Comments on the proof

Kontsevich, Zorich translated the conjecture to a statement on the Lyapunov exponents of the Teichmüller flow.

It is well known that the Lyapunov exponents are of the form

$$2 > 1+\nu_2 \geq \cdots \geq 1+\nu_g \geq 1 = \cdots = 1 \geq 1-\nu_g \geq \cdots \geq 1-\nu_2 \geq 0$$
$$\geq -1+\nu_g \geq \cdots \geq -1+\nu_g \geq -1 = \cdots = -1 \geq -1-\nu_g \geq \cdots \geq -1-\nu_2 > -2.$$

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