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Stochastic Dynamics of Deterministic Systems

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Contents

1. Physical Measures

Average sojourn times, that is, the fraction of time spent by orbits in different parts of the phase space, are the most basic statistical data describing a dynamical system's behaviour. A probability μ in the phase space is called a physical measure if it provides this information

$$\mu(U) = \text{ fraction of time spent by the orbit of } z \text{ inside } U, \tag{1.1}$$

for typical domains U (this will be made precise in a while) *for a sizable set of orbits*. More precisely, this should be true with *positive probability* when the initial point z is picked at random, in some physically meaningful sense. As usual, we postulate physical chance to correspond to some smooth (Lebesgue) probability measure in the phase space.

1.1 Definitions and Examples

Let $f : M \to M$ be a measurable map on some metric space M. Let μ be a probability measure defined on the σ-algebra of M, invariant under f:

$$\mu(f^{-1}(B)) = \mu(B) \quad \text{for every measurable set } B \subset M.$$

We use δ_p to represent the Dirac measure supported on a point p in M.

Definition 1.1.1. *The* basin *of μ is the set $B(\mu)$ of points $z \in M$ such that,*

$$\frac{1}{n}\sum_{j=0}^{n-1} \delta_{f^j(z)} \to \mu \quad \textit{in the weak* sense,} \quad \textit{as } n \to \infty. \tag{1.2}$$

In other words, $z \in B(\mu)$ (one also says that z is a *generic point* for μ) if and only if the time average

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \varphi(f^j(z))$$

exists and coincides with the space average $\int \varphi \, d\mu$, for every continuous function $\varphi : M \to \mathbb{R}$. An equivalent formulation is, see for instance [124, Theorem 6.4],

$$\mu(U) = \lim_{n\to\infty} \frac{1}{n} \#\{0 \le j < n : f^j(z) \in U\}$$

for every measurable subset U of M whose boundary ∂U has zero measure for μ. This is the precise meaning of (1.1).

In what follows we suppose that M has a smooth structure, e.g., a finite-dimensional manifold, possibly branched and/or with boundary. Then it supports a distinguished class of measures, namely those generated by some volume form. In precise terms, we call *Lebesgue measure* to any measure m on the Borel σ-algebra of M such that for every $p \in M$ there exists a volume form ω_p on a neighbourhood V_p of p, so that

$$m(B) = \int_B d\omega_p \quad \text{for every measurable set } B \subset V_p.$$

All Lebesgue measures in M are equivalent, in the sense that they all have the same zero measure sets, and so it is often irrelevant to distinguish between them. So, except where otherwise specified, we use "Lebesgue measure" to mean any measure in the Lebesgue class.

Definition 1.1.2. *An f-invariant Borel probability measure μ is a* physical, *or* Sinai-Ruelle-Bowen (SRB) measure *for f if its basin $B(\mu)$ has positive Lebesgue measure.*

The previous definitions extend naturally to continuous-time dynamical systems, i.e. systems described by flows or by semi-flows. Let X^t, $t \ge 0$ be a semi-flow on M: X^0 is the identity, and each X^t, $t \ge 0$, is a measurable transformation on M, with $X^{t+s} = X^t \circ X^s$ for every t, s. A probability measure μ is invariant under the semi-flow if it is invariant under every map X^t, $t \ge 0$.

Definition 1.1.3. *The* basin *of μ is the set $B(\mu)$ of points $z \in M$ such that, given any continuous function $\varphi : M \to \mathbb{R}$,*

$$\frac{1}{T}\int_0^T \varphi(X^t(z))\,dt \to \int \varphi\,d\mu \quad as \quad T \to +\infty. \tag{1.3}$$

We say that μ is a physical, *or* Sinai-Ruelle-Bowen (SRB) measure *for X^t if its basin has positive Lebesgue measure.*

Example 1.1.1. Suppose the map $f : M \to M$ admits an invariant probability measure μ that is absolutely continuous with respect to Lebesgue measure, and ergodic. Then μ is a physical measure for f, as a simple consequence of Birkhoff's ergodic theorem. Indeed, the theorem states that $B(\mu)$ has nonzero (even full) measure for μ, and so it must have nonzero Lebesgue measure.

For the same reasons, ergodic absolutely continuous invariant probability measures of a semi-flow are physical measures for the semi-flow.

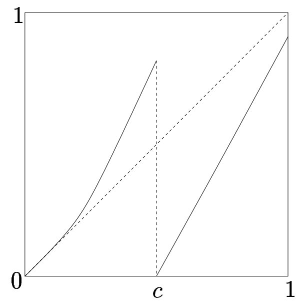

Fig. 1.1. A map with a neutral fixed point

Example 1.1.2. Let $f : [0,1] \to [0,1]$ be a piecewise C^2 map with a neutral fixed point at the origin, as in Figure 1.1. That is, we suppose that $f(0) = 0$ and $f'(0) = 1$, but the second derivative $f''(0)$ is nonzero. On the other hand, $|f'(z)| > 1$ for every $z \neq 0$, including $z = c^{\pm}$. We shall see in Section 3.5 that the orbit of Lebesgue almost every point $z \in [0,1]$ spends almost all the time in an arbitrarily small neighbourhood of the origin: given any $\delta > 0$,

$$\frac{1}{n}\#\{j \in \{0,1,\ldots,n-1\} : |f^j(z)| < \delta\} \to 1, \qquad \text{as } n \to \infty.$$

It follows that, given any continuous function $\varphi : [0,1] \to \mathbb{R}$ and any $\varepsilon > 0$, we have

$$\Big|\frac{1}{n}\sum_{j=0}^{n-1}\varphi(f^j(z)) - \varphi(0)\Big| < \epsilon$$

for every large n. So, for every continuous function φ and Lebesgue almost every point z,

$$\lim_{n\to+\infty}\frac{1}{n}\sum_{j=0}^{n-1}\varphi(f^j(z)) = \varphi(0) = \int \varphi\, d\delta_0 \,.$$

This means that the Dirac measure at zero is the unique SRB measure of f.

Now we describe an example, due to Bowen, of a flow in the plane for which time averages fail to converge for a whole open set of points. In particular, there is no physical measure whose basin intersects this open set. Similar arguments apply to the time-1 map $f = X^1$ of this flow.

Example 1.1.3. The flow is described in Figure 1.2. A main feature is the existence of a double saddle-connection between saddle-points A and B. We denote by L the region bounded by the separatrices that form this connection. Let $-\lambda_A < 0 < \sigma_A$ and $-\lambda_B < 0 < \sigma_B$ be the eigenvalues of the flow at A and B, respectively. We assume that

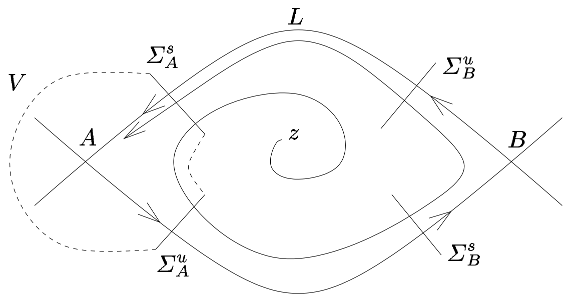

Fig. 1.2. A flow without physical measure

$$\frac{\lambda_A \lambda_B}{\sigma_A \sigma_B} > 1,$$

to ensure that the boundary of L attracts the orbits of all points $z \in L$ that are close enough to it. Then those orbits must visit, alternately, the vicinities of A and B. Fix cross sections Σ^s_A, Σ^u_A close to A and intersecting its stable and unstable separatrices, respectively. Similarly, let Σ^s_B, Σ^u_B be cross sections intersecting the stable and unstable separatrices of B. Fix z and let

$$\cdots < T^s_A(j) < T^u_A(j) < T^s_B(j) < T^u_B(j) < T^s_A(j+1) < \cdots$$

be the successive times at which the orbit of z intersects these cross-sections. Then $\tau_A(j) = T^u_A(j) - T^s_A(j)$ and $\tau_B(j) = T^u_B(j) - T^s_B(j)$ correspond to the successive times spent by the orbit near each of the saddles. It is an elementary exercise to check that

- $\tau_A(j) \approx \tau_B(j)$ and $\tau_B(j) \approx \tau_A(j+1)$, where $\approx$ means that the quotients of the two expressions are bounded by some constant independent of j;
- both sequences $\tau_A(j)$ and $\tau_B(j)$ increase exponentially fast with j, at the rate $(\lambda_A \lambda_B)/(\sigma_A \sigma_B)$;
- the transition times $T^s_B(j) - T^u_A(j)$ and $T^s_A(j+1) - T^u_B(j)$ are bounded by some constant independent of j.

As a consequence, each visit time is comparable to the total time elapsed thus far: there exists $c > 0$ such that

$$\tau_A(j) \geq c T^s_A(j) \quad \text{and} \quad \tau_B(j) \geq c T^s_B(j),$$

for every j. Now we may easily conclude that the time averages $T^{-1} \int \delta_{X^t(z)}\, dt$ do not converge as $T \to +\infty$, for any point $z \in L$ close to the boundary of L. Indeed, suppose otherwise, and let μ be the limit. Let V be some neighbourhood of A as in Figure 1.2. Up to slightly modifying V we may suppose that its boundary has zero μ-measure. Then $\mu(V) = \lim_{T\to+\infty} \tau_V(T)$, where

$$\tau_V(T) = \frac{1}{T} \int \mathcal{X}_V(X^t(z))\, dt$$

is the fraction of the time interval $[0,T]$ spent by z in V. Now,

$$\tau_V(T_A^s(j)) \geq \frac{\tau_A^s(j)}{T_A^s(j)} \geq c \qquad \text{but} \qquad \tau_V(T_A^u(j+1)) \leq \frac{1}{1+c}\,\tau_V(T_A^s(j))$$

for every j. This implies that $\tau_V(T)$ has no limit as $T \to +\infty$, and so we have reached a contradiction.

Such examples show that *SRB measures need not exist* for *all* systems. Existence results are usually difficult, and are known only for certain classes of systems. In particular, it is unknown in which generality do the basins of physical measures cover at least a full Lebesgue measure subset of the phase-space M. We will return to this fundamental problem later.

Right now, let us suppose that a map $f : M \to M$ does have some SRB measure μ. Let m denote Lebesgue measure restricted to the basin of μ, and normalized so as to be a probability. By definition,

$$\frac{1}{n}\sum_{j=0}^{n-1} \varphi(f^j(z)) \to \int \varphi \, d\mu$$

for every $z \in B(\mu)$, and every continuous function $\varphi : M \to \mathbb{R}$. Suppose, for the sake of simplicity, that M is compact. Then the sequence on the left is bounded in norm by $\sup|\varphi|$. As a direct consequence of the dominated convergence theorem,

$$\int \frac{1}{n}\sum_{j=0}^{n-1} \varphi \circ f^j \, dm \to \int \int \varphi \, d\mu \, dm = \int \varphi \, d\mu.$$

The expression on the left is precisely the integral of φ with respect to the measure $n^{-1}\sum_{j=0}^{n-1} f_*^j m$. In other words, we have proved that

$$\frac{1}{n}\sum_{j=0}^{n-1} f_*^j m \;\to\; \mu \quad \text{in the weak}^* \text{ topology.} \tag{1.4}$$

This simple observation suggests that SRB measures might be found as limits or, at least, accumulation points of the averages of forward iterates $f_*^j m$ of Lebesgue measure, possibly restricted to some subset of the phase-space M and normalized.

It is a well-known consequence of the Banach-Alaoglu theorem that the space of probability measures on a compact metric space is compact with respect to the weak* topology. See [76, Section I.8], for instance. Therefore, accumulation points

$$\mu = \lim_{k\to\infty} \frac{1}{n_k}\sum_{j=0}^{n_k-1} f_*^j m \tag{1.5}$$

always exist. Moreover, assuming that the map f is continuous on M, the push-forward operator $f_* : \eta \mapsto f_*\eta$ is also continuous, relative to the weak* topology in the space of Borel measures in M.

Using this fact, one concludes readily that any such accumulation point μ is an invariant measure for f. In fact,

$$f_*\mu = \lim_{k\to\infty} \frac{1}{n_k} \sum_{j=0}^{n_k-1} f_*^{j+1} m = \lim_{k\to\infty} \frac{1}{n_k} \sum_{j=0}^{n_k-1} f_*^j m + \lim_{k\to\infty} \frac{1}{n_k} \left(f_*^{n_k} m - m\right).$$

The first limit on the right is μ, by assumption, and the second one is identically zero: given any continuous function φ,

$$\frac{1}{n_k} \int (\varphi \circ f^{n_k} - \varphi)\, dm \to 0 \quad \text{as } n_k \to \infty.$$

This shows that $f_*\mu = \mu$, which is just the same as saying that μ is invariant.

On the other hand, there is no a priori reason for a measure μ as in (1.5) to have particularly interesting properties: recall for instance Example 1.1.3. Indeed, to be able to conclude that such a μ is an SRB measure one must keep a fair control of the sequence

$$\mu_n = \frac{1}{n} \sum_{j=0}^{n-1} f_*^j m, \tag{1.6}$$

which is where most of the difficulty lies.

In the remaining of this chapter we present some important cases where such a control is possible: the measures μ_n are all absolutely continuous with respect to Lebesgue measure, with uniform bounds on the *densities* (Radon-Nikodym derivatives). More general situations will appear in subsequent chapters.

1.2 Uniformly Expanding Maps

In this section we prove that any uniformly expanding map on a compact (connected) manifold with Hölder continuous Jacobian admits a unique physical measure μ. Moreover, the basin of μ is a full Lebesgue measure subset of the manifold. See Theorem 1.2.1 for the precise statement.

1.2.1 Definitions and Basic Properties

Let M be a compact manifold and $f : M \to M$ be a C^1 map.

Definition 1.2.1. *We say that f is* (uniformly) expanding *if there exist constants $C > 0$ and $\sigma > 1$ such that*

$$\|Df^n(x)v\| \geq C\sigma^n \|v\| \quad \textit{for every } x \in M,\ v \in T_xM,\ \textit{and } n \geq 1. \tag{1.7}$$

Here $\|\cdot\|$ denotes an arbitrary Riemannian norm on the manifold M: since all norms are equivalent, (1.7) holds for $\|\cdot\|$ if and only if it holds for any other norm, apart from the fact that the constants may vary. As a matter of fact, up to choosing a convenient norm, we may always suppose that $C = 1$.

Indeed, let (1.7) hold for some norm $\|\cdot\|$, and constants C, σ. Given any $1 < \sigma_* < \sigma$, fix $N \geq 1$ large enough so that $C(\sigma/\sigma_*)^N \geq 1$. Then, consider the Riemannian norm $\|\cdot\|_*$ defined by

$$\|v\|_*^2 = \sum_{j=0}^{N-1} \sigma_*^{-2j} \|Df^j(x)v\|^2,$$

for each $x \in M$ and $v \in T_x M$. Direct substitution gives

$$\|Df(x)v\|_* \geq \sigma_* \|v\|_* \quad \text{for every } x \in M \text{ and } v \in T_x M. \tag{1.8}$$

A Riemannian norm as in (1.8) is said to be *adapted* to f. It also follows from these remarks that the set of expanding maps is open in the C^1 topology: if f satisfies (1.8) then, up to slightly reducing $\sigma_* > 1$, so does any map g in some C^1 neighbourhood.

Example 1.2.1. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map such that $F(\mathbb{Z}^n) \subset \mathbb{Z}^n$. Then there exists a unique map f on the n-dimensional torus $M = \mathbb{R}^n/\mathbb{Z}^n$ such that $f \circ \pi = \pi \circ F$, where $\pi : \mathbb{R}^n \to M$ is the canonical projection. If all the eigenvalues $\lambda_1, \ldots, \lambda_n$ of F have norm larger than 1 then this map f is expanding: any $1 < \sigma < \inf_i |\lambda_i|$ will do in (1.7).

According to [115], every expanding map on the n-torus is topologically conjugate to a linear model f as in Example 1.2.1. More generally, cf. [50], a manifold admits expanding maps if and only if it is an infranilmanifold, and then any such map is topologically conjugate to an algebraic expanding endomorphism. In general, the conjugacy is a *singular map*: it does not preserve the class of sets with zero Lebesgue measure.

Definition 1.2.2. *Let X, Y be metric spaces and $f : X \to Y$ be a continuous map. Given $C > 0$ and $0 < \nu \leq 1$, we say that f is* (C, ν)-Hölder *if*

$$d_Y(f(x_1), f(x_2)) \leq C d_X(x_1, x_2)^\nu \quad \textit{for every } x_1, x_2 \textit{ in } X.$$

When $\nu = 1$ we also say that f is C-Lipschitz. *In general, f is* ν-Hölder *if it is (C, ν)-Hölder for some $C > 0$.*

The following theorem summarizes results of existence and uniqueness of SRB measures that we prove in Subsections 1.2.2 and 1.2.3.

Theorem 1.2.1. *Let $f : M \to M$ be a uniformly expanding map on a compact manifold M. Assume that there exists $0 < \nu_0 \leq 1$ such that the logarithm $M \ni x \mapsto \log|\det Df(x)|$ of the Jacobian of f is ν_0-Hölder.*

Then f admits a unique invariant measure μ which is absolutely continuous with respect to Lebesgue measure. Moreover, μ is ergodic, its support coincides with M, and its basin $B(\mu)$ is a full Lebesgue measure subset of M. In particular, μ is the unique SRB-measure of f.

The assumption of expansiveness is used in the proof of this theorem through the consequences provided by the following proposition. We fix, once and for all, a Riemannian norm $\|\cdot\|$ adapted to f, and denote $d(\cdot\,,\cdot)$ the corresponding distance on the manifold.

Proposition 1.2.1. *Let M be a compact manifold and $f : M \to M$ be a C^1 expanding map. Then there exists $k \geq 1$ such that every point $y \in M$ has exactly k pre-images under f. Moreover, there exists $\rho_0 > 0$ such that given any pre-image x of a point $y \in M$ there exists a C^1 map $h : B(y, \rho_0) \to M$ with $f \circ h = \mathrm{id}$, $h(y) = x$, and*

$$d(h(y_1), h(y_2)) \leq \sigma^{-1}\, d(y_1, y_2) \quad \textit{for every } y_1, y_2 \in B(y, \rho_0).$$

Proof. We only sketch the arguments, as they are quite standard. Clearly, (1.7) implies that the derivative Df is an isomorphism at every point. So, given any $x \in M$ there exists $\rho_0 > 0$ such that f maps some neighbourhood $V(x)$ of x diffeomorphically onto the ball of radius ρ_0 around $y = f(x)$. By compactness, ρ_0 may be chosen independent of x. Then the number of pre-images of any $y \in M$ must be finite and even bounded. It also follows that the set of points with exactly n pre-images is open, for every $n \geq 0$. So, by connectedness, the number of pre-images must be the same for every $y \in M$. Finally, let us denote $h = (f \mid V(x))^{-1}$. Since the norm is adapted to f,

$$\|Dh(z)\| = \|Df(h(z))^{-1}\| \leq \sigma^{-1}$$

for every z in the domain of h, and so h contracts distances by a factor σ^{-1}, as stated. □

Maps h as in the statement are called *(local) inverse branches* of f. More generally, we can define inverse branches h^n of f^n, $n \geq 1$, as follows. Given $y \in M$ and $x \in f^{-n}(y)$, let $h_1, \ldots, h_n$ be inverse branches of f with

$$h_j(f^{n-j+1}(x)) = f^{n-j}(x)$$

for every $1 \leq j \leq n$. Since each h_j is a contraction, its image is contained in a ball of radius less than ρ_0 around $f^{n-j}(x)$. Then $h^n = h_n \circ \cdots \circ h_1$ is well-defined on the ball of radius ρ_0 around y. Clearly, $f^n \circ h^n = \mathrm{id}$ and $h^n(y) = x$.

Remark 1.2.1. The conclusion of Theorem 1.2.1 is, generally, not true if one drops the assumption that the Jacobian is Hölder continuous: a uniformly expanding map which is just C^1 may fail to have any invariant probability measure absolutely continuous with respect to Lebesgue measure. As a matter of fact, [100] proves that for a generic subset (countable intersection of open dense subsets) of C^1 expanding maps on any compact manifold there is no invariant absolutely continuous probability measure.

1.2.2 Upper Bounds on the Densities

It is easy to see that the pre-image of a zero Lebesgue measure set under an expanding map f also has zero Lebesgue measure. It follows that if a probability measure ν is absolutely continuous with respect to Lebesgue measure, then the same is true for its push-forward $f_*\nu$ under f. Let m be Lebesgue measure on M, normalized so that $m(M) = 1$. Then, in particular, $f_*^n m$ is absolutely continuous with respect to m for every $n \geq 1$.

We prove in Proposition 1.2.2 that if f is an expanding map with Hölder continuous Jacobian, as in the statement of Theorem 1.2.1, then the densities $d(f_*^n m)/dm$ are bounded by some constant independent of $n \geq 1$. From this we deduce that any accumulation point of the sequence (1.6) is absolutely continuous with respect to Lebesgue measure, with density bounded by that same constant. In particular, f has some invariant measure μ that is absolutely continuous with respect to Lebesgue measure.

The main step is the following result of bounded distortion, which is also the only place where the assumption of Hölder continuity is needed in the proof.

Lemma 1.2.1. *There exists $C_1 > 0$ such that given any $n \geq 1$, any $y \in M$, and any inverse branch $h^n : B(y, \rho_0) \to M$ of f^n,*

$$\frac{|\det Dh^n(y_1)|}{|\det Dh^n(y_2)|} \leq \exp(C_1 d(y_1, y_2)^{\nu_0}) \leq \exp(C_1 (2\rho_0)^{\nu_0})$$

for every $y_1, y_2 \in B(y, \rho_0)$.

Proof. Let us write h^n as a composition $h^n = h_n \circ \cdots \circ h_1$ of inverse branches of f. We also denote $h^i = h_i \circ \cdots \circ h_1$ for $1 \leq i < n$, and $h^0 = \mathrm{id}$. Then

$$\log \frac{|\det Dh^n(y_1)|}{|\det Dh^n(y_2)|} = \sum_{i=1}^{n} \log |\det Dh_i(h^{i-1}(y_1))| - \log |\det Dh_i(h^{i-1}(y_2))| .$$

Note that $\log |\det Dh_i| = -\log |\det Df| \circ h_i$ and, by assumption, $\log |\det Df|$ is (C_0, ν_0)-Hölder for some $C_0 > 0$. Moreover, cf. Proposition 1.2.1, each h_j is a σ^{-1}-contraction. Then,

$$\log \frac{|\det Dh^n(y_1)|}{|\det Dh^n(y_2)|} \leq \sum_{i=1}^{n} C_0\, d(h^i(y_1), h^i(y_2))^{\nu_0} \leq \sum_{i=1}^{n} C_0\, \sigma^{-i\nu_0} d(y_1, y_2)^{\nu_0} .$$

So, to prove the lemma it is enough to take $C_1 = C_0 \sum_{i=1}^{\infty} \sigma^{-i\nu_0}$. □

Proposition 1.2.2. *There exists $C_2 > 0$ such that $(f^n_* m)(B) \le C_2 m(B)$ for every measurable set $B \subset M$ and every $n \ge 1$.*

Proof. It is no restriction to take B contained in some ball $B_0 = B(z, \rho_0)$ of radius ρ_0 around a point $z \in M$. Lemma 1.2.1 implies that

$$\frac{m(h^n(B))}{m(h^n(B_0))} = \frac{\int_B |\det Dh^n|\, dm}{\int_{B_0} |\det Dh^n|\, dm} \le \exp(C_1(2\rho_0)^{\nu_0}) \frac{m(B)}{m(B_0)},$$

for each inverse branch h^n of f^n at z. Moreover, $(f^n_* m)(B) = m(f^{-n}(B))$ is the sum of $m(h^n(B))$ over all inverse branches, and analogously for B_0. So, we get that

$$\frac{(f^n_* m)(B)}{(f^n_* m)(B_0)} \le \exp(C_1(2\rho_0)^{\nu_0}) \frac{m(B)}{m(B_0)}.$$

Of course, $(f^n_* m)(B_0) \le (f^n_* m)(M) = 1$. Moreover, the Lebesgue measure of the balls of fixed radius ρ_0 is bounded from zero by some $\alpha_0 > 0$ that depends only on ρ_0. Now it suffices to take $C_2 = \exp(C_1(2\rho_0)^{\nu_0})/\alpha_0$. □

Lemma 1.2.2. *Let ν be a probability measure on a compact metric space X, and $\varphi : X \to [0, +\infty)$ be integrable with respect to ν. Let μ_i, $i \ge 1$, be a sequence of probability measures on X converging to some μ, in the weak* sense. If $\mu_i \le \varphi\nu$ for every $i \ge 1$ then $\mu \le \varphi\nu$.*

Proof. Let B be any measurable set. For each $\varepsilon > 0$, let K_ε be a compact subset of B such that $\mu(B \setminus K_\varepsilon)$ and $(\varphi\nu)(B \setminus K_\varepsilon)$ are both less than ε. Then let A_ε be the open neighbourhood of K_ε defined by $A_\varepsilon = \{z : d(z, K_\varepsilon) < r\}$, where $r > 0$ is small enough so that the measure of $A_\varepsilon \setminus K_\varepsilon$ is less than ε for both μ and $\varphi\nu$. Changing r if necessary, we may suppose that the boundary of A_ε has zero μ-measure (there are at most countably many exceptional values of r). Then $\mu = \lim \mu_i$ implies $\mu(A_\varepsilon) = \lim \mu_i(A_\varepsilon) \le (\varphi\nu)(A_\varepsilon)$. Making $\varepsilon \to 0$ we get $\mu(B) \le (\varphi\nu)(B)$. □

Now we apply this lemma to our situation, with $\varphi \equiv C_2$, $\nu = m$, and $\mu_i = n_i^{-1} \sum_{j=0}^{n_i - 1} f^j_* m$ for any subsequence $(n_i)_i$ such that $(\mu_i)_i$ converges to some measure μ. We immediately get

Corollary 1.2.1. *Every accumulation point μ of $n^{-1} \sum_{j=0}^{n-1} f^j_* m$ is an f-invariant measure absolutely continuous with respect to Lebesgue measure.*

1.2.3 Ergodicity and Uniqueness

Now we show that the probability measure μ constructed above is ergodic and so, recall Example 1.1.1, is a physical measure for f. We also get that the basin of μ is a full Lebesgue measure subset of M. In particular, the physical measure is unique.

We begin by fixing some partition $\mathcal{P}_0 = \{U_1, \ldots, U_s\}$ of M into regions with nonempty interior and diameter less than ρ_0. Then, for each $n \geq 1$, we let $\mathcal{P}_n$ be the partition of M consisting of the images of each of the U_i, $1 \leq i \leq s$, under corresponding inverse branches of f^n. The diameter of $\mathcal{P}_n$, defined as the supremum of the diameters of its elements, is less than $\rho_0 \sigma^{-n}$.

Lemma 1.2.3. *Let $\mathcal{P}_n$, $n \geq 1$, be a sequence of partitions in a compact metric space with diameters converging to zero as $n \to \infty$. Let ν be a probability measure in that space, and B be any measurable subset such that $\nu(B) > 0$. Then there are $V_n \in \mathcal{P}_n$, for $n \geq 1$, so that*

$$\nu(V_n) > 0 \quad \text{and} \quad \frac{\nu(B \cap V_n)}{\nu(V_n)} \to 1 \quad \text{as } n \to \infty.$$

Proof. Given any $0 < \varepsilon < \nu(B)$, let K_ε be some compact subset of B with $\nu(B \setminus K_\varepsilon) < \varepsilon$. As the diameter of the partitions converges to zero, the measure of the union $A_{\varepsilon,n}$ of all the elements of $\mathcal{P}_n$ that intersect K_ε satisfies $\nu(A_{\varepsilon,n} \setminus K_\varepsilon) < \varepsilon$ for every large enough n. If we had

$$\nu(K_\varepsilon \cap V_n) \leq \frac{\nu(B) - \varepsilon}{\nu(B) + \varepsilon} \nu(V_n)$$

for every $V_n \in \mathcal{P}_n$ that intersects K_ε, it would follow that

$$\nu(K_\varepsilon) \leq \frac{\nu(B) - \varepsilon}{\nu(B) + \varepsilon} \nu(A_{\varepsilon,n}) \leq \frac{\nu(B) - \varepsilon}{\nu(B) + \varepsilon} (\nu(K_\varepsilon) + \varepsilon) \leq \nu(B) - \varepsilon,$$

a contradiction. So, there must be some $V_n \in \mathcal{P}_n$ with

$$\nu(B \cap V_n) \geq \nu(K_\varepsilon \cap V_n) > \frac{\nu(B) - \varepsilon}{\nu(B) + \varepsilon} \nu(V_n)$$

and this also implies $\nu(V_n) > 0$. The statement follows, taking $\varepsilon \to 0$. □

We say that $A \subset M$ is an *invariant set* of $f : M \to M$ if $f^{-1}(A) = A$, and we call A *forward invariant* if $f(A) = A$. Note that invariant sets are also forward invariant, because f is surjective.

Lemma 1.2.4. *Let $A \subset M$ be a forward invariant set of a $C^{1+\nu_0}$ expanding map f such that $m(A) > 0$. Then A has full Lebesgue measure in some $U_i \in \mathcal{P}_0$, that is, there exists $1 \leq i \leq s$ so that $m(U_i \setminus A) = 0$.*

Proof. By Lemma 1.2.3, there exist $V_n \in \mathcal{P}_n$ such that $m(V_n \setminus A)/m(V_n)$ converges to zero as $n \to \infty$. Let $U_{i(n)} = f^n(V_n)$. Applying Lemma 1.2.1 to the inverse branch of f^n mapping $U_{i(n)}$ to V_n, we conclude that

$$\frac{m(U_{i(n)} \setminus A)}{m(U_{i(n)})} \leq \frac{m(f^n(V_n \setminus A))}{m(f^n(V_n))} \leq \exp\left(C_1 (2\rho_0)^{\nu_0}\right) \frac{m(V_n \setminus A)}{m(V_n)}$$

also converges to zero. Since $\mathcal{P}_0$ is finite, there must exist $1 \leq i \leq s$ such that $i(n) = i$ for infinitely many values of n. Then $m(U_i \setminus A) = 0$. □

Corollary 1.2.2. *Any $C^{1+\nu_0}$ expanding map $f : M \to M$ has some ergodic absolutely continuous invariant measure.*

Proof. As a consequence of the lemma, there exist at most $\#\mathcal{P}_0$ two-by-two disjoint invariant sets with positive Lebesgue measure. It follows that M can be partitioned into finitely many minimal positive Lebesgue measure invariant sets $A_1, \dots, A_s$, $s \le \#\mathcal{P}_0$: minimality means there are no invariant subsets $B_i \subset A_i$ with $0 < m(B_i) < m(A_i)$. Given any f-invariant absolutely continuous measure μ, there is some i such that $\mu(A_i) > 0$. Then the normalized restriction μ_i of μ to A_i,

$$\mu_i(B) = \frac{\mu(B \cap A_i)}{\mu(A_i)}$$

is invariant, absolutely continuous, and ergodic (because A_i is minimal). □

This argument also gives that there exist only finitely many measures as in the statement. The last step in the proof of Theorem 1.2.1 is to show that, in fact, such a measure is unique. This requires the following strong *topological mixing* property.

Lemma 1.2.5. *Given any nonempty open set $U \subset M$, there exists $N \ge 1$ such that $f^N(U) = M$.*

Proof. Let $x \in U$ and $r > 0$ be such that the ball of radius r around x is contained in U. Given any $n \ge 1$, suppose that $f^n(U)$ does not cover the whole manifold. Then, there is some curve γ connecting $f^n(x)$ to a point $y \in M \setminus f^n(U)$, and γ may be taken with length less than $\operatorname{diam} M + 1$. By lifting γ through the local diffeomorphism f^n we get a curve γ_n connecting x to a point $y_n \in M \setminus U$. Then $r \le \operatorname{length}(\gamma_n) \le \sigma^{-n}(\operatorname{diam} M + 1)$, which gives an upper bound on n. Thus, $f^n(U) = M$ for every large enough n, as stated. □

Corollary 1.2.3. *If $A \subset M$ is a forward invariant set with positive Lebesgue measure, then A has full Lebesgue measure in the whole manifold M.*

Proof. Let U be the interior of a set U_i as given by Lemma 1.2.4, and let $N \ge 1$ be such that $f^N(U) = M$. Then $m(U \setminus A) = 0$, and so $M \setminus A = f^N(U) \setminus f^N(A) \subset f^N(U \setminus A)$ also has zero Lebesgue measure. □

The following statement completes the proof of Theorem 1.2.1.

Corollary 1.2.4. *Let μ be an absolutely continuous invariant measure of f. Then μ is ergodic and its basin $B(\mu)$ has full Lebesgue measure in M. Moreover, the support of μ is the whole manifold M.*

Proof. If A is an f-invariant subset of M then, by the previous corollary, either A or A^c have zero Lebesgue measure. So either $\mu(A) = 0$ or $\mu(A^c) = 0$. This proves ergodicity. Then $B(\mu)$ is an invariant set with positive Lebesgue measure, and so it must have full Lebesgue measure. Similarly, as the support of μ is a compact forward invariant subset with positive Lebesgue measure, it must coincide with M. □

In particular, the map f has a unique absolutely continuous invariant measure μ. We shall see in Corollary 3.3.1 that the density $d\mu/dm$ may be taken Hölder continuous and bounded away from zero on M.

1.3 Piecewise Expanding Maps

In quick terms, we call a transformation $f : M \to M$ piecewise expanding if the ambient manifold M or, at least, a full Lebesgue measure subset of it, can be partitioned into countably many domains restricted to which the transformation is expanding and sufficiently differentiable. Precise definitions will appear later. Figure 1.3 describes some simple examples we have in mind.

Example 1.3.1. We say that $f : [0, 1] \to [0, 1]$ is a *tent map* if it is continuous and there exists $c \in (0, 1)$ such that

- the derivative Df is constant and larger than one in norm, in each of the intervals $[0, c)$ and $(c, 1]$.

See Figure 1.3. More generally, one may consider maps that are affine and expanding on each interval (c_{i-1}, c_i), $1 \le i \le N$, for some finite sequence of points $0 = c_0 < c_1 < \cdots < c_N = 1$.

The next class of examples play a key role in the theory of Lorenz-like attractors of flows, see [2], [51], and Section 7.1.

Example 1.3.2. We call $f : [0, 1] \to [0, 1]$ a *Lorenz-like map* if there exist $c \in (0, 1)$ and $\sigma > 1$ such that

- f is C^2 on each of intervals $[0, c)$ and $(c, 1]$, with a discontinuity at c;
- $|Df(x)| > \sigma$ for every $x \in [0, 1] \setminus \{c\}$;
- the left and the right derivatives at c are both infinite, and $1/|Df|$ extends to c as a Lipschitz function on $[0, 1]$.

See Figure 1.3. This can also be generalized, to include maps with any number of singular points (any number of regularity intervals).

Besides their intrinsic interest, there are several reasons for studying piecewise expanding maps. For one thing, they provide a fairly simple setting for dealing with difficulties that are common to much more complicated systems. This point will be illustrated in a little while. At least as important, they are

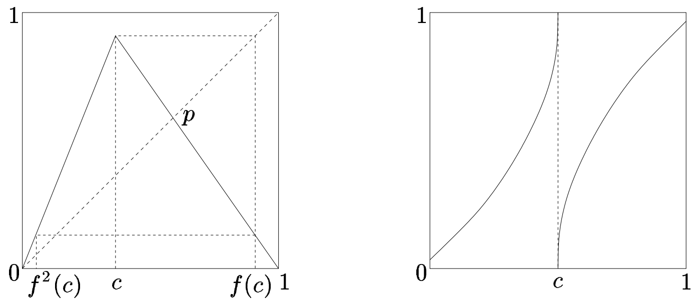

Fig. 1.3. Tent map and Lorenz-like map

often found in the course of studying other dynamical systems: properties of the system can be understood through constructing and analyzing certain piecewise expanding maps that are associated to it. Important examples appear in Chapters 6 and 7.

In this section we only deal with one-dimensional maps, that is, we always take M to be either the circle S^1 or the compact interval $I = [0,1]$. Throughout, m denotes some normalized Lebesgue measure in M. The higher dimensional case is discussed in Section 1.4.

1.3.1 Definitions and Statements

Let $f : M \to M$ be so that there exist $C > 0$, $\sigma > 1$, and a family $\mathcal{P}^1$ of two-by-two disjoint intervals covering a full Lebesgue measure subset of M, such that f is C^2 restricted to each $\xi \in \mathcal{P}^1$, and

$$|Df^n| \geq C\sigma^n \quad \text{for } n \geq 1, \tag{1.9}$$

at any point where the derivative exists. For each $n \geq 1$, let $\mathcal{P}^n$ be the family of *regularity intervals* of f^n. That is, the elements of $\mathcal{P}^n$ are the maximal intervals η such that $f^j(\eta)$ is contained in some atom of $\mathcal{P}^1$, for every $0 \leq j \leq n-1$.

Suppose that $\log|Df \mid \xi|$ is Hölder continuous for every $\xi \in \mathcal{P}^1$, with uniform Hölder constants (this holds, for instance, if $\mathcal{P}^1$ is finite and every $f \mid \xi$ admits a C^2 extension to the boundary). Then, the same arguments as in Lemma 1.2.1 give that the inverse branches $h^n_\eta = (f^n \mid \eta)^{-1}$ of f^n have uniformly bounded distortion: there exists $K > 0$ such that

$$\sup\Big\{\frac{|Dh^n_\eta(y_1)|}{|Dh^n_\eta(y_2)|} : y_1, y_2 \in f^n(\eta)\Big\} \leq K \tag{1.10}$$

for every $\eta \in \mathcal{P}^n$ and $n \geq 1$. In particular, each measure $f^n_*(m \mid \eta)$ is absolutely continuous with respect to Lebesgue measure on $f^n(\eta)$, with density bounded by some uniform constant.

Since $f_*^n m$ is the sum of the $f_*^n(m \mid \eta)$ over all $\eta \in \mathcal{P}^n$, one may hope to show that the measures $f_*^n m$, $n \geq 1$, have uniformly bounded densities, which would imply that f has some invariant measure absolutely continuous with respect to Lebesgue measure. This would follow from the proof of Proposition 1.2.2, if one knew that the Lebesgue measure of the intervals $f^n(\eta)$, $\eta \in \mathcal{P}^n$, $n \geq 1$, is uniformly bounded away from zero. However, this is generally *not* the case, even if f has finitely many regularity intervals.

This fact is a main source of difficulties, and we shall return to it in a while. Before that, let us briefly discuss a special class of maps for which a lower bound for the measure of the $f^n(\eta)$ does exist.

Markov Expanding Maps. Let $f : M \to M$ be as before: it satisfies, and $\log |Df \mid \xi|$ is Hölder continuous for every $\xi \in \mathcal{P}^1$. Suppose, moreover, that

(M1) the image $f(\xi)$ of every $\xi \in \mathcal{P}^1$ coincides with some union of elements of $\mathcal{P}^1$, up to a zero Lebesgue measure set;
(M2) there exists $\delta > 0$ such that $m(f(\xi)) \geq \delta$ for any $\xi \in \mathcal{P}^1$.

The first condition implies that $f^n(\eta)$ contains the image $f(\xi)$ of some $\xi \in \mathcal{P}^1$, for every $\eta \in \mathcal{P}^n$. Then (M2) gives $m(f^n(\eta)) \geq \delta$ for every $n \geq 1$. So, the proof of Proposition 1.2.2 carries on to this case. It follows, as in Corollary 1.2.1, that f has absolutely continuous invariant measures.

Moreover, we can combine this condition (M2) with the distortion bound (1.10) to conclude, as in Subsection 1.2.3, that any f-invariant set A with positive Lebesgue measure must have $m(A) \geq \delta$. As a consequence, f has finitely many ergodic absolutely continuous invariant probabilities, and any absolutely continuous invariant measure is a linear combination of the ergodic ones. Finally, if f is transitive (in the sense that for any $\xi, \eta \in \mathcal{P}^1$, there exists n such that $\xi \cap f^n(\eta)$ has positive Lebesgue measure) then it has a unique absolutely continuous invariant probability.

These facts are often referred to as the Folklore theorem. See e.g. [24] for references and a discussion of the origin of this statement.

Observe that the proof still works if we just assume $\log |D(f \mid \xi)|$ to be Hölder continuous, with uniform constants. In fact, it also extends to higher dimensions, assuming $\log |\det Df|$ is Hölder continuous on each regularity domain, always with uniform constants.

Example 1.3.3. The *Gauss map* $G : [0,1] \to [0,1]$ is defined by

$$G(x) = 1/x - [1/x] \quad \text{for } x \neq 0, \quad \text{and} \quad G(0) = 0.$$

The family of regularity intervals is $\mathcal{P}^1 = \{(1/(n+1), 1/n) : n \geq 1\}$. Note that $|DG(x)| = 1/x^2 \geq 1$ wherever the derivative is defined. Moreover,

$$|DG(x)| > 2 \;\text{ if }\; x \leq 2/3 \quad \text{and} \quad |DG^2(x)| \geq 4 \;\text{ if }\; x > 2/3.$$

This implies (1.9), with $\sigma = 2$ and $C = 1/2$. Finally, given any $1/(n+1) < x < y < 1/n$,

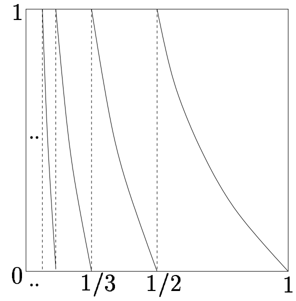

Fig. 1.4. The Gauss map

$$\big|\log|DG(x)| - \log|DG(y)|\big| \le 2(n+1)(y-x) \le 4(y-x)^{1/2}.$$

So $\log|DG|$ is $(4, 1/2)$-Hölder on each element of $\mathcal{P}^1$. Properties (M1) and (M2) are clear, with $\delta = 1$.

Now we go back to general, possibly non-Markov, piecewise expanding maps in dimension one. As we already mentioned, the "chopping" that takes place at the boundary of the regularity intervals may cause the length of the iterates $f^n(\eta)$, $\eta \in \mathcal{P}^n$, to be arbitrarily small. Then, conceivably, one might have several small intervals $f^n(\eta)$ piling-up in the same region, thus causing the density of $f_*^n m$ to grow unbounded as $n \to \infty$.

That this does not actually occur was first proved by [68], for maps with finitely many regularity intervals: the densities

$$\phi_n = \frac{d(f_*^n m)}{dm}$$

have *uniformly bounded variation* and, hence, they are uniformly bounded. Existence of some absolutely continuous invariant measure is a direct consequence, cf. Corollary 1.2.2.

Building on this approach, [69] showed that f admits some ergodic absolutely continuous invariant measure, and the number of such measures is finite. It was observed by [126] that similar arguments apply under a weaker regularity assumption, see (E2) below, that allows for maps with unbounded derivative as in Example 1.3.2. Then [**?**] extended these results to maps with infinitely many regularity intervals.

Definition 1.3.1. *We call* $f : M \to M$ *a* piecewise expanding map*, on either* $M = S^1$ *or* $M = [0,1]$*, if there exists a countable family* $\mathcal{P}^1$ *of two-by-two disjoint intervals covering a full Lebesgue measure subset of* M*, such that*

(E1) the restriction of f *to each* $\xi \in \mathcal{P}^1$ *is a* C^1 *monotonic map, and the function* $\xi \ni x \mapsto 1/|Df(x)|$ *has bounded variation;*

(E2) there exist constants $C > 0$ and $\sigma > 1$ such that $|Df^n(x)| \geq C\sigma^n$ for every $n \geq 1$, and every $x \in M$ for which the derivative is defined.

We call $\mathcal{P}^1$ a *partition into regularity intervals* of f. We say that f has finitely many regularity intervals if the partition $\mathcal{P}^1$ may be chosen finite. The boundary points of the regularity intervals $\xi \in \mathcal{P}^1$ that are not on the boundary of M are called *singular points* of f.

The following theorem is proved in Subsections 1.3.3 and 1.3.4.

Theorem 1.3.1. *Let f be a piecewise expanding map of the circle or the interval, with finitely many regularity intervals.*

Then f has some ergodic invariant probability measure absolutely continuous with respect to Lebesgue measure, and the number of such measures is bounded by the number of singular points of f. The union of their basins is a full Lebesgue measure subset of M. Moreover, any absolutely continuous invariant measure μ can be written $\mu = \varphi m$ where φ has bounded variation.

It is not difficult to see that this result can not hold in the general infinite case. The following counterexample is due to [?], another had been given by [68].

Example 1.3.4. Let $f : [0,1] \to [0,1]$ be given by $f(0) = 0$ and

$$f(x) = 2x - 2^{-j+1} \quad \text{for} \quad x \in (2^{-j}, 2^{-j+1}] \ \text{ and each } \ j \geq 1.$$

Then $f(x) \leq x$ for every $x \in [0,1]$, and the equality holds if and only if x is in $\mathrm{Fix}(f) = \{2^{-k} : k \geq 0\} \cup \{0\}$. Then, by Poincaré's recurrence theorem, every f-invariant measure is supported in $\mathrm{Fix}(f)$, and so it is a linear combination of Dirac measures on fixed points of the map. In particular, f has no absolutely continuous invariant measure. We leave it to the reader to check that Lebesgue almost every orbit of f converges to 0, and so the Dirac measure at zero is the unique SRB measure.

On the other hand, [?] also provides a natural condition under which most of the conclusion of Theorem 1.3.1 does extend to piecewise expanding maps with infinitely many regularity intervals. For each $\xi \in \mathcal{P}^1$, let $\hat{g}_\xi : M \to \mathbb{R}$ be the function defined by

$$\hat{g}_\xi(x) = \frac{1}{|Df(x)|} \quad \text{if } x \in \xi \qquad \text{and} \qquad \hat{g}_\xi(x) = 0 \quad \text{otherwise.} \tag{1.11}$$

Conditions (E1) and (E2) in the definition imply that each $\hat{g}_\xi$ has bounded variation. So, the next result generalizes Theorem 1.3.1.

Theorem 1.3.2. *Let f be a piecewise expanding map of the circle or the interval. Assume that, for some choice of a partition $\mathcal{P}^1$,*

$$\sum_{\xi \in \mathcal{P}^1} \operatorname{var} \hat{g}_\xi < \infty. \tag{1.12}$$

Then f has some ergodic invariant probability measure absolutely continuous with respect to Lebesgue measure, and there are finitely many such measures. The union of their basins covers a full Lebesgue measure subset of M. Moreover, if μ is any absolutely continuous invariant measure then $\mu = \varphi m$ where φ has bounded variation.

The proof of this theorem is given in Subsection 1.3.5, where we also describe a few examples and applications.

1.3.2 Bounded Variation Functions

Let us begin by recalling the definition and some elementary properties of the notion of variation of real functions defined on the circle or the interval. See, for instance, [?] for more information.

Definition 1.3.2. *Let $\varphi : M \to \mathbb{R}$ and $\eta = [a, b]$ be a compact interval in M. The variation of φ on η is*

$$\operatorname*{var}_{\eta} \varphi = \sup \sum_{i=1}^{n} |\varphi(x_{i-1}) - \varphi(x_i)|$$

where the supremum is over all finite sequences $a = x_0 < x_1 < \cdots < x_n = b$, $n \geq 1$, with $<$ representing an arbitrary orientation on η.

The variation $\operatorname{var}_\eta \varphi$ of φ on an arbitrary connected subset η of M (including $\eta = M = S^1$) is the supremum of its variations over all compact intervals contained in η. We represent the variation $\operatorname{var}_M \varphi$ over the whole ambient manifold M simply as $\operatorname{var} \varphi$.

Definition 1.3.3. *A function $\varphi : M \to \mathbb{R}$ has* bounded variation *if* $\operatorname{var} \varphi$ *is finite. Given any connected subset η of M, φ has* bounded variation on η *if* $\operatorname{var}_\eta \varphi < \infty$.

The following properties are direct consequences of the definition.

Lemma 1.3.1. *Let $\varphi_1, \varphi_2 : M \to \mathbb{R}$ and η be a connected subset of M.*

1. $\operatorname*{var}_{\eta}(\varphi_1 + \varphi_2) \leq \operatorname*{var}_{\eta} \varphi_1 + \operatorname*{var}_{\eta} \varphi_2$;
2. $\operatorname*{var}_{\eta}(\varphi_1 \varphi_2) \leq \operatorname*{var}_{\eta} \varphi_1 \sup_{\eta} |\varphi_2| + \sup_{\eta} |\varphi_1| \operatorname*{var}_{\eta} \varphi_2$;
3. $\sup_{\eta} \varphi_1 \leq \operatorname*{var}_{\eta} \varphi_1 + \inf_{\eta} \varphi_1 \leq \operatorname*{var}_{\eta} \varphi_1 + \dfrac{1}{m(\eta)} \displaystyle\int_{\eta} \varphi_1 \, dm$;
4. $\operatorname*{var}_{\eta} |\varphi_1| \leq \operatorname*{var}_{\eta} \varphi_1$;
5. $\operatorname*{var}_{\eta}(\varphi_1 \circ h) = \operatorname*{var}_{h(\eta)} \varphi_1$ *if $h : \eta \to h(\eta)$ is a homeomorphism.*

The claims in the next lemma also follow directly from the definition. See for instance [?] for proofs.

Lemma 1.3.2. *Suppose $\varphi : M \to \mathbb{R}$ has bounded variation on some interval $\eta \subset M$. Then*

1. *the restriction of φ to η can be written as the difference $\varphi_1 - \varphi_2$ of two nondecreasing functions;*
2. *φ has at most countably many discontinuity points;*
3. *the lateral limits $\lim_{x \to z^\pm} \varphi(x)$ exist at every point $z \in \eta$ (for points on the boundary consider only the limit from the inside of η).*

Now we prove Helly's theorem: sets of functions which are uniformly bounded and have uniformly bounded variation are relatively compact in $L^1(m)$.

Lemma 1.3.3. *Let $K_1, K_2 > 0$ and $\psi_n : M \to \mathbb{R}$, $n \geq 1$, be a sequence of functions on M such that $\sup \psi_n \leq K_1$ and $\operatorname{var} \psi_n \leq K_2$ for every $n \geq 1$.*

Then there exists a subsequence $(\psi_{n_k})_k$ and a function $\psi_0 : M \to \mathbb{R}$ with $\sup |\psi_0| \leq K_1$ and $\operatorname{var} \psi_0 \leq K_2$ such that ψ_{n_k} converges to ψ_0 as $k \to \infty$, m-almost everywhere and in $L^1(m)$.

Proof. We consider $M = [0,1]$, the case of the circle is analogous. Write

$$\psi_n^+(x) = \operatorname*{var}_{[0,x]} \psi_n \quad \text{and} \quad \psi_n^- = \psi_n^+ - \psi_n.$$

Then $(\psi_n^-)_n$ and $(\psi_n^+)_n$ are uniformly bounded sequences of nondecreasing functions. Choose $(n_k)_k$ so that $\psi_{n_k}^\pm(q)$ converges to some real number $\phi_0^\pm(q)$ as $k \to \infty$, for every rational $q \in [0,1]$. Clearly, $\phi_0^\pm(q_1) \leq \phi_0^\pm(q_2)$ whenever $q_1 \leq q_2$. Then, extend $\phi_0^\pm$ to nondecreasing functions in the whole $[0,1]$ by setting

$$\phi_0^\pm(x) = \inf\{\phi_0^\pm(q) : q \in [x,1] \cap \mathbb{Q}\}.$$

We claim that $\psi_{n_k}^\pm(x)$ converges to $\phi_0^\pm(x)$ as $k \to \infty$, for every continuity point x of $\phi_0^\pm$ (a co-countable set). Indeed, given any such x and any $\delta > 0$, we may fix rational numbers $q_1 \leq x \leq q_2$ such that

$$\phi_0^\pm(x) - \delta \leq \phi_0^\pm(q_1) \leq \phi_0^\pm(x) \leq \phi_0^\pm(q_2) \leq \phi_0^\pm(x) + \delta.$$

Then, for every sufficiently large k,

$$\phi_0^\pm(x) - 2\delta \leq \phi_0^\pm(q_1) - \delta \leq \psi_{n_k}^\pm(q_1) \leq \psi_{n_k}^\pm(x)$$

and, analogously, $\psi_{n_k}^\pm(x) \leq \phi_0^\pm(x) + 2\delta$. This proves the claim.

Next, let $\psi_0^\pm$ be right-continuous functions coinciding with $\phi_0^\pm$ at every point of continuity of $\phi_0^\pm$, and define $\psi_0 = \psi_0^+ - \psi_0^-$. It follows that ψ_{n_k} converges to ψ_0 except, possibly, on a countable set of points E. In particular, $\psi_{n_k} \to \psi_0$ m-almost everywhere and in $L^1(m)$. Finally,

$$|\psi_0(x)| = \lim_k |\psi_{n_k}(x)| \le \sup_k \sup \psi_{n_k} \qquad \text{and}$$

$$\sum_{j=1}^{s} |\psi_0(x_j) - \psi_0(x_{j-1})| = \lim_k \sum_{j=1}^{s} |\psi_{n_k}(x_j) - \psi_{n_k}(x_{j-1})| \le \sup_k \operatorname{var} \psi_{n_k},$$

for every x and $x_0 \le x_1 \le \cdots \le x_s$ in $[0,1] \setminus E$. Since ψ_0 is right-continuous, this implies that $\sup |\psi_0| \le K_1$ and $\operatorname{var} \psi_0 \le K_2$. □

It is useful to extend the notion of bounded variation to elements of the space $L^1(m)$, in the following way.

Definition 1.3.4. *The* variation $\operatorname{var}_\eta[\varphi]$ *of an element* $[\varphi] \in L^1(m)$ *on an interval* $\eta \subset M$ *is the infimum of the variations* $\operatorname{var}_\eta \varphi$ *on* η *taken over all representatives of* $[\varphi]$. *We say that* $[\varphi]$ *has* bounded variation on η *if* $\operatorname{var}_\eta[\varphi]$ *is finite.*

Remark 1.3.1. Suppose $[\varphi]$ has bounded variation. It is not difficult to check that there is always some representative φ that realizes the infimum in the definition of $\operatorname{var}_\eta[\varphi]$. Indeed, this happens if and only if φ is continuous at the boundary points of η, and $\varphi(z)$ is between $\lim_{x \to z^-} \varphi(x)$ and $\lim_{x \to z^+} \varphi(x)$ for every point z in the interior of η. In particular, such a function φ also realizes the L^∞-norm of $[\varphi]$, that is, $\sup |\varphi| = \|[\varphi]\|_\infty$. Moreover, the variation of φ on any $[a,b] \subset \eta$ coincides with the supremum of $\sum_{i=1}^{n} |\varphi(x_i) - \varphi(x_{i-1})|$ taken only over the sequences $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ such that φ is continuous at every x_i.

The following lemma will not be used in the present section, we include it only for reference in Subsection 1.4.1.

Lemma 1.3.4. *For any* $[\varphi] \in L^1(m)$ *with bounded variation and any interval* $\eta \in M$,

$$\operatorname*{var}_\eta [\varphi] = \sup \left\{ \Big| \int_\eta \varphi \, D\omega \, dm \Big| : \omega \in C_0^1(\eta) \quad \textit{with} \quad \sup |\omega| \le 1 \right\},$$

where $C_0^1(\eta)$ *is the space of all* C^1 *functions on* M *that are zero on the boundary of* η.

Proof. Let φ be a representative of $[\varphi]$ such that $\operatorname{var}_\eta \varphi = \operatorname{var}_\eta[\varphi] < \infty$. Let a and b be the endpoints of η. If $a \notin \eta$ then, according to Lemma 1.3.2, we can always extend φ to $\eta \cup \{a\}$, in such a way that the extension is continuous at the point a. Similarly for b. Moreover, this extension is also a representative of $[\varphi]$, and its variation on $[a,b]$ is equal to $\operatorname{var}_\eta \varphi$. Therefore, it is no restriction to suppose that η is compact, i.e. $\eta = [a,b]$.

The lemma is a consequence of claims (1) and (2) below.

(1) Given any sequence $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$, such that φ is continuous at x_i for every $0 \le i \le n$, and given any $\varepsilon > 0$, there exists $\omega \in C^1_0(\eta)$ with $\sup|\omega| \le 1$, satisfying

$$\Big|\int_\eta \varphi\, D\omega\, dm\Big| \ge \sum_{i=1}^n |\varphi(x_i) - \varphi(x_{i-1})| - \varepsilon.$$

Fix $\delta > 0$ such that $|\varphi(x) - \varphi(x_i)| \le \varepsilon/2n$ for every x such that $|x - x_i| \le \delta$, and every $0 \le i \le n$. Let ω be any C^1 function on η such that

- $\omega(x_i) = 0$ for $0 \le i \le n$, and ω is monotone (either nonincreasing or nondecreasing) on $[x_{i-1}, x_{i-1} + \delta]$ and on $[x_i - \delta, x_i]$, for all $1 \le i \le n$;
- $\omega|[x_{i-1} + \delta, x_i - \delta] \equiv \operatorname{sgn}(\varphi(x_{i-1}) - \varphi(x_i))$ for every $1 \le i \le n$.

Here sgn denotes the usual sign function $\operatorname{sgn}(z) = z/|z|$, with $\operatorname{sgn}(0) = 0$. Let $1 \le i \le n$ be fixed. Then

$$\int_{x_{i-1}}^{x_i} \varphi\, D\omega\, dm = \int_{x_{i-1}}^{x_{i-1}+\delta} \varphi\, D\omega\, dm + \int_{x_i-\delta}^{x_i} \varphi\, D\omega\, dm.$$

If $\operatorname{sgn}(\varphi(x_{i-1}) - \varphi(x_i)) = 1$ then $D\omega \ge 0$ in the first integral, and $D\omega \le 0$ in the second one. It follows that the integral of $\varphi\, D\omega$ on $[x_{i-1}, x_i]$ is bounded from below by

$$\begin{aligned}&\big(\varphi(x_{i-1}) - \frac{\varepsilon}{2n}\big) \int_{x_{i-1}}^{x_{i-1}+\delta} D\omega\, dm + \big(\varphi(x_i) + \frac{\varepsilon}{2n}\big) \int_{x_i-\delta}^{x_i} D\omega\, dm \\ &\qquad = \big(\varphi(x_{i-1}) - \frac{\varepsilon}{2n}\big) - \big(\varphi(x_i) + \frac{\varepsilon}{2n}\big) = |\varphi(x_i) - \varphi(x_{i-1})| - \frac{\varepsilon}{n}\,.\end{aligned}$$

Analogously, we get the same conclusion also when the sign is -1 or zero. Then, adding over all $1 \le i \le n$,

$$\int_a^b \varphi\, D\omega\, dm \ge \sum_{i=1}^n |\varphi(x_i) - \varphi(x_{i-1})| - \varepsilon.$$

This proves (1), which implies the inequality $\le$ in the statement.

(2) Given any function $\omega \in C^1_0(\eta)$ with $\sup|\omega| \le 1$, and given any $\varepsilon > 0$, there exists a sequence $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$, satisfying

$$\sum_{i=1}^n |\varphi(x_i) - \varphi(x_{i-1})| \ge \Big|\int_\eta \varphi\, D\omega\, dm\Big| - \varepsilon.$$

Let $n \ge 1$ and $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ be arbitrary. We use $\mathcal{E}$ to represent various expressions converging to zero as $\sup_i |x_i - x_{i-1}|$ converges to zero. On the one hand,

$$\int \varphi \, D\omega \, dm = \sum_{i=1}^{n} \varphi(x_i) D\omega(x_i)(x_i - x_{i-1}) + \mathcal{E} \, .$$

Since ω is C^1 and $\omega(x_0) = \omega(x_n) = 0$, the term on the right can also be written as

$$\sum_{i=1}^{n} \varphi(x_i)\big(\omega(x_i) - \omega(x_{i-1})\big) + \mathcal{E} = \sum_{i=2}^{n-1} \big(\varphi(x_{i-1}) - \varphi(x_i)\big)\omega(x_{i-1}) + \mathcal{E} \, .$$

As $\sup|\omega| \le 1$, it follows that

$$\Big| \int \varphi \, D\omega \, dm \Big| \le \sum_{i=2}^{n-1} |\varphi(x_i) - \varphi(x_{i-1})| + \mathcal{E} \le \sum_{i=1}^{n} |\varphi(x_i) - \varphi(x_{i-1})| + \mathcal{E} \, .$$

This means that any sequence with sufficiently small $\sup_i |x_i - x_{i-1}|$ satisfies the inequality in (2). □

1.3.3 Absolutely Continuous Invariant Measures

Let f be an expanding map of the circle or the interval, and $\mathcal{P}^1$ be a corresponding partition into regularity intervals. Here, as well as in the next subsection, we assume that $\mathcal{P}^1$ is finite. However, *finiteness is used only in Proposition 1.3.1 and in Corollary 1.3.3.*

For every $n \ge 1$, let $\mathcal{P}^n$ be the partition into regularity intervals of f^n, defined by $\mathcal{P}^n(x) = \mathcal{P}^n(y)$ if and only if $\mathcal{P}^1(f^j(x)) = \mathcal{P}^1(f^j(y))$ for all $0 \le j < n$. For each $n \ge 1$ and $\eta \in \mathcal{P}^n$, we define $g_\eta^n : M \to \mathbb{R}$ by

$$g_\eta^n(y) = \frac{1}{|Df^n|} \circ (f^n \mid \eta)^{-1}(y) \quad \text{ if } y \in f^n(\eta),$$

and $g_\eta^n(y) = 0$ otherwise. For simplicity, we also write $g_\eta = g_\eta^1$ for $\eta \in \mathcal{P}^1$.

Observe that, cf. part 5 of Lemma 1.3.1,

$$\sup g_\eta = \sup \hat{g}_\eta \quad \text{and} \quad \operatorname{var} g_\eta \le \operatorname{var} \hat{g}_\eta \, , \tag{1.13}$$

where $\hat{g}_\eta$ was defined in (1.11). Typically, f maps η homeomorphically onto $f(\eta)$ and, using Lemma 1.3.1, the two variations in (1.13) coincide; the inequality may be strict only if $f(\eta)$ is one of the sets S^1, $[0,1]$, $[0,1)$, or $(0,1]$.

In general, given any $\varphi : M \to \mathbb{R}$ we consider $\varphi \circ (f^n \mid \eta)^{-1}$ as a function on M, identically zero in the complement of $f^n(\eta)$.

Lemma 1.3.5. *For every integrable function $\varphi : M \to \mathbb{R}$, and every $n \ge 1$, the iterate of Lebesgue measure under f^n can be written $f_*^n(\varphi m) = \varphi_n m$, with*

$$\varphi_n = \sum_\eta g_\eta^n \cdot \big(\varphi \circ (f^n \mid \eta)^{-1}\big)$$

where the sum is over all $\eta \in \mathcal{P}^n$ such that $m(\eta) > 0$.

Proof. Let $B \subset M$ be an arbitrary measurable set. By definition,

$$f_*^n(\varphi m \mid \eta)(B) = \varphi m(f^{-n}(B) \cap \eta) = \int_{f^{-n}(B)\cap\eta} \varphi(x)\, dm(x).$$

Changing to the variable $y = (f^n \mid \eta)(x)$, we find

$$f_*^n(\varphi m \mid \eta)(B) = \int_{B\cap f^n(\eta)} \frac{\varphi}{|Df^n|} \circ (f^n \mid \eta)^{-1}(y)\, dm(y).$$

The term on the right can be rewritten as $\int_B g_\eta^n \cdot \big(\varphi \circ (f^n \mid \eta)^{-1}\big)\, dm$, since the functions in this last integral are identically zero outside $f^n(\eta)$. This proves that the density of each $f_*^n(\varphi m \mid \eta)$ is given by $g_\eta^n \cdot \big(\varphi \circ (f^n \mid \eta)^{-1}\big)$, for every $\eta \in \mathcal{P}^n$. On the other hand,

$$f_*^n(\varphi m) = \sum_{\eta\in\mathcal{P}^n} f_*^n(\varphi m \mid \eta),$$

and the intervals η with $m(\eta) = 0$ play no role in this sum, since the corresponding term $f_*^n(\varphi m)$ is identically zero. The conclusion of the lemma follows by summing over all the $\eta \in \mathcal{P}^n$ having nonzero Lebesgue measure.

□

Remark 1.3.2. By definition, for every $n \geq 1$ we have

$$\int \varphi_n\, dm = \int 1\, d(f_*^n(\varphi m)) = \int 1\, d(\varphi m) = \int \varphi\, dm.$$

Let $|\varphi|_n$ be the sequence one obtains as in the previous lemma, when φ is replaced by $|\varphi|$. Then $-|\varphi|_n \leq \varphi_n \leq |\varphi|_n$, since $-|\varphi| \leq \varphi \leq |\varphi|$. In particular,

$$\int |\varphi_n|\, dm \leq \int |\varphi|_n\, dm = \int |\varphi|\, dm \quad \text{for every } n \geq 1. \tag{1.14}$$

Lemma 1.3.6. *There exist $C_1 > 0$ and $0 < \lambda_1 < 1$ such that*

$$\sup g_\eta^n \leq C_1\lambda_1^n \quad \textit{and} \quad \operatorname{var} g_\eta^n \leq C_1\lambda_1^n \quad \textit{for every } n \geq 1 \textit{ and } \eta \in \mathcal{P}^n.$$

Proof. The first claim is a direct consequence of the definition of g_η^n and the expansivity condition (E2): it is enough to take $C_1 > 1/C$ and $\lambda_1 > 1/\sigma$.

Next, given $\eta \in \mathcal{P}^n$ and $0 \leq j < n$, let $\xi_j \in \mathcal{P}^j$, $\eta_j \in \mathcal{P}^1$, $\zeta_j \in \mathcal{P}^{n-j-1}$ be defined by

$$\eta \subset \xi_j, \qquad f^j(\eta) \subset \eta_j, \qquad f^{j+1}(\eta) \subset \zeta_j.$$

By definition, there exists some constant $C_2 > 0$ such that, for every $\xi \in \mathcal{P}^1$,

$$\begin{aligned}\operatorname{var} g_\xi &\leq \operatorname*{var}_{f(\xi)} \frac{1}{|Df|} \circ (f \mid \xi)^{-1} + 2 \sup_{f(\xi)} \frac{1}{|Df|} \circ (f \mid \xi)^{-1} \\ &\leq \operatorname*{var}_{\xi} \frac{1}{|Df|} + 2 \sup_{\xi} \frac{1}{|Df|} \leq C_2,\end{aligned}$$

(the supremum term bounds the variation of g_ξ at the boundary points of $f(\xi)$). Since $g_\eta^n = g_{\xi_j}^j \, g_{\eta_j} \, g_{\zeta_j}^{n-j-1}$ for every $0 \le j < n$, using induction and Lemma 1.3.1 we obtain

$$\operatorname{var} g_\eta^n \le \sum_{j=0}^{n-1} \sup g_{\xi_j}^j \;\operatorname{var} g_{\eta_j} \;\sup g_{\zeta_j}^{n-j-1} \le \sum_{j=0}^{n-1} \frac{C_2}{C^2 \sigma^{n-1}} = (C_2 C^{-2} \sigma) n \sigma^n .$$

Fixing $\lambda_1 > 1/\sigma$ and choosing $C_1 > \sup\{C_2 C^{-2} \sigma \, n \, (\lambda_1 \sigma)^{-n} : n \ge 1\}$, we get the second claim. □

The crucial ingredient for most results in this section is the following result of [68], stating that the variations of the densities φ_n tend to decrease as n grows.

Proposition 1.3.1. *There are $C_0 > 0$ and $0 < \lambda_0 < 1$ such that, given any bounded variation function $\varphi : M \to \mathbb{R}$,*

$$\operatorname{var} \varphi_n \le C_0 \, \lambda_0^n \;\operatorname{var} \varphi + C_0 \int |\varphi| dm$$

for every $n \ge 1$, where φ_n is as in Lemma 1.3.5.

Proof. Combining Lemma 1.3.6 with properties in Lemma 1.3.1 we find

$$\begin{aligned} \operatorname{var} \varphi_n &\le \sum_\eta \operatorname{var} g_\eta^n \sup_\eta |\varphi| + \sup g_\eta^n \big(\operatorname*{var}_\eta \varphi + 2 \sup_\eta |\varphi| \big) \\ &\le \sum_\eta 3 C_1 \lambda_1^n \sup_\eta |\varphi| + C_1 \lambda_1^n \operatorname*{var}_\eta \varphi. \end{aligned} \tag{1.15}$$

Note that the expression in parentheses is an upper bound for the variation of $\varphi \circ (f^n \mid \eta)^{-1}$, taking into account the jumps at the boundary of $f^n(\eta)$. Using the third property in Lemma 1.3.1 we get

$$\begin{aligned} \operatorname{var} \varphi_n &\le \sum_\eta 4 C_1 \lambda_1^n \operatorname*{var}_\eta \varphi + 3 C_1 \lambda_1^n \frac{1}{m(\eta)} \int_\eta |\varphi| \, dm \\ &\le 4 C_1 \lambda_1^n \operatorname{var} \varphi + K(n) \int |\varphi| \, dm , \end{aligned} \tag{1.16}$$

where $K(n) = 3 C_1 \lambda_1^n \sup\{1/m(\eta) : \eta \in \mathcal{P}^n\}$. Recall that we only have to deal with intervals $\eta \in \mathcal{P}^n$ with positive Lebesgue measure.

This is close to what we want, but we still have to explain why $K(n)$ can be replaced by some constant independent of n. For that, we fix $N \ge 1$ such that $4 C_1 \lambda_1^N \le 1/2$, and we denote $K_0 = \max\{K(n) : 1 \le n \le N\}$. Then, given any $n \ge 1$, we write $n = qN + r$ with $q \ge 0$ and $1 \le r \le N$. Using the previous bound with φ replaced by $\varphi_{n-N}, \ldots, \varphi_{n-qN}$, and φ, respectively, and recalling (1.14),

$$\begin{aligned}\operatorname{var}\varphi_n &\le K_0 \int |\varphi_{n-N}|\, dm + \frac{1}{2}\operatorname{var}\varphi_{n-N}\\ &\le \left(1+\cdots+2^{-q+1}\right) K_0 \int |\varphi| dm + \frac{1}{2^q}\operatorname{var}\varphi_r\\ &\le \left(1+\cdots+2^{-q}\right) K_0 \int |\varphi| dm + \frac{1}{2^q} 4C_1\lambda_1^r \operatorname{var}\varphi.\end{aligned}$$

To finish the proof of the proposition, choose $C_0 \ge \max\{2K_0, 4C_1\}$ and $\lambda_0 \ge \max\{2^{-1/N}, \lambda_1\}$. □

Remark 1.3.3. Let ν and μ_n, $n \ge 1$, be finite measures on a compact metric space, such that μ_n is absolutely continuous with respect to ν for every $n \ge 1$. If the Radon-Nikodym derivatives $d\mu_n/d\nu$ converge in $L^1(\nu)$ to some function ψ then μ_n converges to $\mu = \psi\nu$ in the weak* topology. Indeed, given any continuous function $\varphi : M \to \mathbb{R}$,

$$\Big|\int \varphi\, d\mu_n - \int \varphi\, d\mu\Big| \le \int |\varphi|\, \Big|\frac{d\mu_n}{d\nu} - \psi\Big|\, d\nu \le \sup|\varphi|\, \Big\|\frac{d\mu_n}{d\nu} - \psi\Big\|_1,$$

and the last term converges to zero as $n \to \infty$.

In particular, the densities ϕ_n of the iterates $f_*^n m$ of Lebesgue measure are uniformly bounded: by Proposition 1.3.1 and Remark 1.3.2,

$$\sup\phi_n \le \operatorname{var}\phi_n + \int \phi_n\, dm \le (C_0+1)\int 1\, dm = C_0 + 1, \qquad (1.17)$$

for every $n \ge 1$. Hence, as in Corollary 1.2.2, we may conclude that f has some absolutely continuous invariant measure. In fact, we can prove more:

Corollary 1.3.1. *The map f has some absolutely continuous invariant probability measure μ whose density $d\mu/dm$ has bounded variation.*

Proof. Let ϕ_n be as above, and define $\psi_n = n^{-1}\sum_{j=0}^{n-1}\phi_j$ for each $n \ge 1$. Then $\mu_n = n^{-1}\sum_{j=0}^{n-1} f_*^j m$ can be written as $\mu_n = \psi_n m$. Proposition 1.3.1 implies that $\operatorname{var}\phi_j \le C_0$ for every j, and so $\operatorname{var}\psi_n \le C_0$ for every $n \ge 1$. Moreover, by (1.17), we have $\sup|\psi_n| \le C_0 + 1$ for every $n \ge 1$. This means that we may apply Lemma 1.3.3 to conclude that there exists a subsequence $(\psi_{n_k})_k$ converging in $L^1(m)$ to some function φ_0 with $\operatorname{var}\varphi_0 \le C_0$. In particular, cf. Remark 1.3.3, $(\mu_{n_k})_k$ converges to $\mu = \varphi_0\, m$ in the weak* topology. This ensures that μ is invariant under f. □

1.3.4 Bounding the Number of Physical Measures

Now we show that a piecewise expanding map has finitely ergodic absolutely continuous invariant measures. This was proved under a C^2 assumption by [69], see also [67]. Then [126] observed that the bounded variation condition (E1) in Definition 1.3.1 suffices for the argument. The main step is the following consequence of Proposition 1.3.1.

Proposition 1.3.2. *Any absolutely continuous probability measure ν of a piecewise expanding map can be written $\nu = \theta\, m$ where θ has bounded variation.*

Proof. By assumption, one may write $\nu = \psi\, m$ for some integrable function ψ with L^1-norm $\|\psi\|_1 = \int |\psi|\, dm$ equal to 1. We want to prove that ψ coincides Lebesgue almost everywhere with some bounded variation function θ.

Let $(\xi_l)_l$ be some sequence of bounded variation functions converging to ψ in $L^1(m)$. It is no restriction to suppose that $\|\xi_l\|_1 \leq 2$ for all l. For each $n \geq 1$ and $l \geq 1$, let

$$f_*^n(\psi m) = \psi_n m \quad \text{and} \quad f_*^n(\xi_l m) = \xi_{l,n} m$$

where ψ_n and $\xi_{l,n}$ are obtained from ψ and ξ_l, respectively, as in Lemma 1.3.5. Note that, since $\nu = \psi m$ was taken invariant, $f_*^n(\psi\, m) = \psi\, m$ and so $\psi_n = \psi$ Lebesgue almost everywhere. Applying Proposition 1.3.1 to ξ_l we get that

$$\operatorname{var} \xi_{l,n} \leq C_0 \lambda_0^n \operatorname{var} \xi_l + C_0 \int |\xi_l|\, dm \leq 3\, C_0$$

for every large enough n. Moreover, by Lemma 1.3.1 and Remark 1.3.2,

$$\sup |\xi_{l,n}| \leq \operatorname{var} \xi_{l,n} + \int |\xi_{l,n}|\, dm \leq 3\, C_0 + \int |\xi_l|\, dm \leq 3\, C_0 + 2.$$

Hence, the sequence $(\xi_{l,n})_n$ satisfies the assumptions of Lemma 1.3.3, for each fixed l. As a consequence, there exists some function θ_l with $\sup |\theta_l| \leq 3C_0 + 2$ and $\operatorname{var} \theta_l \leq 3C_0$, and there exists a subsequence $(n_k)_k$ such that $(\xi_{l,n_k})_k$ converges to θ_l in $L^1(m)$.

On the one hand, using (1.14) for the function $\varphi = \xi_l - \psi$ we get that

$$\|\theta_l - \psi\|_1 = \lim_{k \to \infty} \|\xi_{l,n_k} - \psi\|_1 = \lim_{k \to \infty} \|\xi_{l,n_k} - \psi_{n_k}\|_1 \leq \|\xi_l - \psi\|_1\,,$$

for every l. This implies that θ_l converges to ψ in $L^1(m)$, since ξ_l does. On the other hand, we may apply Lemma 1.3.3 to the sequence $(\theta_l)_l$, to conclude that some subsequence $(\theta_{l_j})_j$ converges in $L^1(m)$ to a function θ with $\operatorname{var} \theta \leq 3C_0$. This implies that $\psi = \theta$ Lebesgue almost everywhere. Therefore, $\nu = \psi\, m = \theta\, m$. □

Remark 1.3.4. We even obtained a uniform bound $3C_0$ for the variation of any density $\psi \in L^1(m)$ of any absolutely continuous invariant probability measure.

Lemma 1.3.7. *Given any f-invariant set $A \subset M$ with positive Lebesgue measure, there exists some absolutely continuous f-invariant probability measure ν_A such that $\nu_A(A) = 1$.*

Proof. Let $\hat{A}$ be the set of points $x \in A$ such that every neighbourhood of x intersects A in a positive Lebesgue measure subset. By Lebesgue's differentiation theorem, $\hat{A}$ contains a full Lebesgue measure subset of A, and so $m(\hat{A}) = m(A) > 0$. Moreover, $\hat{A}$ is close to being invariant, in the following sense. Let $x \in \hat{A}$ and suppose that x is not a singular point. Then f is a local diffeomorphism near x. In particular, since every neighbourhood of x intersects A in a positive Lebesgue measure subset, the same is true for $f(x)$. In other words, $f(x)$ is also in $\hat{A}$. Thus, we have shown that $\hat{A}$ is contained in the union of $f^{-1}(\hat{A})$ with the (finite) set of singular points. Then, $\hat{A}$ is also contained in the union of $f^{-n}(\hat{A})$ with some finite set, for every $n \geq 1$.

Let $(m \mid \hat{A})$ represent the normalized restriction of Lebesgue measure to $\hat{A}$, and consider the sequence of probability measures

$$\mu_{A,n} = \frac{1}{n} \sum_{j=0}^{n-1} f_*^j(m \mid \hat{A}).$$

By definition, $(m \mid \hat{A}) = \varphi m$ where $\varphi = \mathcal{X}_{\hat{A}} / m(\hat{A})$. Let φ_n be the density of $f_*^n(m \mid \hat{A})$, and ϕ_n be the density of $f_*^n m$, as given in Lemma 1.3.5. Then, cf. (1.17),

$$\varphi_j = \frac{1}{m(\hat{A})}\, \phi_j \leq \frac{C_0 + 1}{m(\hat{A})} \quad \text{for every } j \geq 0.$$

This implies that $\mu_{A,n}$ admits a density bounded by $(C_0 + 1)/m(\hat{A})$ for every $n \geq 1$. It follows from Lemma 1.2.2 that every accumulation point of the sequence $\mu_{A,n}$ is an absolutely continuous invariant measure. Let μ_A be any of these invariant measures.

The property of almost invariance of $\hat{A}$ we proved before implies that $f_*^j(m \mid \hat{A})(\hat{A}) \geq (m \mid \hat{A})(\hat{A}) = 1$ for every $j \geq 1$. This gives $\mu_{A,n}(\hat{A}) = 1$ for every $n \geq 1$, and so $\mu_A(\operatorname{clos}(\hat{A})) = 1$. On the other hand, by Proposition 1.3.2, we have $\mu_A = \theta_A m$ for some function θ_A with bounded variation. Since bounded variation functions have at most countably many discontinuity points, there exist some open interval $J \subset M$ and some $\delta > 0$ such that $\theta_A(x) > \delta$ for every $x \in J$. This implies that the restrictions of μ_A and m to J are equivalent measures. As the closure of $\hat{A}$ has full μ_A-measure, the set $J \setminus \operatorname{clos}(\hat{A})$ has zero μ_A-measure and, hence, zero Lebesgue measure. In particular, $\hat{A}$ has some point in J. By the definition of $\hat{A}$, it follows that $A \cap J$ has positive Lebesgue measure. Then, $\mu_A(A) \geq \delta m(A \cap J) > 0$.

Finally, let ν_A be the normalized restriction of μ_A to A:

$$\nu_A(B) = \frac{\mu_A(B \cap A)}{\mu_A(A)} \quad \text{for any measurable set } B.$$

Then $\nu_A(A) = 1$ and ν_A is absolutely continuous with respect to Lebesgue measure, since it is absolutely continuous with respect to μ_A. Moreover, ν_A is invariant under f, because A and μ_A are f-invariants. □

Corollary 1.3.2. *Every f-invariant set $A \subset M$ with positive Lebesgue measure has full Lebesgue measure in the neighbourhood of some singular point: there are $\varepsilon > 0$ and a singular point c of f such that $m([c-\varepsilon, c+\varepsilon] \setminus A) = 0$.*

Proof. Let ν be any absolutely continuous invariant measure such that $\nu(A) = 1$. By Proposition 1.3.2, $\nu = \theta m$ for some function θ with bounded variation. In particular, there exists an open interval $J \subset M$ such that the infimum of θ on J is strictly positive. Since $\nu(J \setminus A) = 0$, in view of our choice of ν, this implies that $m(J \setminus A) = 0$.

Now we consider the iterates $f^n(J)$, $n \geq 1$, of the interval J. The expansivity condition (E2) in Definition 1.3.1 implies that

$$\operatorname{length}(f^n(J)) \geq C\sigma^n \operatorname{length}(J)$$

as long as $f^j(J)$ does not intersect the singular set of f, for any $0 \leq j \leq n-1$. So, since $\sigma > 1$ whereas the term on the left is bounded by the diameter of M, there must be a first time $N \geq 1$ such that $f^N(J)$ contains some singular point c. In particular, $f^N \mid J$ is a diffeomorphism onto its image. Then, $f^N(J)$ is an open interval and $m(f^N(J) \setminus A) = m(f^N(J \setminus A)) = 0$. □

The last step in the proof of Theorem 1.3.1 is

Corollary 1.3.3. *The map f has some ergodic absolutely continuous invariant probability measure, and the number of such measures does not exceed the number s of singular points of f. Moreover, their basins cover a full Lebesgue measure subset of M.*

Proof. It follows from Corollary 1.3.2 that there are at most s two-by-two disjoint f-invariant sets with positive Lebesgue measure. As a consequence, the manifold M can be partitioned into $r \leq s$ invariant sets $A_1, \ldots, A_r$, such that $m(A_i) > 0$ for every $1 \leq i \leq r$, and which are minimal: there is no invariant set $B_i \subset A_i$ with $0 < m(B_i) < m(A_i)$. As we have seen, for each $1 \leq i \leq r$ there exists an absolutely continuous invariant measure ν_i such that $\nu_i(A_i) = 1$. The fact that A_i is minimal implies that ν_i is ergodic.

Moreover, given any absolutely continuous invariant measure μ we may write $\mu = \sum_i \mu(A_i)\mu_i$, where the sum is over all the values of i such that $\mu(A_i) > 0$, and μ_i denotes the normalized restriction of μ to A_i. Since ν_i and μ_i are both ergodic, either they coincide or they are mutually singular. The second possibility would contradict the assumption that A_i is minimal, and so we must have $\mu_i = \nu_i$. This proves that $\nu_1, \ldots, \nu_r$ are precisely the ergodic absolutely continuous invariant measures of f.

Finally, let E be the complement of $B(\nu_1) \cup \cdots \cup B(\nu_r)$ in M. Then E is f invariant, and $\nu_i(E) = 0$ for every $1 \leq i \leq r$. As a consequence, E has zero measure with respect to any absolutely continuous invariant measure of f. By Lemma 1.3.7 the set E must have zero Lebesgue measure. □

Example 1.3.5. In particular, maps with a unique singular point have a unique absolutely continuous invariant probability measure, and it is ergodic. This includes the tent maps and the Lorenz-like maps as in Figure 1.3.

On the other hand, piecewise expanding maps with more than one singular point may have several physical measures:

Example 1.3.6. Let $g : [0,1] \to [0,1]$ be the full tent map $g(x) = 1-2|x-1/2|$. Define $f : [0,1] \to [0,1]$ so that its restrictions to both $[0,1/2]$ and $(1/2,1]$ be rescaled copies of g. In precise terms,

$$f(x) = \frac{1}{2}g(2x) \text{ if } x \in [0,1/2] \quad \text{and} \quad f(x) = \frac{1}{2}g(2x-1) + \frac{1}{2} \text{ if } x \in (1/2,1].$$

Then, both $2\mathcal{X}_{[0,1/2]}m$ and $2\mathcal{X}_{[1/2,1]}m$ are invariant probability measures, ergodic and absolutely continuous.

This shows that to ensure uniqueness one needs some assumption of indivisibility of the dynamics. Observe that indivisibility was needed also in the case of smooth expanding maps, treated in Section 1.2, but we did not have to assume it because in that case it comes for free from the other hypotheses, cf. Lemma 1.2.5.

Definition 1.3.5. *We say that a piecewise expanding map $f : M \to M$ is* transitive *if there exists some compact $I_* \subset M$ such that*

(1) $f(I_*) \subset I_*$ and the orbit $f^n(x)$, $n \geq 0$, of Lebesgue almost every $x \in M$ intersects the interior of I_*;
(2) given any pair of intervals V_1 and V_2 contained in I_*, there exists $n \geq 0$ such that $f^n(V_1) \cap V_2$ has positive Lebesgue measure.

Corollary 1.3.4. *If f is transitive then it has a unique absolutely continuous invariant probability measure μ. Moreover, the support of μ coincides with I_*.*

Proof. As part of the proof of Corollary 1.3.3, we showed that any absolutely continuous invariant measure can be written as a linear combination of finitely many ergodic ones $\nu_1, \ldots, \nu_s$. Therefore, to obtain the first claim we only have to prove that there exists at most one of these ergodic measures.

Let $1 \leq i \leq s$. We also know, from Corollary 1.3.2, that there exists some interval U_i such that $m(U_i \setminus B(\nu_i)) = 0$. Transitivity implies that for Lebesgue almost any point $x \in U_i \cap B(\nu_i)$ there exists $n \geq 1$ such that $f^k(x)$ is in the interior of I_*. Take x such that neither iterate $f^j(x)$, $j \geq 0$, is a singular point. Then f^k is a local diffeomorphism near x. Hence, there exists some open neighbourhood $V_i \subset U_i$ of x, such that $f^k(V_i) \subset I_*$. Moreover,

$$m(f^k(V_i) \setminus B(\nu_i)) = m(f^k(V_i \setminus B(\nu_i))) = 0.$$

Consequently, $m(W_i \setminus B(\nu_i)) = 0$ where $W_i = \cup_{n=k}^{\infty} f^n(V_i)$. On the other hand, the second condition in Definition 1.3.5 implies that $W_i \cap W_j$ has positive Lebesgue measure, for any $1 \le j \le s$. In particular, $B(\nu_i) \cap B(\nu_j)$ must be nonempty, which implies that $\nu_i = \nu_j$. This proves uniqueness and ergodicity.

The proof of the second claim is similar. Since the density of μ has bounded variation, there exists an interval U such that $d\mu/dm > 0$ on U. Then, U is contained in the support of μ. As before, we can find $k \ge 1$ and an open interval $V \subset U$ such that $f^k(V) \subset I_*$. On the one hand, $W = \cup_{n=k}^{\infty} f^n(V)$ is contained in the support of μ. On the other, the properties in Definition 1.3.5 imply that W is a dense subset of I_*. Thus, I_* is contained in the support. Finally, since μ is ergodic and I_* is an invariant set with nonzero μ-measure, $\mu(M \setminus I_*) = 0$. As I_* is compact, this implies that no point outside I_* is in the support of μ. □

Let $f : I \to I$ be a tent map as in Figure 1.3, with c denoting the singular point. We already know that f has a unique absolutely continuous invariant measure μ. As an application of the previous corollary, we show that if the derivative Df is large enough then the support of μ coincides with the interval $I_* = [f^2(c), f(c)]$. Moreover, in that case the map is transitive. Neither of these conclusions is true, in general, if the derivative is only larger than 1.

Lemma 1.3.8. *If f is a tent map with $|Df(x)| \ge \sigma > \sqrt{2}$ then f is transitive, with $I_* = [f^2(c), f(c)]$ and a strong form of property (2): for any interval $J \subset I_*$ there exists $N \ge 1$ such that $f^N(J) = I_*$.*

Proof. The first condition in Definition 1.3.5 is easy: since $|Df| > 1$, no orbit $f^n(x)$ with $x \ne 0, 1$ can remain forever in $[0, f^2(c)] \cup [f(c), 1]$. We are left to prove the last statement in the lemma.

We claim that $f^n(J)$ must eventually contain the fixed point $p > c$ of f. Indeed, otherwise one would be able to construct a sequence of intervals

$$J = J_0 \supset J_1 \supset \cdots \supset J_n \supset \cdots$$

in the following way.

- If $f^{n-1}(J_{n-1})$ does not contain c, take $J_n = J_{n-1}$. Observe that in this case $m(f^n(J_n)) = \sigma\, m(f^{n-1}(J_{n-1}))$.
- If $f^{n-1}(J_{n-1})$ does contain c, then take $J_n \subset J_{n-1}$ such that $f^{n-1}(J_n)$ coincides with the largest of the two intervals

$$f^{n-1}(J_{n-1}) \cap [f^2(c), c] \quad \text{or} \quad f^{n-1}(J_{n-1}) \cap [c, f(c)].$$

  Then $m(f^n(J_n)) \ge (\sigma/2) m(f^{n-1}(J_{n-1}))$.

Moreover, in this last case $f^n(J_n)$ can not contain c, since we suppose that $p \notin f^{n+1}(J_n)$. This means that for the construction of J_{n+1} we fall in the first case: $J_{n+1} = J_n$. In particular,

$$m(f^{n+1}(J_{n+1})) \geq \sigma m(f^n(J_n)) \geq \frac{\sigma^2}{2} m(f^{n-1}(J_{n-1})).$$

As we suppose $\sigma^2 > 2$, it follows that the sequence $m(f^n(J_n))$ is unbounded, a contradiction. This proves that $p \in f^{n_1}(J_{n_1})$ for some $n_1 \geq 0$, as we claimed.

Then, $p \in f^n(J)$ for every $n \geq n_1$ and, by expansivity, $[c,p] \subset f^{n_2}(J)$ for some $n_2 \geq n_1$. It follows that $[f^3(c), f(c)] \subset f^{n_2+3}(J)$. On the other hand, it is easy to check that $f^3(c) < p$ for all $\sigma > \sqrt{2}$. Now there are two cases to consider. If $f^3(c) \leq c$, we get $[f^2(c), f(c)] \subset f^{n_2+4}(J)$, as we wanted to prove. Otherwise, there must be some odd number $k > 3$ such that $f^k(c) < c$ and $[f^k(c), f(c)] \subset f^{n_2+k}(J)$. Then $[f^2(c), f(c)] \subset f^{n_2+k+1}(J)$ and this also proves that I_* is contained in some iterate of J. □

Example 1.3.7. A similar argument applies if f is a Lorenz-like map with $|Df(x)| \geq \sigma > \sqrt{2}$ for all $x \neq c$. See [51]. Assuming $f(c^-) = 1$, $f(c^+) = 0$, as in Figure 1.3, then for any subinterval J of $I_* = [0,1]$ there exists $N \geq 1$ such that $f^N(J) = I_*$. Therefore, the map is transitive. Furthermore, the support of the absolutely continuous invariant measure is the whole interval $[0,1]$.

1.3.5 Maps With Infinitely Many Branches

In this subsection we prove Theorem 1.3.2, and present a few useful applications. Throughout, f is a piecewise expanding map as in the theorem: there exists a partition $\mathcal{P}^1$ into regularity intervals of f, such that

$$\sum_{\eta \in \mathcal{P}^1} \operatorname{var} \hat{g}_\eta < \infty.$$

As was pointed out before, finiteness of regularity intervals intervened only in two occasions while we were proving Theorem 1.3.1: in Proposition 1.3.1 and in Corollary 1.3.3. Our first step is to extend the proposition to the present situation. For this we need the following lemma.

Let $\hat{g}_\xi$ be the weight functions introduced in (1.11), for each $\xi \in \mathcal{P}^1$. More generally, given any $i \geq 1$ and any interval J contained in an element η of $\mathcal{P}^i$, we define $\hat{g}^i_J : M \to \mathbb{R}$ by

$$\hat{g}^i_J(x) = \frac{1}{|Df^i(x)|} \quad \text{if } x \in J \quad \text{and} \quad \hat{g}^i_J(x) = 0 \quad \text{otherwise.}$$

It is clear from the definitions that the variation of these functions depends monotonically on the domain: if $J \subset L$ then $\operatorname{var} \hat{g}^i_J \leq \operatorname{var} \hat{g}^i_L$.

Lemma 1.3.9. *For every $n \geq 1$,*

$$\sum_{\eta \in \mathcal{P}^n} \operatorname{var} g^n_\eta \leq \sum_{\eta \in \mathcal{P}^n} \operatorname{var} \hat{g}^n_\eta < \infty.$$

Proof. Using part 5 of Lemma 1.3.1, one checks easily that $\operatorname{var} g_\eta^n \le \operatorname{var} \hat{g}_\eta^n$ for every $\eta \in \mathcal{P}^n$. In fact, these variations can be different only if f^n maps η onto the whole M. Compare (1.13). The first inequality is a direct consequence.

Next, given $n \ge 1$ and any $\eta \in \mathcal{P}^{n+1}$,

$$\hat{g}_\eta^{n+1} = \big(\hat{g}_{f^n(\eta)} \circ f^n\big)\hat{g}_\eta^n.$$

So, using Lemma 1.3.1 once more,

$$\operatorname{var} \hat{g}_\eta^{n+1} \le \operatorname{var} \hat{g}_{f^n(\eta)} \sup \hat{g}_\eta^n + \sup \hat{g}_{f^n(\eta)} \operatorname{var} \hat{g}_\eta^n .$$

Observe that $\sup \hat{g}_\eta^n \le \operatorname{var} \hat{g}_\eta^n$, because $\inf \hat{g}_\eta^n = 0$, and analogously for $\hat{g}_{f^n(\eta)}$. Let $\xi \in \mathcal{P}^n$ and $\zeta \in \mathcal{P}^1$ be defined by $\eta \subset \xi$ and $f^n(\eta) \subset \zeta$. Then, the previous inequality can be replaced by

$$\operatorname{var} \hat{g}_\eta^{n+1} \le 2 \operatorname{var} \hat{g}_{f^n(\eta)} \operatorname{var} \hat{g}_\eta^n \le 2 \operatorname{var} \hat{g}_\zeta \operatorname{var} \hat{g}_\xi^n .$$

Since each pair $(\xi, \zeta) \in \mathcal{P}^n \times \mathcal{P}^1$ determines $\eta \in \mathcal{P}^{n+1}$ uniquely, we obtain

$$\sum_{\eta \in \mathcal{P}^{n+1}} \operatorname{var} \hat{g}_\eta^{n+1} \le 2 \sum_{\zeta \in \mathcal{P}^1} \operatorname{var} \hat{g}_\zeta \sum_{\xi \in \mathcal{P}^n} \operatorname{var} \hat{g}_\xi^n .$$

Now the claim in the lemma follows immediately, by induction on n. □

Now we can prove, following [**?**], that Proposition 1.3.1 remains valid in the generality of Theorem 1.3.2.

Proposition 1.3.3. *There are $C_0 > 0$ and $0 < \lambda_0 < 1$ such that, given any bounded variation function $\varphi : M \to \mathbb{R}$,*

$$\operatorname{var} \varphi_n \le C_0 \lambda_0^n \operatorname{var} \varphi + C_0 \int |\varphi| \, dm$$

for any $n \ge 1$, where φ_n is as in Lemma 1.3.5.

Proof. Using Lemmas 1.3.5 and 1.3.1 we get, as in (1.15),

$$\operatorname{var} \varphi_n \le \sum_\eta \operatorname{var} g_\eta^n \sup_\eta |\varphi| + \sup g_\eta^n (\operatorname*{var}_\eta \varphi + 2 \sup |\varphi|),$$

recall that the sum is over the intervals $\eta \in \mathcal{P}^n$ that have positive Lebesgue measure. Moreover, $\sup g_\eta^n \le \operatorname{var} g_\eta^n$ because $\inf g_\eta = 0$. So, the previous inequality may be replaced by

$$\operatorname{var} \varphi_n \le \sum_\eta \operatorname{var} g_\eta^n (3 \sup_\eta |\varphi| + \operatorname*{var}_\eta \varphi). \tag{1.18}$$

The terms involving variation pose no problem: using Lemma 1.3.6,

$$\sum_{\eta} \operatorname{var} g_{\eta}^{n} \operatorname*{var}_{\eta} \varphi \leq C_1 \lambda_1^n \sum_{\eta} \operatorname*{var}_{\eta} \varphi \leq C_1 \lambda_1^n \operatorname{var} \varphi.$$

One would like to replace supremum by variation and integral in the remaining terms, using part 3 of Lemma 1.3.1, as we did before in (1.16). The problem is that the measure $m(\eta)$ of these intervals is no longer bounded away from zero.

To bypass this, we split the sum into two parts. Given any finite subset $\mathcal{Q}^n$ of $\mathcal{P}^n$, we may estimate the sum corresponding to the intervals $\eta \in \mathcal{Q}^n$ in the same way as in the finite case. On the other hand, using the summability in Lemma 1.3.9, we may choose $\mathcal{Q}^n$ in such a way that the total contribution of the remaining terms is much smaller than $\sup |\varphi| \leq \operatorname{var} \varphi + \int |\varphi| \, dm$.

More precisely, we begin by fixing a finite subset $\mathcal{Q}^n$ of $\mathcal{P}^n$ such that

$$\sum_{\eta \notin \mathcal{Q}^n} \operatorname{var} g_{\eta}^{n} \leq C_1 \lambda_1^n.$$

Then,

$$\begin{aligned}\sum_{\eta \notin \mathcal{Q}^n} 3 \operatorname{var} g_{\eta}^{n} \sup_{\eta} |\varphi| &\leq 3 \sup |\varphi| \sum_{\eta \notin \mathcal{Q}^n} \operatorname{var} g_{\eta}^{n} \leq 3C_1 \lambda_1^n \sup |\varphi| \\ &\leq 3C_1 \lambda_1^n \operatorname{var} \varphi + 3C_1 \lambda_1^n \int |\varphi| \, dm.\end{aligned}$$

On the other hand, compare (1.16)

$$\begin{aligned}\sum_{\eta \in \mathcal{Q}^n} 3 \operatorname{var} g_{\eta}^{n} \sup_{\eta} |\varphi| &\leq \sum_{\eta \in \mathcal{Q}^n} 3C_1 \lambda_1^n \big(\operatorname*{var}_{\eta} \varphi + \frac{1}{m(\eta)} \int_{\eta} |\varphi| \, dm \big) \\ &\leq 3C_1 \lambda_1^n \operatorname{var} \varphi + K(n) \int |\varphi| \, dm,\end{aligned}$$

with $K(n) = 3C_1 \lambda_1^n \sup\{1/m(\eta) : \eta \in \mathcal{Q}^n\}$. So, 1.18 leads to

$$\operatorname{var} \varphi_n \leq 7C_1 \lambda_1^n \operatorname{var} \varphi + 2K(n) \int |\varphi| \, dm.$$

Now we only have to remove the dependence on n of the integral term, and this can be done in the same way as in the proof of Proposition 1.3.1. Fixing a large enough integer $N \geq 1$ so that $7C_1 \lambda_1^N \leq 1/2$, we get that it is enough to take $\lambda_0 \geq \max\{2^{-1/N}, \lambda_1\}$ and $C_0 \geq \max\{4K_0, 7C_1\}$, where $K_0 = \max\{K(n) : 1 \leq n \leq N\}$. □

Combining this proposition with Lemma 1.3.3 we immediately get the analog of Corollary 1.3.1:

Corollary 1.3.5. *The map f has some absolutely continuous invariant measure whose density has bounded variation.*

It would not be difficult to obtain the remaining claims in Theorem 1.3.2 at this point. The idea is to check that invariant densities have uniformly bounded variation, cf. Remark 1.3.4, and to deduce that the Lebesgue measure of any invariant measurable set is uniformly bounded away from zero, if it is nonzero. Once that is done, the fact that ergodic absolutely continuous probability measures exist and are finitely many follows in very much the same way as in the smooth case, cf. Corollary 1.2.2. Instead of detailing these arguments right away, we postpone the proof to Subsection 1.4.2, where analogous facts are obtained in more generality.

In the rest of the present section we check the assumptions of Theorem 1.3.2 for a class of piecewise expanding transformations, that we call maps with long branches. An application to maps with neutral fixed points is given in Example 1.3.8, and will be further discussed in Section 3.5.

Definition 1.3.6. *A piecewise expanding map $f : M \to M$ has* long branches *if there exist $\delta > 0$, $K > 0$, and a partition $\mathcal{P}^1$ into regularity intervals such that*

(a) the restriction $f \mid \eta$ of f to each $\eta \in \mathcal{P}^1$ is C^2, and

$$\frac{|D^2(f \mid \eta)|}{|D(f \mid \eta)|^2} \le K \quad \textit{for every } \eta \in \mathcal{P}^1.$$

(b) the image of every interval $\eta \in \mathcal{P}^1$ has Lebesgue measure $m(f(\eta)) \ge \delta$.

If f has long branches then it satisfies the assumptions of Theorem 1.3.2:

Proposition 1.3.4. *If $\mathcal{P}^1$ satisfies (a) and (b) above, for some $\delta, K > 0$, then $\sum_{\eta \in \mathcal{P}^1} \operatorname{var} \hat{g}_\eta < \infty$. Therefore, f has some ergodic absolutely continuous invariant measure, and there are finitely many such measures.*

Proof. Let η be any element of $\mathcal{P}^1$. Condition (a) can be rewritten as

$$\left| D\left(\frac{1}{D(f \mid \eta)}\right)\right| \le K$$

which implies

$$\operatorname*{var}_\eta \frac{1}{|D(f \mid \eta)|} \le \operatorname*{var}_\eta \frac{1}{D(f \mid \eta)} = \int_\eta \left| D\left(\frac{1}{D(f \mid \eta)}\right)\right| \, dm \le K m(\eta).$$

On the other hand, by the mean value theorem there exists some $x_\eta \in \eta$ such that

$$\frac{1}{|Df(x_\eta)|} = \frac{m(\eta)}{m(f(\eta))} \le \frac{1}{\delta} m(\eta).$$

In particular,

$$\sup_\eta \frac{1}{|D(f \mid \eta)|} \leq \operatorname{var}_\eta \frac{1}{|D(f \mid \eta)|} + \inf_\eta \frac{1}{|D(f \mid \eta)|} \leq \big(K + \frac{1}{\delta}\big) m(\eta).$$

Then,

$$\operatorname{var} \hat{g}_\eta \leq \operatorname{var}_\eta \frac{1}{|D(f \mid \eta)|} + 2 \sup_\eta \frac{1}{|D(f \mid \eta)|} \leq \big(3K + \frac{2}{\delta}\big) m(\eta),$$

and so $\sum_{\eta \in \mathcal{P}^1} \operatorname{var} \hat{g}_\eta \leq (3K + 2/\delta)$.

The last part of the statement is now a consequence of Theorem 1.3.2. □

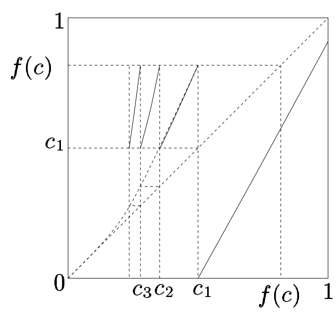

Fig. 1.5. Inducing near a neutral fixed point

Example 1.3.8. Let $f : [0,1] \to [0,1]$ be a map with a neutral fixed point as in Example 1.1.2. From f we construct a new map $\hat{f} : [0,1] \to (0,1]$, as follows. See Figure 1.5. Let $h_0 : (0, f(c)) \to (0, c)$ be the inverse of $f \mid (0, c)$. Then let $c_1 = c$ and $c_{j+1} = h_0(c_j)$, for each $j \geq 1$. Finally, define $\hat{f}$ by

$$\hat{f} \mid (c_1, 1] = f \mid (c_1, 1] \quad \text{and} \quad \hat{f} \mid (c_{j+1}, c_j) = f^j \mid (c_{j+1}, c_j) \quad \text{for each } j \geq 1.$$

For completeness, we also set $\hat{f}(x) = f(c)$ for any $x \in \{0, \ldots, c_3, c_2, c_1\}$, although this is quite arbitrary. Clearly, $\hat{f}$ is C^2 in the interior of each element of

$$\mathcal{P}^1 = \{(c_{j+1}, c_j] : j \geq 1\} \cup \{(c_1, 1]\}.$$

Let $\sigma = \inf\{|Df(x)| : x \in (c_2, c_1) \cup (c_1, 1]\}$. Then $\sigma > 1$, and $|D\hat{f}(x)| \geq \sigma$ at every point x where the derivative is defined. Moreover,

$$\hat{f}\big((c_{j+1}, c_j]\big) = (c, f(c)] \quad \text{for} \quad j \geq 1 \quad \text{and} \quad \hat{f}\big((c_1, 1]\big) = (0, f(1)].$$

So, $\hat{f}$ satisfies (b) in Definition 1.3.6, with $\delta = \min\{|f(c) - c|, |f(1)|\}$. It is not difficult to check that $\hat{f}$ also satisfies condition (a), but we postpone that to Section 3.5, see Lemma 3.5.2. Altogether, this shows that $\hat{f}$ is a piecewise expanding map with long branches, and so Theorem 1.3.2 applies to it.

1.4 Piecewise Expanding Maps in Higher Dimensions

The ergodic theory of piecewise expanding maps in higher dimensions is presently much less satisfactory than in the one-dimensional case, despite the progress attained over the last two decades.

The first existence results for absolutely continuous invariant measures (apart from the Markov case) appeared in [62, 63]. Other constructions were proposed e.g. in [18] and [48]. See this last paper for an account of results obtained in the eighties. More recently, the scope of these results was considerably extended in works such as [1, 4, 28, 114, 121, 33].

In essentially all the cases, the authors consider some notion of variation for functions in higher dimensional domains, and prove a Lasota-Yorke type of inequality, as in Proposition 1.3.1. Ergodic and spectral properties of the system can then be deduced along the lines of the one-dimensional case.

We discuss this approach in the next subsection, sketching an application to a class of multidimensional piecewise expanding maps with long branches.

1.4.1 The Bounded Variation Approach

The theory of bounded variation functions on higher dimensional domains is presented in [47] and [42]. Here we quote some main notions and facts that are more directly relevant for our purposes.

First, we give a definition of variation for functions on domains of $\mathbb{R}^d$, any $d \geq 1$. Instead of Definition 1.3.2, that depends strongly on the order structure of 1-dimensional manifolds, our starting point is Lemma 1.3.4.

Let U be an open subset in some Euclidean space $\mathbb{R}^d$, and $\varphi : U \to \mathbb{R}$ be a Lebesgue integrable function. Given a C^1 vector field $\omega : U \to \mathbb{R}^d$, we denote by $\operatorname{div} \omega$ its *divergent*. That is

$$\operatorname{div} \omega = \frac{\partial \omega_1}{\partial x_1} + \cdots + \frac{\partial \omega_d}{\partial x_d} \qquad \text{if} \qquad \omega = (\omega_1, \ldots, \omega_d).$$

Definition 1.4.1. *The* variation *of $\varphi : U \to \mathbb{R}$ on U is*

$$\operatorname*{var}_U \varphi = \sup \left\{ \Big| \int_U (\varphi \operatorname{div} \omega)\, dm \Big| : \omega \in C_0^1(U) \quad \textit{with} \quad \sup \|\omega\| \leq 1 \right\}$$

where $C_0^1(U)$ is the space of C^1 vector fields $\omega : U \to \mathbb{R}^d$ whose support is a compact subset of U. A function φ has bounded variation on U *if* $\operatorname{var}_U \varphi < \infty$.

Clearly, the variation of a function depends only on its L^1 class, that is, functions that coincide Lebesgue almost everywhere have the same variation. The space of L^1 classes with bounded variation on U is denoted $\mathrm{BV}(U)$.

The next proposition provides a useful criterium for deciding whether a function $\varphi : U \to \mathbb{R}$ has bounded variation by, basically, reducing the problem to dimension 1. Assume that φ has compact support. Then it may be extended to the whole $\mathbb{R}^d$, with the same support and $\operatorname{var}_{\mathbb{R}^d} \varphi = \operatorname{var}_U \varphi$. Hence we may just as well take U to coincide with $\mathbb{R}^d$. Given $1 \le i \le d$ and $\hat{x} = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_d) \in \mathbb{R}^{d-1}$, let

$$\varphi_{i,\hat{x}} : \mathbb{R} \to \mathbb{R} \quad \text{with} \quad \varphi_{i,\hat{x}}(x) = \varphi(x_1, \dots, x_{i-1}, x, x_{i+1}, \dots, x_d).$$

We represent by $\operatorname{var}_{[a,b]} [\varphi_{i,\hat{x}}]$ the variation of the L^1 class of $\varphi_{i,\hat{x}}$ over any compact interval $[a, b]$, recall Definition 1.3.4. Moreover, m_{d-1} denotes Lebesgue measure in $\mathbb{R}^{d-1}$.

Proposition 1.4.1. *Let $\varphi : \mathbb{R}^d \to \mathbb{R}$ be an integrable function with compact support. Then $\hat{x} \mapsto \operatorname{var}_{[a,b]} [\varphi_{i,\hat{x}}]$ is measurable, for every $1 \le i \le d$ and any real numbers $a < b$. Moreover, φ has bounded variation in $\mathbb{R}^d$ if and only if*

$$\int_K \operatorname*{var}_{[a,b]} [\varphi_{i,\hat{x}}] \, dm_{d-1}(\hat{x}) < \infty$$

for every $1 \le i \le s$, every $a < b$, and any compact subset K of $\mathbb{R}^{d-1}$.

For a proof see Lemma 1 and Theorem 2 in [42, Section 5.10].

Example 1.4.1. Let $\alpha > 0$ and $\varphi : \mathbb{R}^d \to \mathbb{R}$ be defined by

$$\varphi(x) = \|x\|^{-\alpha} \quad \text{if} \quad 0 < \|x\| \le 1 \quad \text{and} \quad \varphi(x) = 0 \quad \text{otherwise.}$$

Then φ has bounded variation in $\mathbb{R}^d$ if and only if $\alpha < d - 1$. Indeed, for any $1 \le i \le d$ and $\hat{x} \ne 0$, the function $\varphi_{i,\hat{x}}$ is monotone increasing on $(-\infty, 0]$ and monotone decreasing on $[0, +\infty)$. So, given any $a < -1 < 1 < b$,

$$\operatorname*{var}_{[a,b]} [\varphi_{i,\hat{x}}] = 2 \, \varphi_{i,\hat{x}}(0) = 2 \, \|\hat{x}\|^{-\alpha}.$$

Now it suffices to note that, if K is the closed unit ball in $\mathbb{R}^{d-1}$,

$$\int_K 2 \, \|\hat{x}\|^{-\alpha} \, dm_{d-1}(\hat{x}) < \infty \quad \text{if and only if} \quad \alpha < d - 1.$$

This example shows that bounded variation functions in dimension higher than 1 need not be bounded. On the other hand, the next proposition (Sobolev's inequality, see Theorem 1.28 of [47]) ensures that functions with bounded variation on a d-dimensional domain are in an L^p space, where p is determined by the dimension d. It also follows from Example 1.4.1 that the expression of p in the proposition can not be improved.

Proposition 1.4.2. *There exists $C(d) > 0$, depending only on d, such that, for any bounded variation function $\varphi : U \to \mathbb{R}$ with compact support, we have*

$$\Big(\int_U |\varphi|^p \, dm \Big)^{1/p} \le C(d) \, \operatorname*{var}_U \varphi \qquad \text{where} \quad p = \frac{d}{d-1} \, .$$

In particular, $\mathrm{BV}(U) \subset L^p(U, m)$.

Proposition 1.4.3. *If* $\varphi_n : U \to \mathbb{R}$, $n \geq 1$, *are bounded variation functions converging to* $\varphi : U \to \mathbb{R}$ *in* $L^1(U, m)$ *then*

$$\operatorname*{var}_U \varphi \leq \liminf_n \operatorname*{var}_U \varphi_n$$

In other words, the variation is lower semi-continuous, with respect to the L^1-norm. Proposition 1.4.3 is contained in Theorem 1.9 of [47]. It is not difficult to deduce, cf. Remark 1.12 in [47], that the expression

$$\|\varphi\|_{\mathrm{BV}} = \operatorname*{var}_U \varphi + \|\varphi\|_1 \tag{1.19}$$

defines a complete norm in the space $\mathrm{BV}(U)$.

Also related to Proposition 1.4.3, we have the following important extension of Lemma 1.3.3, for domains with Lipschitz boundary in any dimension: sets of functions with uniformly bounded variation are relatively compact with respect to the L^1-norm.

Proposition 1.4.4. *Suppose the domain* U *is bounded, and its boundary is Lipschitz continuous. Let* $K_1, K_2 > 0$ *and* φ_n, $n \geq 1$, *be a sequence in* $L^1(U, m)$ *such that* $\|\varphi_n\|_1 \leq K_1$ *and* $\operatorname{var}_U \varphi_n \leq K_2$ *for every* $n \geq 1$. *Then there exists a subsequence* $(n_k)_k$ *such that* $(\varphi_{n_k})_k$ *converges in* $L^1(U, m)$ *to a function* φ_0 *with* $\|\varphi_0\|_1 \leq K_1$ *and* $\operatorname{var}_U \varphi_0 \leq K_2$.

A proof of this last fact can be found in Theorem 1.19 of [47]. The bounds on $\|\varphi_0\|_1$ and $\operatorname{var}_U \varphi_0$ follow from the L^1 convergence and Proposition 1.4.3.

These results show that some of the main tools from the one-dimensional case remain valid for bounded variation functions in any dimension. On the other hand, this notion of variation is very sensitive to the geometry of the domain. For instance, cf. Example 1.4 in [47], if $E \subset U$ is a compact domain bounded by a C^2 hypersurface, then

$$\operatorname*{var}_U \mathcal{X}_E = m_{d-1}(\partial E), \tag{1.20}$$

where $\mathcal{X}_E$ is the characteristic function $\mathcal{X}_E$ of E and $m_{d-1}(\partial E)$ denotes the $(d-1)$-dimensional Hausdorff measure of the boundary of E. So, even characteristic functions of open sets may have unbounded variation.

This observation is at the origin of serious difficulties one encounters in higher dimensions. Not surprisingly, the cases of high dimensional piecewise expanding maps one has been able to treat depend on restrictive conditions on the geometry of the domains of smoothness of the map, e.g. their boundaries should be fairly regular. At least in some cases, see the discussion in [96], such conditions are an artifact of this method, and not really necessary for the existence of absolutely continuous invariant measures. Alternative notions of variation have been proposed, but they also require technical restrictions on the shapes of the smoothness domains.

Maps with Long Branches. Here we outline an application of the previous ideas to a class of piecewise expanding maps in any dimension that generalizes the one-dimensional maps with long branches treated in Proposition 1.3.4. This is due to [48], for maps with finitely many regularity domains, and [4] in the general case.

Let R be a bounded region in $\mathbb{R}^d$. We say that $f : R \to R$ is a C^2 *piecewise expanding map* if there is a partition $\mathcal{P}^1$ of R into domains η such that

(E1) the boundary of η is piecewise C^2 and has finite $(d-1)$-dimensional Hausdorff measure
(E2) the restriction of f to the interior of η is a C^2 diffeomorphism onto its image, and it admits a C^2 extension to the closure of η;
(E3) there is $\sigma > 1$ such that $\|Df(x)^{-1}\| \le \sigma^{-1}$ for every point x where the derivative is defined.

We say that a C^2 piecewise expanding map $f : R \to R$ has *long branches* if it satisfies two additional properties, (D) and (G), resemblant of parts (a) and (b) of Definition 1.3.6. The first one is a condition of bounded distortion:

(D) There is some $K > 0$ such that

$$\frac{\|D(Jf_\eta^{-1})\|}{|Jf_\eta^{-1}|} \le K, \quad \text{for every } \eta \in \mathcal{P}^1,$$

where $Jf_\eta^{-1} = \det D(f \mid \eta)^{-1}$ is the Jacobian of the inverse of $(f \mid \eta)$.

(G) is a geometric requirement on the images $f(\eta)$ of the regularity domains: they should not be too small (sizes uniformly bounded away from zero), and the angles at the border corners should also be bounded from below. More precisely, we suppose that

(G) There are $\alpha > 0$, $\delta > 0$, and for each $\eta \in \mathcal{P}^1$ there is a C^1 unitary vector field H_η on the boundary of $f(\eta)$, such that
1. $|\sin \text{angle}(v, H_\eta(x))| \ge \alpha$ for every $x \in \partial f(\eta)$ and any vector v tangent to $\partial f(\eta)$ at x;
2. the segments $[x, x + \delta H_\eta(x)]$, $x \in \partial f(\eta)$, are two-by-two disjoint and their union is a neighbourhood of the boundary in $f(\eta)$.

By convention, a vector field is C^1 on the boundary of $f(\eta)$ if its restriction to each smooth component of the boundary is C^1 on that component. Moreover, the tangent space of $f(\eta)$ at a corner point is the union of the tangent spaces of all the smooth components that contain that point.

Theorem 1.4.1. *Let $f : R \to R$ be a C^2 piecewise expanding map with long branches, i.e., f satisfies (E1)–(E3), (D), (G). Assume that*

$$\sigma > 1 + \alpha^{-1}.$$

Then f has some invariant probability measure absolutely continuous with respect to Lebesgue measure in R.

Let m be the d-dimensional Lebesgue measure on $\mathbb{R}^d$, normalized so that $m(R) = 1$. As in Lemma 1.3.5, given any integrable function $\varphi : R \to \mathbb{R}$ and any $n \geq 1$, there exists $\varphi_n : R \to \mathbb{R}$ such that $f_*^n(\varphi m) = \varphi_n m$. In fact, we may take

$$\varphi_n = \sum_\eta g_\eta^n \cdot \big(\varphi \circ (f^n \mid \eta)^{-1}\big) \tag{1.21}$$

where the sum is over all the regularity domains η of f^n with $m(\eta) > 0$, and

$$g_\eta^n(y) = \frac{1}{|\det Df^n|} \circ (f^n \mid \eta)^{-1}(y) = Jf_\eta^{-n}(y) \qquad \text{if } y \in f^n(\eta),$$

with $g_\eta^n(y) = 0$ otherwise. Moreover, as in (1.14),

$$\int_R |\varphi_n| \, dm \leq \int_R |\varphi| \, dm \quad \text{for every } n \geq 1. \tag{1.22}$$

The main step in the proof of Theorem 1.4.1 is the following version of Proposition 1.3.1:

Proposition 1.4.5. *Suppose f is a C^2 piecewise expanding map with long branches. Then there exists $C_0 > 0$ such that*

$$\operatorname*{var}_R \varphi_1 \leq \lambda \operatorname*{var}_R \varphi + C_0 \int_R |\varphi| \, dm, \qquad \lambda = \sigma^{-1}(1 + \alpha^{-1}),$$

for every $\varphi \in \mathrm{BV}(R)$.

Then, an absolutely continuous invariant measure for f can be found as follows. Fix $\varphi \equiv 1$. By Proposition 1.4.5 and (1.22),

$$\operatorname*{var}_R \varphi_n \leq \lambda \operatorname*{var}_R \varphi_{n-1} + C_0 \int_R |\varphi_{n-1}| \, dm \leq \lambda \operatorname*{var}_R \varphi_{n-1} + C_0$$

for every $n \geq 1$. Recall that we are assuming $\lambda < 1$. It follows that the sequence $(\varphi_n)_n$ has uniformly bounded variation: for every $n \geq 1$,

$$\operatorname*{var}_R \varphi_n \leq \lambda^n \operatorname*{var}_R \varphi + C_0 \big(1 + \cdots + \lambda^{n-1}\big) \leq \frac{C_0}{1 - \lambda}.$$

Therefore, the variation of the sequence $\psi_n = n^{-1} \sum_{j=0}^{n-1} \varphi_j$ is also uniformly bounded, by the same constant. Using Proposition 1.4.4, we conclude that $(\psi_n)_n$ has some subsequence $(\psi_{n_k})_k$ converging in $L^1(R, m)$ to a a bounded variation function ψ_0. In particular the sequence of measures

$$\frac{1}{n_k} \sum_{j=0}^{n_k - 1} f_*^j m = \psi_{n_k} m$$

converges to $\mu = \psi_0 m$, an invariant absolutely continuous measure. This completes our sketch of the proof of Theorem 1.4.1.

1.4.2 Finiteness of Physical Measures

In this subsection we prove that, quite in general, a Lasota-Yorke type of inequality suffices to ensure that there are finitely many ergodic absolutely continuous invariant measures. We also need the map to be piecewise regular, but not necessarily piecewise expanding. The result applies to maps in any dimension and with any number of regularity domains, including the situations in Subsection 1.3.5 and in Theorem 1.4.1 as special cases. The main idea in the proof is the approximation argument used by [69] to obtain Proposition 1.3.2.

Let M be a compact domain in $\mathbb{R}^d$, whose boundary is contained in a finite union of C^2 hypersurfaces. We assume that $f : M \to M$ satisfies

(P1) there exists an open set $U \subset M$ such that $M \setminus U$ has zero Lebesgue measure and f is a local C^1 diffeomorphism at every point of U;
(P2) there exist constants $C_0 > 0$ and $0 < \lambda_0 < 1$ such that for any bounded variation function $\varphi : M \to \mathbb{R}$ we have $f_*^n(\varphi m) = \varphi_n m$ for some function $\varphi_n : M \to \mathbb{R}$ with

$$\operatorname*{var}_M \varphi_n \le C_0 \lambda_0^n \operatorname*{var}_M \varphi + C_0 \int |\varphi| \, dm.$$

Theorem 1.4.2. *If f satisfies (P1) and (P2), then it admits ergodic absolutely continuous invariant probability measures $\nu_1, \ldots, \nu_s$ such that*

1. *the union of the basins of $\nu_1, \ldots, \nu_s$ has full Lebesgue measure in M;*
2. *every absolutely continuous invariant measure μ is a linear combination of $\nu_1, \ldots, \nu_s$;*

Moreover, the density $d\mu/dm$ of any such measure μ has bounded variation.

The main step is the following proposition.

Proposition 1.4.6. *Given any function $\psi : M \to \mathbb{R}$ with $\int |\psi| \, dm = 1$, there exists a subsequence $(n_k)_k$ and a function $\theta : M \to \mathbb{R}$ with $\operatorname{var}_M \theta \le 4C_0$, such that*

$$\frac{d}{dm}\left(\frac{1}{n_k} \sum_{j=0}^{n_k-1} f_*^j(\psi m)\right) \to \theta \quad \text{in} \quad L^1(M, m).$$

Proof. Let $(\xi_l)_l$ be some sequence of bounded variation functions converging to ψ in $L^1(M,m)$. We may suppose that every ξ_l has L^1-norm less than 2. For each $n \ge 1$ and $l \ge 1$, let

$$f_*^n(\psi m) = \psi_n m \quad \text{and} \quad f_*^n(\xi_l m) = \xi_{l,n} m$$

where ψ_n and $\xi_{l,n}$ are obtained from ψ and ξ_l, respectively, as in (1.21). Let us fix l for a while. Assumption (P2) implies that

$$\operatorname*{var}_M \xi_{l,n} \le C_0 \lambda_0^n \operatorname*{var}_M \xi_l + C_0 \int |\xi_l| \, dm \le 3\, C_0$$

for every large enough n. So, increasing n if necessary,

$$\operatorname*{var}_M \left(\frac{1}{n} \sum_{j=0}^{n-1} \xi_{l,j} \right) \le \frac{1}{n} \sum_{j=0}^{n-1} \operatorname*{var}_M \xi_{l,j} \le 4C_0.$$

It follows from Proposition 1.4.4 that there exists a function θ_l and a sequence $(m(l,i))_i \to \infty$ such that

$$\frac{1}{m(l,i)} \sum_{j=0}^{m(l,i)-1} \xi_{l,j} \to \theta_l$$

as $i \to \infty$. Moreover, by Proposition 1.4.3, we have $\operatorname{var} \theta_l \le 4C_0$.

Then, using Proposition 1.4.4 for the sequence θ_l, we conclude that there exists a subsequence $(l_k)_k$ such that θ_{l_k} converges in $L^1(M,m)$ to some function θ with $\operatorname{var}_M \theta \le 4C_0$. It follows, by a triangular inequality argument, that there exists a subsequence $n_k = m(l_k, i_k)$, $k \ge 1$, such that

$$\frac{1}{n_k} \sum_{j=0}^{n_k-1} \xi_{l_k,j} \to \theta$$

in $L^1(M,m)$ as $k \to \infty$. On the other hand, as $\|\xi_{l,j} - \psi_j\|_1 \le \|\xi_l - \psi\|_1$ for every j,l,

$$\Big\| \frac{1}{n_k} \sum_{j=0}^{n_k-1} (\xi_{l_k,j} - \psi_j) \Big\|_1 \le \frac{1}{n_k} \sum_{j=0}^{n_k-1} \|\xi_{l_k} - \psi\|_1 = \|\xi_{l_k} - \psi\|_1 \,,$$

and the last term goes to zero as $k \to \infty$. This implies that

$$\frac{1}{n_k} \sum_{j=0}^{n_k-1} \psi_j \to \theta \quad \text{in} \quad L^1(M,m),$$

as claimed. □

Corollary 1.4.1. *Any absolutely continuous probability measure μ of a piecewise expanding map can be written as $\mu = \theta\, m$ where θ has $\operatorname{var}_M \theta \le 4C_0$.*

Proof. By assumption $\mu = \psi m$ for some $\psi \in L^1(M,m)$ with $\psi \ge 0$ and $\int \psi \, dm = 1$. Proposition 1.4.6 states that a subsequence

$$\frac{1}{n_k} \sum_{j=0}^{n_k-1} \psi_j$$

converges in $L^1(M,m)$ to some function θ whose variation is bounded by $4C_0$. Now, $\psi_n = \psi$ for every n because μ is invariant. This implies that $\psi = \theta$. □

Lemma 1.4.1. *Given any f-invariant set $A \subset M$ with positive Lebesgue measure, there exists some absolutely continuous f-invariant probability measure ν_A such that $\nu_A(A) = 1$.*

Proof. Let $(m \mid A)$ represent the normalized restriction of Lebesgue measure to A. In other words, $(m \mid A) = \psi m$ where $\psi = \mathcal{X}_A / m(A)$. Let us consider the sequence of probability measures

$$\mu_{A,n} = \frac{1}{n} \sum_{j=0}^{n-1} f_*^j (m \mid A).$$

By Proposition 1.4.6, there exists a function θ_A with $\operatorname{var}_M \theta_A \le 4C_0$ such that some subsequence

$$\frac{d\mu_{A,n_k}}{dm} \to \theta_A \quad \text{in} \quad L^1(m),$$

as $k \to \infty$. By Remark 1.3.3, the sequence μ_{A,n_k} converges to $\nu_A = \theta_A m$ in the weak* sense. Moreover, ν_A is an absolutely continuous invariant measure for f. Since A is assumed to be invariant, $f_*^j(m \mid A)(A) = (m \mid A)(A) = 1$ for every $j \ge 1$. This gives $\mu_{A,n}(A) = 1$ for every $n \ge 1$. Finally, L^1 convergence of the densities implies that

$$\nu_A(A) = \int_A \theta_A \, dm = \lim_k \int_A \frac{d\mu_{A,n_k}}{dm} \, dm = \lim_k \mu_{A,n_k}(A) = 1.$$

So, ν_A does satisfy the conclusion of the lemma. □

It is worth pointing out that the argument in Lemma 1.3.7 does not carry on to higher dimensions. This is because the support of functions with bounded variation may have empty interior, see the following example.

Example 1.4.2. Let U be an open subset of $\mathbb{R}^d$ containing the closed unit ball $B_1(0)$ around the origin. Let $\{q_n : n \in \mathbb{N}\}$ be a countable dense subset of $B_1(0)$, and B_n be the open ball of radius 2^{-n-1} around each q_n. Define $E_n = B_1(0) \setminus \cup_{j=1}^n B_j$ and $E = B_1(0) \setminus \cup_{j=1}^\infty B_j$. Note that E is nonempty, in fact it has positive d-dimensional Lebesgue measure:

$$m_d(E) \ge \sigma(d) - \sum_{j=1}^{\infty} 2^{-(j+1)d} \sigma(d) \ge \frac{\sigma(d)}{2},$$

where $\sigma(d)$ is the Lebesgue measure of the unit ball in $\mathbb{R}^d$. Clearly, E, E_n are compact, and E is nowhere dense. According to (1.20)

$$\operatorname{var} \mathcal{X}_{E_n} \le \omega(d) + \sum_{j=1}^{n} 2^{-jd} \omega(d) \le 2\omega(d).$$

for every $n \ge 1$, with $\omega(d)$ denoting the $(d-1)$-dimensional measure of the unit sphere in $\mathbb{R}^d$. Then, $\operatorname{var} \mathcal{X}_E \le \liminf_n \operatorname{var} \mathcal{X}_{E_n} \le 2\omega(d)$, as a consequence of Proposition 1.4.3.

Corollary 1.4.2. *If $A \subset M$ is an f-invariant set with positive Lebesgue measure then $m(A) \geq (4C_0C(d))^{-1/d}$.*

Proof. Let ν_A be an absolutely continuous invariant measure giving full weight to A, as in Lemma 1.4.1, and $\theta_A = d\nu_A/dm$. Let $p = d/(d-1)$ and $q = 1/d$, with $p = \infty$ in the case $d = 1$. By Sobolev's inequality Proposition 1.4.2,

$$\|\theta_A\|_p \leq C(d) \operatorname*{var}_M \theta_A \leq 4C_0C(d).$$

Combining this with Hölder's inequality we get

$$1 = \|\theta_A\|_1 \leq \|\theta_A\|_p \, m(A)^{1/q} \leq 4C_0C(d) \, m(A)^d,$$

as we claimed. □

Corollary 1.4.2 implies that there are finitely many two-by-two disjoint f-invariant sets with positive Lebesgue measure. As an immediate consequence, M can be partitioned into finitely many minimal f-invariant sets:

Corollary 1.4.3. *There exist f-invariant sets $A_1, \ldots, A_s$ such that*

1. *$m(A_i) > 0$ for every $1 \leq i \leq s$, and $M = A_1 \cup \cdots \cup A_s$;*
2. *there are no f-invariant sets $B_i \subset A_i$ with $0 < m(B_i) < m(A_i)$.*

Finally, let $\nu_1, \ldots, \nu_s$ be absolutely continuous invariant measures with $\nu_i(A_i) = 1$ for $i = 1, \ldots, s$, as in Lemma 1.4.1. The fact that A_i is minimal implies that ν_i is ergodic and $B(\nu_i)$ has full measure in A_i. Moreover, any absolutely continuous invariant measure μ can be written as $\mu = \sum_i \mu(A_i)\nu_i$. To see this, write $\mu = \sum_i \mu(A_i)\mu_i$, where the sum is over the values of i such that $\mu(A_i) > 0$ and μ_i is the normalized restriction of μ to A_i. Each μ_i is also an ergodic measure, because A_i is minimal. Consequently, either $\mu_i = \nu_i$ or there exists some invariant subset $B \subset A_i$ with $\mu_i(B) = 0$ and $\nu_i(B) = 1$. The last case would imply $0 < m(B) < m(A_i)$, contradicting the minimality. So, we must have $\mu_i = \nu_i$ for every i.

This shows that these measures ν_i satisfy all the conclusions of Theorem 1.4.2. We have finished the proof of the theorem.

2. Hyperbolic and Partially Hyperbolic Attractors

The notion of hyperbolic dynamical system was introduced some forty years ago by Smale, see [119], and was developed through the work of several mathematicians, specially until the mid-eighties. Some of the main results are reviewed in Appendix A.1. Although not as general a property as was once thought (many systems can not be approximated by hyperbolic ones), hyperbolicity did provide a powerful framework for analyzing complicated dynamical behaviour: hyperbolic systems often exhibit very rich dynamics (infinite number of periodic points, combined with stochastic behaviour of typical orbits), which is now well understood.

In the early seventies, Sinai [117], Ruelle and Bowen [**?**, 25, 22], brought ideas from statistical mechanics into play, to build an ergodic theory of hyperbolic systems. In particular, every hyperbolic attractor of a C^2 diffeomorphism supports a unique physical measure μ, and Lebesgue almost every orbit that converges to the attractor is in the basin of μ. We present a geometric proof of this result, relying on a construction of invariant measures absolutely continuous along uniformly expanding foliations that applies to much more general *partially hyperbolic* attractors [94]. In the hyperbolic case there exists exactly one probability measure supported on the attractor, and it is the unique physical measure.

This approach can be applied to some partially hyperbolic cases but, in general, the ergodic theory of partially hyperbolic attractors is very much open. This is a very active topic of current research, a review of some recent developments is included in Section 7.4.

2.1 Hyperbolic Sets

Here we recall the definitions and some basic facts about hyperbolic sets of diffeomorphisms, specially attractors, that are needed for the next sections. We include a few short proofs, together with references to the longer ones. More information can be found e.g. in [84, 116, **?**, **?**].

2.1.1 Definitions and Examples

The next definition involves the notion of *splitting* $E^1 \oplus E^2$ of the tangent space $T_\Lambda M$ of the manifold M over a subset Λ. By that we understand a map $x \mapsto (E^1_x, E^2_x)$ associating to each point $x \in \Lambda$ two complementary subspaces of the tangent space $T_x M$. We always assume that the subspaces E^1_x have constant dimension at every point $x \in \Lambda$, then the same is true for the E^2_x.

We call the splitting *continuous* if given any $p \in \Lambda$ there exist continuous vector fields $X_1, \dots, X_u, Y_1, \dots, Y_s$ on a neighbourhood $U_p \subset \Lambda$ of p, linearly independent at every point and such that E^1_x is the subspace generated by $X_1(x), \dots, X_u(x)$ and E^2_x is the subspace generated by $Y_1(x), \dots, Y_s(x)$, for every $x \in U_p$.

Definition 2.1.1. *Let $f : M \to M$ be a C^1 diffeomorphism and Λ be a compact subset of M that is invariant under f, that is, $f(\Lambda) = \Lambda$. We say that $\Lambda \subset M$ is a* (uniformly) hyperbolic set *for f if there exists a continuous splitting $T_\Lambda M = E^u \oplus E^s$ of the tangent space of M over Λ such that*

1. *the splitting is* invariant *under the derivative Df: for every $x \in \Lambda$*
$$Df(x)^{-1} \cdot E^u_x = E^u_{f^{-1}(x)} \quad \textit{and} \quad Df(x) \cdot E^s_x = E^s_{f(x)};$$
2. *the subbundle E^u is* expanding *and the subbundle E^s is* contracting *for Df: there are constants $C > 0$ and $\lambda < 1$ such that*
$$\|Df^{-n}(x)|E^u_x\| \le C\lambda^n \quad \textit{and} \quad \|Df^n(x)|E^s_x\| \le C\lambda^n$$
for every $x \in \Lambda$ and every $n \ge 1$.

Clearly, this last condition is independent of the choice of the Riemannian norm $\|\cdot\|$ on M, which affects only the value of the constant $C > 0$. According to the next proposition, we can always find a Riemannian norm on the manifold M for which $C = 1$. Such a norm is said to be *adapted* to f on Λ.

Proposition 2.1.1. *Let Λ be a hyperbolic set for a diffeomorphism f. Then, given any $\lambda_* \in (\lambda, 1)$, there exists a Riemannian norm $\|\cdot\|_*$ on M such that*
$$\|Df(x)^{-1}v^u\|_* \le \lambda_* \|v^u\|_* \quad \textit{and} \quad \|Df(x)v^s\|_* \le \lambda_* \|v^s\|_*$$
for every $v^u \in E^u_x$, $v^s \in E^s_x$, and $x \in \Lambda$.

Proof. Fix $\lambda < \lambda_+ < \lambda_* < 1$ and $N \ge 1$ large enough so that $C(\lambda/\lambda_+)^N < 1$. Given any vector $v = v^u + v^s$ in $E^u_x \oplus E^s_x$, define
$$\|v\|^2_+ = \|v^u\|^2_+ + \|v^s\|^2_+ ,$$
with

$$\|v^u\|_+^2 = \sum_{j=0}^{N-1} \lambda_+^{-2j} \|Df^{-j}(x)v^u\|^2 \quad \text{and} \quad \|v^s\|_+^2 = \sum_{j=0}^{N-1} \lambda_+^{-2j} \|Df^{j}(x)v^s\|^2.$$

It is easy to see that this defines a continuous norm on Λ, and

$$\|Df(x)^{-1}v^u\|_+ \le \lambda_+\|v^u\|_+ \quad \text{and} \quad \|Df(x)v^s\|_+ \le \lambda_+\|v^s\|_+ \tag{2.1}$$

for any $v^u \in E^u_x$, $v^s \in E^s_x$, and $x \in \Lambda$. In general, the subbundles E^s and E^u do not admit smooth extensions to a neighbourhood of Λ, and so $\|\cdot\|_+$ may fail to extend to a smooth norm on M. However, this can be easily solved. Let $\|\cdot\|_*$ be any C^∞ Riemannian norm on M whose restriction to Λ is uniformly close to $\|\cdot\|_+$: if the two norms are close enough then (2.1) remains valid with $\|\cdot\|_*$ in the place of $\|\cdot\|_+$, and λ_* in the place of λ_+. □

Example 2.1.1. (Linear Anosov maps) Let $A \in \mathrm{Sl}(d, \mathbb{Z})$, that is, A is a linear isomorphism of $\mathbb{R}^d$, $d \ge 2$, with determinant equal to ± 1 and whose matrix relative to the canonical basis of $\mathbb{R}^d$ has integer coefficients. Then A preserves the lattice $\mathbb{Z}^d$, and so there exists a unique smooth map f from the d-dimensional torus $M = \mathbb{R}^d/\mathbb{Z}^d$ to itself satisfying $\pi \circ A = f \circ \pi$, where $\pi : \mathbb{R}^d \to M$ is the canonical projection. Besides, f is a diffeomorphism: its inverse may be obtained through the same construction, with A^{-1} in the place of A.

Now, suppose the isomorphism A is *hyperbolic*: all its eigenvalues have norm different from 1. For example, this is the case for the 2-dimensional isomorphism

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

Let $\hat{E}^u$ be the direct sum of the (generalized) eigenspaces of A corresponding to the eigenvalues with norm larger than 1, and $\hat{E}^s$ be defined analogously, for eigenvalues with norm smaller than 1. Given any point $w \in M$, choose $z \in \mathbb{R}^d$ such that $\pi(z) = w$, and then let

$$E^u_w = D\pi(z) \cdot \hat{E}^u \quad \text{and} \quad E^s_w = D\pi(z) \cdot \hat{E}^s.$$

These objects do not depend on the choice of z, and so this defines subbundles $E^u = (E^u_w)_{w \in M}$ and $E^s = (E^s_w)_{w \in M}$ of the tangent space of M. Moreover, $E^u \oplus E^s$ is a hyperbolic splitting for f: the derivative Df leaves both E^u and E^s invariant, while expanding the vectors in E^u and contracting the vectors in E^s.

Indeed, let $0 < \lambda < 1$ be fixed close enough to 1 so that no eigenvalue of A has norm in the interval $[\lambda, \lambda^{-1}]$. Let $\|\cdot\|_e$ be any norm in $\mathbb{R}^d$, and endow M with the Riemannian metric $\|\cdot\|$ defined by $\|D\pi(z)v\| = \|v\|_e$ for every $z \in \mathbb{R}^d$ and $v \in \mathbb{R}^d$. Then

$$\|Df^{-n} \mid E^u_w\| = \|A^{-n} \mid \hat{E}^u\|_e \quad \text{and} \quad \|Df^n(w) \mid E^s_w\| = \|A^n \mid \hat{E}^s\|_e$$

are less than $C\lambda^n$ for all $w \in M$ and $n \geq 1$, as long as the constant C is fixed sufficiently large. This proves that the ambient manifold $\Lambda = M$ is a hyperbolic set for f.

A fundamental property of hyperbolic sets is that they are a *robust* feature of the system: if a diffeomorphism f has a hyperbolic set Λ then any other diffeomorphism g in a C^1 neighbourhood has a hyperbolic set Λ_g, close to Λ. Furthermore, the dynamics of g on Λ_g is topologically equivalent (conjugate) to that of f on Λ. That is the content of the following theorem, see [116, Theorem 8.3].

Theorem 2.1.1. *Let Λ be a hyperbolic set for a diffeomorphism $f : M \to M$. Then there is a neighborhood $\mathcal{N}$ of f in $\mathrm{Diff}^1(M)$, and there is a continuous map $\phi : \mathcal{N} \longrightarrow \mathrm{Emb}(\Lambda, M)$ such that $\phi(f)$ is the inclusion map of Λ in M and $\Lambda_g = \phi(g)(\Lambda)$ is a hyperbolic set for every $g \in \mathcal{N}$. Moreover,*

$$\phi(g) \circ (f \mid \Lambda) = (g \mid \Lambda_g) \circ \phi(g).$$

$\mathrm{Emb}(\Lambda, M)$ denotes the space of continuous one-to-one maps from Λ to M, endowed with the topology of uniform convergence. For $r \geq 1$, $\mathrm{Diff}^r(M)$ is the space of C^r diffeomorphisms on M, with the C^r topology. We call Λ_g the *hyperbolic continuation* of Λ for g. Similarly, we call $\tilde{x} = \phi(g)(x)$ the *hyperbolic continuation* of $x \in \Lambda$ for g.

Example 2.1.1 is somewhat special in that we were able to exhibit the hyperbolic splitting explicitly, which is hardly ever possible. Fortunately, in order to prove that an invariant set is hyperbolic it suffices to have some reasonable approximation of the invariant subbundles E^u and E^s. The precise formulation uses the notion of stable and unstable cone fields.

Cone fields. Let $E \oplus F = T_K M$ be a splitting of the tangent space $T_K M$ over some subset $K \subset M$. The splitting needs not be continuous, but we assume that the angle between E_x and F_x is bounded away from zero. Given $a > 0$, the *cone field of width a around E* is the family $C_a(E) = \big(C_a(E,x)\big)_{x \in K}$ defined by

$$C_a(E,x) = \{v_1 + v_2 \in E_x \oplus F_x : \|v_2\| \leq a\|v_1\|\}.$$

Definition 2.1.2. *$C_a(E)$ is an* unstable cone field *for f on K if*

1. *$C_a(E)$ is* forward invariant*: there exists $\theta < 1$ such that*
$$Df(x) \cdot C_a(E,x) \subset C_{\theta a}(E, f(x)) \tag{2.2}$$
for every $x \in K \cap f^{-1}(K)$.
2. *there exist $C > 0$ and $\sigma > 1$ such that,*
$$\|Df^n(x)v\| \geq C\sigma^n \|v\| \tag{2.3}$$
for every $v \in C_a(E,x)$, $n \geq 1$, and $x \in K \cap f^{-1}(K) \cap \cdots \cap f^{-n+1}(K)$.

A cone field is *stable*, respectively *backward invariant*, for f if it is unstable, respectively forward invariant for f^{-1}.

Formally, besides the splitting $E \oplus F$ and the constant $a > 0$, the definition of the cone field $C_a(E)$ depends also on the choice of a Riemannian norm $\|\cdot\|$ on the tangent space over K. This is usually clear from the context and, to avoid overloading our statements unnecessarily, we mention this dependence explicitly only if doing otherwise might result in ambiguity.

Proposition 2.1.2. *Let Λ be a compact invariant set of a diffeomorphism f. Suppose there is a splitting $T_\Lambda M = E \oplus F$ and there are constants $a > 0$ and $b > 0$ such that $C_a(E)$ is an unstable cone field and $C_b(F)$ is a stable cone field for f on Λ. Then Λ is a hyperbolic set for f.*

The converse is also true: if Λ is a hyperbolic set with splitting $E^u \oplus E^s$ then any cone field with sufficiently small width (relative to an adapted norm) around E^u is an unstable cone field for f and, analogously, any cone field with small width around E^s is a stable cone field for f on Λ. Indeed, both the proposition and this converse are contained in a more general result, Proposition 2.2.1, that we shall prove later on.

It is important to observe that the conditions in Definition 2.1.2 are *open*: if $C_a(E)$ is an unstable cone field for f, then it is also an unstable cone field for any other diffeomorphism C^1 near it. This leads to

Corollary 2.1.1. *Given a hyperbolic set Λ of a diffeomorphism $f : M \to M$, there exists a neighbourhood U of Λ in M, and a neighbourhood $\mathcal{N}$ of f in $\mathrm{Diff}^1(M)$, such that if $g \in \mathcal{N}$ and Γ is any compact subset of U that is invariant under g, then Γ is a hyperbolic set for g.*

The proof of the corollary goes as follows, see e.g. [116, Proposition 7.6]. Start by extending the splitting $T_\Lambda M = E^u \oplus E^s$ continuously to a neighbourhood U of Λ. The extended subbundles may not be Df-invariant but, assuming the neighbourhood U is small enough, small cones fields around them are still unstable/stable cone fields for f in U. Moreover, the same is true for any C^1-nearby map g. Then any g-invariant subset contained in U admits stable and unstable cone fields and so, by the proposition, it is hyperbolic.

Example 2.1.2. We say that $f : M \to M$ is an *Anosov diffeomorphism* if the whole manifold M is a hyperbolic set for f. See [7]. A special case are the hyperbolic automorphisms of the d-torus T^d constructed in Example 2.1.1. As an application of the previous corollary, the class of Anosov diffeomorphisms is open in the C^1 topology.

Anosov diffeomorphisms on tori are always topologically conjugate to a hyperbolic automorphism as in Example 2.1.1, cf. [45, 78]. More generally, Anosov diffeomorphisms may be constructed on infranilmanifolds [119], and then they are topologically conjugate to algebraic models. It is not known whether these diffeomorphisms may exist on other manifolds. On the other hand, by [44, 82] Anosov diffeomorphisms such that either E^u or E^s have dimension 1 exist only on topological tori.

Example 2.1.3. (Solenoids) Let $S^1 = \mathbb{R}/\mathbb{Z}$, D^2 be the closed unit disk in the complex plane, and let Q be the solid torus $Q = S^1 \times D^2$. Given constants $0 < \lambda < \rho < 1/(2\pi)$, let $f : Q \to Q$ be the map given by

$$f(\theta, z) = (2\theta \bmod \mathbb{Z}, \rho e^{2\pi i\theta} + \lambda z). \tag{2.4}$$

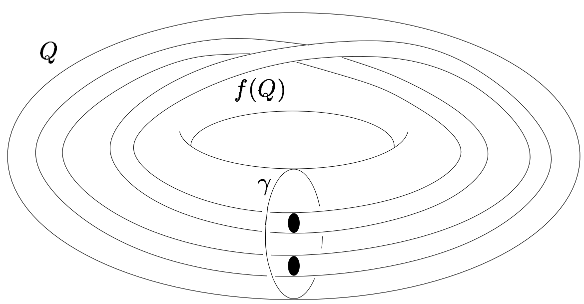

Fig. 2.1. A solenoid

Geometrically, f acts on the solid torus by contracting along the D^2 direction, and stretching and wrapping the image twice around the S^1 direction. See Figure 2.1. The assumptions on ρ and λ ensure that f is an embedding of Q strictly into itself: $f(Q) \subset \text{inter}(Q)$. Then the set

$$\Lambda = \bigcap_{n \geq 0} f^n(Q)$$

of those points whose orbit is defined for all times (both positive and negative) is a hyperbolic set for f. Indeed, as the reader may check,

$$C_a^u(p) = \{(\dot{\theta}, \dot{z}) \in T_p(S^1 \times D^2) : |\dot{z}| \leq a|\dot{\theta}|\}$$

is an unstable cone field, and

$$C_b^s(p) = \{(\dot{\theta}, \dot{z}) \in T_p(S^1 \times D^2) : |\dot{\theta}| \leq b|\dot{z}|\}$$

is a stable cone field for f on Q, if $a = 1$ and b is sufficiently small.

2.1.2 Stable and Unstable Manifolds

Among the most important geometric properties of hyperbolic sets is the existence of invariant foliations (or laminations) that are dynamically defined. That is the subject of this subsection.

Definition 2.1.3. *The* stable manifold $W^s(x)$ *of* $x \in M$ *is the set of points* $y \in M$ *whose forward orbit is asymptotic to that of* x*:*

$$\lim_{n\to+\infty} d(f^n(x), f^n(y)) = 0.$$

Given $\varepsilon > 0$*, the* local stable manifold *of size* ε *of a point* $x \in M$ *is the set* $W^s_\varepsilon(x)$ *of points* $y \in M$ *such that*

$$\lim_{n\to+\infty} d(f^n(x), f^n(y)) = 0 \quad \textit{and} \quad d(f^n(x), f^n(y)) \le \varepsilon \ \textit{ for all } n \ge 0.$$

It follows immediately from the definitions that $y \in W^s(x)$ if and only if $f^n(y)$ is in the local stable manifold of $f^n(x)$ for some $n \ge 0$. That is,

$$W^s(x) = \bigcup_{n\ge 0} f^{-n}\big(W^s_\varepsilon(f^n(x))\big).$$

In general these sets are not submanifolds of M, in fact they may have a very complicated geometric structure. However, according to the next theorem, if x belongs to some hyperbolic set then $W^s_\varepsilon(x)$ is a disk C^r embedded in M. Then, $W^s(x)$ is a C^r immersed submanifold.

We represent by $\mathrm{Emb}^s(N, M)$ the space of C^s embeddings of a manifold N in M, for any integer $s \ge 1$. See [116, Theorem 6.2] for a proof of the next result.

Theorem 2.1.2. *Let* Λ *be a hyperbolic set for a* C^r *diffeomorphism* f *in* M*. Provided* $\varepsilon > 0$ *is small enough, every local stable manifold* $W^s_\varepsilon(x)$*,* $x \in \Lambda$*, is a disk* C^r *embedded in* M*, with* $T_x W^s_\varepsilon(x) = E^s_x$*. Moreover,*

$$W^s_\varepsilon(x) = \{y \in M : d(f^n(x), f^n(y)) \le \varepsilon \textit{ for all } n \ge 0\},$$

and there are $C > 0$ *and* $\lambda < 1$ *such that* $d(f^n(x), f^n(y)) \le C\lambda^n d(x,y)$ *whenever* $y \in W^s_\varepsilon(x)$*.*

In addition, $W^s_\varepsilon(x)$ *varies continuously with the point* $x \in \Lambda$*: given any* $p \in \Lambda$ *there exists a neighbourhood* V_p *of* p *inside* Λ*, and a continuous map*

$$\Phi_p : V_p \to \mathrm{Emb}^r(W^s_\varepsilon(p), M),$$

such that $\Phi_p(p)$ *is the inclusion of* $W^s_\varepsilon(p)$ *in* M*, and every* $W^s_\varepsilon(x)$*,* $x \in V_p$*, is given by the image of* $W^s_\varepsilon(p)$ *under* $\Phi_p(x)$*.*

The *local unstable manifold* of size ε, denoted $W^u_\varepsilon(x)$, and the *unstable manifold*, $W^u(x)$, of a point $x \in M$ are defined in the same way as the local stable manifold and the stable manifold, respectively, replacing f by its inverse. By Theorem 2.1.2 applied to f^{-1}, local unstable manifolds of points in a hyperbolic set are C^r embedded disks, and the unstable manifolds are C^r immersed.

Definition 2.1.4. *Let Λ be a hyperbolic set for a diffeomorphism f in M. The* stable set *$W^s(\Lambda)$ is the set of points $x \in M$ such that*

$$W^s(\Lambda) = \{x \in M : \lim_{n \to +\infty} d(f^n(x), \Lambda) = 0\}.$$

The unstable set *$W^u(\Lambda)$ of Λ is defined similarly, taking $n \to -\infty$.*

Clearly, $W^s(\Lambda)$ contains the union of the stable manifolds $W^s(x)$ of all the points $x \in \Lambda$. In general, this inclusion may be strict. However, this will never happen if the hyperbolic set satisfies an additional assumption, called local product structure: in that case any orbit that approaches Λ is asymptotic to some orbit inside Λ. This and other important properties of hyperbolic sets with local product structure are reviewed in the next subsection.

2.1.3 Local Product Structure

Let Λ be a hyperbolic set for a C^1 diffeomorphism $f : M \to M$. It follows from Theorem 2.1.2 that if $\varepsilon > 0$ is sufficiently small then the local stable manifold $W^s_\varepsilon(x)$ and the local unstable manifold $W^u_\varepsilon(y)$ of points $x, y \in \Lambda$ intersect in at most one point. Moreover, the intersection is nonempty if the points x and y are close enough to each other.

Definition 2.1.5. *We say that Λ has* local product structure *if there exist $0 < \delta < \varepsilon$ such that $W^s_\varepsilon(x) \cap W^u_\varepsilon(y)$ is in Λ whenever $d(x, y) < \delta$.*

Given $\alpha > 0$, an *α-pseudo-orbit* of f is a sequence $\{x_j : j_1 < j < j_2\}$ in M, with $-\infty \le j_1 < j_2 \le +\infty$, such that

$$d(f(x_j), x_{j+1}) \le \alpha \quad \text{for every } j_1 < j < j_2 - 1.$$

The next result, known as the *shadowing lemma*, expresses the remarkable fact that every pseudo-orbit in a hyperbolic set with local product structure is close to some true orbit. See e.g. [116, Proposition 8.20] for a proof.

Theorem 2.1.3. *Suppose Λ is a hyperbolic set with local product structure. Then, given any $\beta > 0$ there exists $\alpha > 0$ such that for any α-pseudo-orbit $\{x_j : j_1 < j < j_2\}$ in Λ there exists some point $y_0 \in \Lambda$ such that*

$$d(x_j, f^j(y_0)) \le \beta \quad \textit{for every } j_1 < j < j_2\,.$$

Given some small $\varepsilon > 0$, fix $0 < \beta < \varepsilon/2$ and let $\alpha > 0$ be as in the theorem. Moreover, fix $\gamma > 0$ smaller than $\alpha/2$ and $\varepsilon/2$, and small enough so that $d(f(x), f(y)) \le \alpha/2$ for any pair of points with $d(x, y) \le \gamma$. Let U be the γ-neighbourhood of Λ, and suppose y is such that $f^j(y) \in U$ for any $j \ge 0$. Then, for each $j \ge 0$ we can find a point $x_j \in \Lambda$ such that $d(f^j(y), x_j) \le \gamma$. The sequence $(x_j)_j$ constructed in this way is an α-pseudo-orbit:

$$d(f(x_j), x_{j+1}) \leq d(f(x_j), f^{j+1}(y)) + d(f^{j+1}(y), x_{j+1}) \leq \frac{\alpha}{2} + \gamma \leq \alpha$$

for every $j \geq 0$. Assuming Λ has local product structure, we may use the previous theorem to find $x \in \Lambda$ such that $d(f^j(x), x_j) \leq \varepsilon/2$ for every $j \geq 0$. Then $d(f^j(x), f^j(y)) \leq \varepsilon$ for every $j \geq 0$ and so, cf. Theorem 2.1.2, y is in the local stable manifold of size ε of x. This proves the proposition we state next. A dual result for unstable manifolds follows immediately, by considering the inverse map f^{-1}.

Proposition 2.1.3. *Suppose Λ is a hyperbolic set with local product structure. Then, given any $\varepsilon > 0$ there exists a neighbourhood U_ε of Λ such that*

$$\{y \in M : f^n(y) \in U_\varepsilon \text{ for all } n \geq 0\} \subset \bigcup_{x \in \Lambda} W^s_\varepsilon(x).$$

Consequently, $W^s(\Lambda) = \cup_{x \in \Lambda} W^s(x)$.

An invariant set Λ of a diffeomorphism $f : M \to M$ is *isolated* if there exists a neighbourhood U of it such that the set of points whose orbits never leave U neither in the future nor in the past is, precisely, Λ:

$$\Lambda = \bigcap_{n \in \mathbb{Z}} f^n(U).$$

It is easy to see that isolated hyperbolic sets have local product structure: by definition, $W^s_\varepsilon(x) \cap W^u_\varepsilon(y)$ remains in the ε-neighbourhood for all times, future and past, hence it is in Λ. Conversely, cf. [116, Proposition 8.22],

Proposition 2.1.4. *Suppose Λ is a hyperbolic set with local product structure. Then Λ is an isolated set, and so is its hyperbolic continuation Λ_g for any nearby map g. More precisely, one may choose a neighbourhood U of Λ so that $\Lambda_g = \cap_{n \in \mathbb{Z}} g^n(U)$ for any g in a C^1 neighbourhood of f.*

We say that an invariant set Λ of a transformation f is *transitive* if there exists some $z \in \Lambda$ whose forward orbit $\{f^n(z) : n \geq 0\}$ is dense in Λ. The following is a useful characterization of transitivity in terms of denseness of stable and unstable manifolds. We denote

$$W^s(\mathcal{O}(x)) = \bigcup_{n \in \mathbb{Z}} W^s(f^n(x)) \quad \text{and} \quad W^u(\mathcal{O}(x)) = \bigcup_{n \in \mathbb{Z}} W^u(f^n(x))$$

the stable manifold and the unstable manifold of the orbit of a point x.

Proposition 2.1.5. *Suppose Λ is a hyperbolic set with local product structure. If Λ is transitive then $W^s(\mathcal{O}(x)) \cap \Lambda$ and $W^u(\mathcal{O}(x)) \cap \Lambda$ are dense in Λ for every $x \in \Lambda$. Conversely, if Λ contains some periodic point p such that $W^s(\mathcal{O}(p)) \cap \Lambda$ and $W^u(\mathcal{O}(p)) \cap \Lambda$ are dense in Λ, then it is transitive.*

Proof. Suppose Λ is transitive, and let x and y be arbitrary points in Λ. We are going to show that there are points of $W^s(\mathcal{O}(x)) \cap \Lambda$ arbitrarily close to y. Let $z \in \Lambda$ be a point whose forward orbit is dense in Λ. In particular, there exists $m \geq 1$ such that $d(x, f^m(z)) < \delta$, where δ is as in Definition 2.1.5. Then $W^s(f^m(z))$ intersects $W^u(x)$ in some point $w \in \Lambda$. Using once more the fact that the forward orbit of z is dense, we find $n \geq 1$ such that $f^{m+n}(z)$ is arbitrarily close to y. Since $d(f^{m+n}(z), f^n(w))$ goes to zero as n increases, $f^n(w)$ can also be made arbitrarily close to y. By construction, $w \in W^s(\mathcal{O}(x)) \cap \Lambda$ and so we have shown that the latter set is dense in Λ. The same argument proves denseness of $W^u(\mathcal{O}(x)) \cap \Lambda$.

The main step to prove the converse is to show that, given any pair U and V of open subsets of Λ, some forward iterate $f^n(U)$ of U intersects V. Let p be a periodic point as in the statement, and $k \geq 1$ be its period. By the assumption, there exist iterates p_1, p_2 of p, and points $x \in W^s(p_1) \cap \Lambda \cap U$ and $y \in W^u(p_2) \cap \Lambda \cap V$. The claim we are proving is not affected if we replace U or V by any of their iterates. So, we may always suppose $p_1 = p_2 = p$. Then $f^{jk}(x)$ and $f^{-jk}(y)$ converge to p as $j \to +\infty$. In particular, if j is large enough then $d(f^{jk}(x), f^{-jk}(y)) < \delta$, and this implies that $W^u_\varepsilon(f^{jk}(x))$ intersects $W^s_\varepsilon(f^{-jk}(y))$ in a point $w_j \in \Lambda$. Then, by the definition of local unstable manifolds and Theorem 2.1.2, $f^{-jk}(w_j)$ is in $W^u_\varepsilon(x) \cap \Lambda$ and converges to x as $j \to \infty$. Analogously, $f^{jk}(w_j)$ is in $W^s_\varepsilon(y) \cap \Lambda$ and converges to y as $j \to \infty$. In particular, $f^{-jk}(w_j) \in U$ and $f^{jk}(w_j) \in V$, and so $f^{2jk}(U) \cap V$ is nonempty, for all large j. This completes the proof of the claim.

The proposition follows by a Baire type argument. Indeed, let $(V_k)_k$ be some countable basis for the topology of Λ. The claim implies that $\cup_{n=0}^{\infty} f^{-n}(V_k)$ is dense in Λ, for every k. Hence, $\cap_k \cup_{n=0}^{\infty} f^{-n}(V_k)$ is nonempty. Let z be any point in this latter set. Then, for any k there exists $n \geq 0$ such that $f^n(z) \in V_k$. This means that the forward orbit of z is dense in Λ. □

2.1.4 Hyperbolic Attractors

Definition 2.1.6. *A transitive hyperbolic set Λ is a* hyperbolic attractor *for $f : M \to M$ if there exists an open neighbourhood U of Λ such that*

$$f\big(\operatorname{clos}(U)\big) \subset U \qquad \text{and} \qquad \Lambda = \bigcap_{n \geq 0} f^n(U). \tag{2.5}$$

A set Λ is a hyperbolic repeller *for f if it is a hyperbolic attractor for f^{-1}.*

Remark 2.1.1. In a sense, the first condition in (2.5) is unnecessary. Indeed, if Λ is any compact invariant set of a homeomorphism f admitting a neighbourhood V such that $\Lambda = \cap_{n \geq 0} f^n(V)$, then there exists an open set U with $\Lambda \subset U \subset V$ and $f(\operatorname{clos}(U)) \subset U$. In particular, $\Lambda = \cap_{n \geq 0} f^n(U)$. The proof of this is left as an exercise (alternatively, see [116, Lemma 2.9]).

If f is a transitive Anosov map then $\Lambda = M$ is a hyperbolic attractor (and a hyperbolic repeller) for f. The solenoids in Example 2.1.3 are transitive sets and, thus, they are hyperbolic attractors for the corresponding maps.

Clearly, hyperbolic attractors are isolated hyperbolic sets, and so they have local product structure. Thus, by Proposition 2.1.4, the hyperbolic continuation of a hyperbolic attractor is also a hyperbolic attractor. Similar facts hold for repellers.

Definition 2.1.7. *The* basin *of a hyperbolic attractor Λ is the set $B(\Lambda)$ of points $z \in M$ such that $d(f^n(z), \Lambda) \to 0$ as $n \to +\infty$.*

By Proposition 2.1.3, $B(\Lambda) = W^s(\Lambda)$ coincides with the union of all the stable manifolds of points in Λ. It is easy to see that the basin also equals the union of all backward iterates $f^{-n}(U)$, for any neighbourhood U as in Definition 2.1.6. The property that $W^s(\Lambda)$ contains a neighbourhood of Λ characterizes attractors among the transitive hyperbolic sets:

Proposition 2.1.6. *Let Λ be a transitive hyperbolic set for $f : M \to M$. The following conditions are equivalent:*

(1) Λ is a hyperbolic attractor for f;
(2) $W^s(\Lambda)$ contains a neighbourhood of Λ in M;
(3) $W^u(p) \subset \Lambda$ for every $p \in \Lambda$.

Proof. We already noted that (1) implies (2). To prove that (2) implies (3), let $W^s_\varepsilon(\Lambda)$ denote the union of all local stable manifolds $W^s_\varepsilon(x)$ with $x \in \Lambda$. Then $f^{-n}(W^s_\varepsilon(\Lambda))$, $n \geq 1$, is a nondecreasing sequence of subsets of $W^s(\Lambda)$. Moreover, by Proposition 2.1.3, their union coincides with the whole $W^s(\Lambda)$ which, by assumption, contains a neighbourhood of Λ. Then, a simple compactness argument shows that $f^{-n}(W^s_\varepsilon(\Lambda))$ must contain a neighbourhood of Λ for some n. Since f is a homeomorphism and Λ is invariant, it follows that $W^s_\varepsilon(\Lambda)$ itself contains a neighbourhood of Λ. Now let $p \in \Lambda$ and q be any point in $W^u(p) \cap \Lambda$. It follows from the definitions that any point r in a sufficiently small neighbourhood of q inside $W^u(p)$ is in $W^u_\varepsilon(q)$, and the previous remarks imply that it is also in $W^s_\varepsilon(x)$ for some $x \in \Lambda$. So, by local product structure, any such r is in Λ. This shows that $W^u(p) \cap \Lambda$ is open in $W^u(p)$. Since Λ is compact, $W^u(p) \cap \Lambda$ is also closed in $W^u(p)$. Therefore, $W^u(p)$ must be contained in Λ.

We are left to prove that (3) implies (1). Let V be a small neighbourhood of Λ. The same argument as we used for proving Proposition 2.1.3 gives that, assuming V is small enough, any point y such that $f^{-n}(y) \in V$ for every $n \geq 0$ must be in the (local) unstable manifold of some point $p \in \Lambda$ and so, in view of the assumption, $y \in \Lambda$. In other words, we have shown that $\cap_{n \geq 0} f^n(V) \subset \Lambda$. The reverse inclusion is clear. So, in view of Remark 2.1.1, the proof of the proposition is complete. □

2.2 Partial Hyperbolicity

Here we introduce an important extension of the notions discussed in the previous section: partially hyperbolic sets and attractors. As we shall comment upon later, partial hyperbolicity is closely related to *robustness* of the dynamics, e.g., attractors that can not be destroyed by any small perturbation of the system.

2.2.1 Definitions and Examples

Definition 2.2.1. *Let Λ be a compact invariant set for a C^1 diffeomorphism $f : M \to M$, and $T_\Lambda M = E^1 \oplus E^2$ be a continuous Df-invariant splitting of the tangent space over Λ. We say that the splitting is* dominated *if there are constants $C > 0$ and $0 < \lambda < 1$ such that*

$$\sup_{x\in\Lambda}\left(\|Df^{-n}(f^n(x)) \mid E^1_{f^n(x)}\| \, \|Df^n(x) \mid E^2_x\|\right) \le C\lambda^n \tag{2.6}$$

for every $n \ge 1$.

Whenever we speak of a splitting of the tangent space it is implicit that the subspaces E^1_x and E^2_x have constant dimensions over their domain. The condition in the definition may be written, equivalently, as

$$\sup_{x,v_1,v_2} \frac{\|Df^n(x)v_2\|}{\|v_2\|}\frac{\|v_1\|}{\|Df^n(x)v_1\|} \le C\lambda^n \tag{2.7}$$

where the supremum is over all nonzero vectors $v_1 \in E^1_x$ and $v_2 \in E^2_x$, and all $x \in \Lambda$. So, roughly speaking, the splitting is dominated if E^1 is more expanding/less contracting than E^2.

Remark 2.2.1. A dominated splitting is unique when it exists, provided we fix dimensions. More precisely, suppose the tangent space at some point $x \in M$ admits two decompositions satisfying (2.7)

$$T_x M = E^1_x \oplus E^2_x = F^1_x \oplus F^2_x,$$

with $\dim E^i_x = \dim F^i_x$ for $i = 1, 2$. Then $E^1_x = F^1_x$ and $E^2_x = F^2_x$. To prove this, suppose F^2_x is not contained in E^2_x. Then there is some $w_2 \in F^2_x$ that can be written as $w_2 = v_1 + v_2$, with $v_1 \in E^1_x$, $v_2 \in E^2_x$, and v_1 nonzero. We check two complementary cases, to show that they both lead to a contradiction. If there is $w_1 \in F^1_x$ with $w_1 = v_1 + v_3$ for some $v_3 \in E^2_x$ then, by the domination property for $E^1_x \oplus E^2_x$, we have $\|Df^n(x)w_1\| \approx \|Df^n(x)v_1\| \approx \|Df^n(x)w_2\|$ for all large n. This contradicts the domination property for $F^1_x \oplus F^2_x$. If, on the contrary, no vector w_1 in F^1_x projects down to v_1 then, since F^1_x has the same dimension as E^1_x, there must be some nonzero vector w_1 in the intersection of F^1_x and E^2_x. In that case, $\|Df^n(x)w_1\| \ll \|Df^n(x)v_1\| \approx \|Df^n(x)w_2\|$, which also contradicts the domination property for $F^1_x \oplus F^2_x$. This proves that, F^2_x is contained in E^2_x. Hence, since the two subspaces have the same dimension, they must coincide. A dual argument, using f^{-1}, proves that $F^1_x = E^1_x$.

Remark 2.2.2. The requirement of continuity in Definition 2.2.1 is superfluous: any Df-invariant splitting $T_\Lambda M = E^1 \oplus E^2$ that satisfies (2.6) is continuous, as long as E^1 and E^2 have constant dimensions. To see this, we only have to show that given a sequence $(x_n)_n$ in Λ converging to $x \in \Lambda$ and such that $E^1_{x_n}$ and $E^2_{x_n}$ converge to subspaces F^1_x and F^2_x of $T_x M$, respectively, then $F^1_x = E^1_x$ and $F^2_x = E^2_x$. Since the domination condition is closed, it remains valid in the limit: (2.6) holds for any $v_1 \in F^1_x$ and $v_2 \in F^2_x$. So, the claim is a direct consequence of the previous remark.

Definition 2.2.2. *Let Λ be a compact invariant set for $f : M \to M$. We say that Λ is* partially hyperbolic *if there is a dominated splitting $T_\Lambda M = E^1 \oplus E^2$ such that either*

1. *E^1 is expanding: $\|Df^{-n}(x) \mid E^1_x\| \leq C\lambda^n$ for every $x \in \Lambda$ and $n \geq 1$,*
2. *or E^2 is contracting: $\|Df^{n}(x) \mid E^2_x\| \leq C\lambda^n$ for every $x \in \Lambda$ and $n \geq 1$.*

In the first case we write $E^1 = E^u$, $E^2 = E^{cs}$, and say that the partially hyperbolic set Λ is of *type* $E^u \oplus E^{cs}$. In the second one we write $E^1 = E^{cu}$, $E^2 = E^s$, and say that Λ is of *type* $E^{cu} \oplus E^s$. A set Λ is partially hyperbolic of type $E^u \oplus E^{cs}$ for f if and only if it is of type $E^{cu} \oplus E^s$ for f^{-1}.

Definition 2.2.3. *A diffeomorphism $f : M \to M$ is* partially hyperbolic *(of type $E^u \oplus E^{cs}$, respectively $E^{cu} \oplus E^s$) if the manifold M is a partially hyperbolic set for f (of type $E^u \oplus E^{cs}$, respectively $E^{cu} \oplus E^s$).*

Hyperbolic sets are also partially hyperbolic, of both types $E^u \oplus E^{cs}$ and $E^{cu} \oplus E^s$. Other examples will be given in a little while. Before that, let us make a few comments about these definitions.

Remark 2.2.3. Suppose $T_\Lambda M = E^1 \oplus E^2$ is a dominated splitting for some f^N, $N \geq 2$. Clearly, the same is true for $T_\Lambda M = Df \cdot E^1 \oplus Df \cdot E^2$. So, by Remark 2.2.1, the subbundles E^1 and E^2 must be Df-invariant. Given any $n \in \mathbb{Z}$, there is $0 \leq r < N$ such that $n - r$ is a multiple of N. This, and the fact that $\|Df^r(x)v\|/\|v\|$ is uniformly bounded away from zero and infinity for all $x \in \Lambda$, $v \in T_x M$, $0 \leq r < N$, ensures that $T_\Lambda M = E^1 \oplus E^2$ is a dominated splitting for f too. Moreover, E^1 is Df-expanding (respectively, E^2 is Df-contracting) if it is Df^N-expanding (respectively, Df^N-contracting). Conversely, if $T_\Lambda M = E^1 \oplus E^2$ is a dominated splitting (respectively, Λ is partially hyperbolic) for f then the same is true for any iterate f^N. Note that $C\lambda^N < 1$ if N is large enough. This means that for f^N we may take constants $\lambda_N = C\lambda^N$ and $C_N = 1$.

Remark 2.2.4. In the terminology of [54], the domination property (2.6) is a condition of *eventual relative* normal hyperbolicity. *Immediate* (as opposed to eventual) domination corresponds to taking the constant $C = 1$. A splitting is dominated for f if and only if it is immediately dominated for f^N, any large N, cf. Remark 2.2.3. *Absolute* (as opposed to relative) domination corresponds to replacing the left hand side of (2.6) by

$$\sup_{x\in\Lambda} \|Df^{-n}(f^n(x)) \mid E^1_{f^n(x)}\| \sup_{x\in\Lambda} \|Df^n(x) \mid E^2_x\|.$$

See the discussion in [54, Section 5] concerning the relations between these notions. Domination and partial hyperbolicity are sometimes defined in the immediate and/or absolute senses, but that is unnecessarily restrictive.

There is another way in which partial hyperbolicity is sometimes defined through a stronger condition: existence of a dominated splitting into *three* invariant subbundles. In this regard we follow Brin-Pesin [26], who required only two invariant subbundles, and we call that stronger condition *strong partial hyperbolicity.* That is, we call a compact invariant set Λ strongly partially hyperbolic for f if there exists a continuous splitting

$$T_\Lambda M = E^u \oplus E^c \oplus E^s$$

of the tangent bundle into three Df-invariant subbundles, such that E^u is uniformly expanding, E^s is uniformly contracting, and both splittings

$$E^1 = E^u,\ E^2 = E^c \oplus E^s \qquad \text{and} \qquad E^1 = E^u \oplus E^c,\ E^2 = E^s$$

are dominated.

Example 2.2.1. (Linear automorphisms) Let $d \geq 3$ and $A \in \mathrm{Sl}(d,\mathbb{Z})$ be such that the spectrum of A splits into three nonempty subsets, contained in

$$\{\lambda : b < |\lambda|\}, \quad \{\lambda : a < |\lambda| < b\}, \quad \{\lambda : |\lambda| < a\},$$

for some $0 < a < 1 < b$. Let $\mathbb{R}^d = \hat{F}^u \oplus \hat{F}^c \oplus \hat{F}^s$ be the corresponding splitting of $\mathbb{R}^d$ into invariant subspaces: A preserves the three subspaces, the eigenvalues of $A \mid \hat{F}^u$ have norm larger than b, those of $A \mid \hat{F}^s$ have norm smaller than a, and the eigenvalues of $A \mid \hat{F}^c$ are all in (a, b). Let $M = T^d$ be the d-dimensional torus, $\pi : \mathbb{R}^d \to M$ be the canonical projection, and f be the diffeomorphism induced in M by A, cf. Example 2.1.1. Then $F^u = D\pi \cdot \hat{F}^u$, $F^c = D\pi \cdot \hat{F}^c$, $F^s = D\pi \cdot \hat{F}^s$ are continuous subbundles of the tangent space of M. It is easy to see that both

$$E^1 = F^u, \quad E^2 = F^c \oplus F^s, \quad \text{and} \quad E^1 = F^u \oplus F^c, \quad E^2 = F^s,$$

define splittings of TM as in Definition 2.2.2, respectively of type $E^u \oplus E^{cs}$ and $E^{cu} \oplus E^s$. This construction is valid even if A is hyperbolic (in which case f is Anosov).

Example 2.2.2. (Time-1 maps of hyperbolic flows) Let $(X^t)_{t\in\mathbb{R}}$ be an Anosov flow on a compact manifold M: $(X^t)_t$ is a C^1 flow without singularities such that there exists a continuous splitting $TM = E^s \oplus E^0 \oplus E^u$ of the tangent bundle of M into three subbundles that are invariant under DX^t, for every $t \in \mathbb{R}$; moreover, E^u is expanding and E^s is contracting: there are $C > 0$ and $0 < \lambda < 1$ such that

$$\|DX^{-t} \mid E^u\| \le C\lambda^t, \quad \|DX^t \mid E^s\| \le C\lambda^t, \quad \text{for } t > 0,$$

and E^0 is the one-dimensional subbundle generated by the vector field. Let $f = X^1$ be the time-1 map of the flow. Then M is a strongly partially hyperbolic set for f, with splitting $E^u \oplus E^0 \oplus E^s$. More generally, a hyperbolic set of a flow is strongly partially hyperbolic for the corresponding time 1 diffeomorphism.

Now we extend Proposition 2.1.2 to partially hyperbolic systems. The terminology is as in Definition 2.1.2.

Proposition 2.2.1. *Let Λ be a compact invariant set for $f : M \to M$.*

1. *Λ admits a dominated splitting if and only if there exists a splitting $T_\Lambda M = E \oplus F$ and a constant $a > 0$ such that the cone field $C_a(E)$ is forward invariant with respect to some positive iterate f^N.*
2. *Λ is partially hyperbolic of type $E^u \oplus E^{cs}$ if and only if $C_a(E)$ may be taken to be an unstable cone field for f^N.*

Here the splitting $E \oplus F$ needs not be continuous, but it is assumed that E_x and F_x have constant dimensions, and the angle between them is bounded away from zero: we use that there exists $K_0 > 0$ such that

$$\frac{1}{K_0} \max\{\|e\|, \|f\|\} \le \|e + f\| \le 2 \max\{\|e\|, \|f\|\}, \tag{2.8}$$

for every $e \in E_x$, $f \in F_x$, and $x \in \Lambda$.

Proof. The 'only if' part is easy. If $T_\Lambda M = E^1 \oplus E^2$ is a dominated splitting then any cone field $C_a(E^1)$ centered around E^1 is forward invariant for f^N if N is large enough. Moreover, the cone field is unstable for f^N if the subbundle E^1 is expanding. The verifications are left as an exercise.

Let us prove that a forward invariant cone field is also a sufficient condition for existence of a dominated splitting. Assume $T_\Lambda M = E \oplus F$, $a > 0$, $\theta < 1$, and $N \ge 1$, are given, as in the statement. In particular,

$$Df^N(x) \cdot C_a(E, x) \subset C_{\theta a}(E, f^N(x)) \tag{2.9}$$

for every $x \in \Lambda$. For each $n \ge 1$, let $\mathcal{L}_n(x)$ denote the set of linear maps from E_x to F_x such that graph(L_x) is contained in $Df^{nN}(x)C_a(E, f^{-nN}(x))$. A linear map from E_x to F_x is completely described by the images of the vectors in some fixed basis of E_x. Using this correspondence, we may identify each $\mathcal{L}_n(x)$ with a compact subset of F_x^d, $d = \dim E_x$. As a consequence of

(2.9), the sequence $\mathcal{L}_n(x)$, $n \geq 1$, is nonincreasing for every $x \in \Lambda$. Clearly, $Df^{nN}(x)E_{f^{-nN}(x)}$ is always contained in $Df^{nN}(x)C_a(E, f^{-nN}(x))$ and, in particular, it is transverse to F_x. Hence, $Df^{nN}(x)E_{f^{-nN}(x)}$ may be written as the graph of some linear map from E_x to F_x, which proves that $\mathcal{L}_n(x)$ is always nonempty. These remarks show that the intersection of the $\mathcal{L}_n(x)$ over all $n \geq 0$ is nonempty for all $x \in \Lambda$. In other words, there exists some $L(x) : E_x \to F_x$ such that

$$\operatorname{graph}(L(x)) \subset Df^{nN}(x)C_a(E, f^{-nN}(x))$$

for all $n \geq 1$. Let us take a closer look at the case when x is a periodic point, say $f^{kN}(x) = x$. Then $Df^{kN}(x)$ is a linear isomorphism of T_xM, admitting a strictly invariant cone $C_a(E,x)$. An elementary reasoning, using e.g. the canonical Jordan form, shows that there exist $T_xM = G^1 \oplus G^2$ and $\rho > 0$ such that G^1 is the sum of all generalized eigenspaces of $Df^{kN}(x)$ corresponding to eigenvalues with norm strictly larger than ρ, G^2 is the sum of the generalized eigenspaces corresponding to eigenvalues with norm strictly smaller than ρ, and $G^1 \subset C_a(E,x)$, whereas G^2 intersects $C_a(E,x)$ only at the zero vector. Then, the intersection of all the $Df^{nN}(x)C_a(E, f^{-nN}(x))$ is just G^1. In particular, in this case $L(x)$ is uniquely determined and its graph G^1 is invariant under $Df^{kN}(x)$.

Let K be a subset of Λ containing exactly one point in each orbit of f^N through Λ. For each $x \in K$ and $j \in \mathbb{Z}$, set

$$E^1_{f^{jN}(x)} = Df^{jN}(x)\operatorname{graph}(L(x)).$$

In view of the previous remarks about periodic points, this is well-defined. By (2.9) and the choice of $L(\cdot)$, E^1_y is contained in $C_{\theta a}(E,y)$ for every $y \in \Lambda$. It is also clear from the construction that the subbundle E^1 of $T_\Lambda M$ defined in this way is invariant under Df^N. On the other hand, (2.9) may be rewritten as

$$Df^{-N}(x) \cdot C_{1/(\theta a)}(F,x) \subset C_{1/a}(F, f^{-N}(x))$$

where $C_b(F,x)$ denotes the cone of width b around F. So, a similar argument gives another Df^N-invariant subbundle E^2 of $T_\Lambda M$ such that E^2_x is contained in $C_{1/a}(F,x)$ for all $x \in \Lambda$. Since the cones $C_{\theta a}(E,x)$ and $C_{1/a}(F,x)$ intersect only at the zero vector, the same is true for the subspaces E^1_x and E^2_x. Then $E^1_x \oplus E^2_x = T_xM$ for all $x \in \Lambda$, because E^1_x, E^2_x have the same dimensions as E_x, F_x, respectively.

Now we show that the Df^N-invariant splitting $T_\Lambda M = E^1 \oplus E^2$ we have constructed so far is a dominated splitting for f^N. It is easy to see that there exists $\delta > 0$ such that

$$u \in C_{\theta a}(E,x) \text{ and } \|v\| \leq \delta\|u\| \quad \Rightarrow \quad u + v \in C_a(E,x) \tag{2.10}$$

for every $x \in \Lambda$. Indeed, it suffices to choose $\delta > 0$ sufficiently small so that $(1 + 2K_0\delta)\theta \leq (1 - 2K_0\delta)$, with K_0 as in (2.8). We also need the following fact, whose proof we postpone until Subsection 3.2.2:

Lemma 2.2.1. *There exist constants* $K_1 > 0$ *and* $\lambda_1 < 1$ *such that*

$$e_1 + e_2 \in Df^{nN}(x)C_a(E,x) \quad \Rightarrow \quad \|e_2\| \leq K_1\lambda_1^n\|e_1\|$$

for any $e_1 \in E_x^1$, $e_2 \in E_x^2$, $x \in \Lambda$, *and* $n \geq 1$.

Let v_1 and v_2 be arbitrary nonzero vectors in E_1^x and E_x^2, respectively. By (2.10), $v = v_1/\|v_1\| + \delta v_2/\|v_2\|$ is in the cone $C_a(E,x)$. So, Lemma 2.2.1 applies to its image $Df^{nN}(x)v$:

$$\left\|Df^{nN}(x)\delta\frac{v_2}{\|v_2\|}\right\| \leq K_1\lambda_1^n\left\|Df^{nN}(x)\frac{v_1}{\|v_1\|}\right\|,$$

for all $n \geq 1$. This proves the domination (2.7) for f^N, with $C = K_1/\delta$ and $\lambda = \lambda_1$. Then, by Remark 2.2.3, $T_\Lambda M = E^1 \oplus E^2$ is also a dominated splitting for f. This concludes the proof of part 1 of the proposition. Finally, if $C_a(E)$ is an unstable cone field for f^N then the subbundle E^1 we have constructed is expanding for f^N, and so also for f. Hence, in that case Λ is partially hyperbolic. This proves part 2. □

The following result is deduced in the same way as Corollary 2.1.1:

Corollary 2.2.1. *If* Λ *is a partially hyperbolic set (respectively, an invariant set with a dominated splitting) for* f, *then there exists a neighbourhood* U *of* Λ *in* M *and a neighbourhood* $\mathcal{N}$ *of* f *in* $\mathrm{Diff}^1(M)$, *such that if* $g \in \mathcal{N}$ *and* Γ *is any compact subset of* U *that is invariant under* g, *then* Γ *is partially hyperbolic (respectively, has a dominated splitting) for* g.

Example 2.2.3. (Fiber bundles over hyperbolic sets) Let $f : M \to M$ be a diffeomorphism with a hyperbolic set Λ, e.g. an Anosov diffeomorphism. Let $T_\Lambda M = E^u \oplus E^s$ be the corresponding splitting of the tangent space over Λ. Fix $\lambda_0 < 1$ and some Riemannian metric on M so that

$$\|Df^{-1} \mid E^u\| < \lambda_0 \quad \text{and} \quad \|Df \mid E^s\| < \lambda_0\,.$$

Let N be a compact manifold, $\pi : N \to M$ be a C^1 fiber bundle over M, and $\tilde{f} : N \to N$ be a C^1 isomorphism of fiber bundles covering f: $\pi \circ \tilde{f} = f \circ \pi$. An interesting special case is $N = M \times P$, for some compact manifold P, $\pi(x,y) = x$, and $\tilde{f}$ a skew-product $\tilde{f}(x,y) = (f(x), g(x,y))$. In general, denote $\tilde{\Lambda} = \pi^{-1}(\Lambda)$. Clearly, $E_z^0 = \mathrm{kernel}\, D\pi(z)$ defines a subbundle E^0 of TN invariant under $D\tilde{f}$. Assume that there exists $\lambda \in (\lambda_0, 1)$ such that

$$\lambda\|v\| \leq \|D\tilde{f}(z)v\| \leq \lambda^{-1}\|v\| \tag{2.11}$$

for all $z \in \tilde{\Lambda}$ and $v \in T_z E_z^0$. Then $\tilde{\Lambda}$ *is a strongly partially hyperbolic set for* $\tilde{f}$. This can be proved using cone fields, as follows.

Let F be the orthogonal complement of the subbundle E^0, with respect to an arbitrary Riemannian metric on N. Then $D\pi(z) \mid F_z$ is an isomorphism onto $T_{\pi(z)}M$ for every $z \in \tilde{\Lambda}$. Up to modifying the Riemannian metric inside each F_z, we may suppose that this isomorphism is an isometry, and we do so. The subbundle $D\tilde{f} \cdot F$ is transverse to E^0 at every point and so, by continuity and compactness, there exists a uniform lower bound for the angle between them. Then there is $K_1 \geq 1$ such that $\|w^0\| \leq K_1 \|w\|$ for any $v \in F_z$, where

$$D\tilde{f}(z)v = w_0 \oplus w \in E^0_{\tilde{f}(z)} \oplus F_{\tilde{f}(z)} .$$

Let E_z^+ and E_z^- be the subspaces of F_z defined by

$$D\pi(z)E_z^+ = E^u_{\pi(z)} \quad \text{and} \quad D\pi(z)E_z^- = E^s_{\pi(z)}.$$

Fix $K > 0$ large enough so that $(\lambda_0/\lambda)K + 2K_1 < K$, then define

$$C_K(E^+, z) = \{v^+ + v^0 + v^- \in E_z^+ \oplus E_z^0 \oplus E_z^- : \\ \|v^0\| \leq K\|v^+\| \text{ and } \|v^-\| \leq \|v^+\|\}.$$

The condition on K ensures that $C_K(E^+)$ is an unstable cone field for $\tilde{f}$. Analogously, the cone field $C_K(E^-)$ obtained interchanging the roles of v^+ and v^- in these definitions is a stable cone field for $\tilde{f}$. Thus, the claim follows from Proposition 2.2.1.

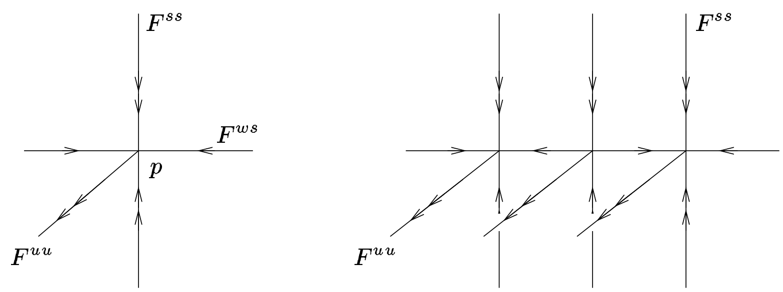

Fig. 2.2. Deforming an Anosov diffeomorphism

Example 2.2.4. Partially hyperbolic sets can also be obtained by deformation of hyperbolic ones, as in the following construction of [74]. Start with an Anosov diffeomorphism f_0 in T^3 that admits an invariant splitting into three subbundles $F^{uu} \oplus F^{ws} \oplus F^{ss}$. Here F^{uu} is expanding, and F^{ws} and F^{ss} are contracting, with F^{ss} dominating F^{ws}: there are $\lambda < 1$ and a Riemannian metric on M such that

$$\|Df_0 \mid F_x^{ss}\| \, \|Df_0^{-1} \mid F^{ws}_{f(x)}\| \leq \lambda$$

for every x. Let p be a fixed point of f_0. Deform f_0 by isotopy in a small neighbourhood of p, as described in Figure 2.2: keep the diffeomorphism essentially unchanged in the directions of F^{uu} and F^{ss}, while modifying it in the direction of F^{ws} so that the fixed point goes through a pitchfork bifurcation, that gives rise to two new fixed points. As shown in [74], this can be done in such a way that the whole $M = T^3$ is a partially hyperbolic *transitive* set for the resulting diffeomorphism f, as well as for any other one in a C^1 neighbourhood of it. Since f has periodic saddles with either 1 or 2 contracting eigenvalues, it can not be an Anosov diffeomorphism.

2.2.2 Invariant Foliations. Absolute Continuity

Partially hyperbolic sets share some of the geometric properties of hyperbolic sets, specially, the existence of invariant foliations tangent to expanding or contracting subbundles. Here the definition of foliation is somewhat unusual, in that we require very little transverse regularity (just continuity). This is well suited for our context, as the holonomy of these invariant foliations is generally not differentiable. Nevertheless, these foliations are always transversely absolutely continuous, if the diffeomorphism is twice differentiable, and they are transversely Hölder continuous. These facts play a key role in the ergodic theory of hyperbolic and partially hyperbolic systems.

By a *foliation* $\mathcal{F}$ on a set $\Lambda \subset M$ we mean a family of two-by-two disjoint C^r immersed submanifolds with constant dimension, $1 \leq r \leq \infty$, called the *leaves* of $\mathcal{F}$, such that every leaf intersects Λ, and every point of Λ is contained in some of the leaves. The foliation is *f-invariant* if $f(\mathcal{F}(p)) = \mathcal{F}(f(p))$ for every $p \in \Lambda$, where $\mathcal{F}(p)$ denotes the leaf that contains p.

All the foliations we deal with are *continuous*, in the sense of Theorem 2.1.2: given any $p \in \Lambda$ there exists a neighbourhood V_p of p in Λ, a disk W_p around p inside $\mathcal{F}(p)$, and a continuous map

$$\Phi_p : V_p \to \mathrm{Emb}^r(W_p, M),$$

such that $\Phi_p(p)$ is the inclusion of W_p in M, and the image $\Phi_p(z)(W_p)$ of the embedding $\Phi_p(z)$ is a neighbourhood of z inside $\mathcal{F}(z)$. When the value of r is relevant, we say that $\mathcal{F}$ is a *continuous foliation with C^r leaves*.

In the next theorem $d_u(\cdot\,,\cdot)$ represents the distance (length of shortest piecewise smooth curve connecting two points) inside a strong-unstable leaf.

Theorem 2.2.1. *Let Λ be a partially hyperbolic set of type $E^u \oplus E^{cs}$ for a C^r diffeomorphism f, any $r \geq 1$. Then there exists a unique f-invariant foliation $\mathcal{F}^u$ on Λ such that $T_p\mathcal{F}^u(p) = E^u_p$ at every point p of Λ. Moreover, $\mathcal{F}^u$ is a continuous foliation with C^r leaves. In addition, the leaves of $\mathcal{F}^u$ are exponentially contracted by backward iterates of f: there are $C > 0$ and $\lambda < 1$ such that, for each pair of points z_1, z_2 in the same leaf of $\mathcal{F}^u$,*

$$d_u(f^{-j}(z_1), f^{-j}(z_2)) \leq C\lambda^j d_u(z_1, z_2), \quad \textit{for any } j \geq 1.$$

A proof is given in [26, Section 2], under the absolute version of the definition of partial hyperbolicity (recall Remark 2.2.4). In [54] a more general argument is used that applies in our context, see Theorem 5.5 and the discussion in page 62 of [54]. See also [116, Appendix IV].

We call $\mathcal{F}^u$ *strong-unstable foliation* of Λ. Dually, given a partially hyperbolic set of type $E^{cu} \oplus E^s$ there exists a unique f-invariant *strong-stable foliation* $\mathcal{F}^s$ tangent to E^s. Furthermore, it is continuous, its leaves are C^r submanifolds, and they are exponentially contracted by forward iterates. This follows from applying Theorem 2.2.1 to the inverse map f^{-1}. When Λ is hyperbolic with splitting $T_\Lambda M = E^u \oplus E^s$, we just call $\mathcal{F}^u$ and $\mathcal{F}^s$ the *unstable foliation* and the *stable foliation* of Λ, respectively.

Definition 2.2.4. *A transitive partially hyperbolic set Λ is a* partially hyperbolic attractor *for $f : M \to M$ if there exists an open neighbourhood U of Λ such that*

$$\operatorname{clos}\big(f(U)\big) \subset U \qquad \textit{and} \qquad \Lambda = \bigcap_{n \ge 0} f^n(U).$$

The following extension of Proposition 2.1.6 is part of [94, Proposition 1].

Proposition 2.2.2. *Suppose Λ is a partially hyperbolic attractor of type $E^u \oplus E^{cs}$. Then it is a union of entire strong-unstable leaves: $\mathcal{F}^u(p)$ is contained in Λ for every $p \in \Lambda$.*

Theorem 2.2.1 guarantees that the leaves as these invariant foliations are as smooth as the diffeomorphism itself. Quite in contrast, the holonomy maps (projections along the leaves from one transverse section to another) are usually not differentiable, regardless of how smooth the diffeomorphism is. Indeed, the holonomy may fail to be Lipschitz, even in the case of real analytic Anosov diffeomorphisms [7]. This lack of transverse regularity is a major source of technical difficulties, specially in high dimensional systems.

However, these invariant foliations of hyperbolic and partially hyperbolic attractors and repellers do have some amount of transverse regularity, that is crucial for the ergodic theory of these systems. This is our subject for the rest of this subsection. According to Theorems 2.2.2 and 2.2.3, these foliations are absolutely continuous: despite not being differentiable, *the holonomy maps send zero Lebesgue sets into zero Lebesgue measure sets.* Here one assumes that the diffeomorphism is of class C^2. Furthermore, by Proposition 2.2.3, the holonomy maps are always Hölder continuous.

In order to give the precise statements, we should explain what we exactly mean by holonomy map of a foliation. This relies on the following useful notion.

Foliated charts. For the time being, let Λ be a hyperbolic attractor and $\mathcal{F}^u$ be its unstable foliation. Let $\varepsilon > 0$ be fixed, small enough so that the conclusion of Theorem 2.1.2 holds for Λ. Given any $p \in \Lambda$, let V_p be a neigh-

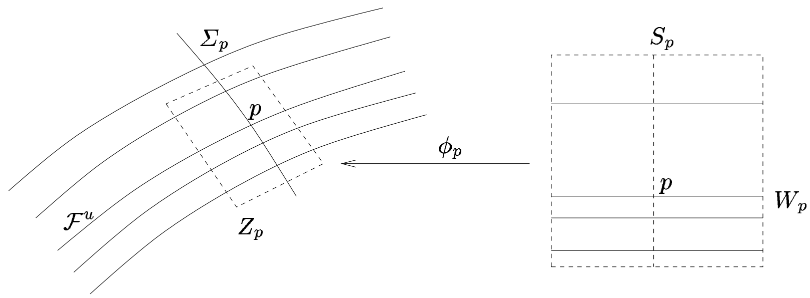

Fig. 2.3. A foliated chart for $\mathcal{F}^u$

bourhood of p inside Λ, and

$$\Phi_p : V_p \to \mathrm{Emb}^r(W^u_\varepsilon(p), M)$$

be a continuous map as in Theorem 2.1.2: $W^u_\varepsilon(z) = \Phi_p(z)(W^u_\varepsilon(p))$ for every $z \in V_p$. Moreover, let Σ_p be some smooth disk transverse to the unstable foliation at p, in the sense that $T_p\Sigma_p \oplus E^u_p = T_pM$. We take Σ_p small enough so that it intersects each local unstable manifold of size ε in not more than one point. Note that, by transversality, Σ_p does intersect the local unstable manifold of any point of Λ close enough to p. We denote, cf. Figure 2.3,

$$W_p = W^u_\varepsilon(p) \cap V_p \quad \text{and} \quad S_p = \Sigma_p \cap V_p\,.$$

Up to replacing V_p by some smaller compact neighbourhood of p inside Λ, we may suppose that W_p is a compact disk around p (inside its strong-unstable leaf) and S_p is also a compact set. Then we define

$$\phi_p : W_p \times S_p \to \Lambda, \qquad \phi_p(x,y) = \Phi_p(y)(x).$$

Observe that ϕ_p does take values in Λ, by the last property in Proposition 2.1.6. Moreover, as a consequence of Theorem 2.1.2,

(F1) ϕ_p is a homeomorphism onto a neighbourhood Z_p of p in Λ, with $\phi_p(x,p) = x$ for every $x \in W_p$;
(F2) $\phi_{p,y} = \phi_p \mid (W_p \times \{y\})$ is a C^r diffeomorphism onto a neighbourhood of y inside $W^u(y)$, for every $y \in S_p$;
(F3) the diffeomorphisms $\phi_{p,y}$ vary continuously with the point $y \in S_p$, in the C^r topology.

We call $\phi_p : W_p \times S_p \to Z_p$ a *foliated chart for the unstable foliation* $\mathcal{F}^u$ at the point p.

The size of W_p, S_p, Z_p is essentially determined by $\varepsilon > 0$. In particular, foliated charts can be constructed at any point p of Λ in such a way that Z_p contains a neighbourhood of p in Λ, with fixed radius. Of course, decreasing ε we can also make Z_p arbitrarily small.

This construction extends verbatim to the case of partially hyperbolic attractors of type $E^u \oplus E^{cs}$, with Theorem 2.2.1 and Proposition 2.2.2 replacing Theorem 2.1.2 and Proposition 2.1.6.

Foliated charts for stable foliations of hyperbolic attractors are obtained along similar lines, using the version of Theorem 2.1.2 for stable manifolds. Let W_p' be a small compact disk around p inside its stable manifold and, for each x close to p let $\Phi_p'(x) : W_p' \to M$ be the map provided by the theorem: $\Phi_p'(x)$ is a C^r diffeomorphism of W_p' onto a neighbourhood of x inside the corresponding stable manifold. Take $S_p' = \Sigma_p'$ a small compact disk around p inside $W^u(p)$ (which is contained in Λ, by Proposition 2.1.6). The homeomorphism

$$\phi_p' : S_p' \times W_p' \to Z_p', \quad \phi_p'(x,y) = \Phi_p'(x)(y)$$

is a *foliated chart for the stable foliation* $\mathcal{F}^s$ at the point p. Observe that the image Z_p' is a neighbourhood of p in the ambient space M.

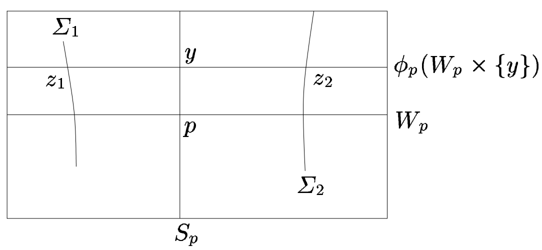

Fig. 2.4. A local holonomy map for $\mathcal{F}^u$

Local holonomy maps. Suppose Λ is a partially hyperbolic attractor of type $E^u \oplus E^{cs}$. Let $p \in M$ and $\phi_p : W_p \times S_p \to Z_p$ be a foliated chart for $\mathcal{F}^u$ at p. Let Σ_1 and Σ_2 be C^1 submanifolds embedded in Z_p, with $\dim \Sigma_i = \dim E^{cs}$ and transverse to the strong-unstable foliation: Σ_i intersects each strong-unstable disk $\phi_p(W_p \times \{y\})$ in at most one point, and this intersection is transverse, for $i = 1, 2$. See Figure 2.4. Reducing Σ_1 if necessary we may suppose that the unstable disk $\phi_p(W_p \times \{y\})$ passing through any $z_1 \in \Sigma_1$ intersects Σ_2 in some point $z_2 = \pi(z_1)$. This is always implicit in the sequel. The *local holonomy map* of $\mathcal{F}^u$ from Σ_1 to Σ_2 is

$$\pi : \Sigma_1 \to \Sigma_2, \quad z_1 \mapsto \pi(z_1) = z_2 \,.$$

Since ϕ_p is a homeomorphism, π is a homeomorphism onto its image.

In particular, this defines local holonomy maps for the unstable foliation of a hyperbolic attractor. Local holonomy maps of the stable foliation $\mathcal{F}^s$ are defined analogously, using corresponding foliated charts $\phi_p' : S_p' \times W_p' \to Z_p'$.

Foliated charts and holonomy maps for hyperbolic and partially hyperbolic repellers are defined analogously, just replacing the diffeomorphism by its inverse.

A first positive result about transverse regularity of invariant foliations:

Proposition 2.2.3. *Suppose Λ is a partially hyperbolic attractor of type $E^u \oplus E^{cs}$ for a C^1 diffeomorphism f. Then every local holonomy map of the strong-unstable foliation of Λ is Hölder continuous.*

We include a proof this proposition in Appendix A.3. See also [99]. In the hyperbolic case the conclusion remains valid for the stable foliation, by the same arguments applied to the inverse diffeomorphism.

With a similar flavour, we have

Proposition 2.2.4. *Suppose Λ is a partially hyperbolic set for a C^2 diffeomorphism f. Then, the corresponding subbundles E^1 and E^2 are Hölder continuous. If Λ is strongly partially hyperbolic then all three invariant subbundles E^u, E^s, and E^c are Hölder continuous.*

See [26, Corollary 2.1] and [53, Theorem 6.4]. We say that a subbundle E is Hölder continuous if, in the neighbourhood of any point, there are Hölder continuous linearly independent vector fields spanning E. An example in [125] shows that the integral foliation of a Hölder continuous subbundle may not be transversely Hölder continuous. Therefore, Proposition 2.2.3 is not a consequence of Proposition 2.2.4, even in the C^2 case.

Remark 2.2.5. If Λ is an attractor of type $E^u \oplus E^{cs}$ and the strong-unstable subbundle has co-dimension 1, that is $\dim E^{cs} = 1$, then one has much more: E^u extends to a C^1 bundle in a neighbourhood of Λ. As a consequence, the local holonomy maps of the strong-unstable foliation are also C^1 in this case. See the last section in [94].

Next, we state the absolute continuity theorem for Anosov systems. This fundamental result was first established by Anosov [7], as a main step in the proof that the geodesic flow on manifolds with negative curvature is ergodic. See also [8]. Here m_{Σ_1} and m_{Σ_2} represent the Riemannian volumes induced by the Riemannian metric on Σ_1 and Σ_2, respectively.

Theorem 2.2.2. *Let f be a C^2 Anosov diffeomorphism. Then any local holonomy map $\pi : \Sigma_1 \to \Sigma_2$ of the stable foliation of f is absolutely continuous. In fact there exists a Hölder continuous function $J\pi : \Sigma_1 \to (0, +\infty)$, bounded from zero and infinity such that*

$$m_{\Sigma_2}(\pi(B)) = \int_B (J\pi)\, dm_{\Sigma_1} ,$$

for any measurable set $B \subset \Sigma_1$.

The same arguments apply, more generally, to the stable foliation of any hyperbolic attractor, and the unstable foliation of any hyperbolic repeller. On the other hand, the C^2 assumption is necessary: there exist C^1 Anosov diffeomorphisms whose stable and unstable foliations are not absolutely continuous [108].

The following extension of the previous theorem to partially hyperbolic diffeomorphisms was obtained by Pugh-Shub [98] and Brin-Pesin [26, Theorem 3.1]. Since they use the absolute version of partial hyperbolicity (recall Remark 2.2.4), we explain in Appendix A.4 how the arguments can be extended to our setting.

Theorem 2.2.3. *Suppose f is a C^2 partially hyperbolic diffeomorphism of type $E^{cu} \oplus E^s$. Then for any local holonomy map $\pi : \Sigma_1 \to \Sigma_2$ of the strong-stable foliation of f there exist positive constants C_1 and C_2 such that*

$$C_1 m_{\Sigma_1}(B) \le m_{\Sigma_2}(\pi(B)) \le C_2 m_{\Sigma_1}(B)$$

for any measurable set $B \subset \Sigma_1$.

As before, the arguments hold for the strong-stable foliation of any partially hyperbolic attractor of type $E^{cu} \oplus E^s$. Taking the inverse map, one gets corresponding results for the strong-unstable foliation of any partially hyperbolic repeller of type $E^u \oplus E^{cs}$.

2.3 Measures Absolutely Continuous Along $\mathcal{F}^u$

In this section we take $f : M \to M$ to be a C^2 diffeomorphism. We prove that every partially hyperbolic attractor Λ of type $E^u \oplus E^{cs}$ supports some invariant probability measure μ that is *absolutely continuous along the strong-unstable foliation* $\mathcal{F}^u$. The precise statement will be given in Theorem 2.3.1, after we have defined this last notion. We shall show that, in many situations, such probabilities are physical measures for f on Λ. In particular, this is the case if the subbundle E^{cs} is contracting, corresponding to Λ being hyperbolic.

We begin by defining absolute continuity of a measure along a foliation, in an abstract setting. Let X, Y be compact metric spaces, and $\mathcal{F}$ denote the partition of $X \times Y$ into "horizontal lines"

$$\mathcal{F} = \{X \times \{y\} : y \in Y\}.$$

Let ν be some probability measure in X. We also denote $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ the canonical projections. Given any measure μ on $X \times Y$, we let $\hat{\mu} = \pi_{Y*}(\mu)$. That is, $\hat{\mu}$ is the measure defined on Y by

$$\hat{\mu}(\xi) = \mu(\pi_Y^{-1}(\xi)) = \mu(X \times \xi) \quad \text{for every measurable set} \quad \xi \subset Y.$$

Definition 2.3.1. *A measure μ on $X \times Y$ is* absolutely continuous with respect to ν along the horizontal *if there exists some measurable function $\rho : X \times Y \to [0, +\infty)$ such that*

$$\mu(B) = \int_B \rho(x, y)\, d\nu(x)\, d\hat{\mu}(y)$$

for every measurable set $B \subset X \times Y$.

In other words, μ is absolutely continuous with respect to ν along the horizontal if and only if μ is absolutely continuous with respect to the product measure $\nu \times \hat{\mu}$. Then, ρ is the Radon-Nikodym derivative of μ relative to $\nu \times \hat{\mu}$. We call $\{\rho(\cdot, y)\nu : y \in Y\}$ *conditional measures* of μ relative to the horizontal foliation $\mathcal{F}$.

Remark 2.3.1. Conditional measures can be defined in much more generality, see Appendix A.2. However, the present setting is sufficient for all our purposes.

Going back to the dynamical context, let Λ be a partially hyperbolic attractor of type $E^u \oplus E^{cs}$ for a C^2 diffeomorphism $f : M \to M$, and let $\mathcal{F}^u$ be the corresponding strong-unstable foliation. The definition of absolute continuity of a measure along $\mathcal{F}^u$ uses the notion of foliated chart introduced in Subsection 2.2.2.

Definition 2.3.2. *A measure μ supported in Λ is* absolutely continuous along $\mathcal{F}^u$ *if for every $p \in \Lambda$ there exists a foliated chart $\phi_p : W_p \times S_p \to Z_p$ for $\mathcal{F}^u$ at p such that the pull-back $\phi_p^* \mu$ of μ under ϕ_p is absolutely continuous with respect to Lebesgue measure along the horizontal.*

The *pull-back* is the measure on $W_p \times S_p$ defined by $\phi_p^* \mu(B) = \mu(\phi_p(B))$, for every measurable subset B.

Let U be a compact disk smoothly embedded inside some leaf of $\mathcal{F}^u$, and m_U be a Riemannian volume on U. That is, m_U is the Lebesgue measure induced on U by the volume element associated to some Riemannian metric of M. In this section we prove the following result of [94].

Theorem 2.3.1. *Any accumulation point of the sequence*

$$\frac{1}{n} \sum_{j=0}^{n-1} f_*^j m_U$$

is an invariant measure for f, absolutely continuous along $\mathcal{F}^u$.

2.3.1 Conditional Measures

As a first step in the proof of Theorem 2.3.1, we obtain a few abstract results about disintegration of a measure into conditional probability measures. Throughout we restrict ourselves to the setting of Definition 2.3.1: μ is a measure on the product $X \times Y$ of two compact metric spaces, and ν is a probability measure in X. For the sake of completeness, a more general discussion is presented in Appendix A.2.

Lemma 2.3.1. *Suppose there exists a measurable function $\psi : X \to [0, +\infty)$ with $\int \psi \, d\nu < \infty$, and a family $\mathcal{R}$ of rectangles $A \times \xi \subset X \times Y$ generating the σ-algebra of all measurable subsets of $X \times Y$, so that*

$$\mu(A \times \xi) \leq \hat{\mu}(\xi) \int_A \psi \, d\nu \quad \text{for every } A \times \xi \in \mathcal{R}.$$

Then μ is absolutely continuous with respect to ν along the horizontal, with $\rho(x, y) \leq \psi(x)$ at $(\nu \times \hat{\mu})$-almost every point $(x, y) \in X \times Y$.

Proof. It is easy to check that the family of measurable subsets $B \subset X \times Y$ for which

$$\mu(B) \leq \int_B \psi(x) \, d\nu(x) \, d\hat{\mu}(y) \tag{2.12}$$

is a σ-algebra. By assumption, this family contains $\mathcal{R}$. Since we also take $\mathcal{R}$ to be generating, (2.12) must hold for every measurable subset B. This implies that μ is absolutely continuous with respect to $(\nu \times \hat{\mu})$, with Radon-Nikodym derivative $\rho = d\mu / d(\nu \times \hat{\mu})$ satisfying $\rho(x, y) \leq \psi(x)$ at $(\nu \times \hat{\mu})$-almost every point. □

Remark 2.3.2. In the same setting, let $\phi : X \to [0, +\infty)$ be any measurable function such that

$$\mu(A \times \xi) \geq \hat{\mu}(\xi) \int_A \phi \, d\nu$$

for every $A \times \xi$ in some generating family of rectangles. Then, by similar arguments, the Radon-Nikodym derivative also satisfies $\rho(x, y) \geq \phi(x)$ at $(\nu \times \hat{\mu})$-almost every point.

Proposition 2.3.1. *Let μ_k, $k \geq 1$, be a sequence of measures on $X \times Y$ converging to some measure μ in the weak* topology. Suppose there exists a measurable function $\psi : X \to [0, +\infty)$ with $\int \psi \, d\nu < \infty$, and a family $\mathcal{R}$ of rectangles $A \times \xi \subset X \times Y$ generating the σ-algebra of measurable subsets of $X \times Y$, so that*

$$\mu_k(A \times \xi) \leq \hat{\mu}_k(\xi) \int_A \psi \, d\nu \quad \text{for every } A \times \xi \in \mathcal{R} \text{ and } k \geq 1.$$

Then μ is absolutely continuous with respect to ν along the horizontal, with density $\rho \leq \psi$ at $(\nu \times \hat{\mu})$-almost every point in $X \times Y$.

Proof. Let $\hat{\mu}_k = \pi_{Y*}(\mu_k)$. By the previous lemma, for each $k \geq 1$ there exists $\rho_k : X \times Y \to [0, +\infty)$ such that $\mu_k = \rho_k(\nu \times \hat{\mu}_k)$ and $\rho_k \leq \psi$ at $(\nu \times \hat{\mu}_k)$-almost every point. In particular,

$$\mu_k(A \times \xi) \leq \hat{\mu}_k(\xi) \int_A \psi \, d\nu$$

for any measurable rectangle $A \times \xi$ in $X \times Y$. Since π_Y is a continuous map, the assumption $\mu_k \to \mu$ implies that $\hat{\mu}_k \to \hat{\mu}$. Then, assuming the boundary of ξ has zero $\hat{\mu}$-measure, $\hat{\mu}_k(\xi)$ converges to $\hat{\mu}(\xi)$ as $k \to \infty$. Let us suppose, furthermore, that $A \subset X$ and $\xi \subset Y$ are open subsets. Then

$$\mu(A \times \xi) \leq \liminf_{k \to \infty} \mu_k(A \times \xi) \leq \liminf_{k \to \infty} \hat{\mu}_k(\xi) \int_A \psi \, d\nu = \hat{\mu}(\xi) \int_A \psi \, d\nu.$$

Now the proposition follows from Lemma 2.3.1, together with the observation that the family of open rectangles $A \times \xi$ such that $\hat{\mu}(\partial \xi) = 0$ generates the σ-algebra of $X \times Y$. □

Proposition 2.3.2. *In the setting of Proposition 2.3.1, let $\phi : X \to [0, +\infty)$ be any measurable function such that*

$$\mu_k(A \times \xi) \geq \hat{\mu}_k(\xi) \int_A \phi \, d\nu \quad \text{for any } A \times \xi \in \mathcal{R} \text{ and } k \geq 1.$$

Then the derivative ρ satisfies $\rho \geq \phi$ at $(\nu \times \hat{\mu})$-almost every point.

Proof. This is similar to the previous proposition. According to Remark 2.3.2, we have $\rho_k \geq \phi$ at $(\nu \times \hat{\mu}_k)$-almost every point. In particular,

$$\mu_k(A \times \xi) \geq \hat{\mu}_k(\xi) \int_A \phi \, d\nu$$

for any measurable rectangle $A \times \xi$ in $X \times Y$. Taking $A \subset X$ and $\xi \subset Y$ to be closed, and the boundary of ξ to have zero measure for $\hat{\mu}$, we conclude that

$$\mu(A \times \xi) \geq \limsup_k \mu_k(A \times \xi) \geq \limsup_k \hat{\mu}_k(\xi) \int_A \phi \, d\nu = \hat{\mu}(\xi) \int_A \phi \, d\nu.$$

Since these closed rectangles generate the σ-algebra of $X \times Y$, the proposition follows from Remark 2.3.2. □

2.3.2 Distortion Along Strong-Unstable Leaves

In what follows we suppose that some Riemannian metric has been chosen on M: determinants and lengths of curves are always meant with respect to this metric. This also determines a Riemannian volume on each of the leaves of the strong-unstable foliation, that we denote m_u. We denote m_B the restriction of m_u to an arbitrary measurable subset B of some strong-unstable leaf. The constant $\lambda < 1$ is taken as in Definitions 2.2.1 and 2.2.2.

For $j \in \mathbb{Z}$ and $y \in \Lambda$, we let $J^u f^j$ be the norm of the Jacobian of f restricted to the unstable subspace E^u_y :

$$J^u f^j(y) = |\det(Df^j(y) \mid E^u_y)|.$$

Let U be a compact disk contained in some leaf of $\mathcal{F}^u$.

Lemma 2.3.2. *For any $n \geq 1$, we have $f^n_* m_U = (J^u f^{-n})\, m_{f^n(U)}$.*

Proof. By definition, given any measurable subset B of $f^n(U)$,

$$f^n_* m_U(B) = \int_{f^{-n}(B)} 1\, dm_U .$$

Changing variables $y = (f^n \mid U)(x)$ in the integral, we get

$$f^n_* m_U(B) = \int_B (J^u f^{-n})\, dm_{f^n(U)}$$

for any measurable subset B, as we claimed. □

Definition 2.3.3. *Given points x and y in the same strong-unstable leaf F,*

$$d_u(x,y) = \inf\{\text{length}(\alpha) : \alpha \text{ a piecewise } C^1 \text{ curve in } F \text{ connecting } x \text{ to } y\}.$$

Clearly, $d_u(\cdot\,,\cdot)$ defines a distance on each leaf F of the foliation $\mathcal{F}^u$. It is also easy to check that this distance is uniformly contracted by backward iterates of f (as asserted by Theorem 2.2.1). Indeed, since the derivative Df is uniformly expanding along the tangent bundle $E^u = T\mathcal{F}^u$, given any piecewise C^1 curve α connecting x to y inside the leaf F, the length of $f^{-j}(\alpha)$ is less than $C\lambda^j \,\text{length}(\alpha)$, for every $j \geq 1$. Therefore,

$$d_u(f^{-j}(x), f^{-j}(y)) \leq C\lambda^j d_u(x,y) \quad \text{for every } j \geq 1. \tag{2.13}$$

For completeness, we also set $d_u(x,y) = \infty$ when x, y are in different strong-unstable leaves.

Proposition 2.3.3. *Given $L > 0$ there exists $K > 0$ such that*

$$\frac{J^u f^{-n}(y_1)}{J^u f^{-n}(y_2)} \leq K$$

for every $n \geq 1$ and any $y_1, y_2 \in \Lambda$ such that $d_u(y_1, y_2) \leq L$.

Proof. Since we suppose the diffeomorphism f to be C^2, the tangent bundle $E^u = T\mathcal{F}^u$ is ν_0-Hölder for some $0 < \nu_0 \leq 1$. Recall Proposition 2.2.4. As a consequence, the map

$$\varphi^u : \Lambda \to \mathbb{R}, \quad \varphi^u(x) = \log|\det(Df(x) \mid E^u_x)|,$$

is (C_0, ν_0)-Hölder for some constant $C_0 > 0$. In particular, its restriction to each strong-unstable leaf is (C_0, ν_0)-Hölder with respect to the d_u-distance on the leaf. Using the chain rule,

$$\log \frac{J^u f^{-n}(y_1)}{J^u f^{-n}(y_2)} = \sum_{j=1}^{n} \varphi^u(f^{-j}(y_2)) - \varphi^u(f^{-j}(y_1)).$$

By Hölder continuity and (2.13), this is less than

$$\sum_{j=1}^{n} C_0 d_u(f^{-j}(y_1), f^{-j}(y_2))^{\nu_0} \leq \sum_{j=1}^{n} C_0 \big(C\lambda^j d_u(y_1, y_2)\big)^{\nu_0}.$$

So, since we suppose that $d_u(y_1, y_2) \leq L$,

$$\log \frac{J^u f^{-n}(y_1)}{J^u f^{-n}(y_2)} \leq C_0 \big(CL\big)^{\nu_0} \sum_{j=1}^{n} \lambda^{j\nu_0}.$$

Thus, we may take $K = \exp\Big(C_0 \big(CL\big)^{\nu_0} \sum_{j=1}^{\infty} \lambda^{j\nu_0}\Big)$. □

This yields the following uniform control for the distortion of iterates of the diffeomorphism restricted to leaves of $\mathcal{F}^u$.

Corollary 2.3.1. *Given any $L > 0$ there exists $K > 0$ so that, for $n \geq 1$ and any domain D in $f^n(U)$ with d_u-diameter less than L, we have*

$$\frac{1}{K} \frac{m_{f^n(U)}(B)}{m_{f^n(U)}(D)} \leq \frac{f^n_* m_U(B)}{f^n_* m_U(D)} \leq K \frac{m_{f^n(U)}(B)}{m_{f^n(U)}(D)}$$

for any measurable subset B of D.

Proof. This is a direct consequence of Lemma 2.3.2 and Proposition 2.3.3, with the same constant K as in the proposition:

$$\frac{f^n_* m_U(B)}{f^n_* m_U(D)} = \frac{\int_B (J^u f^{-n})\, dm_{f^n(U)}}{\int_D (J^u f^{-n})\, dm_{f^n(U)}} \leq K \frac{m_{f^n(U)}(B)}{m_{f^n(U)}(D)},$$

and the lower inequality is obtained in the same way. □

As another consequence of Proposition 2.3.3, we obtain the following result about positive Lebesgue measure subsets of strong-unstable leaves: forward iterates of the set fill-in an arbitrarily large fraction of some d_u-ball with given radius. The d_u-ball of radius $r > 0$ around a point $q \in \Lambda$ is denoted $B^u_r(q)$. We use the following simple observation.

Remark 2.3.3. According to Theorem 2.2.1, $\mathcal{F}^u$ is a continuous foliation with C^2 leaves. In particular, its leaves have bounded curvature restricted to each foliated chart. Since Λ can be covered by finitely many foliated charts, e.g. because it is compact, it follows that the leaves of $\mathcal{F}^u$ have uniformly bounded curvature over the whole attractor Λ.

Proposition 2.3.4. *Let $r > 0$ be fixed. Let F be a leaf of the foliation $\mathcal{F}^u$, and A be a subset of F such that $m_u(A) > 0$. Then there exist $p_n \in f^n(A)$, $n \geq 1$, such that*

$$\frac{m_u\big(f^n(A) \cap B_r^u(p_n)\big)}{m_u\big(B_r^u(p_n)\big)} \to 1 \quad \textit{as } n \to \infty.$$

Proof. It is no restriction to suppose that A is compact. For each $n \geq 1$, let Q_n be a maximal finite subset of $f^n(A)$ such that the open d_u-balls $B_r^u(q)$ of radius r around the points of Q_n are two-by-two disjoint. Then the d_u-balls of radius $2r$ around the points $q \in Q_n$ cover $f^n(A)$. Since the curvature of the strong-unstable leaves is uniformly bounded, cf. Remark 2.3.3, there exists $\kappa > 0$, depending only on r, such that

$$1 \leq \frac{m_u(B_{2r}^u(q))}{m_u(B_r^u(q))} \leq \kappa$$

for any point $q \in \Lambda$. This last statement follows from volume comparison theorems in Riemannian geometry, see for instance [30, Theorem 3.10]. Then, taking $L = 2r$ in Corollary 2.3.1,

$$1 \leq \frac{m_u\big(f^{-n}(B_{2r}^u(q))\big)}{m_u\big(f^{-n}(B_r^u(q))\big)} \leq K\kappa \tag{2.14}$$

for any $n \geq 1$ and $q \in Q_n$. Let us show that, for n large, there exists $q \in Q_n$ such that A fills-in a large fraction of $f^{-n}(B_r^u(q))$. More precisely,

Claim:

$$\min_{q \in Q_n} \frac{m_u\big(f^{-n}(B_r^u(q)) \setminus A\big)}{m_u\big(f^{-n}(B_r^u(q))\big)} \to 0 \quad \text{as } n \to \infty. \tag{2.15}$$

Proof. We prove the claim by contradiction. Suppose there exists $\delta > 0$ such that, for every $q \in Q_n$ and arbitrarily large $n \geq 1$,

$$m_u\big(f^{-n}(B_r^u(q)) \setminus A\big) \geq \delta m_u\big(f^{-n}(B_r^u(q))\big).$$

Since the $B_r^u(q)$, $q \in Q_n$ are two-by-two disjoint, adding over q we get

$$m_u\left(\bigcup_{q \in Q_n} f^{-n}(B_r^u(q)) \setminus A\right) \geq \delta m_u\left(\bigcup_{q \in Q_n} f^{-n}(B_r^u(q))\right). \tag{2.16}$$

On the one hand, the d_u-diameter of $f^{-n}(B^u_r(q))$ is less than $2rC\lambda^n$, which converges to zero as $n \to \infty$. Since $f^{-n}(q) \in A$ for all $q \in Q_n$, it follows that the union of the $f^{-n}(B^u_r(q))$ is contained in a small neighbourhood of the compact set A if n is large. Thus, the left hand side of (2.16) goes to zero as $n \to \infty$. On the other hand, by (2.14), the right hand side is bounded from below by

$$\frac{\delta}{K\kappa} m_u \left(\bigcup_{q \in Q_n} f^{-n}(B^u_{2r}(q)) \right) \geq \frac{\delta}{K\kappa} m_u(A) > 0.$$

We have reached a contradiction, so the claim is proved. □

For each $n \geq 1$ we pick p_n to be a point in Q_n where the minimum in (2.15) is attained. Then, using Corollary 2.3.1 once more,

$$\frac{m_u\big(B_r(p_n) \setminus f^n(A)\big)}{m_u\big(B_r(p_n)\big)} \leq K \frac{m_u\big(f^{-n}(B_r(p_n)) \setminus A\big)}{m_u\big(f^{-n}(B_r(p_n))\big)} \to 0 \quad \text{as } n \to \infty.$$

This proves the proposition. □

2.3.3 Proof of the Existence Theorem

Now we are in a position to prove Theorem 2.3.1: given any compact disk U embedded inside a strong-unstable leaf, every accumulation point of

$$\mu_n = \frac{1}{n} \sum_{j=0}^{n-1} f^j_* m_U$$

is absolutely continuous along $\mathcal{F}^u$. For this, we want to show that, given any accumulation point μ of μ_n and any point p in Λ, there exists a foliated chart ϕ_p at p such that the pull-back $\phi^*_p \mu$ is absolutely continuous along the horizontal.

We fix $\mu = \lim_k \mu_{n_k}$ and the point p in all that follows. The choice of a foliated chart $\phi_p : W_p \times S_p \to Z_p$ is rather arbitrary: we only require that the boundary of Z_p have zero μ-measure:

$$\mu(\partial Z_p) = 0, \tag{2.17}$$

which can always be obtained, replacing W_p and S_p by slightly smaller sets if necessary. We show that, for any such chart, $\phi^* \mu$ is indeed absolutely continuous along the horizontal.

Definition 2.3.4. *We say that a connected component γ of $f^j(U) \cap Z_p$* crosses Z_p *if $\phi_p^{-1}(\gamma)$ is a graph over W_p, that is, $\phi_p^{-1}(\gamma)$ projects homeomorphically onto W_p under the canonical projection $\pi_1 : W_p \times S_p \to W_p$.*

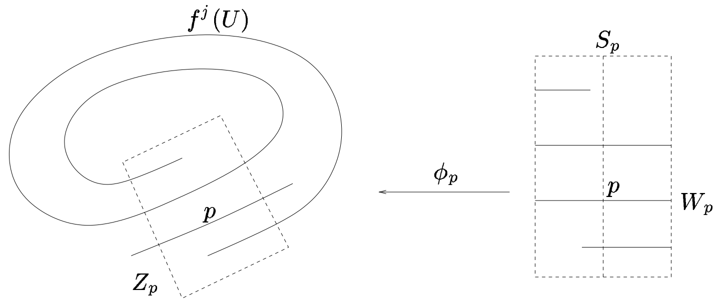

Fig. 2.5. Crossing and noncrossing components

Each μ_n, $n \geq 1$, is supported in the union of the iterates $f^j(U)$ over all $0 \leq j \leq n-1$. We denote Γ_j^c the union of the connected components of $f^j(U) \cap Z_p$ that cross Z_p, and Γ_j^{nc} the union of all the other components. Then we write the restriction of μ_n to Z_p as

$$(\mu_n \mid Z_p) = \mu_n^c + \mu_n^{nc},$$

where μ_n^c is the part of the measure μ_n that sits on components crossing Z_p, and μ_n^{nc} is the part of μ_n sitting on noncrossing components:

$$\mu_n^c = \frac{1}{n}\sum_{j=0}^{n-1}(f_*^j m_U) \mid \Gamma_j^c \quad \text{and} \quad \mu_n^{nc} = \frac{1}{n}\sum_{j=0}^{n-1}(f_*^j m_U) \mid \Gamma_j^{nc}.$$

Firstly, we prove that the total mass of μ_n^{nc} goes to zero as n goes to infinity.

Lemma 2.3.3. *We have* $\mu_n^{nc}(M) = \mu_n^{nc}(Z_p) \to 0$ *as* $n \to \infty$.

Proof. Let $j \geq 0$ and z be any point in Γ_j^{nc}. Recall that $\Gamma_j^{nc} \subset Z_p \cap f^j(U)$. Since Z_p can be written as a disjoint union

$$Z_p = \bigcup_{y \in S_p} \phi_p(W_p \times \{y\}),$$

there exists a unique $y \in S_p$ such that $z \in \phi_p(W_p \times \{y\})$. Then the connected component of $Z_p \cap f^j(U)$ which contains z is a subset of the strong-unstable disk $\phi_p(W_p \times \{y\})$. Since this component does not cross Z_p, the disk $f^j(U)$ can not contain $\phi_p(W_p \times \{y\})$. Therefore, there exists some $z_0 \in \phi_p(W_p \times \{y\})$ that is on the boundary of $f^j(U)$. In particular, $d_u(z, z_0) \leq \delta_0$, where $\delta_0 > 0$ is any upper bound for the d_u-diameter of the $\phi_p(W_p \times \{y\})$ over all $y \in S_p$ and $p \in \Lambda$. In other words, we have proved that Γ_n^{nc} is contained in the d_u-neighbourhood of radius δ_0 of the boundary of $f^j(U)$ inside the corresponding strong-unstable leaf.

Since Df expands distances uniformly along strong-unstable leaves, cf. (2.13), we conclude that $f^{-j}(\Gamma_j^{nc})$ is contained in the d_u-neighbourhood $N(\partial U, C\lambda^j\delta_0)$ with radius $C\lambda^j\delta_0$ of the boundary of U. Therefore,

$$f_*^j m_U(\Gamma_j^{nc}) = m_U(f^{-j}(\Gamma_j^{nc})) \leq m_U(N(\partial U, C\lambda^j\delta_0))$$

for every $j \geq 0$. The last term $m_U(N(\partial U, C\lambda^j\delta_0))$ converges to $m_U(\partial U) = 0$ as $j \to \infty$, because $\lambda < 1$. So $f_*^j m_U(\Gamma_j^{nc})$ converges to zero as well, as $j \to \infty$. As a consequence,

$$\mu_n^{nc}(M) = \frac{1}{n}\sum_{j=0}^{n-1} f_*^j m_U(\Gamma_j^{nc}) \to 0 \quad \text{as } j \to \infty,$$

as claimed in the lemma. □

It follows that the fraction of the μ_n that is supported on noncrossing components has no effect on the limit measure μ:

Corollary 2.3.2. *We have* $\phi_p^*\mu = \lim_k \phi_p^*\mu_{n_k}^c$.

Proof. Let B be any measurable subset of $W_p \times S_p$ whose boundary ∂B has zero $\phi_p^*\mu$-measure. In other words, $\mu(\phi_p(\partial B)) = 0$. Since the boundary of $\phi_p(B)$ is contained in $\partial Z_p \cup \phi_p(\partial B)$, assumption (2.17) implies $\mu(\partial\phi_p(B)) = 0$. As a consequence,

$$\mu(\phi_p(B)) = \lim_{k\to\infty} \mu_{n_k}(\phi_p(B)) = \lim_{k\to\infty} \mu_{n_k}^c(\phi_p(B)) + \mu_{n_k}^{nc}(\phi_p(B)).$$

By Lemma 2.3.3, the last term on the right converges to zero. So,

$$\mu(\phi_p(B)) = \lim_{k\to\infty} \mu_{n_k}^c(\phi_p(B))$$

for any subset B as above. This is equivalent to the claim in the corollary. □

From now on we focus our attention on crossing components. For notational simplicity, we write $\eta = \phi_p^*\mu$ and $\eta_n = \phi_p^*\mu_n^c$, and we let $\hat{\eta}$ and $\hat{\eta}_n$ represent the quotient measures on S_p:

$$\hat{\eta}(\xi) = \phi_p^*\mu(W_p \times \xi) \quad \text{and} \quad \hat{\eta}_n(\xi) = \phi_p^*\mu_n^c(W_p \times \xi).$$

Lemma 2.3.4. *There exists $C_1 > 1$, depending only on the diffeomorphism f, such that*

$$\frac{1}{C_1}\frac{m_u(A)}{m_u(W_p)}\hat{\eta}_n(\xi) \leq \eta_n(A \times \xi) \leq C_1 \frac{m_u(A)}{m_u(W_p)}\hat{\eta}_n(\xi)$$

for any measurable sets $A \subset W_p$ and $\xi \subset S_p$.

Proof. We explain how to obtain the upper inequality, the lower one is analogous. The main step is the following

Claim: There exists $C_1 > 1$, depending only on f, such that

$$\frac{f_*^j m_U\big(\phi_p(A \times \xi) \cap \gamma\big)}{f_*^j m_U(\gamma)} \le C_1 \, \frac{m_u(A)}{m_u(W_p)}$$

for every $0 \le j \le n-1$, and every connected component γ of $Z_p \cap f^j(U)$ crossing Z_p and intersecting $\phi_p(A \times \xi)$.

Proof. Since γ crosses Z_p, there exists $y \in S_p$ such that $\gamma = \phi_p(W_p \times \{y\})$. Note that γ intersects $\phi_p(A \times \xi)$ if and only if $y \in \xi$ and, in that case,

$$(A \times \xi) \cap (W_p \times \{y\}) = (A \times \{y\}).$$

Now we use bounded distortion. Let δ_0 be an upper bound for the d_u-diameter of the strong-unstable disks $\phi_p(W_p \times \{y\})$, over all $y \in S_p$ and $p \in \Lambda$. By Corollary 2.3.1 there exists a constant $C_2 = K(\delta_0) > 0$ such that

$$\frac{f_*^j m_U\big(\phi_p(A \times \{y\})\big)}{f_*^j m_U\big(\phi_p(W_p \times \{y\})\big)} \le C_2 \frac{m_{f^j(U)}\big(\phi_p(A \times \{y\})\big)}{m_{f^j(U)}\big(\phi_p(W_p \times \{y\})\big)}. \tag{2.18}$$

Recall that, according to property (F2) of foliated charts in Subsection 2.2.2, $\phi_{p,y} = (\phi_p \mid W_p \times \{y\})$ is a diffeomorphism of $W_p \times \{y\}$ onto γ. Recall also that $m_{f^j(U)}$ is just the restriction of the Riemannian volume m_u to $f^j(U)$. By the mean value theorem,

$$\frac{m_{f^j(U)}\big(\phi_p(A \times \{y\})\big)}{m_{f^j(U)}\big(\phi_p(W_p \times \{y\})\big)} = \frac{|\det D\phi_{p,y}(x_1, y)|}{|\det D\phi_{p,y}(x_2, y)|} \frac{m_u(A)}{m_u(W_p)} \tag{2.19}$$

for some $x_1, x_2 \in W_p$. The quotient of the Jacobians is uniformly bounded:

$$\frac{|\det D\phi_{p,y}(z_1, y)|}{|\det D\phi_{p,y}(z_2, y)|} \le C_3 \tag{2.20}$$

for some $C_3 > 1$ and every z_1, z_2 in W_p. Moreover, by properties (F3) and (F1) of foliated charts, the diffeomorphisms $\phi_{p,y}$ vary continuously with the point y, and $\phi_{p,p}$ is, essentially, the identity on W_p. Since Λ is compact, these facts ensure that the constant C_3 may be chosen depending only on f (neither on y nor on p). Taking $C_1 = C_2 C_3$, the claim follows from (2.18), (2.19), (2.20). □

Going back to proving the lemma, we write

$$\mu_n^c\big(\phi_p(A \times \xi)\big) = \frac{1}{n} \sum_{j=0}^{n-1} \sum_{\gamma} f_*^j m_U(\phi_p(A \times \xi) \cap \gamma), \tag{2.21}$$

where the last sum is over the components γ of $Z_p \cap f^j(U)$ crossing Z_p that intersect $\phi_p(A \times \xi)$. As noted in the proof of the Claim, any connected component crossing Z_p may be written as $\gamma = \phi_p(W_p \times \{y\})$. Moreover, it intersects $\phi_p(A \times \xi)$ if and only if $y \in \xi$. Clearly, the last condition does not depend on the set A. So, replacing A by W_p in (2.21) gives

$$\mu_n^c\big(\phi_p(W_p \times \xi)\big) = \frac{1}{n}\sum_{j=0}^{n-1}\sum_{\gamma} f_*^j m_U(\gamma) \tag{2.22}$$

where the last sum runs over the same subset of connected components γ of $Z_p \cap f^j(U)$ as in (2.21). Now, using the Claim to compare the sums in (2.21) and (2.22) term by term, we find

$$\frac{\eta_n(A \times \xi)}{\hat{\eta}_n(\xi)} = \frac{\mu_n^c\big(\phi_p(A \times \xi)\big)}{\mu_n^c\big(\phi_p(W_p \times \xi)\big)} \le C_1 \frac{m_u(A)}{m_u(W_p)},$$

which is what we wanted to prove. □

Corollary 2.3.3. *If $\mu = \lim_k \mu_{n_k}$, $p \in \Lambda$, and $\phi_p : W_p \times S_p \to Z_p$ are as above, then $\phi_p^*\mu$ is absolutely continuous with respect to normalized Lebesgue measure $m_u/m_u(W_p)$ on W_p along the horizontal. Moreover, the density ρ is bounded away from zero and infinity: $1/C_1 \le \rho \le C_1$.*

Proof. This follows from Propositions 2.3.1 and 2.3.2, with $X = W_p$, $Y = S_p$, $\phi_p^*\mu_{n_k}^c$ in the role of μ_k, $\eta = \phi_p^*\mu$ in the role of μ, $\nu = m$, $\psi = C_1$, $\phi = 1/C_1$, and $\mathcal{R}$ being the family of all measurable rectangles $A \times \xi \subset W_p \times S_p$. Lemma 2.3.4 states that the assumptions of the two propositions are satisfied. Recall also Corollary 2.3.2.

The propositions give that $1/C_1 \le \rho \le C_1$ on a subset with full $m \times \hat{\eta}$-measure. Then, modifying the values of ρ on the complement if necessary (since the complement has measure zero, the new function is again a density for $\phi_p^*\mu$), we may suppose that $1/C_1 \le \rho \le C_1$ everywhere. □

Theorem 2.3.1 is contained in the following proposition, that summarizes the main facts we proved in this section.

Proposition 2.3.5. *For any accumulation point μ of $\mu_n = n^{-1}\sum_{j=0}^{n-1} f_j^* m_U$, and any point $p \in \Lambda$, there exist foliated charts ϕ_p for the strong-unstable foliation $\mathcal{F}^u$ at p, so that $\phi_p^*\mu$ is absolutely continuous with respect to normalized Lebesgue measure along the horizontal, with density ρ uniformly bounded away from zero and infinity.*

Remark 2.3.4. The conclusion of Theorem 2.3.1 is also valid for the accumulation points of

$$\frac{1}{n}\sum_{j=0}^{n-1} f_*^j(\varphi\,\mathrm{Leb}),$$

where φ is any nonnegative integrable function supported in a sufficiently small neighbourhood of Λ, and Leb is Lebesgue measure in the ambient manifold M. See Theorem 3 of [94].

We close this section with the following useful lemma. An invariant probability measure has *positive densities along* $\mathcal{F}^u$ if it is absolutely continuous along $\mathcal{F}^u$, and there exists a family of foliated charts $\phi_p : W_p \times S_p \to Z_p$ such that the interiors of the corresponding Z_p cover Λ, and for which the density ρ of $\phi_p^*\mu$ may be taken strictly positive. The measures constructed in the previous proposition have positive densities along $\mathcal{F}^u$.

Lemma 2.3.5. *Suppose μ is an invariant probability measure supported in Λ with positive densities along $\mathcal{F}^u$. Then the support of μ consists of entire orbits of strong-unstable leaves.*

Proof. Let q be any point in the support of μ. By the assumption, there exists a foliated chart $\phi_p : W_p \times S_p \to Z_p$ such that Z_p is a neighbourhood of q inside Λ, and the density ρ of $\eta = \phi_p^*\mu$ is strictly positive. Let r be any point in W_p and in the interior of Z_p, and let V be a neighbourhood of r contained in Z_p. Fix measurable sets $A \subset W_p$ and $B \subset S_p$ such that $A \times B$ is a neighbourhood of $\phi_p^{-1}(r)$ contained in $\phi_p^{-1}(V)$. Then,

$$\hat{\eta}(B) = \phi_p^*(W_p \times B) > 0$$

because $W_p \times B$ is a neighbourhood of $\phi_p^{-1}(q)$, and q is in the support of μ. Then,

$$\mu(V) \geq \phi_p^*(A \times B) = \int_{A \times B} \rho \, dm \, d\hat{\eta} > 0$$

since ρ is positive and A has positive Lebesgue measure. This proves that r is in the support of μ. In this way we have shown that the support of μ is open in the strong-unstable leaf that passes through q. Clearly, the support is also closed in $\mathcal{F}^u(q)$. So, it must contain the whole strong-unstable leaf. Finally, as the support is an f-invariant set, it contains the whole orbit of $\mathcal{F}^u(q)$. □

2.4 The Theorem of Sinai-Ruelle-Bowen

Now we prove the following result from [117, 22, ?]:

Theorem 2.4.1. *Suppose that Λ is a hyperbolic attractor for a C^2 diffeomorphism $f : M \to M$ of a compact manifold M. Then there exists a unique invariant probability measure μ supported in Λ that is absolutely continuous along $\mathcal{F}^u$. This measure is ergodic and its support coincides with Λ. Moreover, μ is a physical measure for f, and $B(\mu)$ is a full Lebesgue measure subset of the basin $B(\Lambda)$ of Λ.*

Let us begin by giving a sketch of the proof. A similar approach can be applied to certain partially hyperbolic (nonhyperbolic) attractors, to prove existence and finiteness of SRB measures, see Section 7.4.

First, we prove that any invariant measure absolutely continuous along the unstable foliation splits into finitely many ergodic components, that are also absolutely continuous along $\mathcal{F}^u$. For this purpose we introduce an equivalence relation, the *accessibility relation*, such that time averages are constant on each equivalence class. This is defined in the (full measure) set of regular points, which are the points at which forward and backward time averages exist and coincide, for any continuous function. Accessibility is the smallest equivalence relation such that regular points in the same stable or unstable manifold, or in the same orbit, are in the same equivalence class.

There are only finitely many accessibility classes having positive weight for some invariant measure μ absolutely continuous along $\mathcal{F}^u$. The ergodic components of any such μ are its normalized restrictions to those accessibility classes. As part of the proof we also get that ergodic measures absolutely continuous along $\mathcal{F}^u$ have the SRB property. This is based on the fact that the stable foliation $\mathcal{F}^s$ is absolutely continuous, cf. Theorem 2.2.2. Using, for the first time, the assumption that f is transitive on Λ, we show that the accessibility class is, actually, unique. In this way we conclude that μ is ergodic and unique. Moreover, its basin fills-in a full Lebesgue measure subset of the whole basin of the attractor. Finally, by Lemma 2.3.5 and Proposition 2.1.5, the support of μ must coincide with the whole attractor Λ.

Now let us present the detailed proof.

Definition 2.4.1. *A point $x \in M$ is* regular *if the forward and backward time averages*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \varphi(f^j(x)) = \lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \varphi(f^{-j}(x))$$

exist and coincide, for every continuous function $\varphi : M \to \mathbb{R}$.

Note that, according to the ergodic theorem of Birkhoff, the set $\mathcal{R}$ of regular points has full measure with respect to any probability measure that is invariant under f.

Definition 2.4.2. *Given $x, y \in \mathcal{R}$, we set $x \approx y$ if there exist $N \geq 1$ regular points $x = z_0, z_1, \dots, z_{N-1}, z_N = y$, and integers $k_1, \dots, k_N$, such that*

$$z_i \in W^u(f^{k_i}(z_{i-1})) \cap W^s(f^{k_i}(z_{i-1}))$$

for every $i = 1, \dots, N$.

See Figure 2.6, corresponding to a situation with $k_1 = \cdots = k_N = 0$. It is easy to check that $\approx$ is an equivalence relation. We refer to the equivalence

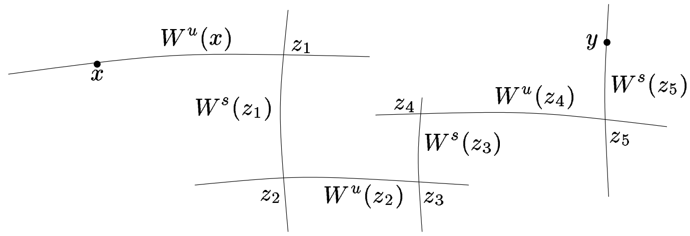

Fig. 2.6. The accessibility relation

classes as *accessibility classes*. Note that, by definition, they are invariant sets for f.

The usefulness of this notion stems from the following simple observation.

Lemma 2.4.1. *The time averages of any continuous function* $\varphi : M \to \mathbb{R}$ *are constant on each accessibility class.*

Proof. Suppose x and y are in the same stable set, that is, $d(f^j(x), f^j(y)) \to 0$ as $j \to +\infty$. Then $|\varphi(f^j(x)) - \varphi(f^j(y))| \to 0$ as $j \to +\infty$, and so the two points have the same forward time average:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x)) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(y)).$$

Similarly, points in the same unstable set have the same backward time averages. It is also clear that if x and y are iterates of each other then they have the same time averages, both in the future and in the past. Now suppose $x \approx y$, and let $x = z_0, z_1, \ldots, z_N = y$ be as in Definition 2.4.2. Since all these points are assumed to be regular, the previous remarks show that they all have the same time averages for φ. □

Corollary 2.4.1. *If an invariant measure* μ *gives positive weight to an accessibility class* $\mathcal{A}$ *then its normalized restriction to* $\mathcal{A}$ *is an ergodic measure, and it does not depend on* μ*.*

Proof. $\mathcal{A}$ has full measure for this normalized restriction $\mu_{\mathcal{A}}$, and we have shown in the lemma that time averages are constant on it. So

$$\mu_{\mathcal{A}} = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j(z)} \quad \text{for any } z \in \mathcal{A}.$$

This proves that μ_A is ergodic and independent of μ. □

Everything we said so far in this subsection holds for general homeomorphisms in metric spaces. Now we focus again on the context of C^2 diffeomorphisms admitting a hyperbolic attractor Λ. We use $\phi'_p : S'_p \times W'_p \to Z'_p$

as a generic notation for a foliated chart of the stable foliation $\mathcal{F}^s$ of Λ at the point p. Recall that, by definition, W'_p is contained in the local stable manifold and S'_p is contained in the local unstable manifold of p.

In what follows we consider only accessibility classes that intersect some unstable leaf in a subset with positive Lebesgue measure. As we shall see in a while, in Corollary 2.4.2, the union of these classes has full measure for any probability measure absolutely continuous along $\mathcal{F}^u$.

The next lemma states that the stable set $W^s(\mathcal{A})$ of any of these accessibility classes $\mathcal{A}$ contains Lebesgue almost all points inside some unstable disk with uniform radius.

Lemma 2.4.2. *There exists $r > 0$ such that, given any accessibility class $\mathcal{A}$ in Λ that intersects some unstable leaf in a positive m_u-measure subset, there exists $p_{\mathcal{A}} \in \Lambda$ such that m_u-almost every point z in the d_u-ball of radius r around $p_{\mathcal{A}}$ is in the stable manifold of some point of $\mathcal{A}$.*

Proof. We fix $r > 0$ so that the d_u-ball of radius r around any point $p \in \Lambda$ is contained in the interior of Z'_p, where Z'_p is the image of some foliated chart for the stable foliation $\mathcal{F}^s$ at p. Let F be any leaf of $\mathcal{F}^u$ for which $A = \mathcal{A} \cap F$ has positive m_u-measure. Note that $f^n(A) \subset \mathcal{A}$ for every $n \geq 1$, because $\mathcal{A}$ is invariant. So, according to Proposition 2.3.4, there exist points $p_n \in \mathcal{A}$ such that

$$\frac{m_u\big(B^u_r(p_n) \setminus \mathcal{A}\big)}{m_u\big(B^u_r(p_n)\big)} \to 0 \quad \text{as } n \to \infty. \tag{2.23}$$

By compactness of Λ, we may suppose that the sequence $(p_n)_n$ converges to some point $p_{\mathcal{A}} \in \Lambda$. In particular, because $\mathcal{F}^u$ is a continuous foliation, the $B^u_r(p_n)$ converge to the d_u-ball of radius r around $p_{\mathcal{A}}$. In view of our choice of r, $B^u_r(p_{\mathcal{A}})$ is contained in Z'_p, and so is $B^u_r(p_n)$ for every large n. It follows from (2.23) and the absolute continuity property of $\mathcal{F}^s$, Theorem 2.2.2, that the local stable manifolds through points in $\mathcal{A} \cap B^u_r(p_n)$ intersect $B^u_r(p_{\mathcal{A}})$ in a subset G_n such that

$$\frac{m_u(B^u_r(p_{\mathcal{A}}) \setminus G_n)}{m_u(B^u_r(p_{\mathcal{A}}))} \leq C_0 \frac{m_u(B^u_r(p_n) \setminus \mathcal{A})}{m_u(B^u_r(p_n))} \to 0$$

as $n \to \infty$. Let $G = \cup_n G_n$. Then G has full m_u-measure in $B^u_r(p_{\mathcal{A}})$, and any $z \in G$ is in the stable manifold of some point in $\mathcal{A}$. □

Lemma 2.4.3. *Given $r > 0$ there exists $s > 0$ such that the following holds. Let $\mathcal{A}$ be an accessibility class and $p \in \Lambda$ be such that m_u-almost every point in the d_u-ball of radius r around p is in the stable manifold of some point of $\mathcal{A}$. Then the same is true for*

1. *Lebesgue almost every point in the s-neighbourhood $B_s(p)$ of p in the ambient manifold M,*

2. and m_u-almost every point in $F \cap B_s(p)$, for any unstable leaf F.

Proof. We choose $s > 0$ small enough so that, given any point $p \in \Lambda$, there exists a foliated chart $\phi'_p : S'_p \times W'_p \to Z'_p$ for the stable foliation at p such that Z_p contains $B_s(p)$. We reduce s if necessary, so that $\pi_1(B_s(p)) \subset B^u_r(p)$ for every $p \in \Lambda$, where π_1 is the projection onto S'_p along the stable leaves inside Z'_p. In other words, $\pi_1 = \phi_p \circ \tilde{\pi}_1 \circ \phi_p^{-1}$, where $\tilde{\pi}_1 : S'_p \times W'_p \to S'_p$ is the canonical projection.

Claim: Let Σ be a C^1 embedded disk in $B_s(p)$ and transverse to $\mathcal{F}^s$. Then m_Σ-almost every point in Σ is in the stable manifold of some point of $\mathcal{A}$.

Proof. Let $\pi : \Sigma \to S'_p$ be the local holonomy map of $\mathcal{F}^s$ from Σ to S'_p. Our choice of s ensures that the image $\tilde{\Sigma}$ of π is contained in $B^u_r(p)$. By Theorem 2.2.2, both π and its inverse π^{-1} map zero Lebesgue measure sets into zero Lebesgue measure sets. So, the assumption implies that $\pi(z)$ is in the stable manifold of a point in $\mathcal{A}$, for m_Σ-almost every point. Of course, z and $\pi(z)$ are in the same stable manifold, so the conclusion follows. □

Part 2 of the lemma is a particular case of the Claim. To prove part 1 it suffices to consider any C^1 foliation $\mathcal{G}$ of $B_s(p)$ by disks transverse to $\mathcal{F}^s$. The Claim applies to each of the leaves of $\mathcal{G}$ and so, by Fubini's theorem, Lebesgue almost every point in that s-neighbourhood is in the stable manifold of some point of the accessibility class $\mathcal{A}$. □

Corollary 2.4.2. *1. There exist only finitely many accessibility classes $\mathcal{A}_1, \ldots, \mathcal{A}_N$ intersecting some unstable leaf in a positive m_u-measure subset.*
2. The ergodic decomposition of any invariant measure μ absolutely continuous along $\mathcal{F}^u$ is given by $\mu = \sum_i \mu(\mathcal{A}_i)\mu_i$ where the sum is over the values of i such that $\mu(\mathcal{A}_i) > 0$, and each μ_i is the normalized restriction of μ to $\mathcal{A}_i$.

Proof. Suppose there are infinitely many distinct accessibility classes $\mathcal{A}_i$, $i \geq 1$. Then, combining Lemmas 2.4.2 and 2.4.3, there is $s > 0$ and there are points $p_i \in \Lambda$, $i \geq 1$, such that the union of the stable manifolds of the points in $\mathcal{A}_i$ contains a full Lebesgue measure subset of the s-neighbourhood of p_i, for every $i \geq 1$. By compactness, we may suppose that the p_i converge to some $p \in \Lambda$. Then, for every large i and j, the neighbourhoods $B_s(p_i)$ and $B_s(p_j)$ intersect each other in a set with positive Lebesgue measure. In particular, there exists $q \in B_s(p_i) \cap B_s(p_j)$ that is in the stable manifolds of points $q_i \in \mathcal{A}_i$ and $q_j \in \mathcal{A}_j$. This implies that q_i and q_j are in the same accessibility class, contradicting the assumption that $\mathcal{A}_i$ and $\mathcal{A}_j$ are different. Part 1 of the corollary is proved.

Now we prove part 2. In view of Corollary 2.4.1, we only have to show that the union of all the accessibility classes as in Lemma 2.4.2 has full measure for μ. In equivalent terms, it suffices to prove that any measurable subset B

of Λ with $\mu(B) > 0$ intersects at least one accessibility class $\mathcal{A}$ that contains a positive m_u-measure subset of some unstable leaf.

Indeed, given any measurable set $B \subset \Lambda$ with $\mu(B) > 0$, there exist $p \in \Lambda$ and a foliated chart $\phi_p : W_p \times S_p \to Z_p$ for $\mathcal{F}^u$ at p such that $B \cap Z_p$ has positive μ-measure (consider some finite covering of Λ by images of foliated charts). As the set $\mathcal{R}$ of regular points has full measure, we also have $\mu(B \cap Z_p \cap \mathcal{R}) > 0$. This can be rewritten as

$$\phi_p^* \mu\big(\phi_p^{-1}(B \cap Z_p \cap \mathcal{R})\big) > 0.$$

Then, since $\phi_p^* \mu$ is absolutely continuous along the horizontal in $W_p \times S_p$, there exists $y \in S_p$ such that $\phi_p^{-1}(B \cap Z_p \cap \mathcal{R}) \cap (W_p \times \{y\})$ has positive m_u-measure. Taking the image under the embedding $\phi_p \mid (W_p \times \{y\})$,

$$m_u\big((B \cap Z_p \cap \mathcal{R}) \cap \phi_p(W_p \times \{y\})\big) > 0.$$

The set $(B \cap Z_p \cap \mathcal{R}) \cap \phi_p(W_p \times \{y\})$ is contained in the accessibility class of any of its points, since it is contained in an unstable disk $\phi_p(W_p \times \{y\})$. The last inequality shows that this accessibility class intersects the unstable disk in a subset with positive m_u-measure. □

Corollary 2.4.3. *Any ergodic measure μ absolutely continuous along the unstable foliation is a physical measure for f.*

Proof. By Corollary 2.4.2, μ is ergodic if and only if there is an accessibility class $\mathcal{A}_i$ so that $\mu(\mathcal{A}_i) = 1$. Moreover, cf. Corollary 2.4.1, in that case μ is given by the limit of $n^{-1} \sum_{j=0}^{n-1} \delta_{f^j(z)}$ as $n \to \infty$, for any point $z \in \mathcal{A}_i$. By Lemma 2.4.3, there exists $p \in \Lambda$ such that Lebesgue almost every point in the s-neighbourhood of p_i is in the stable manifold of some point of $\mathcal{A}_i$. Since points in the same stable manifold have the same forward time averages, it follows that

$$\mu = \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j(w)}$$

for a full Lebesgue measure subset of $B_s(p_i)$. This means that the basin of μ contains that subset of $B_s(p_i)$, and so it has positive Lebesgue measure. □

Observe that Theorem 2.3.1, together with the second part of Corollary 2.4.2, imply that there does exist at least one accessibility class as in Lemma 2.4.2. Now we use the assumption that f is transitive on Λ to show that, in fact, such an accessibility class is unique. Thus, f has a unique physical measure supported in Λ.

Lemma 2.4.4. *Let $p \in \Lambda$ and $\mathcal{A}$ be any accessibility class as in Lemma 2.4.2. Then m_u-almost every point in $B_r^u(p)$ is in the stable manifold of some point of $\mathcal{A}$.*

Proof. Let $\phi'_p : S'_p \times W'_p \to Z'_p$ be a foliated chart for the stable foliation at p, such that Z'_p contains the r-neighbourhood of p (recall the choice of r in Lemma 2.4.2). By the second part of Lemma 2.4.3, there is $p_{\mathcal{A}} \in \Lambda$ such that m_u-almost every point in $F \cap B_s(p_{\mathcal{A}})$ is in the stable manifold of some point of $\mathcal{A}$, for any unstable leaf F intersecting the s-neighbourhood $B_s(p_{\mathcal{A}})$. The fact that f is transitive on Λ implies that there exist points $q_k \to p_{\mathcal{A}}$ and times $n_k \to \infty$ such that $f^{n_k}(q_k) \to p$. In particular, q_k is in the $s/2$-neighbourhood of $p_{\mathcal{A}}$ and $B_r(f^{n_k}(q_k))$ is contained in Z'_p, for every large k. Then, m_u-almost every point in the d_u-ball of radius $s/2$ around q_k is in the stable manifold of some point in $\mathcal{A}$. By the invariance of $\mathcal{A}$, the same is true for m_u-almost every point in the f^{n_k}-image of this ball. Increasing n_k if necessary, this image contains the d_u-ball $B^u_r(f^{n_k}(q_k))$. So, we have shown that p is accumulated by points $f^{n_k}(q_k)$ such that m_u-almost every point in their d_u-balls of radius r are in the stable manifold of some point in $\mathcal{A}$. The same argument as in Lemma 2.4.2 shows that this remains true for the d_u-ball of radius r around p. □

Corollary 2.4.4. *There exists exactly one accessibility class that intersects some unstable manifold in a positive m_u-measure set.*

Proof. In view of the remarks preceding Lemma 2.4.4, we only have to prove uniqueness. Let $p \in \Lambda$, and $\mathcal{A}$, $\mathcal{B}$, be any accessibility classes as in the hypothesis. By the previous lemma, the stable sets of $\mathcal{A}$ and $\mathcal{B}$ fill-in full measure subsets of $B^u_r(p)$. In particular, they must intersect each other, and this implies that $\mathcal{A} = \mathcal{B}$. □

Corollary 2.4.5. *There exists a unique invariant probability measure μ absolutely continuous along $\mathcal{F}^u$, and μ is ergodic. Moreover, the basin of μ contains a full Lebesgue measure subset of the basin of Λ. Consequently, μ is also the unique SRB measure of f in Λ.*

Proof. Uniqueness and ergodicity follow from Corollaries 2.4.2 and 2.4.4:

$$\mu = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j(z)} \tag{2.24}$$

for any z in the unique accessibility class $\mathcal{A}$. Lemmas 2.4.3 and 2.4.4 imply that Lebesgue almost every point in the s-neighbourhood of any $p \in \Lambda$ is in the stable manifold of some $z \in \mathcal{A}$. So, (2.24) is true also for Lebesgue almost every point in the s-neighbourhood of the attractor Λ. Since the basin $B(\Lambda)$ is the union of the pre-images of this neighbourhood, we have (2.24) for Lebesgue almost every point in $B(\Lambda)$. The last statement is an immediate consequence. □

We have already observed that Lemma 2.3.5 and Proposition 2.1.5 imply that the support of μ coincides with Λ. The proof of Theorem 2.4.1 is complete.

3. Convergence to Equilibrium and Decay of Correlations

In this chapter we view the dynamics of a system $f : M \to M$ at the level of density distributions (*ensembles*, in the Physics literature), rather than individual points. The main observation is that, while individual orbits may behave in a very complicated way, it is often the case that the iterates $f_*^j(\varphi m)$ of an initial density distribution φm converge to some equilibrium as $j \to \infty$. For instance, if there is a physical measure μ and the density φ is supported inside the basin $B(\mu)$, then one has at least a Cesaro limit:

$$\frac{1}{n}\sum_{j=0}^{n-1} f_*^j(\varphi m) \to \mu \int \varphi \, dm.$$

So, the question becomes whether this can be replaced by actual limit. Moreover, in the affirmative case, how fast is the convergence ?

A related problem concerns the decay of correlations. Here one is given an invariant probability measure μ, for example a physical measure. Then one wants to known whether the iterates $f_*^j(\varphi \mu)$ converge to $\mu \int \varphi \, dm$ and, if so, how fast this happens. The "distance" between these measures can be expressed by the *correlations*

$$C_j(\varphi, \psi) = \int \psi \, d\Big(f_*^j(\varphi \mu) - \mu \int \varphi \, d\mu\Big) = \int (\psi \circ f^j)\varphi \, d\mu - \int \psi \, d\mu \int \varphi \, d\mu.$$

Decay of $C_j(\varphi, \psi)$ to zero as $j \to \infty$ may be seen as a sort of memory dissipation in the system: information contained in the initial density is gradually forgotten, the value of ψ at time $j \gg 1$ is, essentially, uncorrelated to ("independent" from) the value of φ at time zero.

An important tool is the transfer operator, that describes the action of the dynamical system on absolutely continuous measures. These operators were implicit in the previous chapters, but now we are to exploit their properties as linear operators, specially spectral properties, to deduce ergodic properties of the system. A main theme is that exponential convergence corresponds to existence of a gap in the spectrum of the transfer operator.

In Sections 3.1 and 3.2 we develop general tools for analyzing the spectrum of transfer operators. Applications to concrete cases appear in Sections 3.3 and 3.4. In Section 3.5 we describe some general approaches to subexponential decay, starting from Example 1.1.2. Finally, in Section 3.6 we prove exponential convergence for hyperbolic attractors of diffeomorphisms.

3.1 Transfer Operators

Let M be a manifold or, more generally, a subset of some manifold, endowed with a Riemannian metric. We denote by m the corresponding Riemannian volume, normalized so that $m(M) = 1$. We always suppose that the map $f : M \to M$ is absolutely continuous with respect to Lebesgue measure:

$$m(A) = 0 \quad \Rightarrow \quad m\big(f^{-1}(A)\big) = 0. \tag{3.1}$$

In other words, the measure $f_* m$ is absolutely continuous with respect to Lebesgue measure m. For instance, cf. Lemma 3.1.2 below, this is the case if f is almost everywhere a local diffeomorphism. It follows from (3.1) that the push-forward $f^*(\varphi m)$ of any absolutely continuous measure $\varphi\, m$ is also absolutely continuous. This allows us to introduce the following linear operator:

Definition 3.1.1. *The* transfer operator $\mathcal{L} = \mathcal{L}_f : L^1(m) \to L^1(m)$ *of* f *is the transformation assigning to each* $\varphi \in L^1(m)$ *the density* $\mathcal{L}\varphi$ *of the push-forward* $f_*(\varphi\, m)$*:*

$$(\mathcal{L}\varphi)\, m = f_*(\varphi\, m). \tag{3.2}$$

Recall that densities are uniquely defined up to zero Lebesgue measure sets only. If two functions φ_1 and φ_2 represent the same element of $L^1(m)$, that is, if they coincide Lebesgue almost everywhere, then $\varphi_1 m = \varphi_2 m$. Thus, $\mathcal{L}\varphi$ is well-defined in $L^1(m)$. We shall check in Lemma 3.1.1 that $\mathcal{L}$ does take values in $L^1(m)$, moreover, it is a bounded operator with respect to the L^1-norm.

Before proceeding, let us point out that transfer operators may be defined in more generality, e.g. with other reference measures ν in the place of Lebesgue measure. Such operators play a central part in the theory of equilibrium states [22, ?], and dynamical ζ-functions [112, 9].

3.1.1 Definitions and Basic Facts

The notation $L^1(m)$ corresponds to two slightly different objects, depending on whether we consider its elements to be represented by real-valued or by complex-valued functions. The distinction is irrelevant for most purposes, in general we do not worry about making it explicit. However, in the present chapter we always consider the *complex* version of $L^1(m)$, which is a Banach space over $\mathbb{C}$.

In Lemma 3.1.1 we collect some basic properties of $\mathcal{L}$. In the sequel we shall often use them without explicit reference to the lemma.

Lemma 3.1.1. *1.* $\int (\mathcal{L}\varphi)\psi\, dm = \int \varphi(\psi \circ f)\, dm$ *for any* $\varphi \in L^1(m)$ *and any* $\mathcal{L}\varphi$*-integrable function* ψ*. In particular,* $\int (\mathcal{L}\varphi)\, dm = \int \varphi\, dm$*.*

2. $\mathcal{L}$ *is a nonnegative operator, with* $|\mathcal{L}\varphi| \le \mathcal{L}|\varphi|$ *and* $\|\mathcal{L}\varphi\|_1 \le \|\varphi\|_1$ *for any* $\varphi \in L^1(m)$*. In particular,* $\mathcal{L}$ *is a bounded operator from* $L^1(m)$ *to itself.*

Proof. Part 1 is a restatement of the definition of the transfer operator:

$$\int \psi(\mathcal{L}\varphi)dm = \int \psi\, d\big(f_*(\varphi m)\big) = \int (\psi \circ f)\varphi\, dm. \tag{3.3}$$

For $\psi = 1$ this gives $\int (\mathcal{L}\varphi)\, dm = \int \varphi\, dm$. Next, suppose $\varphi \ge 0$. Then

$$f_*(\varphi m)(A) = (\varphi m)(f^{-1}(A)) = \int_{f^{-1}(A)} \varphi\, dm \ge 0$$

for every measurable set A, and so $\mathcal{L}\varphi \ge 0$. This means that the operator $\mathcal{L}$ is nonnegative. Analogously, $\mathcal{L}\varphi$ is real-valued if φ is real-valued. By linearity, it follows that $\Re\mathcal{L} = \mathcal{L}\Re$ and $\Im\mathcal{L} = \mathcal{L}\Im$, where $\Re$ and $\Im$ represent real and imaginary part, respectively. Then, given any $\varphi \in L^1(m)$ and $\theta \in \mathbb{R}$,

$$\mathcal{L}|\varphi| - \Re\big(e^{i\theta}\mathcal{L}\varphi\big) = \mathcal{L}|\varphi| - \Re\mathcal{L}\big(e^{i\theta}\varphi\big) = \mathcal{L}\big(|\varphi| - \Re(e^{i\theta}\varphi)\big) \ge 0,$$

because $|\varphi| \ge \Re(e^{i\theta}\varphi)$. This implies that $\mathcal{L}|\varphi| - |\mathcal{L}\varphi| \ge 0$, because we can always choose θ so that $e^{i\theta}\mathcal{L}\varphi = |\mathcal{L}\varphi|$. As a consequence,

$$\|\mathcal{L}\varphi\|_1 = \int |\mathcal{L}\varphi|\, dm \le \int (\mathcal{L}|\varphi|)\, dm = \int |\varphi|\, dm = \|\varphi\|_1 .$$

Thus, the operator $\mathcal{L}$ maps $L^1(m)$ inside itself, and its norm is at most 1. □

In all the situations we are interested in, f is a local diffeomorphism at Lebesgue almost every point. Then there is an explicit formula for the transfer operator, that is given by the next lemma. Particular cases appeared before, for instance, in Lemma 1.3.5. Here $\det Df$ is the Jacobian of f with respect to the Riemannian metric of M.

Lemma 3.1.2. *Suppose there is a closed subset* S *of* M *with zero Lebesgue measure, such that* f *is a local diffeomorphism at every point* $x \notin S$*. Then, for any* $\varphi \in L^1(m)$ *and Lebesgue almost every* $y \in M$*,*

$$\mathcal{L}\varphi(y) = \sum_{x \notin S: f(x)=y} \frac{\varphi(x)}{|\det Df(x)|}\,.$$

Proof. The assumption means that every $x \notin S$ admits some open neighbourhood W_x such that the restriction $f \mid W_x$ is a diffeomorphism onto its image. Since $M \setminus S$ is a countable union of compact subsets, we may find a countable subcovering $W_{x_1}, W_{x_2}, \ldots$ of $M \setminus S$. Let $V_1 = W_{x_1}$ and

$V_k = W_{x_k} \setminus (V_1 \cup \cdots \cup V_{k-1})$, for $k \geq 2$. Then $V_1, V_2, \ldots$ is a partition of $M \setminus S$ such that each $f \mid V_i$ is a diffeomorphism onto its image. Since $M \setminus S$ has full Lebesgue measure, we may decompose $m = \sum_k (m \mid V_k)$. It follows that $f_*(\varphi m) = \sum_k f_*(\varphi m \mid V_k)$, for any integrable function φ. By definition,

$$\int \psi \, d\big(f_*(\varphi m \mid V_k)\big) = \int_{V_k} (\psi \circ f) \, \varphi \, dm,$$

for any ψ and $k \geq 1$. Changing variables $y = f(x)$ in the second integral,

$$\int \psi \, d\big(f_*(\varphi m \mid V_k)\big) = \int_{f(V_k)} \psi \, \big(\frac{\varphi}{|\det Df|} \circ f^{-1}\big) dm.$$

In other words,

$$f_*(\varphi m \mid V_k) = \big(\frac{\varphi}{|\det Df|} \circ f^{-1}\big) \mathcal{X}_{f(V_k)} m$$

for every $k \geq 1$. Then,

$$\mathcal{L}\varphi = \frac{d}{dm} f_*(\varphi m) = \sum_k \frac{d}{dm} f_*(\varphi m \mid V_k) = \sum_k \big(\frac{\varphi}{|\det Df|} \circ f^{-1}\big) \mathcal{X}_{f(V_k)}$$

Finally, given any $y \in M$,

$$\sum_k \big(\frac{\varphi}{|\det Df|} \circ f^{-1}\big) \mathcal{X}_{f(V_k)}(y) = \sum_{x \notin S : f(x) = y} \frac{\varphi(x)}{|\det Df(x)|}.$$

So, the lemma is proved. □

It follows from the definition of the transfer operator that an absolutely continuous probability measure $\mu = \varphi_0 m$ *is invariant under* f *if and only if the density* φ_0 *is a fixed point of* $\mathcal{L}$*:*

$$f_*(\varphi_0 m) = \varphi_0 m \quad \Leftrightarrow \quad \mathcal{L}\varphi_0 = \varphi_0 .$$

We are going to see that *if* 1 *is the unique point in the spectrum of* $\mathcal{L}$ *with maximum norm and it is a simple eigenvalue, then there is exponential convergence to the equilibrium* $\mu = \varphi_0 m$ *and the system* (f, μ) *has exponential decay of correlations.* These statements are in Proposition 3.1.1 and Corollary 3.1.1. Beforehand, let us recall some elementary facts from spectral theory in Banach spaces. See [39, Chapter VII] for proofs.

Let $L : E \to E$ be a bounded linear operator on a complex Banach space E. The *spectrum* of L is the set $\operatorname{spec}(L)$ of complex numbers z such that $(z \operatorname{id} - L)$ is not an isomorphism. The spectrum is a nonempty compact subset of $\mathbb{C}$, contained in the closed disk around the origin with radius equal to the norm $\|L\|$ of the operator. The *spectral radius* of L is the number

$$\rho(L) = \sup\{|z| : z \in \operatorname{spec}(L)\} = \lim_n \|L^n\|^{1/n}. \tag{3.4}$$

A complex number z is an *eigenvalue* of L if $(z\,\mathrm{id} - L)$ is not injective. The corresponding *eigenspace* is $\operatorname{kernel}(z\,\mathrm{id} - L)$. The *index* of $z \in \mathbb{C}$ is the smallest integer $0 \le k \le \infty$ such that

$$\operatorname{kernel}(z\,\mathrm{id} - L)^k = \operatorname{kernel}(z\,\mathrm{id} - L)^{k+1}.$$

(By convention $T^0 = \mathrm{id}$ for any operator T). Thus, the index is positive if and only if z is an eigenvalue. The *resolvent function* of L is the map $R(L,z) = (z\,\mathrm{id} - L)^{-1}$, defined from $\mathbb{C}\setminus\operatorname{spec}(L)$ to the Banach space of linear operators in E. The resolvent function is analytic in its domain. Moreover,

$$|R(L,z)| \ge \frac{1}{\operatorname{dist}(z, \operatorname{spec}(L))}$$

so that every point in the spectrum is a singularity for $R(L,\cdot)$. If $z \in \operatorname{spec}(L)$ is a pole of order k of $R(L,\cdot)$, then it has index k.

Let $\Sigma_1, \ldots, \Sigma_N$ be a partition of the spectrum of L into open and closed subsets. For each $1 \le j \le N$, define the bounded linear operator

$$\pi_j = \pi(L, \Sigma_j) : E \to E, \qquad \pi_j = \frac{1}{2\pi i} \int_{\gamma_j} R(L,z)\, dz,$$

where γ_j is a finite union of rectifiable Jordan curves disjoint from the spectrum and such that the intersection of $\operatorname{spec}(L)$ with the inside of γ_j is Σ_j. This definition does not depend on the choice of γ_j, because the resolvent function is analytic outside $\operatorname{spec}(L)$. We have

$$\pi_1 + \cdots + \pi_N = \mathrm{id}, \quad \pi_i \pi_j = \begin{cases} \pi_j & \text{if } i = j \\ 0 & \text{if } i \ne j \end{cases}, \quad \text{and} \quad \pi_j L = L \pi_j \tag{3.5}$$

for any $1 \le i, j \le N$. Consequently, E may be decomposed as a direct sum $E = \pi_1(E) \oplus \cdots \oplus \pi_N(E)$. Moreover, L maps every $\pi_j(E)$ inside itself, and the spectrum of $L \mid \pi_j(E)$ coincides with Σ_j. We call $\pi_1(E) \oplus \cdots \oplus \pi_N(E)$ the *spectral decomposition*, and the π_j are the *spectral projections*, associated with the partition $\Sigma_1, \ldots, \Sigma_N$. If some $\pi_j(E)$ has finite dimension then Σ_j consists of a finite number of eigenvalues $\lambda_1, \ldots, \lambda_l$, and

$$\pi_j(E) = \operatorname{kernel}(\lambda_1\,\mathrm{id} - L)^{i_1} \oplus \cdots \oplus \operatorname{kernel}(\lambda_l\,\mathrm{id} - L)^{i_l},$$

where i_s is the index of λ_s for $1 \le s \le l$.

Definition 3.1.2. *We say that an operator $L : E \to E$ has a* spectral gap *if its spectrum can be partitioned as $\Sigma_0 \cup \{1\}$, where*

1. *Σ_0 is contained in a closed disk of radius $\tau < 1$;*
2. *1 is a simple eigenvalue: the image of $\pi_1 = \pi(L, \{1\})$ has dimension 1.*

3.1.2 Consequences of a Spectral Gap

Now we go back to the dynamical setting. Let $\mathcal{L} : L^1(m) \to L^1(m)$ be the transfer operator associated to $f : M \to M$. In what follows we assume that there is a vector subspace E of $L^1(m)$, and a complete norm $\|\cdot\|_E$ in E, such that

(O1) $\mathcal{L}(E) \subset E$ and $\mathcal{L} : E \to E$ is a bounded operator with respect to $\|\cdot\|_E$;
(O2) there exist $p \in [1, \infty]$ and $C_0 > 0$ such that

$$\|\varphi\|_p \leq C_0 \|\varphi\|_E \quad \text{for all } \varphi \in E.$$

We are going to show in Proposition 3.1.1 that *if $\mathcal{L} : E \to E$ has a spectral gap then iterates of φm converge exponentially fast, for any $\varphi \in E$.* Then, in Corollary 3.1.1, we deduce a corresponding statement for decay of correlations.

Before continuing, let us say a few words about the role and choice of the subspace E. As the following example illustrates, it is rather unusual for the transfer operator to have a spectral gap on the whole space $L^1(m)$.

Example 3.1.1. Let $f : S^1 \to S^1$ be the map of the circle $S^1 = \mathbb{R}/\mathbb{Z}$ defined by $f(\theta) = D\theta \mod \mathbb{Z}$, for some integer $D \geq 2$. The transfer operator of f is given by

$$\mathcal{L}\varphi(\theta) = \frac{1}{D} \sum_{k=0}^{D-1} \varphi\left(\frac{\theta + k}{D}\right).$$

For any $\rho \in \mathbb{C}$ with $|\rho| < 1$, the function $\varphi_\rho(\theta) = \sum_{n=0}^{\infty} \rho^n e^{2\pi i D^n \theta}$ is well defined and continuous on S^1. Moreover,

$$\mathcal{L}\varphi_\rho(\theta) = \frac{1}{D} \sum_{k=0}^{D-1} \left(e^{2\pi i(\theta+k)/D} + \sum_{n=1}^{\infty} \rho^n e^{2\pi i D^{n-1}(\theta+k)} \right) = \sum_{n=1}^{\infty} \rho^n e^{2\pi i D^{n-1}\theta}$$

(the sum of the first terms in the parentheses is zero, and the second terms are all equal). The term on the right hand side is equal to $\rho\varphi_\rho$, which means that we have shown that ρ is an eigenvalue of $\mathcal{L}$, with eigenfunction φ_ρ . Since $\|\mathcal{L}\|_1 \leq 1$, we conclude that the spectrum of $\mathcal{L}$ is the whole closed unit disk.

In most applications where the spectral gap property can be checked, E is a subspace of functions with some extra regularity (in the previous example one can take Hölder continuous functions), and the spectral gap property expresses the fact that the transfer operator tends to improve that regularity ($\mathcal{L}\varphi$ tends to have better Hölder constants than φ). It is not surprising that speeds of convergence to equilibrium or decay of correlations be related to regularity of the initial density φ, it just reflects the deterministic nature of the dynamical system. For instance, if φm is concentrated on a small region near some $z \in M$ (in that case φ can not be very regular: if it is Hölder continuous then the Hölder constants are necessarily bad), then $f_*^j(\varphi m)$ is concentrated near $f^j(z)$ for many initial values of j: the system "remembers" the vicinity of the initial state z for a long time.

Let us start to prove the facts claimed above. We begin by finding an explicit expression for the spectral decomposition of the transfer operator.

Lemma 3.1.3. *There is an eigenfunction φ_0 of $\mathcal{L}$ associated to the eigenvalue 1 such that $\varphi_0 \geq 0$ and $\int \varphi_0 \, dm = 1$. Moreover, the spectral projections associated to the partition of* $\operatorname{spec}(\mathcal{L})$ *into Σ_0 and $\Sigma_1 = \{1\}$ are, respectively,*

$$\pi_0(\varphi) = \varphi - \varphi_0 \int \varphi \, dm \quad \textit{and} \quad \pi_1(\varphi) = \varphi_0 \int \varphi \, dm.$$

Proof. For the time being, let $\varphi_0 \neq 0$ be any eigenfunction associated to the eigenvalue 1. As $\pi_1(E)$ is 1-dimensional, $\pi_1(E) = \operatorname{kernel}(\operatorname{id} - L) = \mathbb{C}\varphi_0$. Given any $\varphi \in E$, decompose

$$\varphi = \psi_0 + \alpha\varphi_0\,, \quad \text{with } \psi_0 \in \pi_0(E) \text{ and } \alpha \in \mathbb{C}.$$

Since $\Sigma_0 = \operatorname{spec}(\mathcal{L} \mid \pi_0(E))$ is contained in a disk of radius $\tau < 1$, the sequence $\|\mathcal{L}^n\psi_0\|_E$ goes to zero exponentially fast. Using condition (O2) we get that $\mathcal{L}^n\psi_0$ converges to zero in $L^p(m)$. Then $\mathcal{L}^n\varphi$ converges to $\alpha\varphi_0$ in $L^p(m)$, and so $\int \mathcal{L}^n\varphi \, dm$ converges to $\alpha \int \varphi_0 \, dm$. Since $\int \mathcal{L}^n\varphi \, dm = \int \varphi \, dm$ for all $n \geq 1$, this shows that

$$\int \varphi \, dm = \alpha \int \varphi_0 \, dm. \tag{3.6}$$

In the particular case $\varphi = 1$, we get $\alpha\varphi_0 = \lim \mathcal{L}^n 1 \geq 0$ and $\alpha \int \varphi_0 \, dm = 1$. So, replacing φ_0 by $\alpha\varphi_0$, we may suppose right from the start that $\varphi_0 \geq 0$ and $\int \varphi_0 \, dm = 1$. Now, for general $\varphi \in E$, (3.6) means that $\alpha = \int \varphi \, dm$. Then

$$\pi_1(\varphi) = \alpha\varphi_0 = \varphi_0 \int \varphi \, dm,$$

and it also follows that $\pi_0(\varphi) = (\operatorname{id} - \pi_1)(\varphi) = \varphi - \varphi_0 \int \varphi \, dm$. □

Let $\mu = \varphi_0 m$, where $\varphi_0 \in E$ is as in Lemma 3.1.3. The next proposition says that, for any $\varphi \in E$, the iterates of φm converge exponentially fast to the measure $\mu \int \varphi \, dm$. To state this precisely, we introduce the expression

$$B_n(\varphi, \psi) = \int (\psi \circ f^n) \, \varphi \, dm - \int \psi \, d\mu \int \varphi \, dm, \tag{3.7}$$

defined for any $n \geq 1$, and any measurable functions φ and ψ such that the integrals exist. Observe that $B_n(\varphi, \psi)$ is just the difference between the integrals of ψ with respect to the two measures $f_*^n(\varphi m)$ and $\mu \int \varphi m$:

$$B_n(\varphi, \psi) = \int \psi \, d\big(f_*^n(\varphi m)\big) - \int \psi \, d\big(\mu \int \varphi m\big).$$

Let $C_0 > 0$ and $1 \le p \le \infty$ be as in assumption (O2), and $\tau < 1$ be as in Definition 3.1.2. Take $q \in [1, \infty]$ such that $1/p + 1/q = 1$.

Proposition 3.1.1. *Given any $\tau_1 > \tau$ there exists $C_1 > 0$ such that*

$$|B_n(\varphi, \psi)| \le C_1 \tau_1^n \|\varphi\|_E \|\psi\|_q$$

for any $n \ge 1$, $\varphi \in E$, and ψ in $L^q(m)$.

Proof. By Lemmas 3.1.1 and 3.1.3, $B_n(\varphi, \psi)$ may be rewritten as

$$\int (\psi \circ f^n) \varphi \, dm - \int \psi \, d\mu \int \varphi \, dm = \int \psi \, \big(\mathcal{L}^n \varphi - \varphi_0 \int \varphi \, dm\big) \, dm =$$
$$= \int \psi \, \mathcal{L}^n \big(\varphi - \varphi_0 \int \varphi \, dm\big) \, dm = \int \psi \, \mathcal{L}^n \big(\pi_0(\varphi)\big) \, dm \, .$$

Using the Hölder inequality and condition (O2),

$$\Big| \int \psi \, \mathcal{L}^n \big(\pi_0(\varphi)\big) \, dm \Big| \le \|\psi\|_q \|\mathcal{L}^n \left(\pi_0(\varphi)\right) \|_p \le C_0 \|\psi\|_q \, \|\mathcal{L}^n \left(\pi_0(\varphi)\right) \|_E \, .$$

Since the spectral radius $\rho(\mathcal{L} \mid \pi_0(E))$ is less than τ_1, there exists $K_1 > 0$ such that $\|(\mathcal{L} \mid \pi_0(E))^n\|_E \le K_1 \tau_1^n$ for all $n \ge 1$. Then

$$|B_n(\varphi, \psi)| \le C_0 \|\psi\|_q \, K_1 \tau_1^n \|\pi_0(\varphi)\|_E \le C_1 \tau_1^n \|\psi\|_q \, \|\varphi\|_E \, ,$$

with $C_1 = C_0 K_1 \|\pi_0\|_E$. □

Definition 3.1.3. *Let μ be an f-invariant probability measure, and ϕ and ψ be measurable functions. The* nth correlation, *$n \ge 1$, between ϕ and ψ is defined by*

$$C_n(\phi, \psi) = \int (\psi \circ f^n) \, \phi \, d\mu - \int \psi \, d\mu \int \phi \, d\mu, \tag{3.8}$$

whenever the integrals exist.

Let F_0 be the subspace of functions $\phi \in L^1(\mu)$ such that $\phi\varphi_0$ is in E. According to the next result, existence of a spectral gap for $\mathcal{L} : E \to E$ implies that the correlations decay exponentially fast as $n \to \infty$, for any $\phi \in F_0$. We represent by $\| \cdot \|_{r,\nu}$ the norm in the space $L^r(\nu)$, for any $1 \le r \le \infty$ and any measure ν in M.

Corollary 3.1.1. *Given any $\tau_1 > \tau$ there exists $C_1 > 0$ such that*

$$|C_n(\phi, \psi)| \le C_1 \tau_1^n \|\phi\varphi_0\|_E \|\psi\|_q \tag{3.9}$$

for any $n \ge 1$, $\phi \in F_0$, and $\psi \in L^q(m)$.

Proof. This is a direct consequence of Proposition 3.1.1: by definition,

$$C_n(\phi, \psi) = \int (\psi \circ f^n) \phi \varphi_0 \, dm - \int \psi \, d\mu \int \phi \varphi_0 \, dm = B_n(\phi \varphi_0, \psi),$$

and so $|C_n(\phi, \psi)| \le C_1 \tau_1^n \|\phi \varphi_0\|_E \|\psi\|_q$, as claimed. □

Remark 3.1.1. For some applications, e.g. in Chapter 4, one needs estimates for the rate of decay in terms of a norm $\|\psi\|_{q,\mu}$ rather than $\|\psi\|_q$. Let us mention one relevant situation where that can be easily deduced from (3.9); another result of this kind will appear in Lemma 4.2.3. Assume that φ_0 is bounded from zero, that is, $\varphi_0 \ge \delta$ for some $\delta > 0$. Then

$$\|\psi\|_q^q = \int |\psi|^q \, dm = \int (|\psi|^q / \varphi_0) \, d\mu \le \int |\psi|^q \delta^{-1} \, d\mu = \delta^{-1} \|\psi\|_q^q ,$$

and so (3.9) yields $|C_n(\phi, \psi)| \le \tilde{C}_1 \tau_1^n \|\phi \varphi_0\|_E \|\psi\|_{q,\mu}$, with $\tilde{C}_1 = C_1 \delta^{-1/q}$. Here we supposed $q < \infty$, but the infinite case is even easier: it suffices to assume $\varphi_0 > 0$ to conclude that $\|\psi\|_\infty = \|\psi\|_{\infty,\mu}$.

3.1.3 Quasi-Compact Operators

This subsection is about proving the spectral gap property. In Theorems 3.1.1 and 3.1.2 we quote two general results from functional analysis, respectively, the theorem of Ionescu-Tulcea and Marinescu [58], and the uniform ergodic theorem of Yosida and Kakutani [127]; see [39, Section VIII.8]. Both theorems give sufficient conditions under which the peripherical spectrum of a bounded linear operator consists of a finite number of isolated eigenvalues, all with index 1 and finite-dimensional eigenspaces. We call such operators quasi-compact.

These results hold for general bounded operators in Banach spaces. In Subsection 3.1.4 we shall apply them to transfer operators associated to dynamical systems. As we shall discuss there, once one has quasi-compactness existence of a spectral gap is tantamount to mixing properties of the invariant measure.

Theorem 3.1.1. *Let $(X, \|\cdot\|_X)$ and $(E, \|\cdot\|_E)$ be Banach spaces with $E \subset X$. Let $L : E \to E$ be a linear operator that is bounded with respect to both $\|\cdot\|_E$ and the restriction of $\|\cdot\|_X$ to E. Assume there are $C > 0$, $\lambda < 1$, $N \ge 1$, such that*

1. *if $(\varphi_n)_n$ is a bounded sequence in $(E, \|\cdot\|_E)$ such that $(\|\varphi_n - \varphi\|_X)_n \to 0$ for some $\varphi \in V$, then $\varphi \in E$ and $\|\varphi\|_E \le \sup_n \|\varphi_n\|_E$;*
2. *the sequence of norms $\|L^{Nn}\|_E$, $n \ge 1$, is bounded;*
3. *$\|L^N \varphi\|_E \le \lambda \|\varphi\|_E + C \|\varphi\|_X$ for every $\varphi \in E$;*

4. *for any bounded set B in $(E, \|\cdot\|_E)$, the closure of $L^N(B)$ in $(X, \|\cdot\|_X)$ is compact.*

Then the spectrum of L is contained in the closed unit disk, and it intersects the unit circle in a finite number of eigenvalues $\lambda_1, \ldots, \lambda_k$. Each λ_j is a simple pole of the resolvent function, and $\pi(L, \{\lambda_j\})(E) = \operatorname{kernel}(\lambda_j \operatorname{id} - L)$ has finite dimension.

Remark 3.1.2. [58] proves the case $N = 1$. To get the theorem as we stated it, it suffices to observe that the conclusion holds for an operator L if it holds for L^N. That is because $\operatorname{spec}(L^N) = \operatorname{spec}(L)^N$, see [39, Theorem VII.3.11]. Moreover, if λ is an eigenvalue of L then λ^N is an eigenvalue of L^N, the eigenspace $\operatorname{kernel}(\lambda \operatorname{id} - L)$ is contained in $\operatorname{kernel}(\lambda^N \operatorname{id} - L^N)$, and λ is a simple pole of $R(L, \cdot)$ if λ^N is a simple pole of $R(L^N, \cdot)$.

Next, we state the uniform ergodic theorem [39, Section VIII.8]. A sequence of bounded operators $L_n : E \to E$ in a Banach space $(E, \|\cdot\|_E)$ converges to $L : E \to E$ *in the weak sense* if $h\big(L_n(x)\big) \to h\big(L(x)\big)$ as $n \to \infty$, for every $x \in E$ and every h in the dual space E^*. This defines a topology in the space of bounded operators in E, weaker than the norm topology: if the norms $\|L_n - L\|_E$ converge to zero then L_n converges to L in the weak sense; the converse is false, in general. A linear operator $K : E \to E$ is called *compact* if the closure of $K(B)$ is compact for any bounded set $B \subset E$.

Theorem 3.1.2. *Let $(E, \|\cdot\|_E)$ be a Banach space, and $L : E \to E$ be a bounded operator. Assume that*

1. *$\mathcal{L}^n/n$ converges to zero in the weak sense;*
2. *there exists a compact operator $K : E \to E$ such that $\|L^N - K\| < 1$ for some $N \geq 1$.*

Then the spectrum of L is contained in the closed unit disk, and it intersects the unit circle in a finite number of eigenvalues $\lambda_1, \ldots, \lambda_k$. Each λ_j is a simple pole of the resolvent function, and $\pi(L, \{\lambda_j\})(E) = \operatorname{kernel}(\lambda_j \operatorname{id} - L)$ has finite dimension.

Thus, in the setting of either Theorem 3.1.1 or Theorem 3.1.2, the spectrum of L may be decomposed as $\Sigma_0 \cup \{\lambda_1, \ldots, \lambda_k\}$ where $\lambda_1, \ldots, \lambda_k$ are eigenvalues with norm 1, and Σ_0 is contained in some disk of radius $\tau < 1$. Denoting $\pi_0 = \pi(L, \Sigma_0)$ and $\pi_j = \pi(L, \{\lambda_j\})$ for $1 \leq j \leq k$,

$$\pi_j(E) = \operatorname{kernel}(\lambda_j \operatorname{id} - L) \tag{3.10}$$

has finite-dimension for $1 \leq j \leq k$. On the other hand, the spectral radius of L restricted to $\pi_0(E)$ is smaller than 1. Note that

$$L = L\pi_0 + \sum_{j=1}^{k} L\pi_j = L\pi_0 + \sum_{j=1}^{k} \lambda_j \pi_j ,$$

because (3.10) means that $L\pi_j = \lambda_j \pi_j$ for $1 \le j \le k$. More generally, using the properties in (3.5),

$$L^n = (L\pi_0)^n + \sum_{j=1}^{k} \lambda_j^n \pi_j ,$$

for $n \ge 1$. Thus, L is a quasi-compact operator:

Definition 3.1.4. *A bounded linear operator $L : E \to E$ is* quasi-compact *if there are $k \ge 1$, bounded operators $Q, P_1, \dots, P_k$, and complex numbers $\lambda_1, \dots, \lambda_k$, satisfying*

1. *$QP_j = P_jQ = 0$, $P_iP_j = P_jP_i = 0$, and $P_jP_j = P_j$ for $1 \le i < j \le k$;*
2. *$P_j(E) = \mathrm{kernel}(\lambda_j \,\mathrm{id} - L)$ and has finite-dimension for $1 \le j \le k$;*
3. *the spectral radius $\rho(Q)$ of Q is smaller than $1 = |\lambda_1| = \cdots = |\lambda_k|$;*

and such that $L^n = Q^n + \sum_{j=1}^{k} \lambda_j^n P_j$ for any $n \ge 1$.

Remark 3.1.3. This definition is slightly more restrictive than usual. For instance, [65] allows the index of each λ_j to be larger than 1, requiring only that $P_j(E)$ be finite-dimensional; moreover, the λ_j need not have norm 1. All the situations we deal with fit into the definition given above.

3.1.4 Spectral Gap and Mixing Properties

Let us go back to the dynamical setting of Subsection 3.1.2. That is, we consider the transfer operator $\mathcal{L} : L^1(m) \to L^1(m)$ associated to a transformation $f : M \to M$. Moreover, $E \subset L^1(m)$ and $\|\cdot\|_E$ is a complete norm in E satisfying conditions (O1)-(O2).

Definition 3.1.5. *An f-invariant probability measure ν is* mixing *for f if, given any ϕ and ψ in $L^2(\nu)$,*

$$\int \phi(\psi \circ f^n)\, d\nu \to \int \phi\, d\nu \int \psi\, d\nu \quad \textit{as } n \to \infty. \tag{3.11}$$

That is, ν is mixing if and only if, for any L^2 functions ϕ and ψ, the correlations $C_n(\phi, \psi)$ with respect to ν decay to zero as $n \to \infty$. Every mixing measure is ergodic: if $A \subset M$ is f-invariant, taking $\phi = \psi = \mathcal{X}_A$ in (3.11) gives

$$\nu(A) = \int \mathcal{X}_A(\mathcal{X}_A \circ f^n)\, d\nu \to \int \mathcal{X}_A\, d\nu \int \mathcal{X}_A\, d\nu = \nu(A)^2;$$

this implies that $\nu(A)$ is either 0 or 1, and so ν is ergodic.

The following observation will be useful in the sequel.

Lemma 3.1.4. *If ν is mixing then, given $p, q \in [1, \infty]$ with $1/p + 1/q = 1$, property (3.11) holds for any $\phi \in L^p(\nu)$ and $\psi \in L^q(\nu)$.*

Proof. Let us prove this when $p \le 2 \le q$, the case $q \le 2 \le p$ is similar. Fix any sequence ξ_l of functions in $L^2(\nu)$ converging to ϕ in the norm $\|\cdot\|_{p,\nu}$. By the Hölder inequality,

$$\left|\int \phi(\psi\circ f^n)\,d\nu - \int \xi_l(\psi\circ f^n)\,d\nu\right| \le \|\phi-\xi_l\|_{p,\nu}\|\psi\circ f^n\|_{q,\nu} = \|\phi-\xi_l\|_{p,\nu}\|\psi\|_{q,\nu}\,.$$

The last step uses that ν is f-invariant. We also have

$$\left|\int \phi\,d\nu \int \psi\,d\nu - \int \xi_l\,d\nu \int \psi\,d\nu\right| \le \|\phi-\xi_l\|_{1,\nu}\|\psi\|_{1,\nu} \le \|\phi-\xi_l\|_{p,\nu}\|\psi\|_{q,\nu}\,.$$

Therefore, $|\int \phi(\psi\circ f^n)\,d\nu - \int\phi\,d\nu\int\psi\,d\nu|$ is less than

$$2\|\phi-\xi_l\|_{p,\nu}\|\psi\|_{q,\nu} + \left|\int \xi_l(\psi\circ f^n)\,d\nu - \int \xi_l\,d\nu\int\psi\,d\nu\right|,$$

for every $l \ge 1$ and $n \ge 1$. Fixing l sufficiently large, one can make the first term arbitrarily small. Then, by (3.11) applied to ξ_l and ψ, the last term is also small for any large n. In this way, we conclude that $\int\phi(\psi\circ f^n)\,d\nu$ converges to $\int\phi\,d\nu\int\psi\,d\nu$ as $n\to\infty$. □

Suppose $\varphi_0 \in E$ is a nonnegative fixed point of the transfer operator $\mathcal{L}$, normalized so that $\int\varphi_0\,dm = 1$. Let $\mu = \varphi_0 m$, and E_0 be the subspace of E formed by the functions of the form $\varphi = \phi\varphi_0$.

Lemma 3.1.5. *E_0 is invariant under $\mathcal{L}$, that is, $\mathcal{L}(E_0) \subset E_0$.*

Proof. Let $\varphi = \phi\varphi_0 \in E_0$. Then $\mathcal{L}(\varphi)m = f_*(\varphi m) = f_*(\phi\mu)$. It is easy to see that the measure $f_*(\phi\mu)$ is absolutely continuous with respect to μ: if A is a measurable subset with $\mu(A) = 0$ then $\mu(f^{-1}(A)) = 0$, because μ is f-invariant, and so

$$f_*(\phi\mu)(A) = (\phi\mu)(f^{-1}(A)) = \int_{f^{-1}(A)} \phi\,d\mu = 0.$$

Therefore, there exists ξ such that $\mathcal{L}(\varphi)m = f_*(\phi\mu) = \xi\mu = \xi\varphi_0 m$. This means that $\mathcal{L}(\varphi) = \xi\varphi_0$, and so $\mathcal{L}(\varphi)$ is in E_0. □

Let $\bar{E}_0$ be the closure of E_0 in E. Then $(\bar{E}_0, \|\cdot\|_E)$ is a Banach space satisfying (O1)-(O2): Lemma 3.1.5 implies that $\mathcal{L}(\bar{E}_0) \subset \bar{E}_0$, and all the other conditions follow immediately from $\bar{E}_0 \subset E$.

Proposition 3.1.2. *If $\mathcal{L} : E \to E$ is quasi-compact and $\mu = \varphi_0 m$ is mixing then $\mathcal{L} : \bar{E}_0 \to \bar{E}_0$ has a spectral gap.*

Proof. Let $\pi_0 = \pi(\mathcal{L}, \Sigma_0)$ and $\pi_j = \pi(\mathcal{L}, \{\lambda_j\})$, for $1 \le j \le k$, be the spectral projections. By assumption, 1 is an eigenvalue of $\mathcal{L}$, with eigenfunction φ_0.

So, renumbering the λ_j if necessary, we may suppose $\lambda_1 = 1$. We are going to show that, given any $\varphi \in E_0$,

$$\pi_1(\varphi) = \varphi_0 \int \varphi \, dm \quad \text{and} \quad \pi_j(\varphi) = 0 \quad \text{for } 2 \le j \le k. \tag{3.12}$$

Since $\pi_1, \pi_2, \ldots, \pi_k$ are continuous operators, these equalities remain true for any $\varphi \in \bar{E}_0$. In particular, $\pi_1(\bar{E}_0) = \varphi_0 \mathbb{C} \subset \bar{E}_0$. Moreover, $\pi_0 = \mathrm{id} - \pi_1$ on $\bar{E}_0$, and so $\pi_0(\bar{E}_0)$ is also contained in $\bar{E}_0$. Then $\bar{E}_0 = \pi_0(\bar{E}_0) \oplus \pi_1(\bar{E}_0)$. Recall that $\pi_0(\bar{E}_0)$ and $\pi_1(\bar{E}_0)$ are invariant under $\mathcal{L}$, because the spectral projections commute with the operator. Then

$$\mathrm{spec}(\mathcal{L} \mid \bar{E}_0) = \mathrm{spec}(\mathcal{L} \mid \pi_0(\bar{E}_0)) \oplus \mathrm{spec}(\mathcal{L} \mid \pi_1(\bar{E}_0)).$$

Clearly, $\mathrm{spec}(\mathcal{L} \mid \pi_1(\bar{E}_0)) = \{1\}$. Moreover, given $1 > \tau_1 > \rho(\mathcal{L} \mid \pi_0(E))$,

$$\|\mathcal{L}^n \mid \pi_0(\bar{E}_0)\|_E \le \|\mathcal{L}^n \mid \pi_0(E)\|_E \le \tau_1^n$$

for every large n, and so $\mathrm{spec}(\mathcal{L} \mid \pi_0(\bar{E}_0))$ is contained in the disk of radius τ_1. Therefore, $\mathcal{L} : \bar{E}_0 \to \bar{E}_0$ has a spectral gap, as claimed. Thus, we have reduced the proposition to the claims in (3.12). In what follows we prove those claims.

Let $\varphi = \phi\varphi_0 \in E_0$ and ψ be an arbitrary element of $L^\infty(m)$. By condition (O2), φ is in $L^p(m) \subset L^1(m)$. Then $\phi \in L^1(\mu)$ and $\psi \in L^\infty(\mu)$, since

$$\|\phi\|_{1,\mu} = \int |\phi|\varphi_0 \, dm = \|\varphi\|_1 \quad \text{and} \quad \|\psi\|_{\infty,\mu} \le \|\psi\|_\infty$$

(because μ is absolutely continuous with respect to m). So, cf. Lemma 3.1.4, the hypothesis that μ is mixing implies that

$$\int \psi \, (\mathcal{L}^n \varphi) \, dm = \int (\psi \circ f^n) \varphi \, dm = \int (\psi \circ f^n) \phi \, d\mu \tag{3.13}$$

converges to

$$\int \psi \, d\mu \int \phi \, d\mu = \int \psi\varphi_0 \, dm \int \varphi \, dm = \int \psi \big(\varphi_0 \int \varphi \, dm\big) \, dm, \tag{3.14}$$

as $n \to \infty$. On the other hand, by quasi-compactness,

$$\int \psi \, (\mathcal{L}^n \varphi) \, dm = \int \psi \mathcal{L}^n\big(\pi_0(\varphi)\big) \, dm + \int \psi \sum_{j=1}^{k} \lambda_j^n \pi_j(\varphi) \, dm. \tag{3.15}$$

Since the spectral radius of $\mathcal{L}$ on $\pi_0(E)$ is less than 1, $\|\mathcal{L}^n\big(\pi_0(\varphi)\big)\|_E$ goes to zero as $n \to \infty$. In view of condition (O2), it follows that $\|\mathcal{L}^n\big(\pi_0(\varphi)\big)\|_p \to 0$, and so $\|\mathcal{L}^n\big(\pi_0(\varphi)\big)\|_1 \to 0$. This implies that the first summand on the right hand side of (3.16) goes to zero when $n \to \infty$. So, putting (3.13)-(3.14) and (3.15) together gives that

$$\int \psi \sum_{j=1}^{k} \lambda_j^n \pi_j(\varphi)\, dm \to \int \psi(\varphi_0 \int \varphi\, dm)\, dm \tag{3.16}$$

as $n \to \infty$, for any $\psi \in L^\infty(m)$. Now we are ready to deduce the statements in (3.12). Fix $2 \le j \le k$, and suppose $\pi_j(\varphi)$ is not zero. Then there is a bounded functional $\Phi_j : L^1(m) \to \mathbb{C}$ such that

$$\Phi_j(\pi_j(\varphi)) = 1 \quad \text{and} \quad \Phi_j(\pi_i(\varphi)) = 0 \quad \text{for } i \neq j.$$

Since the dual space of $L^1(m)$ is isomorphic to $L^\infty(m)$, there is $\psi_j \in L^\infty(m)$ such that $\Phi_j(\xi) = \int \psi_j \xi\, dm$ for any $\xi \in L^1(m)$; see e.g. [**?**, Theorem 6.16]. Taking $\psi = \psi_j$ in (3.16) yields

$$\lambda_j^n = \Phi_j\left(\sum_{i=1}^{k} \lambda_i^n \pi_i(\varphi)\right) \to \Phi_j(\varphi_0 \int \varphi\, dm)$$

as $n \to \infty$. However, λ_j is different from 1 because we are considering $j \ge 2$. So, λ_j^n has no limit as $n \to \infty$. This contradiction proves that $\pi_j(\varphi) = 0$ for any $2 \le j \le k$. Moreover, now (3.16) simply means that

$$\int \psi \pi_1(\varphi)\, dm \to \int \psi(\varphi_0 \int \varphi\, dm)\, dm$$

as $n \to \infty$. Then these two expressions must coincide, for any $\psi \in L^\infty(m)$. This implies that $\pi_1(\varphi) = \varphi_0 \int \varphi\, dm$. So the proof of (3.12) is complete. □

Corollary 3.1.2. *Under the hypotheses of Proposition 3.1.2, for any $\tau_1 > \tau$ there exists $C_1 > 0$ such that*

$$|C_n(\phi, \psi)| = |B_n(\phi\varphi_0, \psi)| \le C_1 \tau_1^n \|\phi\varphi_0\|_E \|\psi\|_q$$

for any $n \ge 1$, $\phi \in F_0$, and $\psi \in L^q(m)$.

Proof. Proposition 3.1.2 states that $\mathcal{L} : \bar{E}_0 \to \bar{E}_0$ has a spectral gap. The conclusions follow from Proposition 3.1.1 and Corollary 3.1.1, applied with $\bar{E}_0$ in the place of E. By definition, $\phi \in F_0$ if and only if $\phi\varphi_0 \in E_0$. □

Next, we are going to prove a partial converse to this last result: assuming F_0 is dense in $L^1(\mu)$, mixing is necessary for the conclusion of Corollary 3.1.2. In fact, μ must have an even stronger property, called exactness.

Let $\mathcal{F}$ be the Borel σ-algebra of M, and $\mathcal{F}_n = f^{-n}(\mathcal{F})$, for $n \ge 0$. Since f is a measurable map, the $\mathcal{F}_n$ form a nonincreasing sequence: $\mathcal{F}_{n-1} \supset \mathcal{F}_n$ for every $n \ge 1$.

Definition 3.1.6. *An f-invariant probability measure ν is* exact *for f if the σ-algebra*

$$\mathcal{F}_\infty = \bigcap_{n=0}^{\infty} \mathcal{F}_n$$

is ν-trivial: every element of $\mathcal{F}_\infty$ has either zero or full ν-measure.

Remark 3.1.4. Equivalently, ν is exact if every $\mathcal{F}_\infty$-measurable function is constant ν-almost everywhere. By definition, a function ξ is $\mathcal{F}_\infty$-measurable if it is $\mathcal{F}_n$-measurable for every $n \geq 1$. Moreover, ξ is $\mathcal{F}_n$-measurable if and only if there exists some $\mathcal{F}$-measurable function ξ_n such that $\xi = \xi_n \circ f^n$. Let us explain why. The 'if' part is obvious. To prove the 'only if' part, suppose that ξ is $\mathcal{F}_n$-measurable. Then for each $z \in \mathbb{C}$ there exists $A_z \in \mathcal{F}$ such that $\xi^{-1}(z) = f^{-n}(A_z)$. In general, A_z is not unique. However,

$$B_z = A_z \cap f^n(M) = f^n(\xi^{-1}(z))$$

is uniquely determined. Moreover, $\xi^{-1}(z) = f^{-n}(B_z)$. The sets B_z are two-by-two disjoint and they cover $f^n(M)$. Set $\xi_n \mid B_z \equiv z$ for every $z \in \mathbb{C}$. This defines some function ξ_n on $f^n(M)$. The definition on the complement of $f^n(M)$ is irrelevant in this context, because $M \setminus f^n(M)$ has zero ν-measure (since its pre-image under f^n is empty). Given any measurable subset E of $\mathbb{C}$, take $A_E \in \mathcal{F}$ such that $\xi^{-1}(E) = f^{-n}(A_E)$. Then

$$\xi_n^{-1}(E) = \bigcup_{z \in E} B_z = \bigcup_{z \in E} f^n(\xi^{-1}(z)) = f^n(\xi^{-1}(E)) = A_E \cap f^n(M).$$

So, ξ_n is measurable, cf. [?, Section 1.37]. Finally, let $x \in M$ and $z = \xi(x)$. Then $x \in \xi^{-1}(z)$ and so $f^n(x) \in B_z$. Therefore, $\xi_n(f^n(x)) = z = \xi(x)$. This shows that $\xi = \xi_n \circ f^n$.

The following lemma collects a few simple facts about the σ-algebra $\mathcal{F}_\infty$ that will useful later on. Note that Δ is the symmetric difference of two sets.

Lemma 3.1.6. *Suppose $Z_0 \in \mathcal{F}_\infty$ and, for each $n \geq 1$, let $Z_n \in \mathcal{F}$ be such that $Z = f^{-n}(Z_n)$. For every $n \geq 1$,*

(a) $\mu\big(Z_n \Delta f^{-m}(Z_{m+n})\big) = 0$ for every $m \geq 1$;
(b) there exists $\tilde{Z}_n \in \mathcal{F}_\infty$ such that $\mu(Z_n \Delta \tilde{Z}_n) = 0$.

Proof. Let Z_n' be the union of Z_n and $f^{-m}(Z_{m+n})$. Then

$$f^{-n}(Z_n') = f^{-n}(Z_n) \cup f^{-m-n}(Z_{m+n}) = Z_0 \cup Z_0 = Z_0\,.$$

So, as the measure μ is invariant, $\mu(Z_n') = \mu(Z_0)$. Also by invariance, $\mu(Z_n) = \mu(Z_0)$ and $\mu(f^{-m}(Z_{m+n})) = \mu(Z_0)$. It follows that $Z_n' \setminus Z_n$ and $Z_n' \setminus f^{-m}(Z_{m+n})$ have zero μ measure, which is precisely the claim in (a).

To prove part (b) of the lemma, it suffices to take

$$\tilde{Z}_n = \limsup_{m\to\infty} f^{-m}(Z_{m+n}) = \bigcap_{p\geq 1} \bigcup_{m\geq p} f^{-m}(Z_{m+n}).$$

Indeed, (a) implies that $\mu(Z_n \Delta \tilde{Z}_n) = 0$. Moreover, as the unions form a sequence decreasing with p,

$$\tilde{Z}_n = \bigcap_{p\geq k} \bigcup_{m\geq p} f^{-m}(Z_{m+n}) = f^{-k}\big(\bigcap_{p\geq k} \bigcup_{m\geq p} f^{-m+k}(Z_{m+n})\big) \in \mathcal{F}_k$$

for every $k \geq 1$. This proves that $\tilde{Z}_n \in \mathcal{F}_\infty$. □

Let ν be any f-invariant probability measure. For each $n \geq 0$, let $L^2(\mathcal{F}_n) = L^2(M, \mathcal{F}_n, \nu)$ represent the space of $\mathcal{F}_n$-measurable functions in $L^2(\nu) = L^2(M, \mathcal{F}, \nu)$. Note that $L^2(\mathcal{F}_{n-1}) \supset L^2(\mathcal{F}_n)$ for every $n \geq 1$. Moreover, each $L^2(\mathcal{F}_n)$ is a Banach space with respect to the L^2-norm. So, $L^2(\mathcal{F}_n)$ is a closed subspace of $L^2(\mathcal{F}_{n-1})$, for every $n \geq 1$. If ν is exact then $\cap_{n=0}^{\infty} L^2(\mathcal{F}_n)$ contains only constant functions. These observations are used in the proof of the following

Lemma 3.1.7. *Every exact measure is mixing.*

Proof. Let $\phi, \psi \in L^2(\nu)$. The sequence on the left hand side of (3.11) is bounded by $\|\phi\|_{2,\nu}\|\psi\|_{2,\nu}$. So, to prove (3.11) it is enough to show that for any subsequence $(n_j)_j$ such that $\int \phi(\psi \circ f^{n_j})\, d\nu$ converges the limit is $\int \phi\, d\nu \int \psi\, d\nu$. Clearly, $(\psi \circ f^n)_n$ is a bounded sequence in $L^2(\nu)$. Then, up to replacing $(n_j)_j$ by a convenient subsequence, we may suppose that $(\psi \circ f^{n_j})_j$ is weakly convergent to some $\psi_0 \in L^2(\nu)$:

$$\int \xi(\psi \circ f^{n_j})\, d\nu \to \int \xi \psi_0\, d\nu \tag{3.17}$$

for every $\xi \in L^2(\nu)$. Given $n \geq 0$, we have $\psi \circ f^{n_j} \in L^2(\mathcal{F}_{n_j}) \subset L^2(\mathcal{F}_n)$ for every $n_j \geq n$. Then, using (3.17) with ξ in the orthogonal complement of $L^2(\mathcal{F}_n)$, we may conclude that the weak limit ψ_0 is also in $L^2(\mathcal{F}_n)$. It follows that ψ_0 is in the intersection of all the $L^2(\mathcal{F}_n)$, and so it is constant. Then (3.17) may be rewritten as

$$\int \xi(\psi \circ f^{n_j})\, d\nu \to \int \xi\, d\nu \int \psi_0\, d\nu.$$

Choosing $\xi = \phi$ we get the statement we wanted to prove. □

Proposition 3.1.3. *Let ν be an f-invariant measure. Assume there exists a dense subset F of $L^1(\nu)$, and for each $\phi \in F$ there exists a sequence $(K_n(\phi))_n$ converging to zero, such that*

$$\left|\int \phi(\psi \circ f^n)\, d\nu - \int \phi\, d\nu \int \psi\, d\nu\right| \leq K_n(\phi) \sup|\psi| \tag{3.18}$$

for every $\phi \in F$ and any bounded function ψ. Then ν is exact.

Proof. Let $A \in \mathcal{F}_\infty$. By definition, for each $n \geq 1$ there exists a measurable set A_n such that $A = f^{-n}(A_n)$. Then $\mathcal{X}_A = \mathcal{X}_{A_n} \circ f^n$, and $\nu(A_n) = \nu(A)$ for every $n \geq 1$, as ν is invariant under f. Then,

$$\begin{aligned}\int \phi(\mathcal{X}_A - \nu(A))\, d\nu &= \int \phi(\mathcal{X}_{A_n} \circ f^n)\, d\nu - \nu(A) \int \phi\, d\nu \\ &= \int \phi(\mathcal{X}_{A_n} \circ f^n)\, d\nu - \int \mathcal{X}_{A_n}\, d\nu \int \phi\, d\nu.\end{aligned}$$

So, using the hypothesis (3.18),

$$\left| \int \phi(\mathcal{X}_A - \nu(A))\, d\nu \right| \leq K_n(\phi) \sup |\mathcal{X}_{A_n}| \leq K_n(\phi)$$

for every $n \geq 1$ and $\phi \in F$. The term on the left does not depende on n, whereas the one on the right goes to zero when $n \to \infty$. So, this proves that

$$\int \phi(\mathcal{X}_A - \nu(A))\, d\nu = 0$$

for every $\phi \in F$. Since F is assumed to be dense in $L^1(\nu)$, this equality remains valid for every $\phi \in L^1(\mu)$. Taking $\phi = \mathcal{X}_A - \nu(A)$ we conclude that $\mathcal{X}_A = \nu(A)$ ν-almost everywhere. In other words, $\nu(A)$ is either zero or 1. Thus, ν is exact. □

The next corollary summarizes most of Subsection 3.1.4. The notations are the same as before.

Corollary 3.1.3. *Assume the transfer operator $\mathcal{L} : E \to E$ is quasi-compact, and F_0 is dense in $L^q(\mu)$. Then the following conditions are equivalent:*

(a) $\mu = \varphi_0 m$ is exact;
(b) $\mu = \varphi_0 m$ is mixing;
(c) $C_n(\phi, \psi) \leq C_1 \tau_1^n \|\phi\varphi_0\|_E \|\psi\|_q$ for every $n \geq 1$, $\phi \in F_0$, and $\psi \in L^q(m)$.

Proof. Lemma 3.1.7 contains (a) ⇒ (b). Proposition 3.1.2 and Corollary 3.1.2 give (b) ⇒ (c). Finally, (c) ⇒ (a) follows from Proposition 3.1.3, taking $F = F_0$ and $K_n(\phi) = C_1 \tau_1^n \|\phi\varphi_0\|$, and noting that $\|\psi\|_q \leq \sup |\psi|$ for any ψ and $1 \leq q \leq \infty$. □

3.2 Projective Metrics

Here we describe another approach for proving decay of correlations from spectral properties of the transfer operator. It is based on the concept, due to G. Birkhoff [17], of *projective metric* associated to a convex cone in a vector space. It was first applied to a dynamical context in [43], and substantially developed in [70]. All the results in this section are for general linear maps on vector spaces, they are not specific to transfer operators. The main one is Proposition 3.2.2: any linear operator that leaves a cone strictly invariant must contract the corresponding projective metric.

3.2.1 Invariant Cones

Let E be a vector space. A *cone* in E is a subset $C \subset E \setminus \{0\}$ satisfying

$$v \in C \text{ and } t > 0 \quad \Rightarrow \quad tv \in C.$$

The cone C is *convex* if $t_1 v_1 + t_2 v_2 \in C$ for any $v_1, v_2 \in C$ and any $t_1, t_2 > 0$. The *closure* $\widetilde{C}$ of C is the set of vectors $w \in E$ for which there is $v \in C$ and a sequence of positive real numbers $(t_n)_n \to 0$ such that $(w + t_n v) \in C$ for all $n \geq 1$. We call the cone *proper* if

$$\widetilde{C} \cap (-\widetilde{C}) = \{0\}, \tag{3.19}$$

that is, $w \in \widetilde{C}$ and $-w \in \widetilde{C}$ only for $w = 0$ (clearly, $0 \in \widetilde{C}$ for any cone C). In what follows, cones will always be convex and proper.

Given $v_1, v_2 \in C$, we define

$$\alpha(v_1, v_2) = \sup\{t > 0 : v_2 - tv_1 \in C\}$$
$$\text{and} \quad \beta(v_1, v_2) = \inf\{s > 0 : sv_1 - v_2 \in C\}.$$

Their geometric meaning is illustrated in Figure 3.1. If $v_2 - tv_1 \notin C$ for all $t > 0$ then, by convention, $\alpha(v_1, v_2) = 0$. Similarly, we set $\beta(v_1, v_2) = +\infty$ if $sv_1 - v_2 \notin C$ for all $s > 0$.

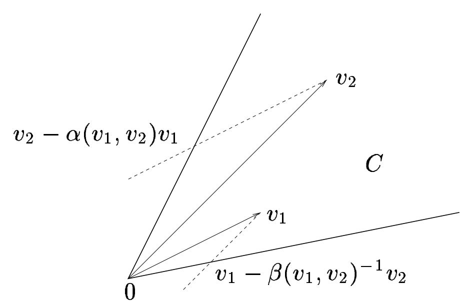

Fig. 3.1. Defining the projective metric of a cone C

Observe that $\alpha(v_1, v_2)$ is always finite. Indeed, $\alpha(v_1, v_2) = +\infty$ would mean that there exists $(t_n)_n \to +\infty$ such that $v_2 - t_n v_1 \in C$ for all $n \geq 1$. Then $s_n = 1/t_n$ would define a sequence of positive numbers $(s_n) \to 0$ such that $s_n v_2 - v_1 \in C$ for all $n \geq 1$. This would give $-v_1 \in \widetilde{C}$, contradicting (3.19). A similar argument proves that $\beta(v_1, v_2) > 0$ for all $v_1, v_2 \in C$: $\beta(v_1, v_2) = 0$ would imply $-v_2 \in \widetilde{C}$. These remarks ensure that the following notion is well defined:

Definition 3.2.1. *Let C be a proper convex cone. Given $v_1, v_2 \in C$, define*

$$\theta(v_1, v_2) = \log \frac{\beta(v_1, v_2)}{\alpha(v_1, v_2)},$$

with $\theta(v_1, v_2) = +\infty$ if either $\alpha(v_1, v_2) = 0$ or $\beta(v_1, v_2) = +\infty$.

We call $\theta(\cdot, \cdot)$ the *projective metric* associated to C. This terminology is justified by the next proposition which, roughly speaking, says that $\theta(\cdot, \cdot)$ induces a distance in the projective quotient of C. The projective quotient of an arbitrary set $A \subset E$ is defined as the quotient space of A under the equivalence relation

$$v_1 \sim v_2 \quad \Leftrightarrow \quad \text{there exists } t > 0 \text{ such that } v_1 = tv_2 .$$

Proposition 3.2.1. *If C is a proper convex cone then*

(a) $\theta(v_1, v_2) = \theta(v_2, v_1)$ *for all* $v_1, v_2 \in C$
(b) $\theta(v_1, v_2) + \theta(v_2, v_3) \geq \theta(v_1, v_3)$ *for all* $v_1, v_2, v_3 \in C$
(c) $\theta(v_1, v_2) \geq 0$ *for all* $v_1, v_2 \in C$
(d) $\theta(v_1, v_2) = 0$ *if and only if there exists* $t > 0$ *such that* $v_1 = tv_2$
(e) $\theta(t_1 v_2, t_2 v_2) = \theta(v_1, v_2)$ *for all* $v_1, v_2 \in C$ *and* $t_1, t_2 > 0$.

Proof. If $\alpha(v_2, v_1) > 0$ then

$$\begin{aligned}\alpha(v_2, v_1) &= \sup\{t > 0 : v_1 - tv_2 \in C\} = \sup\{t > 0 : \frac{1}{t} v_1 - v_2 \in C\} \\ &= \big(\inf\{s > 0 : sv_1 - v_2 \in C\}\big)^{-1} = \beta(v_1, v_2)^{-1}.\end{aligned}$$

On the other hand, if $\alpha(v_2, v_1) = 0$ then

$$v_1 - tv_2 \notin C \text{ for all } t > 0 \quad \Leftrightarrow \quad sv_1 - v_2 \notin C \text{ for all } s > 0,$$

that is, $\beta(v_1, v_2) = +\infty$. Thus $\alpha(v_2, v_1) = \beta(v_1, v_2)^{-1}$ also in this case. Then, interchanging the roles of v_1 and v_2, we also have $\beta(v_2, v_1) = \alpha(v_1, v_2)^{-1}$ for every $v_1, v_2 \in C$. Now part (a) of the proposition follows immediately.

Next, we claim that $\alpha(v_1, v_2)\alpha(v_2, v_3) \leq \alpha(v_1, v_3)$ for all $v_1, v_2, v_3 \in C$. This is obvious if $\alpha(v_1, v_2) = 0$ or $\alpha(v_2, v_3) = 0$, so we may suppose that $\alpha(v_1, v_2) > 0$ and $\alpha(v_2, v_3) > 0$. Then, by definition, there are sequences of positive numbers $(r_n)_n \nearrow \alpha(v_1, v_2)$ and $(s_n)_n \nearrow \alpha(v_2, v_3)$ such that

$$v_2 - r_n v_1 \in C \quad \text{and} \quad v_3 - s_n v_2 \in C \quad \text{for all } n \geq 1.$$

By the convexity of C, we have $v_3 - s_n r_n v_1 \in C$, and so $s_n r_n \leq \alpha(v_1, v_3)$, for all $n \geq 1$. Passing to the limit as $n \to +\infty$ we get the claim. Moreover, a similar argument proves that $\beta(v_1, v_2)\beta(v_2, v_3) \leq \beta(v_1, v_3)$ for all $v_1, v_2, v_3 \in C$. Part (b) of the proposition is an easy consequence.

Part (c) just means that $\alpha(v_1, v_2) \leq \beta(v_1, v_2)$ for all $v_1, v_2 \in C$. To prove this, let t and s be such that $v_2 - tv_1 \in C$ and $sv_1 - v_2 \in C$. Then, by convexity, $(s-t)v_1 \in C$. If $s - t$ were negative, we would have that $-v_1 \in C$, contradicting (3.19). So, we must have $s \geq t$, for any t and s as above. This means that $\alpha(v_1, v_2) \leq \beta(v_1, v_2)$, as we claimed.

Let $v_1, v_2 \in C$ be such that $\theta(v_1, v_2) = 0$. Then $\alpha(v_1, v_2) = \beta(v_1, v_2) = \gamma$ for some $\gamma \in (0, +\infty)$. Then there are sequences $(t_n)_n \nearrow \gamma$ and $(s_n)_n \searrow \gamma$ of positive numbers such that

$$v_2 - t_n v_1 \in C \quad \text{and} \quad s_n v_1 - v_2 \in C, \qquad \text{for all } n \geq 1.$$

Writing $v_2 - t_n v_1 = (v_2 - \gamma v_1) + (\gamma - t_n)v_1$, we conclude that $v_2 - \gamma v_1$ is in the closure $\widetilde{C}$ of C. Analogously, $\gamma v_1 - v_2 \in \widetilde{C}$. So, in view of (3.19), $v_2 - \gamma v_1 = 0$. This proves statement (d) in the proposition.

Finally, let $t_1, t_2 > 0$ and $v_1, v_2 \in C$. From the definitions of $\alpha(\cdot,\cdot)$ and $\beta(\cdot,\cdot)$, we get

$$\alpha(t_1 v_1, t_2 v_2) = \frac{t_2}{t_1}\,\alpha(v_1, v_2) \quad \text{and} \quad \beta(t_1 v_1, t_2 v_2) = \frac{t_2}{t_1}\,\beta(v_1, v_2),$$

and so $\theta(t_1 v_2, t_2 v_2) = \theta(v_1, v_2)$. □

Observe that the projective metric depends monotonically on the corresponding cone. Indeed, let C_1 and C_2 be two proper convex cones in E with $C_1 \subset C_2$, and let $\alpha_i(\cdot,\cdot)$, $\beta_i(\cdot,\cdot)$, and $\theta_i(\cdot,\cdot)$, $i = 1, 2$, be the corresponding objects, as defined above. Clearly, given any $v_1, v_2 \in C_1 \subset C_2$,

$$\alpha_1(v_1, v_2) \leq \alpha_2(v_1, v_2) \quad \text{and} \quad \beta_1(v_1, v_2) \geq \beta_2(v_1, v_2)$$

and so $\theta_1(v_1, v_2) \geq \theta_2(v_1, v_2)$.

More generally, let E_1, E_2 be two vector spaces and $C_i \subset E_i$, $i = 1, 2$, be proper convex cones. Let $L\colon E_1 \to E_2$ be a linear operator and assume that $L(C_1) \subset C_2$. Then

$$\begin{aligned}
\alpha_1(v_1, v_2) &= \sup\{t > 0 : v_2 - t\,v_1 \in C_1\} \\
&\leq \sup\{t > 0 : L(v_2 - t\,v_1) \in C_2\} \\
&= \sup\{t > 0 : L(v_2) - tL(v_1) \in C_1\} = \alpha_2(L(v_1), L(v_2))
\end{aligned}$$

and, analogously, $\beta_1(v_1, v_2) \geq \beta_2(L(v_1), L(v_2))$. Therefore,

$$\theta_1(v_1, v_2) \geq \theta_2(L(v_1), L(v_2)) \quad \text{for all } v_1, v_2 \in C_1. \tag{3.20}$$

In general, the inequality in (3.20) is not strict. However, according to the next proposition this is necessarily so if $L(C_1)$ has finite θ_2-diameter inside C_2: then L is a uniform contraction relative to the projective metrics θ_1, θ_2. Recall that

$$\tanh z = \frac{1 - e^{-2z}}{1 + e^{-2z}} < 1 \quad \text{for every } z \in \mathbb{R}.$$

Proposition 3.2.2. *Let $C_1 \subset E_1$, $C_2 \subset E_2$, and $L : E_1 \to E_2$ be as before. Suppose $D = \sup\{\theta_2(L(v_1), L(v_2)) : v_1, v_2 \in C_1\}$ is finite. Then*

$$\theta_2(L(v_1), L(v_2)) \le \tanh\left(\frac{D}{4}\right) \theta_1(v_1, v_2) \quad \textit{for all } v_1, v_2 \in C_1.$$

Proof. Let $v_1, v_2 \in C_1$. We may suppose $\alpha_1(v_1, v_2) > 0$ and $\beta_1(v_1, v_2) < +\infty$ for otherwise $\theta_1(v_1, v_2) = +\infty$, and there is nothing to prove. Then there are sequences of positive numbers $(t_n)_n \nearrow \alpha_1(v_1, v_2)$ and $(s_n)_n \searrow \beta_1(v_1, v_2)$, such that

$$v_2 - t_n\, v_1 \in C_1 \quad \text{and} \quad s_n\, v_1 - v_2 \in C_1\,.$$

In particular, $\theta_2(L(v_2 - t_n v_1), L(s_n v_1 - v_2)) \le D$ for all $n \ge 1$. Fix any $D_0 > D$. Then we can choose positive numbers T_n and S_n such that

$$\begin{aligned} L(s_n\, v_1 - v_2) - T_n L(v_2 - t_n v_1) \in C_2\,, \\ S_n\, L(v_2 - t_n v_1) - L(s_n\, v_1 - v_2) \in C_2\,, \end{aligned} \tag{3.21}$$

and $\log(S_n/T_n) \le D_0$ for every $n \ge 1$. The first part of (3.21) gives

$$(s_n + t_n T_n) L(v_1) - (1 + T_n) L(v_2) \in C_2$$

which, by the definition of $\beta_2(\cdot, \cdot)$, implies

$$\beta_2(L(v_1), L(v_2)) \le \frac{s_n + t_n T_n}{1 + T_n}\,.$$

Analogously, the second statement in (3.21) gives

$$\alpha_2(L(v_1), L(v_2)) \ge \frac{s_n + t_n S_n}{1 + S_n}.$$

So, $\theta_2(L(v_1), L(v_2))$ can not exceed

$$\log\left(\frac{s_n + t_n T_n}{1 + T_n} \cdot \frac{1 + S_n}{s_n + t_n S_n}\right) = \log\left(\frac{s_n/t_n + T_n}{1 + T_n} \cdot \frac{1 + S_n}{s_n/t_n + S_n}\right)\,.$$

The last term can be rewritten as

$$\begin{aligned} \log\left(\frac{s_n}{t_n} + T_n\right) - \log(1 + T_n) - \log\left(\frac{s_n}{t_n} + S_n\right) + \log(1 + S_n) = \\ = \int_0^{\log(s_n/t_n)} \left(\frac{e^x\, dx}{e^x + T_n} - \frac{e^x\, dx}{e^x + S_n}\right), \end{aligned}$$

and this is not larger than

$$\sup_{x>0} \frac{e^x (S_n - T_n)}{(e^x + T_n)(e^x + S_n)} \log\left(\frac{s_n}{t_n}\right)\,.$$

Now we use the following elementary facts:

$$\sup_{y>0} \frac{y(S_n - T_n)}{(y+T_n)(y+S_n)} = \frac{1-\sqrt{T_n/S_n}}{1+\sqrt{T_n/S_n}} \le \frac{1-e^{-D_0/2}}{1+e^{-D_0/2}} = \tanh\frac{D_0}{4}.$$

Indeed, the supremum is attained at $y = \sqrt{S_n T_n}$, and the inequality is a consequence of $S_n/T_n \le e^{D_0}$. This shows that

$$\theta_2(L(v_1), L(v_2)) \le \tanh\left(\frac{D_0}{4}\right) \log\left(\frac{s_n}{t_n}\right).$$

Note that $\theta(v_1, v_2) = \lim_n \log(s_n/t_n)$, because of our choice of s_n and t_n. So, passing to the limit as $n \to \infty$, and then making $D_0 \to D$, completes the proof. □

3.2.2 Examples

Example 3.2.1. Let $E = \mathbb{R}^2$ and $C = \{(x,y) : y > |x|\}$. Clearly C is a convex cone, and $\widetilde{C} = \{(x,y) : y \ge |x|\}$, so that C is also proper. Its projective quotient is naturally identified with the interval $(-1,1)$ through $(x,1) \sim x$. Given $-1 < x_1 \le x_2 < 1$ we have

$$\begin{aligned}\alpha((x_1,1),(x_2,1)) &= \sup\{t>0 : (x_2,1) - t(x_1,1) \in C\}\\ &= \sup\{t>0 : 1-t > x_2 - tx_1\} = \frac{1-x_2}{1-x_1}\\ \text{and}\quad \beta((x_1,1),(x_2,1)) &= \frac{x_2+1}{x_1+1}.\end{aligned}$$

Hence $\theta((x_1,1),(x_2,1)) = \log R(-1, x_1, x_2, 1)$, where R denotes the *cross-ratio* of four points $a < b \le c < d$ in the real line

$$R(z_1, z_2, z_3, z_4) = \frac{z_3 - z_1}{z_2 - z_1} \cdot \frac{z_4 - z_2}{z_4 - z_3}.$$

It is well known that $d_P(x_1, x_2) = \log R(-1, x_1, x_2, 1)$ is an expression for the Poincaré metric on the interval $(-1,1)$, which is defined as the restriction to $(-1,1)$ of the Poincaré metric on the unit disk $\mathbb{D} = \{x+yi \in \mathbb{C} : x^2+y^2 < 1\}$ in the complex plane. We have shown that the metric induced by θ on the projective quotient of C coincides with d_P, up to the identification above.

There is another way to see this, without actually computing θ: it suffices to check that the metric induced by it on the projective quotient of C is preserved by any Möbius transformation

$$\ell : x \mapsto \frac{ax+b}{cx+d}, \qquad ad - bc \ne 0,$$

that maps $(-1,1)$ onto itself, as this last property characterizes the Poincaré metric (up to a constant factor). To do this, consider the linear isomorphism L of $\mathbb{R}^2$ defined by

$$L(x,y) = (ax+by, cx+dy).$$

The assumption that $\ell\big((-1,1)\big) = (-1,1)$ means that L maps the cone C onto itself. Then, by (3.20) applied both to L and to its inverse, L is an isometry for θ. On the other hand, ℓ corresponds to the map induced by L on the projective quotient. Therefore, ℓ does preserve the metric induced by θ on that projective quotient.

Moreover, similar arguments can applied to the cone

$$C = \{(x,y,z) \in \mathbb{R}^3 : z > \sqrt{x^2+y^2}\},$$

whose projective quotient is naturally identified with the unit disk $\mathbb{D}$, showing that the projective metric associated to C corresponds to the Poincaré metric in $\mathbb{D}$. We leave it to the reader to carry out the details.

Example 3.2.2. Let X be a compact metric space and $E = C^0(X)$ be the space of continuous real-valued functions defined on X. Take

$$C = \{\varphi \in E : \varphi(x) > 0 \text{ for all } x \in X\}.$$

Then C is a proper convex cone, its closure $\widetilde{C}$ being the cone of continuous functions with $\varphi(x) \geq 0$ for all $x \in X$. For any $\varphi_1, \varphi_2 \in C$

$$\begin{aligned} \alpha(\varphi_1,\varphi_2) &= \sup\{t > 0 : (\varphi_2 - t\,\varphi_1)(x) > 0 \text{ for all } x \in X\} \\ &= \inf\left\{\frac{\varphi_2}{\varphi_1}(x) : x \in X\right\} \\ \text{and} \quad \beta(\varphi_1,\varphi_2) &= \sup\left\{\frac{\varphi_2}{\varphi_1}(x) : x \in X\right\}. \end{aligned}$$

Therefore

$$\theta(\varphi_1,\varphi_2) = \log\frac{\sup(\varphi_2/\varphi_1)}{\inf(\varphi_2/\varphi_1)} = \log\sup\left\{\frac{\varphi_2(x)\varphi_1(y)}{\varphi_1(x)\varphi_2(y)} : x,y \in X\right\}. \tag{3.22}$$

Let us prove that the projective metric constructed in this class of examples is complete, as this fact will be useful later.

Proposition 3.2.3. *Let C and θ be as in Example 3.2.2. Any θ-Cauchy sequence $(\varphi_n)_n$ in C is θ-convergent in C. Moreover, if it is taken so that $\int \varphi_n\,d\nu = 1$, for all $n \geq 1$ and for some probability measure ν in X, then $(\varphi_n)_n$ is also uniformly convergent.*

Proof. Let $(\varphi_n)_n$ be a θ-Cauchy sequence: given any $\varepsilon > 0$, there exists $N \geq 1$ such that for all $m,n \geq N$

$$\theta(\varphi_n, \varphi_m) = \log \frac{\sup(\varphi_n/\varphi_m)}{\inf(\varphi_n/\varphi_m)} \le \varepsilon. \tag{3.23}$$

Up to dropping a finite number of terms, if necessary, we may suppose that $\theta(\varphi_n, \varphi_m) \le 1$ for all $m, n \ge 1$. Then, in particular,

$$\frac{1}{e} \le \frac{\varphi_n(x)\varphi_1(y)}{\varphi_n(y)\varphi_1(x)} \le e \quad \text{for all } x, y \in X \text{ and } n \ge 1.$$

As a consequence,

$$\frac{1}{R} \le \frac{\varphi_n(x)}{\varphi_n(y)} \le R \quad \text{for all } x, y \in X \text{ and } n \ge 1, \tag{3.24}$$

where $R = e \sup\{\varphi_1(s)/\varphi_1(t) : s, t \in X\}$. Let ν be any probability measure in X. Up to replacing each φ_n by a convenient multiple $t_n\varphi_n$, $t_n > 0$, we may suppose that $\int \varphi_n \, d\nu = 1$ for all $n \ge 1$. This implies that

$$\inf \varphi_n \le 1 \le \sup \varphi_n \quad \text{and} \quad \inf \frac{\varphi_n}{\varphi_m} \le 1 \le \sup \frac{\varphi_n}{\varphi_m} \tag{3.25}$$

for all $n, m \ge 1$. So, on the one hand, (3.24) implies

$$\frac{1}{R} \le \varphi_n(x) \le R \quad \text{for all } x \in X \text{ and } n \ge 1.$$

On the other hand, (3.23) and the second part of (3.25) give

$$e^{-\varepsilon} \le \inf \frac{\varphi_n}{\varphi_m} \le 1 \le \sup \frac{\varphi_n}{\varphi_m} \le e^{\varepsilon} \tag{3.26}$$

for all $m, n \ge N$. It follows that

$$\sup |\varphi_m - \varphi_n| \le \sup \varphi_m \sup \left| \frac{\varphi_n}{\varphi_m} - 1 \right| \le R(e^{\varepsilon} - 1).$$

This shows that $(\varphi_n)_n$ is a Cauchy sequence with respect to the supremum norm, and so it is uniformly convergent. Let φ_0 be the uniform limit. Then $R^{-1} \le \varphi_0(x) \le R$ for all $x \in X$, so that $\varphi_0 \in C$. Passing to the limit as $m \to \infty$ in (3.26), we get

$$e^{-\varepsilon} \le \inf \frac{\varphi_n}{\varphi_0} \le 1 \le \sup \frac{\varphi_n}{\varphi_0} \le e^{\varepsilon}.$$

for all $n \ge N$. This proves that both $\sup(\varphi_n/\varphi_0)$ and $\inf(\varphi_n/\varphi_0)$ converge to 1. Consequently, $\theta(\varphi_n, \varphi_0)$ converges to zero as $n \to \infty$. In other words, $(\varphi_n)_n$ is θ-convergent to φ_0. □

Remark 3.2.1. The normalization condition in the statement of the proposition can be replaced by $\sup \varphi_n = 1$ for all $n \ge 1$, or else $\inf \varphi_n = 1$ for all $n \ge 1$. This is because (3.25) remains valid in either case.

Projective metrics need nod be complete, in general:

Example 3.2.3. Let X be a compact manifold, C be the cone of continuous positive functions on X, and θ be the corresponding projective metric. Let C_1 be the subset of functions $\varphi \in C$ that are differentiable. It is easy to see that C_1 is a proper convex cone, and its projective metric is given by the same formal expression (3.22) as θ. In other words, θ_1 is the restriction of θ to C_1. Let φ_0 be some positive function on X which is continuous but not differentiable, and let $(\varphi_n)_n$ be a sequence of positive differentiable functions converging uniformly to φ_0. Then $\sup(\varphi_n/\varphi_0)$ and $\inf(\varphi_n/\varphi_0)$ converge to 1 as $n \to \infty$, and so (φ_n) is θ-convergent to φ_0 in C_0. In particular, (φ_n) is θ_1-Cauchy, but it can not be θ_1-convergent, since φ_0 is not in the cone C_1.

As another application of the previous methods, we prove Lemma 2.2.1. The context is that of Proposition 2.2.1. For simplicity, we write $g = f^N$.

Proof. One is tempted to use the projective metrics associated to each $C_a(E, x)$, but this does not make much sense, since these cones are not convex nor proper. Instead, we consider the family of cones defined by

$$C(x, e_1) = C_a(E, x) \cap (\mathbb{R}^+ e_1 \oplus E_x^2)$$

for each $e_1 \in E_x^1$ and $x \in \Lambda$. It is easy to check that every $C(e_1, x)$ is a proper convex cone. We denote by θ_{x,e_1} the corresponding projective metric. Since $C_a(E)$ is a forward invariant cone field for g, and the subbundles E^1 and E^2 are also Dg-invariant,

$$\begin{aligned} Dg(x)C(x, e_1) &\subset C_{\theta a}(E, g(x)) \cap (\mathbb{R}^+ Dg(x)e_1 \oplus E_{g(x)}^2) \\ &\subset C(g(x), Dg(x)e_1) \end{aligned} \tag{3.27}$$

for all x and e_1. In fact, we claim that there exists $D_1 > 0$ such that the diameter of $Dg(x)C(x, e_1)$, relative to the projective metric of $C(g(x), Dg(x)e_1)$, is bounded by D_1 for all $x \in \Lambda$ and e_1. First, we establish this claim.

Let v_1, v_2 be generic vectors in $C_{\theta a}(E, y) \cap (\mathbb{R}^+ \xi_1 \oplus E_y^2)$, for any $y \in \Lambda$ and $\xi_1 \in E_y^1$. By (3.27), to prove (i) we only have to show that $\theta_{y,\xi_1}(v_1, v_2) \le D_1$ for some uniform constant D_1. Since projective metrics are not affected by multiplication by positive constants, we may always take $v_1 = \xi_1 + \eta_1$ and $v_2 = \xi_1 + \eta_2$ for some $\eta_1, \eta_2 \in E_y^2$. The angle between E_y^1 and E_y^2 is bounded away from zero, because the former is contained in $C_{\theta a}(E, y)$ whereas the latter is contained in $C_{1/a}(F, y)$. So, there exists some constant $K_2 > 1$, depending only on a and θ, such that for any $\xi \in E_y^1$ and $\eta \in E_y^2$,

$$\xi + \eta \in C_a(E, y) \quad \Rightarrow \quad \|\eta\| \le K_2 \|\xi\| \;\text{ and }\; \frac{1}{K_2}\|\xi\| \le \|\xi + \eta\| \le K_2 \|\xi\|.$$

In particular, $\|v_1\| \le K_2\|\xi_1\| \le K_2^2\|v_2\|$. Take any $0 < t < \delta/K_2^2$, where $\delta > 0$ is a small constant as in (2.10). Using (2.10) we get that $v_2 - tv_1 \in C_a(E, x)$. Moreover,

$$v_2 - tv_1 = (1-t)\xi_1 + (\eta_2 - t\eta_1)$$

is in $\mathbb{R}^+\xi_1 + E_y^2$. In other words, $v_2 - tv_1 \in C(x, e_1)$ for any such t. This shows that $\alpha(v_1, v_2) \ge \delta/K_2^2$, and a similar argument gives $\beta(v_1, v_2) \le K_2^2/\delta$. Therefore, we may take $D_1 = 2\log(K_2^2/\delta)$.

Now let $\lambda_1 = \exp(-D_1)$. By Proposition 3.2.2, the projective diameter of $Dg^n(y)C(y, \xi_1)$ is less that $D_1\lambda_1^{n-1}$ for all $n \ge 1$, $y \in \Lambda$, and $\xi_1 \in E_y^1$. Clearly, if $e_1 \in E_x^1$ and $e_2 \in E_x^2$ are as in Lemma 2.2.1, that is

$$e_1 + e_2 \in Dg^n(x)C_a(E, g^{-n}(x)),$$

then $e_1 + e_2$ is in $Dg^n(x)C(g^{-n}(x), Dg^{-n}(x)\xi_1)$. Of course, so is e_1. Therefore,

$$\theta_{x,e_1}(e_1, e_1 + e_2) \le D_1\lambda_1^{n-1}.$$

We are left to show that this implies $\|e_2\| \le K_1\lambda_1^n\|e_1\|$ for some uniform constant $K_1 > 0$. We argue as follows. Since $\beta_{x,e_1} \ge 1$, the bound on θ_{x,e_1} implies that

$$\alpha_{x,e_1}(e_1, e_1 + e_2) \le \exp(-D_1\lambda_1^{n-1}).$$

The left hand side is less than $1 - D_2\lambda_1^{n-1}$, where the constant D_2 depends only on D. Let $t = 1 - D_2\lambda_1^{n-1}$. Then,

$$(e_1 + e_2) - te_1 = D_2\lambda_1^{n-1}e_1 + e_2$$

is in $C_a(E, x)$. This implies $\|e_2\| \le K_2D_2\lambda_1^{n-1}\|e_1\|$. So, it suffices to take $K_1 = K_2D_2/\lambda_1$. The proof of Lemma (2.2.1) is now complete. □

3.3 Uniformly Expanding Maps

In this section we prove that uniformly expanding maps on compact manifolds, endowed with the corresponding physical measures, have exponential convergence to equilibrium and exponential decay of correlations in the space of Hölder continuous functions:

Theorem 3.3.1. *Let $f\colon M \to M$ be a $C^{1+\nu_0}$ expanding map on a compact connected manifold M, for some $\nu_0 > 0$, and μ be the physical measure of f. Then μ is exact for f and, given any Hölder continuous function $\varphi : M \to \mathbb{R}$,*

1. *$f_*^n(\varphi m)$ converges to $\mu\int\varphi\,dm$ exponentially fast (Proposition 3.3.3);*
2. *$f_*(\varphi\mu)$ converges to $\mu\int\varphi\,d\mu$ exponentially fast (Proposition 3.3.4).*

This result is part of Ruelle's theory of equilibrium states for expanding maps [?]. Our presentation, based on the invariant cone approach that was introduced in Subsection 3.2.1, is taken from [43] and [70, Section 2].

3.3.1 An Invariant Cone

Let us start the proof of Theorem 3.3.1. In what follows $f : M \to M$ is an expanding map as in the statement of the theorem. Rescaling M if necessary, we may suppose that its diameter is less than 1. We denote by m some Riemannian volume on M, normalized so that $m(M) = 1$. Let $\mathcal{L}$ be the transfer operator of f and $E = C^0(M)$ be the space of continuous real functions on M, endowed with the supremum norm $\|\varphi\|_0 = \sup|\varphi|$.

Lemma 3.3.1. *$\mathcal{L}(E) \subset E$ and $\mathcal{L} : E \to E$ is a bounded operator.*

Proof. By Lemma 3.1.2, the transfer operator $\mathcal{L}$ of f is given by

$$(\mathcal{L}\varphi)(y) = \sum_{i=1}^{k} \frac{\varphi(x_i)}{|\det Df(x_i)|}, \tag{3.28}$$

for each $\varphi \in L^1(m)$ and $y \in M$, where $x_1, \ldots, x_k$ are the pre-images of $y \in M$. The number k of pre-images does not depend on y, and each of the x_i varies continuously with y, as we observed in Proposition 1.2.1. Therefore, $\mathcal{L}\varphi$ is a continuous function if φ is. Moreover,

$$\|\mathcal{L}\varphi\|_0 = \sup_{y\in M} \left| \sum_{i=1}^{k} \frac{\varphi(x_i)}{|\det Df(x_i)|} \right| \le \frac{k}{\inf|\det Df|} \|\varphi\|_0,$$

for any $\varphi \in E$. Thus, $\mathcal{L}$ is a bounded operator on E. □

Let $\rho_0 > 0$ be fixed, as in Proposition 1.2.1. Given $a > 0$ and $0 < \nu \le 1$, we define $C(a, \nu) \subset E$ to be the cone of all continuous functions φ such that $\varphi(x) > 0$ for all $x \in M$, and $\log\varphi$ is (a, ν)-Hölder on ρ_0-neighbourhoods. This last condition means that

$$\frac{\varphi(x_1)}{\varphi(x_2)} \le \exp(a\, d(x_1, x_2)^\nu) \quad \text{for any } x_1, x_2 \in M \text{ with } d(x_1, x_2) \le \rho_0.$$

Remark 3.3.1. If a function ϕ is (a, ν)-Hölder on ρ_0-neighbourhoods then it is (Na, ν)-Hölder on the whole manifold M, for some N that depends only on ρ_0 and the diameter of M (any integer larger than $\operatorname{diam} M/\rho_0$ will do). That is because, given arbitrary points x and y in M, we may choose $z_0 = x$, $z_1, \ldots, z_N = y$, on a minimal geodesic connecting x to y, in such a way that

$$\sum_{i=1}^{N} d(z_{i-1}, z_i) = d(x, y) \quad \text{and} \quad d(z_{i-1}, z_i) \le \rho_0 \text{ for } i = 1, \ldots, N,$$

and then

$$|\phi(y) - \phi(x)| \le \sum_{i=1}^{N} |\phi(z_i) - \phi(z_{i-1})| \le \sum_{i=1}^{N} a\, d(z_{i-1}, z_i)^\nu \le N\, a\, d(x, y)^\nu.$$

Lemma 3.3.2. *For any $a > 0$ and $\nu > 0$, the cone $C(a, \nu)$ is proper and convex. The corresponding projective metric $\theta = \theta_{a,\nu}$ is given by*

$$\theta(\varphi_1, \varphi_2) = \log \frac{\beta(\varphi_1, \varphi_2)}{\alpha(\varphi_1, \varphi_2)}$$

where $\alpha(\varphi_1, \varphi_2)$ is the infimum and $\beta(\varphi_1, \varphi_2)$ is the supremum of

$$\left\{ \frac{\varphi_2}{\varphi_1}(x), \frac{\exp(ad(x,y)^\nu)\varphi_2(x) - \varphi_2(y)}{\exp(ad(x,y)^\nu)\varphi_1(x) - \varphi_1(y)} : x \neq y \text{ and } d(x,y) \leq \rho_0 \right\}. \tag{3.29}$$

Proof. Given any $\varphi_1, \varphi_2 \in C(a, \nu)$ and $t_1, t_2 > 0$,

$$\exp(-ad(x,y)^\nu) \leq \frac{t_1\varphi_1(x) + t_2\varphi_2(x)}{t_1\varphi_1(y) + t_2\varphi_2(y)} \leq \exp(ad(x,y)^\nu)$$

for all $x, y \in M$ with $d(x,y) \leq \rho_0$. So, $t_1\varphi_1 + t_2\varphi_2$ is in $C(a, \nu)$, and this shows that $C(a, \nu)$ is a convex cone. It is also proper, since the closure $\widetilde{C}(a, \nu)$ is contained in the cone of nonnegative functions. By definition, $\alpha(\varphi_1, \varphi_2)$ is the supremum of all the $t > 0$ satisfying

(i) $(\varphi_2 - t\varphi_1)(x) > 0 \Leftrightarrow t < \dfrac{\varphi_2}{\varphi_1}(x)$,

(ii) $\dfrac{(\varphi_2 - t\varphi_1)(x)}{(\varphi_2 - t\varphi_1)(y)} \leq \exp(ad(x,y)^\nu) \Leftrightarrow t \leq \dfrac{\exp(ad(x,y)^\nu)\varphi_2(y) - \varphi_2(x)}{\exp(ad(x,y)^\nu)\varphi_1(y) - \varphi_1(x)}$

and

$\dfrac{(\varphi_2 - t\varphi_1)(x)}{(\varphi_2 - t\varphi_1)(y)} \geq \exp(-ad(x,y)^\nu) \Leftrightarrow t \leq \dfrac{\exp(ad(x,y)^\nu)\varphi_2(x) - \varphi_2(y)}{\exp(ad(x,y)^\nu)\varphi_1(x) - \varphi_1(y)}$

for all $x, y \in M$ with $x \neq y$ and $d(x,y) \leq \rho_0$. Therefore, $\alpha(\varphi_1, \varphi_2)$ equals

$$\inf \left\{ \frac{\varphi_2(x)}{\varphi_1(x)}, \frac{\exp(ad(x,y)^\nu)\varphi_2(x) - \varphi_2(y)}{\exp(ad(x,y)^\nu)\varphi_1(x) - \varphi_1(y)} : x \neq y \text{ and } d(x,y) \leq \rho_0 \right\}.$$

Analogously, $\beta(\varphi_1, \varphi_2)$ is the supremum of this same set. □

The key fact behind Theorem 3.3.1 is that the transfer operator tends to improve the regularity (Hölder constants) of functions:

Proposition 3.3.1. *Given any $0 < \nu \leq \nu_0$, there exists $\lambda_1 < 1$ such that $\mathcal{L}(C(a, \nu)) \subset C(\lambda_1 a, \nu)$ for every sufficiently large $a > 0$.*

Proof. It follows, immediately, from the expression (3.28) that the function $\mathcal{L}\varphi$ is positive if φ is positive. So, we only have to check the second condition of the definition of $C(a, \nu)$. Let $y_1, y_2 \in M$ with $d(y_1, y_2) \leq \rho_0$. By Proposition 1.2.1 we may write

$$f^{-1}(y_j) = \{x_{j1}, \dots, x_{jk}\} \quad \text{for } j = 1, 2,$$

with $d(x_{1i}, x_{2i}) \le \sigma^{-1} d(y_1, y_2)$ for every $1 \le i \le k$. Our assumptions imply that $\log|\det Df|$ is (a_0, ν_0)-Hölder for some $a_0 > 0$. Then $\log|\det Df|$ is also (a_0, ν)-Hölder, because $\nu \le \nu_0$ and the diameter of M is less than 1. So, given any $\varphi \in C(a, \nu)$,

$$\begin{aligned}(\mathcal{L}\varphi)(y_1) &= \sum_{i=1}^{k} \frac{\varphi(x_{1i})}{|\det Df(x_{1i})|} = \sum_{i=1}^{k} \frac{\varphi(x_{2i})}{|\det Df(x_{2i})|} \frac{\varphi(x_{1i})}{\varphi(x_{2i})} \frac{|\det Df(x_{2i})|}{|\det Df(x_{1i})|} \\ &\le \sum_{i=1}^{k} \frac{\varphi(x_{2i})}{|\det Df(x_{2i})|} \exp\left(a\, d(x_{1i}, x_{2i})^\nu + a_0\, d(x_{1i}, x_{2i})^\nu\right) \\ &\le (\mathcal{L}\varphi)(y_2) \exp\left((a + a_0)\sigma^{-\nu}\, d(y_1, y_2)^\nu\right).\end{aligned}$$

Taking $\lambda_1 \in (\sigma^{-\nu}, 1)$ and assuming that $a \ge a_0/(\lambda_1\sigma^\nu - 1)$, we find that

$$(\mathcal{L}\varphi)(y_1) \le (\mathcal{L}\varphi)(y_2) \exp(\lambda_1 a\, d(y_1, y_2)^\nu),$$

as required. □

In the next proposition we also use the cone of positive functions on M

$$C_+ = \{\varphi \in E : \varphi(x) > 0 \text{ for all } x \in M\}.$$

The projective metric θ_+ associated to C_+ was calculated in Example 3.2.2, where we found

$$\theta_+(\varphi_1, \varphi_2) = \log \sup\left\{\frac{\varphi_2(x)}{\varphi_1(x)} \frac{\varphi_1(y)}{\varphi_2(y)} : x, y \in M\right\}. \tag{3.30}$$

Proposition 3.3.2. *The θ-diameter of $C(\lambda_1 a, \nu)$,*

$$D_1 = \sup\{\theta(\varphi_1, \varphi_2) : \varphi_1, \varphi_2 \in C(\lambda_1 a, \nu)\},$$

is finite, for every $a > 0$, $\nu > 0$, and $\lambda_1 < 1$.

Proof. The proof has two steps. First we show that

$$\theta\text{-}\operatorname{diam}(C(\lambda_1 a, \nu)) \le \theta_+\text{-}\operatorname{diam}(C(\lambda_1 a, \nu)) + K'(\lambda_1),$$

for some $K'(\lambda_1) < \infty$. Then, for every $b > 0$, we bound $\theta_+\text{-}\operatorname{diam}(C(b, \nu))$ by some finite expression $K''(b)$. It follows that $\theta\text{-}\operatorname{diam}(C(\lambda_1 a, \nu))$ is less than $K'(\lambda_1) + K''(\lambda_1 a)$.

Let $\varphi_1, \varphi_2 \in C(\lambda_1 a, \nu)$, and $x, y \in M$ be such that $d(x, y) \le \rho_0$. Since,

$$\varphi_i(x) \exp(-a\lambda_1\, d(x, y)^\nu) \le \varphi_i(y) \le \varphi_i(x) \exp(a\lambda_1\, d(x, y)^\nu)$$

for $i = 1, 2$, we have

$$\frac{\exp(ad(x,y)^\nu)\varphi_2(x) - \varphi_2(y)}{\exp(ad(x,y)^\nu)\varphi_1(x) - \varphi_1(y)} \geq \frac{\varphi_2}{\varphi_1}(x)\frac{\zeta - \zeta^{\lambda_1}}{\zeta - \zeta^{-\lambda_1}} \geq \frac{\varphi_2}{\varphi_1}(x)\, K_1\,,$$

where $\zeta = \exp(ad(x,y)^\nu)$ and

$$K_1 = \inf\left\{\frac{z - z^{\lambda_1}}{z - z^{-\lambda_1}} : z > 1\right\}.$$

It is easy to check that $0 < K_1 < 1$. Then, comparing (3.29) and (3.30), we conclude that $\alpha(\varphi_1, \varphi_2) \geq K_1\alpha_+(\varphi_1, \varphi_2)$. Analogously, we get that $\beta(\varphi_1, \varphi_2) \leq K_2\,\beta_+(\varphi_1, \varphi_2)$, with

$$K_2 = \sup\left\{\frac{z - z^{-\lambda_1}}{z - z^{\lambda_1}} : z > 1\right\} \in (1, +\infty).$$

It follows that $\theta(\varphi_1, \varphi_2) \leq \theta_+(\varphi_1, \varphi_2) + \log(K_2/K_1)$ and this concludes the first step of the proof, with $K'(\lambda_1) = \log(K_2/K_1)$.

For the second step we use Remark 3.3.1: given φ_1 and φ_2 in $C(b, \nu)$, their logarithms are (Nb, ν)-Hölder on the whole manifold M. That is,

$$\log\frac{\varphi_2(x)}{\varphi_2(y)} \leq Nb\, d(x,y)^\nu \leq Nb \quad \text{and} \quad \log\frac{\varphi_1(y)}{\varphi_1(x)} \leq Nb\, d(x,y)^\nu \leq Nb$$

for every $x, y \in M$. So, by (3.30), we have $\theta_+(\varphi_1, \varphi_2) \leq 2Nb$. This completes the proof, with $K''(b) = 2Nb$. □

From now on, we fix $0 < \nu \leq \nu_0$, $0 < \lambda_1 < 1$, and $a > 0$ as in the statement of Proposition 3.3.1.

Corollary 3.3.1. *There exists a unique function $\varphi_0 \in C(\lambda_1 a, \nu)$ such that $\mathcal{L}\varphi_0 = \varphi_0$ and $\int \varphi_0\, dm = 1$. Moreover, φ_0 is ν-Hölder and bounded from zero and infinity: there is $R_0 > 0$ such that $1/R_0 \leq \varphi_0 \leq R_0$.*

Proof. At this stage the idea is quite simple. Putting Propositions 3.3.1 and 3.3.2 together, we have that the θ-diameter D_1 of $\mathcal{L}(C(a, \nu))$ in $C(a, \nu)$ is finite. Then, by Proposition 3.2.2, the operator $\mathcal{L}$ contracts the metric θ, with contraction rate $\Lambda_1 = \tanh(D_1/4) < 1$. Using the fact that $\theta \geq \theta_+$, and the projective metric θ_+ is complete, cf. Proposition 3.2.3, we conclude that $\mathcal{L}$ has a fixed point φ_0, unique up to a multiplicative factor.

Next, we fill-in the details of the proof. Let $\varphi_n = \mathcal{L}^n 1$, for $n \geq 1$. Clearly, the constant function 1 is in $C(a, \nu)$. So, $\varphi_n \in C(\lambda_1 a, \nu) \subset C(a, \nu)$ for every $n \geq 1$, by Proposition 3.3.1. Since $\mathcal{L}$ contracts the metric θ, the sequence $(\varphi_n)_n$ is θ-Cauchy. Then it is also θ_+-Cauchy, because $C(a, \nu)$ is contained in C_+, and this implies that $\theta_+ \leq \theta$. On the other hand,

$$\int \varphi_n \, dm = (\varphi_n m)(M) = (f_*^n m)(M) = m(f^{-n}(M)) = 1, \tag{3.31}$$

for all $n \geq 1$. Thus, we may use Proposition 3.2.3, to conclude that $(\varphi_n)_n$ is θ_+-convergent and uniformly convergent to some function $\varphi_0 \in C_+$.

Let us check that this φ_0 has the properties claimed in the statement. Firstly, (3.31) implies that, for each $n \geq 1$, there are points x_n' and xn'' in M such that $\varphi_0(x_n') \leq 1 \leq \varphi_0(x_n'')$. On the other hand, by Remark 3.3.1, $\log \varphi_n$ is (Na, ν)-Hölder on M. It follows that

$$\exp(-Na) \leq \varphi_n(x) \leq \exp(Na), \quad \text{for all } x \in M \text{ and } n \geq 1$$

We take $R_0 = \exp(Na)$. Taking the limit as $n \to \infty$ in the previous expression, we get that $1/R_0 \leq \varphi_0 \leq R_0$. In particular, φ_0 is a positive function. Furtermore, $\log \varphi_0$ is the uniform limit of φ_n. Since the Hölder condition is closed under uniform (even pointwise) limits, it follows that $\log \varphi_0$ is (a, ν)-Hölder on ρ_0-neighbourhoods. This proves that $\varphi_0 \in C(a, \nu)$ and, moreover, φ_0 is $(R_0 Na, \nu)$-Hölder . Since $\mathcal{L}$ is a bounded operator with respect to the sup-norm, by Lemma 3.3.1, φ_0 must be a fixed point of the transfer operator:

$$\mathcal{L}\varphi_0 = \mathcal{L}(\lim_n \mathcal{L}^n 1) = \lim_n \mathcal{L}^{n+1} 1 = \varphi_0 .$$

The fact that the limit is uniform also implies that $\int \varphi_0 \, dm$ is equal to 1. Finally, any fixed point of $\mathcal{L}$ in $C(a, \nu)$ must be a multiple of φ_0, since $\mathcal{L}$ contracts the projective metric θ. □

Let $\mu = \varphi_0 m$, where φ_0 is as in the proposition. Then μ is an f-invariant probability measure, and it is absolutely continuous with respect to Lebesgue measure. We shall show that μ is an exact measure. In particular, μ is ergodic, and so it is a physical measure for f. The fact that φ_0 is positive implies that μ is even equivalent to Lebesgue measure: they have the same zero measure sets. Therefore, Lebesgue almost every point is in the basin $B(\mu)$. This implies that μ is the unique physical measure of f. Thus, the present approach provides an alternative proof of the existence and uniqueness of the physical measure, that we had established in Section 1.2.

3.3.2 Exponential Convergence

Now we prove exponential convergence to equilibrium and exponential decay of correlations in the space of Hölder functions. We start by deriving the following direct consequence of the arguments in the previous subsection. Here $\Lambda_1 = 1 - e^{-D_1}$, with $D_1 > 0$ as in Proposition 3.3.2.

Lemma 3.3.3. *There exists $R_1 > 0$ such that,*

$$\left\| \mathcal{L}^n \varphi - \varphi_0 \int \varphi \, dm \right\|_0 \leq R_1 \Lambda_1^n \int \varphi \, dm$$

for every $\varphi \in C(\lambda_1 a, \nu)$ and $n \geq 1$.

Proof. Let $\varphi \in C(\lambda_1 a, \nu)$. Then $\varphi > 0$ at every point, and so $\int \varphi \, dm$ is also positive. Clearly, the lemma holds for φ if and only if it holds for $\varphi / \int \varphi \, dm$. Thus, it is no restriction to suppose $\int \varphi \, dm = 1$, and we do so in what follows. Then $\int \mathcal{L}^n \varphi \, dm = 1 = \int \varphi_0 \, dm$ for each $n \geq 1$. Since $\mathcal{L}^n \varphi$ and φ_0 are nonnegative, it follows that there exist points x'_n and x''_n in M such that

$$\mathcal{L}^n \varphi(x'_n) \leq \varphi_0(x'_n) \quad \text{and} \quad \mathcal{L}^n \varphi(x''_n) \geq \varphi_0(x''_n).$$

Using the expression of θ_+ in (3.30), we get

$$\frac{\mathcal{L}^n \varphi(x)}{\varphi_0(x)} \leq \frac{\mathcal{L}^n \varphi(x'_n)}{\varphi_0(x'_n)} \exp(\theta_+(\mathcal{L}^n \varphi, \varphi_0)) \leq \exp(\theta_+(\mathcal{L}^n \varphi, \varphi_0)),$$

for any $x \in M$. Since $\mathcal{L}\varphi_0 = \varphi_0$ and $\mathcal{L}$ is a Λ_1-contraction for the metric θ,

$$\theta_+(\mathcal{L}^n \varphi, \varphi_0) \leq \theta(\mathcal{L}^n \varphi, \varphi_0) \leq \theta(\varphi, \varphi_0) \Lambda_1^{n-1} \leq D_1 \Lambda_1^{n-1}.$$

Combining these inequalities,

$$\frac{\mathcal{L}^n \varphi(x)}{\varphi_0(x)} \leq \exp\left(D_1 \Lambda_1^{n-1}\right) \leq 1 + R_2 \Lambda_1^n$$

for every $x \in M$, where the constant R_2 may be chose depending only on D_1 and Λ_1. Analogously,

$$\frac{\mathcal{L}^n \varphi(x)}{\varphi_0(x)} \geq \exp\left(-D_1 \Lambda_1^{n-1}\right) \geq 1 - R_2 \Lambda_1^n.$$

Therefore, given any $x \in M$,

$$\left| \frac{\mathcal{L}^n \varphi(x)}{\varphi_0(x)} - 1 \right| \leq R_2 \Lambda_1^n$$

and so $|\mathcal{L}^n \varphi(x) - \varphi_0(x)| \leq R_0 R_2 \Lambda_1^n$. Thus, we may take $R_1 = R_0 R_2$. □

For $\nu > 0$ and any ν-Hölder function $\varphi : M \to \mathbb{R}$, we denote by $H_\nu(\varphi)$ the smallest multiplicative Hölder constant of φ, that is,

$$H_\nu(\varphi) = \inf\{H > 0 : |\varphi(x) - \varphi(y)| \leq H d(x,y)^\nu \text{ for all } x, y \in M\}.$$

As before, $B_n(\varphi, \psi) = \int (\psi \circ f^n) \varphi \, dm - \int \psi \, d\mu \int \varphi \, dm$, for each $n \geq 0$.

Proposition 3.3.3. *There exists $K_1 = K_1(\nu) > 0$ such that*

$$|B_n(\varphi, \psi)| \leq K_1 \Lambda_1^n \int |\psi| \, dm \left(\int |\varphi| \, dm + H_\nu(\varphi) \right)$$

for any ν-Hölder function $\varphi : M \to \mathbb{R}$, any $\psi \in L^1(m)$, and any $n \geq 0$.

Proof. First, we consider $\varphi \in C(\lambda_1 a, \nu)$. By the definition of $\mathcal{L}$ and $\mu = \varphi_0\, m$,

$$|B_n(\varphi,\psi)| = \left|\int \psi \left(\mathcal{L}^n \varphi - \varphi_0 \int \varphi\, dm\right) dm\right| \le \left(\int |\psi|\, dm\right) \left\|\mathcal{L}^n \varphi - \varphi_0 \int \varphi\, dm\right\|_0 .$$

So, using the previous lemma,

$$|B_n(\varphi,\psi)| \le R_1 \Lambda_1^n \int |\psi|\, dm \int |\varphi|\, dm. \tag{3.32}$$

Now let φ be a general ν-Hölder function, and let $H > H_\nu(\varphi)$. We write $\varphi = \varphi^+ - \varphi^-$, where

$$\varphi^+ = \frac{1}{2}(|\varphi| + \varphi) + B \quad \text{and} \quad \varphi^- = \frac{1}{2}(|\varphi| - \varphi) + B, \quad \text{with} \quad B = \frac{H}{\lambda_1 a}.$$

Clearly, φ^+ and φ^- are (H,ν)-Hölder and positive: $\varphi^\pm \ge B > 0$. Moreover, $\log \varphi^\pm$ are $(\lambda a_1, \nu)$-Hölder: given any $x, y \in M$,

$$\left|\log \varphi^\pm(x) - \log \varphi^\pm(y)\right| \le \frac{|\varphi^\pm(x) - \varphi^\pm(y)|}{\min\{\varphi^\pm(x), \varphi^\pm(y)\}} \le \frac{1}{B} H d(x,y)^\nu \le \lambda a_1 d(x,y)^\nu$$

because of our choice of B. This means that $\varphi^\pm \in C(\lambda_1 a, \nu)$, and so we may apply (3.32) to both functions:

$$\left|B_n(\varphi^\pm,\psi)\right| \le R_1 \Lambda_1^n \int |\psi|\, dm \int \varphi^\pm\, dm.$$

Using $B_n(\varphi,\psi) = B_n(\varphi^+,\psi) - B_n(\varphi^-,\psi)$, we conclude that

$$|B_n(\varphi,\psi)| \le \left|B_n(\varphi^+,\psi)\right| + \left|B_n(\varphi^-,\psi)\right| \le R_1 \Lambda_1^n \int |\psi|\, dm \int (\varphi^+ + \varphi^-)\, dm.$$

Moreover, by the definition of $\varphi^\pm$,

$$\int (\varphi^+ + \varphi^-)\, dm = \int \big(|\varphi| + 2B\big)\, dm = \int |\varphi|\, dm + \frac{2H}{\lambda_1 a}.$$

Since H may be taken arbitrarily close to $H_\nu(\varphi)$, it follows that

$$|B_n(\varphi,\psi)| \le R_1 \Lambda_1^n \int |\psi|\, dm \left(\int |\varphi|\, dm + \frac{2}{\lambda_1 a} H_\nu(\varphi)\right).$$

Therefore, we may take any $K_1 \ge \max\{R_1, 2R_1/(\lambda_1 a)\}$. □

From this one can easily deduce exponential decay of correlations in the space of Hölder continuous functions. Recall that

$$C_n(\varphi,\psi) = \int (\psi \circ f^n)\varphi\, d\mu - \int \psi\, d\mu \int \varphi\, d\mu.$$

Proposition 3.3.4. *There exists $K_2 = K_2(\nu)$ such that*

$$|C_n(\phi, \psi)| \le K_2 \Lambda_1^n \int |\psi|\, dm \left(\int |\phi|\, d\mu + H_\nu(\varphi) \right)$$

for any ν-Hölder function $\varphi : M \to \mathbb{R}$, any $\psi \in L^1(m)$, and any $n \ge 1$.

For the proof we need the following elementary

Lemma 3.3.4. *Given ν-Hölder functions $\phi_i : M \to \mathbb{R}$, $i = 1, 2$, and any probability measure η in M,*

(a) $H_\nu(\phi_1 \phi_2) \le \sup |\phi_2| H_\nu(\phi_1) + \sup |\phi_1| H_\nu(\phi_2)$
(b) $\sup |\phi_1| \le \int |\phi_1|\, d\eta + H_\nu(\phi_1)$.

Proof. Fix $H_1 > H_\nu(\phi_1)$ and $H_2 > H_\nu(\phi_2)$. Given any $x, y \in M$,

$$\begin{aligned} |\phi_1\phi_2(x) - \phi_1\phi_2(y)| &\le |\phi_1(x) - \phi_1(y)||\phi_2(x)| + |\phi_1(y)||\phi_2(x) - \phi_2(y)| \\ &\le H_1 d(x,y)^\nu \sup |\phi_2| + \sup |\phi_1| H_2 d(x,y)^\nu. \end{aligned}$$

This proves claim (a) in the statement, since H_i may be taken arbitrarily close to $H_\nu(\phi_i)$, for each $i = 1, 2$. Claim (b) can be obtained as follows. Since η is a probability, there exists $x_\eta \in M$ such that $|\phi_1(x_\eta)| \le \int |\phi_1|\, d\eta$. Then, for any $x \in M$,

$$|\phi_1(x)| \le |\phi_1(x_\eta)| + H_1 d(x, x_\eta)^\nu \le \int |\phi_1|\, d\eta + H_1\,.$$

In the first step we used $H_1 > H_\nu(\phi_1) \ge H_\nu(|\phi_1|)$; for the second one, recall that we took the diameter of M to be less than 1. This shows that $\sup |\phi_1|$ does not exceed $\int |\phi_1|\, d\eta + H_1$. Claim (b) follows, by taking H_1 arbitrarily close to $H_\nu(\phi_1)$. □

Now Proposition 3.3.4 follows easily:

Proof. Since $C_n(\phi, \psi) = B_n(\phi\varphi_0, \psi)$, we get from Proposition 3.3.3 that

$$|C_n(\phi, \psi)| \le K_1 \Lambda_1^n \int |\psi|\, dm \left(\int |\phi\varphi_0|\, dm + H_\nu(\phi\varphi_0) \right).$$

Of course, $\int |\phi\varphi_0|\, dm = \int |\phi|\, d\mu$. On the other hand, using Lemma 3.3.4 with $\phi_i = \phi$, $\phi_2 = \varphi_0$, and $\eta = \mu$,

$$\begin{aligned} H_\nu(\phi\varphi_0) &\le \sup |\varphi_0|\, H_\nu(\phi) + \sup |\phi|\, H_\nu(\varphi_0) \\ &\le R_0\, H_\nu(\phi) + \left(\int |\phi|\, d\mu + H_\nu(\phi) \right) H_0\,. \end{aligned}$$

Here, $H_0 = H_\nu(\varphi_0)$ was introduced for notational simplicity. Replacing these inequalities in the bound for $|C_n(\phi, \psi)$ obtained above, we find

$$|C_n(\phi,\psi)| \le K_1 \Lambda_1^n \int |\psi|\, dm \left((1 + H_0) \int |\phi|\, d\mu + (R_0 + H_0) H_\nu(\phi) \right).$$

This proves the proposition, for any $K_2 \ge K_1 \max\{1 + H_0\,,\, R_0 + H_0\}$. □

Finally, we prove that μ is an exact measure. The arguments also yield a new proof of the uniqueness statement in Theorem 1.2.1.

Corollary 3.3.2. *The measure μ is exact, and it is the unique f-invariant probability measure absolutely continuous with respect to Lebesgue measure.*

Proof. This is an application of Proposition 3.1.3 in the space $L^1(\mu)$, with the subspace of ν-Hölder continuous functions in the role of F. Recall, e.g. from [**?**, Theorem 3.14], that the space of continuous functions is dense in $L^1(\mu)$. Moreover, see e.g. [89, Proposition I.2.7], every continuous function can be uniformly approximated by smooth (and, hence, ν-Hölder) functions. Therefore, the subspace F of ν-Hölder continuous functions is indeed dense in $L^1(\mu)$. On the other hand, the hypothesis (3.18) is granted by Proposition 3.3.4, with $K_n(\phi) = K_2 \Lambda_1^n \left(\int |\phi|\, d\mu + H_\nu(\varphi) \right)$. Thus, according to Proposition 3.1.3, the measure μ is exact. By Lemma 3.1.7, μ is also mixing and, in particular, ergodic. To prove uniqueness, let μ' be any invariant probability absolutely continuous with respect to m. By Corollary 3.3.1, the measures μ and m are equivalent. So, μ' is also absolutely continuous with respect to μ. Since we have already shown that μ is ergodic, this implies that $\mu' = \mu$. □

3.4 Piecewise Expanding Maps

Let $f : M \to M$ be a piecewise expanding map of the circle $M = S^1$ or the interval $M = [0, 1]$, satisfying the summability condition

$$\sum_{\xi \in \mathcal{P}^1} \operatorname{var} \tilde{g}_\xi < \infty \tag{3.33}$$

As we have seen in Theorem 1.3.2, f has a finite number of physical measures, that are ergodic invariant measures absolutely continuous with respect to Lebesgue measure.

Here we prove a theorem of Hofbauer-Keller [55] and Rychlik [**?**], stating that any physical measure can be decomposed into a finite number of components that are cyclically permuted by f, and are exact for an iterate f^N of f. Moreover, each of these components has exponential decay of correlations for the map f^N, in the space of bounded variation functions.

Theorem 3.4.1. *Let $f : M \to M$ be a piecewise expanding map of the circle, or the interval, satisfying (3.33), and μ be any of its physical measures. Then there are $N \geq 1$ and probability measures $\mu_1, \ldots, \mu_N$ in M, such that $\mu = (\mu_1 + \cdots + \mu_N)/N$,*

$$f_*\mu_1 = \mu_2 , \quad \ldots, \quad f_*\mu_{N-1} = \mu_N , \quad f_*\mu_N = \mu_1 ,$$

and every μ_i is an exact measure for f^N. Moreover, $f_*^{nN}(\phi\mu_i)$ converges to $\mu_i \int \varphi \, d\mu_i$ exponentially fast as $n \to \infty$, for any $1 \leq i \leq s$ and any function ϕ with bounded variation (Proposition 3.4.3).

In general, these physical measures need not be exact for the f itself, as the following example shows. A few sufficient conditions for exactness will be given in Remark 3.4.2.

Example 3.4.1. Let $f : [0,1] \to [0,1]$ be the piecewise expanding map in Figure 3.2: f is affine on each of the intervals bounded by 0, 1/4, 3/4, and 1, with $f(0) = f(1) = 1/2$, $f(1/4) = 1$, and $f(3/4) = 0$. Let μ be any

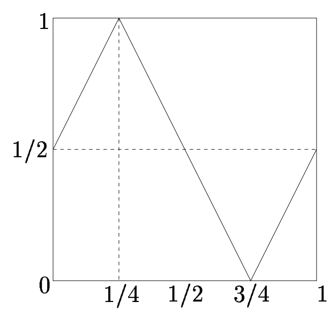

Fig. 3.2. Ergodic nonmixing physical measure

physical measure of f. The basin $B(\mu)$ is an invariant set for f and so, by Corollary 1.3.2, it has full Lebesgue measure in a neighbourhood $[c-\varepsilon, c+\varepsilon]$ of some singular point c of f. Here the singular points are 1/4 and 3/4. If $c = 1/4$ then, iterating once forward, $B(\mu)$ must have full measure in $[1-\varepsilon, 1]$ and, iterating once more, in $[1/2-\varepsilon, 1/2]$. Analogously, if $c = 3/4$ then $B(\mu)$ must have full measure in $[0, \varepsilon]$, in $[1/2, 1/2+\varepsilon]$, and, iterating a third time, in $[1/2-\varepsilon, 1/2]$. This proves that the basin of any physical measure contains a full Lebesgue subset of some interval $[1/2-\varepsilon, 1/2]$. Since basins are two-by-two disjoint, the physical measure μ has to be unique. On the other hand, μ is not mixing for f. Indeed, let $\phi = \mathcal{X}_{[0,1/2)}$. Note that $f^{-1}\big([0,1/2)\big) = (1/2, 1)$ and $f^{-1}\big((1/2,1]\big) = (0, 1/2)$. It follows that

$$\phi \circ f^{2k-1} = \mathcal{X}_{(1/2,1]} = 1 - \phi \quad \text{and} \quad \phi \circ f^{2k} = \mathcal{X}_{[0,1/2)} = \phi,$$

μ-almost everywhere, for any $k \geq 1$. Since μ is f-invariant, we also have $\int \phi \, d\mu = \int (\phi \circ f^n) \, d\mu$ for all n. Hence, $\int \phi \, d\mu = 1/2$. Furthermore,

$$C_n(\phi, \phi) = \int (\phi \circ f^n) \phi \, d\mu - \left(\int \phi \, d\mu\right)^2 = \begin{cases} 0 - (1/2)^2 = -1/4 & \text{if } n \text{ is odd} \\ 1/2 - (1/2)^2 = 1/4 & \text{if } n \text{ is even.} \end{cases}$$

In particular, $C_n(\phi, \phi)$ does not go to zero when $n \to \infty$. This shows that μ is not a mixing measure for f.

We use the approach developed in Subsections 3.1.2–3.1.4 for proving Theorem 3.4.1. Here is an outline of the arguments, details will be given in Subsections 3.4.1–3.4.2. We consider the transfer operator $\mathcal{L}$ acting on the Banach space $E = \mathrm{BV}$ of functions with bounded variation (precise definitions will appear in a while). $\mathcal{L}$ fulfills the assumptions of Theorem 3.1.2, and so it is quasi-compact on the space BV. Actually, the same is true about $\mathcal{L}^n$, for any $n \geq 1$. We also prove that each physical measure can be decomposed into exact measures of some iterate f^N, as stated. As in Proposition 3.1.2, this implies existence of a spectral gap for the transfer operator $\mathcal{L}^N$ of f^N. Exponential decay of correlations follows, along the lines of Corollary 3.1.2.

Theorem 3.1.1 can be used, instead of Theorem 3.1.2, to prove quasi-compactness of the transfer operator. We leave the corresponding verifications as an exercise. A proof of Theorem 3.4.1 based on an invariant cone construction is also possible, at least if the derivative of f is bounded; see [70] and [123, Section 3.2].

3.4.1 Quasi-Compactness

As before, we let m denote a normalized Lebesgue measure on M. By definition, the variation of a complex function $\varphi : M \to \mathbb{R}$, or of a complex-valued element of $L^1(m)$, is the sum of the variations of its real part and imaginary part. Recall that in the real-valued case, the notion of variation was introduced in Definitions 1.3.2 and 1.3.4. In the present section, BV denotes the space of complex-valued elements of $L^1(m)$ with bounded (finite) variation. For each $\varphi \in \mathrm{BV}$, we define the BV-norm

$$\|\varphi\|_{\mathrm{BV}} = \operatorname{var} \varphi + \|\varphi\|_1 . \tag{3.34}$$

Throughout, we suppose that $f : M \to M$ is a piecewise expanding map as in Theorem 3.4.1, and $\mathcal{L} : L^1(m) \to L^1(m)$ is the corresponding transfer operator. Before proceeding, let us point out that every iterate f^n, $n \geq 1$, of f also satisfies the assumptions of the theorem: it is clear that f^n is piecewise expanding, and Lemma 1.3.9 states that f^n satisfies (3.33) if f does. That means that the results we get in this subsection are valid for the transfer operator $\mathcal{L}^n$ of any iterate f^n.

In what follows we consider any element of BV to be represented by some function as in Remark 1.3.1, that is, one that realizes the variation and the

L^∞-norm. We need versions of properties in Lemma 1.3.1 for complex-valued functions. The following are sufficient for our purposes (though, probably, not sharp):

$$\operatorname{var}(\varphi_1 \varphi_2) \le 2 \operatorname{var} \varphi_1 \|\varphi_2\|_\infty + 2 \operatorname{var} \varphi_2 \|\varphi_1\|_\infty \tag{3.35}$$
$$\|\varphi\|_\infty \le \operatorname{var} \varphi + 2\|\varphi\|_1 . \tag{3.36}$$

They follow, in a straightforward way, from parts 2 and 3 of Lemma 1.3.1 applied to the real and imaginary parts, using also $\operatorname{var} \varphi = \operatorname{var} \Re\varphi + \operatorname{var} \Im\varphi$,

$$\|\varphi\|_\infty \le \|\Re\varphi\|_\infty + \|\Im\varphi\|_\infty \le 2\|\varphi\|_\infty ,$$

and the corresponding fact for the L^1-norm.

Lemma 3.4.1. *(a)* $\|\varphi\|_\infty \le 2\,\|\varphi\|_{\mathrm{BV}}$ *for every* $\varphi \in \mathrm{BV}$*;*
(b) BV *is a complex vector space and* $\|\cdot\|_{\mathrm{BV}}$ *is a complete norm in it;*
(c) $\mathcal{L}$ *maps* BV *inside itself, and is a bounded operator relative to* $\|\cdot\|_{\mathrm{BV}}$*;*
(d) the sequence of norms $(\|\mathcal{L}^n\|_{\mathrm{BV}})_n$ *of the iterates of* $\mathcal{L}$ *is bounded.*
(e) BV *is dense in the space* $L^1(m)$*, with respect to the norm* $\|\cdot\|_1$*.*

Proof. Claim (a) follows immediately from (3.36):

$$\|\varphi\|_\infty \le \operatorname{var} \varphi + 2\|\varphi\|_1 \le 2\|\varphi\|_{\mathrm{BV}} .$$

Let us prove part (b). Clearly, $\|\varphi\|_{\mathrm{BV}} = 0$ implies $\|\varphi\|_1 = 0$, and so $\varphi = 0$ in $L^1(m)$. Moreover, for any $t \in \mathbb{C}$ and $\varphi, \varphi_1, \varphi_2$,

$$\operatorname{var}(t\varphi) = |t| \operatorname{var} \varphi \quad \text{and} \quad \operatorname{var}(\varphi_1 + \varphi_2) \le \operatorname{var} \varphi_1 + \operatorname{var} \varphi_2 ,$$

are direct consequences of Lemma 1.3.1 applied to the real and imaginary parts. Together with similar facts for the L^1-norm, this implies that $\|\cdot\|_{\mathrm{BV}}$ is a norm restricted to the subset BV of $\varphi \in L^1(m)$ for which $\|\varphi\|_{\mathrm{BV}}$ is finite. It also follows that BV is a complex vector space.

The statement that the BV-norm is complete can be deduced from Lemma 1.3.3, in the following way. Let $(\varphi_k)_k$ be a Cauchy sequence for $\|\cdot\|_{\mathrm{BV}}$. Then $(\varphi_k)_k$ is also Cauchy for $\|\cdot\|_1$, and so it converges in $L^1(m)$ to some function φ. We claim that $(\varphi_k)_k$ converges to φ in BV too. To prove this, we only have to show that $\operatorname{var}(\varphi_k - \varphi)$ goes to zero when k goes to infinity. The assumption that $(\varphi_k)_k$ is a Cauchy sequence for the BV-norm has two useful consequences. On the one hand, $(\varphi_k)_k$ is bounded for $\|\cdot\|_{\mathrm{BV}}$ and so, using part (a) of this lemma, it is also a bounded sequence in $L^\infty(m)$. So, up to choosing convenient representatives of the φ_k, we may suppose that $K = \sup_{k,x} |\varphi_k(x)|$ is finite. On the other hand, given any $\varepsilon > 0$ we may find $p \ge 1$ such that $\operatorname{var}(\varphi_k - \varphi_n) \le \varepsilon$ for all $n > k > p$. Fix k larger than p, and define $\psi_n = \varphi_k - \varphi_n$ for every $n > p$. Then $\sup |\psi_n| \le 2K$ and $\operatorname{var} \psi_n \le \varepsilon$ for all n. Using Lemma 1.3.3 with $K_1 = 2K$ and $K_2 = \varepsilon$, we conclude that $(\psi_n)_n$ has some subsequence converging in $L^1(m)$ to a function ψ_0 such that

$\operatorname{var}\psi_0 \le \varepsilon$. Concurrently, ψ_n converges in $L^1(m)$ to $\varphi_k - \varphi$. Thus, $\psi_0 = \varphi_k - \varphi$. Hence, we have shown that $\operatorname{var}(\varphi_k - \varphi) \le \varepsilon$ for all $k > p$. This finishes the proof of our claim, and of part (b) of the lemma.

Next, by Proposition 1.3.3 and Lemma 3.1.1,

$$\|\mathcal{L}^n\varphi\|_{\mathrm{BV}} = \operatorname{var}\mathcal{L}^n\varphi + \|\mathcal{L}^n\varphi\|_1 \le C_0\lambda_0^n \operatorname{var}\varphi + (C_0+1)\|\varphi\|_1\,,$$

and so $\|\mathcal{L}^n\varphi\|_{\mathrm{BV}} \le (C_0+1)\,\|\varphi\|_{\mathrm{BV}}$ for every $n \ge 1$ and every $\varphi \in \mathrm{BV}$. This proves part (d) of the lemma. The case $n = 1$ gives that $\mathcal{L}$ preserves the space BV, and that it is a bounded operator with respect to the BV-norm, as stated in (c).

Finally, every L^1 function is $\|\cdot\|_1$-approximated by continuous ones, and every continuous function is uniformly approximated by functions that are smooth, and so have bounded variation in M. See [**?**, Theorem 3.14] and [89, Proposition I.2.7]. Thus BV is dense in $L^1(m)$, as claimed in (e). □

Parts (b) and (c) of this lemma mean that $\mathcal{L} : \mathrm{BV} \to \mathrm{BV}$ satisfies condition (O1) of Subsection 3.1.2. Part (a) ensures that it also satisfies condition (O2), with $p = \infty$ and $C_0 = 2$. Part (d) implies that

$$\rho(\mathcal{L}) = \lim_n \left(\|\mathcal{L}^n\|_{\mathrm{BV}}\right)^{1/n} \le 1.$$

We already know that $\mathcal{L}$ has fixed points, cf. Theorem 1.3.2, so the spectral radius of $\mathcal{L}$ is really equal to 1.

Proposition 3.4.1. *The operator $\mathcal{L} : \mathrm{BV} \to \mathrm{BV}$ is quasi-compact: its spectrum splits as $\operatorname{spec}(\mathcal{L}) = \{\lambda_1 = 1, \lambda_2, \ldots, \lambda_k\} \cup \Sigma_0$, where each λ_i is an eigenvalue on the unit circle, with index 1 and finite multiplicity, and Σ_0 is contained in some disc of radius $\tau < 1$.*

Proof. This is an application of Theorem 3.1.2. Let us verify that the hypotheses of the theorem are satisfied in this context. The first one is an immediate consequence of part (d) in the previous lemma: given any h in the dual space BV^* and any $x \in M$,

$$\left|h\left(\frac{1}{n}\,L^n(x)\right)\right| \le \frac{1}{n}\,\|h\|_{\mathrm{BV}^*}\|\mathcal{L}^n\|_{\mathrm{BV}} \le \frac{C_0+1}{n}\,\|h\|_{\mathrm{BV}^*}\,,$$

which goes to zero when $n \to \infty$. Now we have to check the second hypothesis, namely, that there exist $N \ge 1$ and a compact operator $\mathcal{K} : \mathrm{BV} \to \mathrm{BV}$ such that $\|\mathcal{L}^N - \mathcal{K}\|_{\mathrm{BV}} < 1$. We do that as follows.

Let $\mathcal{P}^n$ be the partition of M into monotonicity intervals of f^n. By Lemmas 1.3.5 and 1.3.6

$$\mathcal{L}^n\varphi = \sum_{\eta\in\mathcal{P}^n} g_\eta^n\left(\varphi\circ(f^n \mid \eta)^{-1}\right),$$

and there are $C_1 > 0$ and $\lambda_1 < 1$ such that both the supremum and the variation of every g_η^n are less than $C_1\lambda_1^n$. Choose some finite subset $\mathcal{Q}^n$ of each $\mathcal{P}^n$ such that

$$\sum_{\eta\notin\mathcal{Q}^n} \operatorname{var} g_\eta^n \leq C_1\lambda_1^n. \tag{3.37}$$

This is possible because of Lemma 1.3.9. Then let us define linear operators $p_n : \mathrm{BV} \to \mathrm{BV}$, $n \geq 1$, by the condition that every $p_n\varphi$ is constant equal to the average of φ on each interval $\eta \in \mathcal{Q}^n$, and is identically zero on every $\eta \notin \mathcal{Q}^n$. More precisely, given any $\varphi \in \mathrm{BV}$,

$$p_n(\varphi)(x) = \begin{cases} E_\eta\varphi = \frac{1}{m(\eta)}\int_\eta \varphi\, dm & \text{for } x\in\eta,\quad \eta\in\mathcal{Q}^n \\ 0 & \text{for } x\in\eta,\quad \eta\in\mathcal{P}^n\setminus\mathcal{Q}^n \end{cases}$$

Since $\mathcal{Q}^n$ is taken finite, the range of p_n has finite dimension, and so the same is true for the composition $\mathcal{L}^n p_n$. In particular, every $\mathcal{L}^n p_n$ is a compact operator. We claim that

$$\|\mathcal{L}^n - \mathcal{L}^n p_n\|_{\mathrm{BV}} \leq 10C_1\lambda_1^n \quad \text{for all } n \geq 1. \tag{3.38}$$

Assuming this fact, it is easy to complete the proof of the proposition. Indeed, it suffices to take $\mathcal{K} = \mathcal{L}^N p_N$ for any $N \geq 1$ large enough so that $10C_1\lambda_1^N < 1$, to have the second condition in Theorem 3.1.2. Then quasi-compactness of the transfer operator follows from the theorem.

We are left to prove (3.38). This involves two series of estimates, first for the variation, and then for the L^∞ and L^1-norms. Most ideas appeared before in Propositions 1.3.1 and 1.3.3.

We start by writing $(\mathcal{L}^n - \mathcal{L}^n p_n)\varphi$ as

$$\sum_{\eta\in\mathcal{Q}^n} g_\eta^n\big([\varphi - E_\eta\varphi]\circ(f^n \mid \eta)^{-1}\big) + \sum_{\eta\notin\mathcal{Q}^n} g_\eta^n\big(\varphi\circ(f^n \mid \eta)^{-1}\big). \tag{3.39}$$

Just as in (1.18), the variation of the first term in (3.39) is bounded by

$$\sum_{\eta\in\mathcal{Q}^n} \operatorname{var} g_\eta^n\big(3\sup_\eta|\varphi - E_\eta\varphi| + \operatorname*{var}_\eta(\varphi - E_\eta\varphi)\big). \tag{3.40}$$

Observe that $\operatorname{var}_\eta(\varphi - E_\eta\varphi) = \operatorname{var}_\eta\varphi$, because $E_\eta\varphi$ is constant on η. We claim that $\sup_\eta|\varphi - E_\eta\varphi|$ is bounded above by $\operatorname{var}_\eta\varphi$. Indeed, suppose first that φ is real-valued. Part 3 of Lemma 1.3.1 and the definition of $E_\eta\varphi$ give, for every $x \in \eta$,

$$(\varphi - E_\eta\varphi)(x) \leq \frac{1}{m(\eta)}\int(\varphi - E_\eta\varphi)\, dm + \operatorname*{var}_\eta(\varphi - E_\eta\varphi) = \operatorname*{var}_\eta\varphi.$$

Applying the same argument to $E_\eta\varphi - \varphi$, we get that $\sup|\varphi - E_\eta\varphi| \leq \operatorname{var}_\eta\varphi$. Finally, this inequality remains true when φ is complex-valued, because the supremum of $|\varphi - E_\eta \varphi|$ is bounded by the sum of the suprema of the norms of its real part and imaginary part. Thus our claim is proved. Replacing these observations in (3.40), we get that the variation of the first term in (3.39) is bounded above by

$$\sum_{\eta \in \mathcal{Q}^n} C_1 \lambda_1^n \cdot 4 \operatorname*{var}_\eta \varphi \le 4 C_1 \lambda_1^n \operatorname{var} \varphi \le 4 C_1 \lambda_1^n \|\varphi\|_{\mathrm{BV}} \tag{3.41}$$

Analogously, the variation of the second term in (3.39) is bounded by

$$\sum_{\eta \notin \mathcal{Q}^n} \operatorname{var} g_\eta^n \, (3 \sup_\eta |\varphi| + \operatorname*{var}_\eta \varphi).$$

Clearly, $\operatorname{var}_\eta \varphi$ is bounded by the variation $\operatorname{var} \varphi$ of φ over the whole M, and analogously for the supremum. Therefore, using (3.36), the expression inside the parentheses is bounded by $4 \operatorname{var} \varphi + 3\|\varphi\|_1 \le 4\|\varphi\|_{\mathrm{BV}}$. Hence, the variation of the second term in (3.39) is less than

$$\sum_{\eta \notin \mathcal{Q}^n} \operatorname{var} g_\eta^n \cdot 4\|\varphi\|_{\mathrm{BV}} \le 4 C_1 \lambda_1^n \|\varphi\|_{\mathrm{BV}}\,. \tag{3.42}$$

The last inequality is due to our choice of $\mathcal{Q}^n$ in (3.37). Combining (3.41) and (3.42), we obtain

$$\operatorname{var}\big((\mathcal{L}^n - \mathcal{L}^n p_n)\varphi\big) \le 11 C_1 \lambda_1^n \|\varphi\|_{\mathrm{BV}}\,. \tag{3.43}$$

Now we need a corresponding estimate for the L^1-norm of $(\mathcal{L}^n - \mathcal{L}^n p_n)\varphi$. On the one hand, the supremum of the first term in (3.39) is less that

$$\sum_{\eta \in \mathcal{Q}^n} \sup g_\eta^n \sup_\eta |\varphi - E_\eta \varphi| \le \sum_{\eta \in \mathcal{Q}^n} C_1 \lambda_1^n \operatorname*{var}_\eta \varphi \le \; C_1 \lambda_1^n \operatorname{var} \varphi. \tag{3.44}$$

On the other hand, the supremum of the second term in (3.39) is less that

$$\sum_{\eta \notin \mathcal{Q}^n} \sup g_\eta^n \sup_\eta |\varphi| \le \sum_{\eta \notin \mathcal{Q}^n} \operatorname{var} g_\eta^n \sup |\varphi| \le \; C_1 \lambda_1^n \big(\operatorname{var} \varphi + 2\|\varphi\|_1 \big). \tag{3.45}$$

The first inequality uses $\sup g_\eta^n \le \inf g_\eta^n + \operatorname{var} g_\eta^n = \operatorname{var} g_\eta^n$. Adding (3.44) and (3.45), we get

$$\|(\mathcal{L}^n - \mathcal{L}^n p_n)\varphi\|_1 \le \sup |(\mathcal{L}^n - \mathcal{L}^n p_n)\varphi| \le 2 C_1 \lambda_1^n \|\varphi\|_{\mathrm{BV}}\,. \tag{3.46}$$

Finally, adding (3.43) and (3.46), we obtain (3.38). This completes the proof of the proposition. □

Remark 3.4.1. Every norm 1 eigenvalue λ_j is a root of unity. More than that, the intersection of spec($\mathcal{L}$) with the unit circle is a union of full cyclic groups, see [**?**, Theorem 1(a)]. In particular, there exists $N \ge 1$ such that 1 is the unique eigenvalue of $\mathcal{L}^N$ on the unit circle. Another interesting fact is that the number of ergodic absolutely continuous invariant probabilities of f is precisely the dimension of the eigenspace kernel(id $-L$) associated to the eigenvalue 1. See [**?**, Theorem 3].

3.4.2 Exactness

We continue to consider $f : M \to M$ a piecewise expanding map as in Theorem 3.4.1. Let $\mu = \varphi_0 m$ be any ergodic absolutely continuous invariant measure for f.

Proposition 3.4.2. *There exist $N \geq 1$, and measurable sets $X_1, \ldots, X_N$, such that $\mu(X_i) = 1/N$ for $1 \leq i \leq N$, and the normalized restrictions μ_i of μ to these X_i satisfy*

1. $\mu = (\mu_1 + \cdots + \mu_N)/N$;
2. $f_* \mu_i = \mu_{i+1}$ *for every* $1 \leq i < N$, *and* $f_* \mu_N = \mu_1$;
3. μ_i *is an exact measure for the iterate* f^N, *for every* $1 \leq i \leq N$.

The first step in the proof of Proposition 3.4.2 is the following lemma, containing a stronger form of results in Subsection 1.4.2. Let $\mathcal{F}$ be the Borel σ-algebra of M. Moreover, $\mathcal{F}_n = f^{-n}(\mathcal{F})$ for $n \geq 0$, and $\mathcal{F}_\infty = \cap_{n=0}^{\infty} \mathcal{F}_n$.

Lemma 3.4.2. *There exists $\delta > 0$ such that, given any set $Z \in \mathcal{F}$ with positive Lebesgue measure, for every n sufficiently large the image $f^n(Z)$ contains a measurable set with Lebesgue measure larger than δ.*

Proof. Let $\psi = \mathcal{X}_Z / m(Z)$. Fix $C_0 > 1$ and $\lambda_0 < 1$ as in Proposition 1.3.3. By part (e) of Lemma 3.4.1, there exist bounded variation functions φ such that

$$\Big| \int \psi \, dm - \int \varphi \, dm \Big| \leq \|\psi - \varphi\|_1 \leq \frac{1}{400 C_0} . \tag{3.47}$$

Let φ be fixed from now on. Since $\int \psi \, dm = \|\psi\|_1 = 1$, we have that $\int \varphi \, dm$ and $\|\varphi\|_1$ are between $1/2$ and $3/2$. By Proposition 1.3.3,

$$\operatorname{var}(\mathcal{L}^n \varphi) \leq C_0 \lambda_0^n \operatorname{var} \varphi + C_0 \|\varphi\|_1 \leq C_0 + 2C_0 = 3C_0$$

if n is large enough. Then, cf. (3.36) and Lemma 3.1.1,

$$\|\mathcal{L}^n \varphi\|_\infty \leq \operatorname{var}(\mathcal{L}^n \varphi) + 2\|\mathcal{L}^n \varphi\|_1 \leq 3C_0 + 2\|\varphi\|_1 \leq 3C_0 + 3 \leq 6C_0 .$$

Now let A_n be the set of points $y \in M$ such that $\mathcal{L}^n \varphi_k(y) \geq 1/4$, and B_n be the set of points $y \in M$ such that $|\mathcal{L}^n \psi(y) - \mathcal{L}^n \varphi(y)| \geq 1/8$. Then,

$$\frac{1}{2} \leq \int \varphi \, dm = \int \mathcal{L}^n \varphi \, dm \leq \frac{1}{4} + 6C_0 m(A_n).$$

Thus, $m(A_n) \geq 1/(25C_0)$. Moreover, $m(B_n) \leq 1/(50C_0)$. Indeed, this last fact is a direct consequence of

$$\|\mathcal{L}^n \psi - \mathcal{L}^n \varphi\|_1 \leq \|\psi - \varphi\|_1 \leq \frac{1}{400 C_0} .$$

It follows that the Lebesgue measure of $A_n \setminus B_n$ is at least $1/(50C_0)$. Observe that $\mathcal{L}^n\varphi(y) \geq 1/8 > 0$ for every $y \in A_n \setminus B_n$. On the other hand, by Lemma 3.1.2,

$$\mathcal{L}^n\varphi(y) = \sum_{x: f^n(x)=y} \frac{\varphi(x)}{|Df(x)|} = \frac{1}{m(Z)} \sum_{x: f^n(x)=y} \frac{\mathcal{X}_Z(x)}{|Df(x)|}$$

is nonzero only if $y \in f^n(Z)$. Therefore, the measurable set $A_n \setminus B_n$ is contained in $f^n(Z)$. This proves the lemma, for any $\delta < 1/(50C_0)$. □

It follows that the ambient M may be partitioned into a finite number of elements of $\mathcal{F}_\infty$ with minimal Lebesgue measure:

Corollary 3.4.1. *There exists a finite partition $\mathcal{Y}$ of M consisting of elements of $\mathcal{F}_\infty$ with positive Lebesgue measure, such that any set $W \in \mathcal{F}_\infty$ coincides with some union of elements of $\mathcal{Y}$, up to a zero Lebesgue measure subset.*

Proof. Fix $\delta > 0$ as given by Lemma 3.4.2. We claim that if $W^1, \ldots, W^s$ is any family of two-by-two disjoint elements of $\mathcal{F}_\infty$ with positive Lebesgue measure, then s can not exceed $1/\delta$. This can be seen as follows. By definition, for any $1 \leq i \leq s$ and $n \geq 1$ there exists $W_n^i \in \mathcal{F}$ such that $W^i = f^{-n}(W_n^i)$. Moreover, according to the previous lemma, for each $1 \leq i \leq s$ and $n \geq 1$ sufficiently large, we may a measurable set $E_n^i \subset f^n(W^i)$ with Lebesgue measure larger than δ. We fix such an $n \geq 1$ once and for all. Note that $E_n^i \subset f^n(W^i) = W_n^i \cap f^n(M)$ for every $1 \leq i \leq s$. Suppose $E_{i,n} \cap E_{j,n}$ is nonempty for some $i \neq j$. Then $W_n^i \cap W_n^j \cap f^n(M)$ is nonempty, and its preimage under f^n is a nonempty set contained in both W^i and W^j. This contradicts our assumptions on $W^1, \ldots, W^s$. Therefore, the sets E_n^i, $1 \leq i \leq s$, must be two-by-two disjoint. It follows that their number s is at most $1/\delta$, as we claimed.

Now let $\mathcal{Y}_1, \mathcal{Y}_2, \ldots$ be the sequence of finite subsets of $\mathcal{F}_\infty$ constructed by the following inductive procedure. $\mathcal{Y}_1$ contains only the set M. Suppose $\mathcal{Y}_n$ has been constructed, and there exists $Y \in \mathcal{Y}_n$ and $W \in \mathcal{F}_\infty$ such that $0 < m(Y \cap W) < m(Y)$. By definition, $\mathcal{Y}_{n+1}$ is obtained replacing Y by the two subsets $Y \cap W$ and $Y \setminus W$ in $\mathcal{Y}_n$. Observe that every $\mathcal{Y}_n$ is a partition of M into elements of $\mathcal{F}_\infty$ with positive Lebesgue measure. Thus, by the claim in the first part of the proof, this construction must stop after not more than $1/\delta$ steps. That is, there exists $r \leq 1/\delta$ such that, given any $W \in \mathcal{F}_\infty$ and $Y \in \mathcal{Y}_r$, either $m(Y \cap W) = 0$ or $m(Y \setminus W) = 0$. Then, up to zero Lebesgue measure, W coincides with the union of those elements Y of $\mathcal{Y}_r$ such that $m(Y \cap W)$ is nonzero. Thus, we may take $\mathcal{Y} = \mathcal{Y}_r$. □

Now we can prove Proposition 3.4.2:

Proof. If there is no $Z_0 \in \mathcal{F}_\infty$ with $0 < \mu(Z) < 1$ then μ is an exact measure for f. We take $N = 1$ and $\phi_1 = \varphi_0$, and the claims in the statement follow immediately.

Now suppose that there is such a subset $Z_0 \in \mathcal{F}_\infty$. Note that $m(Z_0) > 0$, because μ is absolutely continuous with respect to Lebesgue measure m. By Corollary 3.4.1, we may suppose that Z_0 has minimal Lebesgue measure, meaning that it has no subset $Z \in \mathcal{F}_\infty$ with $0 < m(Z) < m(Z_0)$: otherwise, it suffices to replace Z_0 by some element of the partition $\mathcal{Y}$ contained in it. Using absolute continuity once more, it follows that there is no subset $Z \in \mathcal{F}_\infty$ of Z_0 with $0 < \mu(Z) < \mu(Z_0)$. For each $n \geq 1$, let $Z_n \in \mathcal{F}$ be such that $Z_0 = f^{-n}(Z_n)$. Then $\mu(Z_n) = \mu(Z_0) > 0$ for every $n \geq 1$. Consequently, the Z_n can not be two-by-two disjoint: there exist $p \geq 0$ and $N \geq 1$ such that $Z_p \cap Z_{p+N}$ has positive μ-measure. We fix N smallest such that this happens. By Lemma 3.1.6,

$$\mu(Z_0 \cap Z_N) = \mu(f^{-p}(Z_p \cap Z_{p+N})) = \mu(Z_p \cap Z_{p+N}) > 0,$$

and $\mu(Z_0 \cap \tilde{Z}_N) = \mu(Z_0 \cap Z_N) > 0$. Clearly, $Z_0 \cap \tilde{Z}_N$ is in $\mathcal{F}_\infty$. So, since we took Z_0 with minimal measure, $\mu(Z_0 \cap Z_N) = \mu(Z_0 \cap \tilde{Z}_N) = \mu(Z_0)$. In other words, $Z_0 = f^{-N}(Z_N)$ coincides with Z_N up to a subset with zero μ-measure.

Let $X = X_1 \cup \cdots \cup X_N$, where $X_i = f^{-N+i}(Z_N)$ for each $1 \leq i \leq N$. Clearly, $\mu(X_i) = \mu(Z_N)$ for all $1 \leq i \leq N$. Due to our choice of N, the intersection $X_i \cap X_j$ has zero μ-measure for any $i \neq j$. Moreover, $f^{-1}(X) = X$, up to a set with zero μ-measure. Since μ is an ergodic measure for f, it follows that X has full μ-measure. Moreover, $\mu(X_i) = 1/N$ for every $1 \leq i \leq N$. Let $\mu_i = N\mathcal{X}_{X_i}\mu$ be the normalized restriction of μ to each X_i. Claim 1 in the proposition is a direct consequence of the previous observations: $1 = \mathcal{X}_X = \mathcal{X}_{X_1} + \cdots + \mathcal{X}_{X_N}$ at μ-almost every point. Claim 2, simply, corresponds to the fact that $X_i = f^{-1}(X_{i+1})$ for $1 \leq i < N$, and $X_N = Z_N$ coincides with $Z_0 = f^{-1}(X_1)$ up to a zero μ-measure set. In particular, we get that $f_*^N \mu_i = \mu_i$ for every $1 \leq i \leq N$. In other words, each probability μ_i is f^N-invariant. Moreover, μ_i is an exact measure for f^N. Before explaining this, let us point out that the two σ-algebras

$$\mathcal{F}_\infty(f) = \bigcap_{n=0}^{\infty} f^{-n}(\mathcal{F}) \quad \text{and} \quad \mathcal{F}_\infty(f^N) = \bigcap_{n=0}^{\infty} f^{-nN}(\mathcal{F}) \tag{3.48}$$

coincide, since the sequence $\mathcal{F}_n = f^{-n}(\mathcal{F})$ is monotone nonincreasing. Now, suppose there was $W_i \in \mathcal{F}_\infty$ such that $0 < \mu_i(W_i) < 1$. Since X_i has full measure for μ_i, we may suppose that $W_i \subset X_i$, and $0 < \mu(W_i) < \mu(X_i)$. Then, $W_0 = f^{-i}(W_i)$ is contained in $Z_0 = f^{-i}(X_i)$. Clearly, $\mu(W_0) = \mu(W_i)$ is positive and strictly less than $\mu(Z_0) = \mu(Z_i)$. On the other hand, from $W_i \in \mathcal{F}_{nN}$ we get that $W_0 \in \mathcal{F}_{nN+i}$, for all $n \geq 1$. Therefore, $W_0 \in \mathcal{F}_\infty$. This contradicts our choice of Z_0 having minimal measure. Hence, there is no such $W_i \in \mathcal{F}_\infty$, which proves that the measure μ_i is exact for f^N. □

Proposition 3.4.2 gives the decomposition of μ into exact components that was claimed in Theorem 3.4.1. To complete the proof of the theorem, we only have to prove the exponential convergence of the iterates $f_*^{nN}(\phi\mu_i)$. This will be done in the next proposition. Define

$$B_n^{N,i}(\varphi,\psi) = \int (\psi \circ f^{nN})\varphi\, dm - \int \psi\, d\mu_i \int \varphi\, dm \quad \text{and}$$
$$C_n^{N,i}(\varphi,\psi) = \int (\psi \circ f^{nN})\varphi\, d\mu_i - \int \psi\, d\mu_i \int \varphi\, d\mu_i\,,$$

whenever the integrals make sense.

Proposition 3.4.3. *There exist* $C_2 > 0$ *and* $\tau_2 < 1$ *such that, given any* $1 \le i \le s$, *we have*

$$|C_n^{N,i}(\phi,\psi)| = |BC_n^{N,i}(\phi\varphi_i,\psi)| \le C_2\tau_2^n \|\phi\|_{\mathrm{BV}}\|\psi\|_1$$

for any $n \ge 1$, $\phi \in \mathrm{BV}$, *and* $\psi \in L^1(m)$.

Proof. As pointed out at the beginning of Subsection 3.4.1, all the results therein extend to any iterate f^n, $n \ge 1$. In particular, Lemma 3.4.1 and Proposition 3.4.1 give that the transfer operator

$$\mathcal{L}^N : (\mathrm{BV}, \|\cdot\|_{\mathrm{BV}}) \to (\mathrm{BV}, \|\cdot\|_{\mathrm{BV}})$$

of f^N satisfies conditions (O1)-(O2) of Subsection 3.1.2 (with $p = \infty$) and is quasi-compact. Hence, we are in a position to use Proposition 3.1.2 and Corollary 3.1.2, with $E = \mathrm{BV}$, and $\mathcal{L}^N$ and μ_i in the place of $\mathcal{L}$ and μ, respectively. We conclude that

$$|C_n^{N,i}(\phi,\psi)| = |B_n^{N,i}(\phi\varphi_i,\psi)| \le C_1\tau_1^n \|\phi\varphi_i\|_{\mathrm{BV}}\|\psi\|_1$$

for any $n \ge 1$, $\phi \in F_0$, and $\psi \in L^1(m)$. Here φ_i is a density of μ_i. By definition, F_0 is the space of functions $\phi \in L^1(\mu_i)$ such that $\phi\varphi_i$ is in BV. To conclude the proof we only have to find a constant $K > 0$ such that $\|\phi\varphi_i\|_{\mathrm{BV}} \le K\|\phi\|_{\mathrm{BV}}$ for any $\phi \in \mathrm{BV}$. This can be done as follows. Firstly, since f^N is a piecewise expanding map satisfying (3.33), the density φ_i must have bounded variation. Recall Proposition 1.3.2 and Theorem 1.4.2. Then, by (3.35) and (3.36)

$$\operatorname{var}(\phi\varphi_i) \le 2\|\phi\|_\infty \operatorname{var}\varphi_i + 2\operatorname{var}\phi\|\varphi_i\|_\infty \le 4\|\varphi_i\|_{\mathrm{BV}}\|\phi\|_{\mathrm{BV}}$$

At this point it suffices to choose $K \ge 4\|\varphi_i\|_{\mathrm{BV}}$. Finally, take $C_2 = KC_1$ and $\tau_2 = \tau_1$. Since we are dealing with a finite number of measures μ_i only, we may suppose that C_1, K, and C_2 are the same for all $1 \le i \le s$. □

The proof of Theorem 3.4.1 is complete.

Remark 3.4.2. Let μ be a physical measure for a piecewise expanding map $f : M \to M$ as in Theorem 3.4.1. Suppose μ is ergodic for every iterate f^n of

f. Then μ is exact for f (this is equivalently to exactness for f^N, cf. (3.48)). Indeed, each μ_i in the decomposition given by the theorem is absolutely continuous with respect to μ, being a normalized restriction of μ. Moreover, μ and μ_i are both invariant under f^N. So, ergodicity implies that $\mu_i = \mu$ for all $1 \le i \le s$. Therefore, μ is exact, and we may take $N = 1$ in the theorem.

Another relevant situation where we have exactness is when the map f is topologically mixing, in the following strong sense: given any nonempty open set $U \subset M$, there exists $k \ge 1$ such that $f^k(U) = M$ (and then $f^n(U) = M$ for all $n > k$). Indeed, let $\mu_i = \varphi_i m$ be as in Theorem 3.4.1. As we have seen in the proof of Proposition 3.4.3, the density φ_i has bounded variation. Consequently, $\{\varphi_i > 0\}$ contains some interval U. Then it contains $f^{nN}(U)$ for every $n \ge 1$, because μ_i is f^N-invariant. Consequently, $\{\varphi_i > 0\}$ is the whole M. On the other hand, Lebesgue almost every point of $\{\varphi_i > 0\}$ is in the basin of μ_i, relative to the map f^N, because μ_i is ergodic for f^N. We have shown that Lebesgue almost every point of M is in the basin of every one of the measures μ_i. This implies that $\mu_1 = \cdots = \mu_N = \mu$. Therefore, μ is exact and we may take $N = 1$.

3.5 Subexponential Mixing

In this section we discuss decay of correlations slower than exponential. We use interval maps with neutral fixed points as a motivating example.

3.5.1 Maps With Neutral Fixed Points

Let us consider the following class of systems, that contains Example 1.1.2. In what follows, $f : M \to M$ is a piecewise C^2 transformation of the interval or the circle, with finitely many smoothness domains, satisfying

(N1) f has a fixed point p such that $Df(p) = 1$, and there are positive constants α_1 α_2, α_3, and d such that

$$\alpha_1 |x-p|^d \le Df(x) - 1 \le \alpha_2 |x-p|^d \quad \text{and} \quad |D^2 f(x)| \le \alpha_3 |x-p|^{d-1}$$

for all $x \ne p$ in a neighbourhood of p ($D^2 f(p)$ needs not exist);

(N2) $|Df(x)| > 1$ for all $x \ne p$; in fact, $|Df|$ is bounded from 1, and $|D^2 f|$ is bounded above, outside any neighbourhood of p;

(N3) there exists a half-neighbourhood $U = (p - \varepsilon, p)$ or $U = (p, p + \varepsilon)$ of p such that for every singular point c of f there exists $l \ge 1$ and some half-neighbourhood of c that is mapped onto U by f^l.

We are going to prove

Theorem 3.5.1. *Suppose $f : M \to M$ satisfies conditions (N1), (N2), (N3).*

1. *If $0 < d < 1$ then f admits a unique f-invariant probability measure μ_0 absolutely continuous with respect to Lebesgue measure. Moreover, μ_0 is ergodic, and its basin is a full Lebesgue measure subset of M.*
2. *If $d \geq 1$ then f has no finite absolutely continuous invariant measures, although it has σ-finite ones. The Dirac measure on the neutral fixed point $p = 0$ is the physical measure of f, and its basin is a full Lebesgue measure subset of M. In particular, the Lyapunov exponent $\lim_{n\to\infty} \frac{1}{n} \log |Df^n(x)|$ is zero at Lebesgue almost every $x \in M$.*

For instance, if f is C^2 near p then (N1) holds for $d = 1$ if and only if $D^2 f(p)$ is not zero. Moreover, d can not be less than 1 if f is C^2. It is no restriction to suppose $M = [0,1]$. For definiteness, we consider $p = 0$ but the case when p is an interior point is analogous, see Remark 3.5.1.

The proof of Theorem 3.5.1 has two main steps. First, we construct a piecewise expanding map $\hat{f}$ *induced from* f: it is defined on a full Lebesgue measure subset of $[0,1]$, and for each x in the domain there exists an integer $n(x) \geq 1$ so that

$$\hat{f}(x) = f^{n(x)}(x).$$

Then we invoke the results of Section 3.4 to get a unique $\hat{f}$-invariant probability $\hat{\mu}$ absolutely continuous with respect to Lebesgue measure. In the second part of the proof, we use $\hat{\mu}$ to obtain an f-invariant measure μ that is also absolutely continuous. We check that μ is finite if and only if $d < 1$ and, in that case, the normalization μ_0 of μ is the unique physical measure of f.

The construction of the induced map $\hat{f}$ goes as follows. Let c be the singular point of f closest to $p = 0$, and ξ_0 be the interval bounded by p and c. The remaining smoothness intervals of f are denoted $\xi_1, \ldots, \xi_N$. For each $0 \leq i \leq N$, represent by h_i the inverse branch of f on ξ_i, that is, $h_i = (f|\xi_i)^{-1}$. Write $c_1 = c$. Moreover, $c_{j+1} = h_0(c_j)$ and $\eta_j = (c_{j+1}, c_j) \subset \xi_0$ for each $j \geq 1$. This is well-defined, since $h_0 : f(\xi_0) \to \xi_0$ and $f(\xi_0)$ contains contains ξ_0. Finally, set

$$\hat{f}(x) = \begin{cases} f(x) & \text{for } x \in \xi_1 \cup \cdots \cup \xi_N \\ f^j(x) & \text{for } x \in \eta_j\,, \quad j \geq 1. \end{cases}$$

Figure 3.3 describes a case with $N = 1$, corresponding to Example 1.1.2.

Remark 3.5.1. When p is in the interior of M, let c^l and c^r be the two nearest singular points, to each side of p. For $* = l$ and $* = r$, let ξ_0^* be the interval bounded by p and c^*, and h_0^* be the inverse branch of f on ξ_0^*. Let $c_1^* = c^*$ and, for each $j \geq 1$, $c_{j+1}^* = h_0^*(c_j^*)$ and $\eta_j^* = (c_{j+1}^*, c_j^*) \subset \xi_0^*$. Finally, $\hat{f} = f^j$ on $\eta_j^l \cup \eta_j^r$, for any $j \geq 1$, and $\hat{f} = f$ in the complement of $\xi_0^l \cup \xi_0^r$. Up to straightforward adaptations, all the arguments that follow extend to this case.

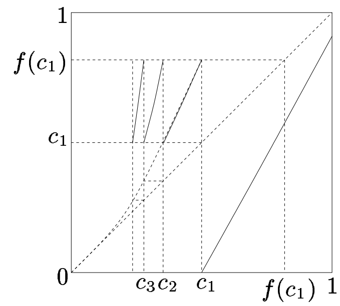

Fig. 3.3. Inducing a piecewise expanding map

Lemma 3.5.1. *There exist positive constants* $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ *such that*

$$\frac{\gamma_1}{j^{1/d}} \le c_j \le \frac{\gamma_2}{j^{1/d}} \qquad \textit{and} \qquad \frac{\gamma_3}{j^{1+1/d}} \le c_j - c_{j+1} \le \frac{\gamma_4}{j^{1+1/d}}$$

for every $j \ge 1$.

Proof. It follows from (N1) that $Dh_0(y) - 1 \approx -y^d$ for any y close to zero. Then, there are constants $\beta_2 > \beta_1 > 0$ such that

$$y - \beta_1 y^{d+1} \ge h_0(y) \ge y - \beta_2 y^{d+1}$$

for every y in some neighbourhood of zero. Fix $k \ge 1$ such that c_j is in that neighbourhood for all $j \ge k$. Choose $\gamma_1 > 0$ small and $\gamma_2 > 0$ large enough so that

$$\frac{\gamma_1}{j^{1/d}} \le c_j \le \frac{\gamma_2}{j^{1/d}} \quad \text{for every } 1 \le j \le k.$$

We are going to show, by induction on j, that these inequalities remain valid for all j, with one extra condition on $\gamma_1\,\gamma_2$. Assuming $c_j \ge \gamma_1/j^{1/d}$, then

$$c_{j+1} = h_0(c_j) \ge h_0\left(\frac{\gamma_1}{j^{1/d}}\right) \ge \left(\frac{\gamma_1}{j^{1/d}}\right) - \beta_2\left(\frac{\gamma_1}{j^{1/d}}\right)^{d+1}$$

Fix constants $K_2 > 1 > K_1 > 0$ such that $1 + K_1 x \le (1+x)^{1/d} \le 1 + K_2 x$ for all $0 \le x \le 1$. The last term in the previous inequality may be rewritten

$$\frac{\gamma_1}{(j+1)^{1/d}}\left(\frac{j+1}{j}\right)^{1/d}\left(1 - \beta_2\gamma_1^d\frac{1}{j}\right) \ge \frac{\gamma_1}{(j+1)^{1/d}}\left(1 + K_1\frac{1}{j}\right)\left(1 - \beta_2\gamma_1^d\frac{1}{j}\right)$$

The product of the two last factors is larger than 1, if we assume that γ_1 was chosen small enough so that $2\beta_2\gamma_1^d < K_1$. Then $c_{j+1} \ge \gamma_1/(j+1)^{1/d}$. The proof of the upper bound is analogous, assuming $\beta_1\gamma_2^d > K_2$. This proves the first claim in the statement. The second claim is an immediate consequence:

$$c_j - c_{j+1} \geq \beta_1 c_j^{d+1} \geq \frac{\gamma_3}{j^{1+1/d}} \quad \text{and} \quad c_j - c_{j+1} \leq \beta_2 c_j^{d+1} \leq \frac{\gamma_4}{j^{1+1/d}}$$

with $\gamma_3 = \beta_1 \gamma_1^{d+1}$ and $\gamma_4 = \beta_2 \gamma_2^{d+1}$. □

Lemma 3.5.2. *$\hat{f}$ is a piecewise expanding map with long branches (recall Definition 1.3.6).*

Proof. Let $\sigma = \inf\{|Df(x)| : x \in (c_2, 1]\}$. Then, $\sigma > 1$ and $|D\hat{f}(x)| \geq \sigma$ at every point. Take $\mathcal{P}^1 = \{\xi_1, \ldots, \xi_N, \eta_1, \eta_2, \ldots\}$ as a partition into regularity intervals: clearly, $\hat{f}$ is C^2 on each ξ_i and each η_j. By construction, we have $\hat{f}(\eta_j) = f(\eta_1)$ for every $j \geq 1$. This implies that the set of all images of the smoothness intervals is finite, and then condition (b) of Definition 1.3.6 is automatic. Let us check condition (a). Assumption (N2) ensures that

$$\frac{D^2 \hat{f} \mid \xi_i}{(D\hat{f})^2 \mid \xi_i} = \frac{D^2 f \mid \xi_i}{(Df)^2 \mid \xi_i}$$

is bounded for every $1 \leq i \leq N$. So, we only have to check that $D^2\hat{f}/(D\hat{f})^2$ is also bounded in ξ_0. As a first step, we claim that there exists $K_1 > 0$ such that, given any $j \geq 1$ and any $x, y \in \eta_j$,

$$\frac{1}{K_1} \leq \frac{Df^j(x)}{Df^j(y)} \leq K_1 .$$

Let us explain why this is so. By assumptions (N1), (N2) and Lemma 3.5.1,

$$|D(\log Df)(z)| = \frac{|D^2 f|}{|Df|}(z) \leq \frac{\alpha_3 \gamma_2^{d-1}}{i^{1-1/d}}$$

for every $z \in \eta_i$ and $i \geq 1$. Moreover, if $x, y \in \eta_j$ then

$$|f^{j-i}(x) - f^{j-i}(y)| \leq m(\eta_i) \leq \frac{\gamma_4}{i^{1+1/d}}$$

for any $1 \leq i \leq j$. So,

$$\begin{aligned}
\left| \log \frac{Df^j(x)}{Df^j(y)} \right| &\leq \sum_{i=1}^{j} \left| \log Df(f^{j-i}(x)) - \log Df(f^{j-i}(y)) \right| \\
&\leq \sum_{i=1}^{j} \frac{\alpha_3 \gamma_2^{d-1}}{i^{1-1/d}} \cdot \frac{\gamma_4}{i^{1+1/d}} = \sum_{i=1}^{j} \frac{\alpha_3 \gamma_2^{d-1} \gamma_4}{i^2} .
\end{aligned}$$

This proves our claim, for $K_1 = \exp(\alpha_3 \gamma_2^{d-1} \gamma_4 \sum_{i=1}^{\infty} i^{-2})$. Now, for any $j \geq 1$ and $x \in \eta_j$,

$$\frac{D^2\hat{f}}{(D\hat{f})^2}(x) = \frac{1}{D\hat{f}(x)} D(\log D\hat{f})(x) = \frac{1}{D\hat{f}(x)} \sum_{l=0}^{j-1} D\big(\log(Df \circ f^l)\big)(x)$$

$$= \frac{1}{Df^j(x)} \sum_{l=0}^{j-1} \frac{D^2 f}{Df}(f^l(x))\, Df^l(x) = \sum_{l=0}^{j-1} \frac{(D^2 f/Df)(f^l(x))}{Df^{j-l}(f^l(x))}.$$

Since $f^l(x) \in \eta_{j-l}$, the numerator is bounded by $\alpha_3 \gamma_2^{d-1}(j-l)^{-1+1/d}$. On the other hand, by the claim and Lemma 3.5.1,

$$\frac{1}{|Df^{j-l}(f^l(x))|} \le K_1 \frac{m(\eta_{j-l})}{m(f^l(\eta_{j-l}))} \le \frac{K_1}{\ell} \frac{\gamma_4}{(j-l)^{1+1/d}}, \tag{3.49}$$

where ℓ is the length of $(c_1, f(c_1)] = f^l(\eta_{j-l})$. Let $K_2 = (K_1/\ell)\alpha_3\gamma_2^{d-1}\gamma_4$. Then

$$\left|\frac{D^2\hat{f}}{(D\hat{f})^2}(x)\right| \le \sum_{l=0}^{j-1} K_2 (j-l)^{-2} \le \sum_{l=1}^{\infty} K_2 i^{-2}.$$

This completes the proof of the lemma. □

According to Proposition 1.3.4, the map $\hat{f}$ is in the conditions of Theorem 1.3.2. In particular, it has some ergodic absolutely continuous invariant measure. The proposition also guarantees that such measures are finitely many. However, under assumption (N3) we have uniqueness:

Lemma 3.5.3. *Any $\hat{f}$-invariant subset of M has either zero or full Lebesgue measure. Consequently, $\hat{f}$ has a unique invariant probability measure $\hat{\mu}$ absolutely continuous with respect to Lebesgue measure. The density $\hat{\varphi} = d\hat{\mu}/dm$ is bounded on M, and bounded from zero in a neighbourhood of $p = 0$.*

Proof. The first statement follows from a variation of arguments in Subsections 1.3.4 and 1.4.2. Let A be an invariant subset with positive Lebesgue measure. By Lemma 1.4.1 and Corollary 1.4.1 and, it fills-in some interval up to a zero Lebesgue measure subset. Cf. Corollary 1.3.2, this interval may be taken to be a neighbourhood of some singular point of $\hat{f}$. Then, in view of (N3), A fills-in a full Lebesgue measure subset of $(0, \delta)$, for any small δ. If the complement of A had positive Lebesgue measure, we could apply the same argument to it, thus reaching a contradiction. So, A has full Lebesgue measure.

In particular, the basin of any ergodic absolutely continuous invariant measure is a full Lebesgue measure subset, and so there exists only one such measure. Since any absolutely continuous invariant measure is a convex combination of ergodic ones, cf. Theorem 1.3.2, this implies the uniqueness claim. The rest of the statement is proved in a similar way. By Corollary 1.4.1, the density $\hat{\varphi}$ has bounded variation, and so it is bounded. Moreover, there exists some interval where $\hat{\varphi}$ is positive and bounded from zero. Mapping this interval forward, and arguing as in the first part of the proof, we conclude that $\hat{\varphi}$ is bounded from zero in $(0, \delta)$, for any small δ. □

Now we move toward the second part of the proof of Theorem 3.5.1. From $\hat{\mu}$ we find an invariant (possibly infinite) measure μ for the initial map f:

$$\mu = \sum_{i=1}^{N} (\hat{\mu}|\xi_i) + \sum_{j=1}^{\infty} \sum_{i=0}^{j-1} f_*^i(\hat{\mu}|\eta_j). \tag{3.50}$$

This is an example of the following general construction.

Lemma 3.5.4. *Let $g : M \to M$ be measurable, and $\hat{g}$ be a map induced from g: there exists a measurable function $n(\cdot)$ with values in $\mathbb{N}$ such that $\hat{g}(x) = g^{n(x)}(x)$ for all x. Suppose $\hat{g}$ preserves some probability measure $\hat{\nu}$. Then*

$$\nu = \sum_{j=1}^{\infty} \sum_{i=0}^{j-1} g_*^i\big(\hat{\nu}|\{x : n(x) = j\}\big) = \sum_{i=0}^{\infty} g_*^i\big(\hat{\nu}|\{x : n(x) > i\}\big)$$

is a (finite or infinite) invariant measure for g. Moreover, $\nu(M) = \int n\, d\hat{\nu}$, so that ν is finite if and only if the inducing time $n(\cdot)$ is $\hat{\nu}$-integrable.

Proof. It is clear that ν is σ-additive, since it is given by a countable sum of measures. Invariance is proved as follows. First,

$$g_*\nu = \sum_{j=1}^{\infty} \sum_{i=1}^{j} g_*^i\big(\hat{\nu} \mid \{n = j\}\big) = \nu + \sum_{j=1}^{\infty} g_*^j\big(\hat{\nu} \mid \{n = j\}\big) - \sum_{j=1}^{\infty} \big(\hat{\nu} \mid \{n = j\}\big)$$

By the definition of $\hat{g}$,

$$\sum_{j=1}^{\infty} g_*^j\big(\hat{\nu} \mid \{n = j\}\big) = \sum_{j=1}^{\infty} \hat{g}_*\big(\hat{\nu} \mid \{n = j\}\big) = \hat{g}_*\big(\sum_{j=1}^{\infty} \hat{\nu} \mid \{n = j\}\big) = \hat{g}_*\hat{\nu}\,.$$

Since $\hat{\nu}$ is $\hat{g}$-invariant, this is the same as $\hat{\nu} = \sum_{j=1}^{\infty} \big(\hat{\nu} \mid \{n = j\}\big)$. Therefore, $g_*\nu = \nu$. Finally,The last part of the lemma is immediate:

$$\nu(M) = \sum_{j=1}^{\infty} \sum_{i=0}^{j-1} \hat{\nu}\big(\{n = j\} \cap g^{-i}(M)\big) = \sum_{j=1}^{\infty} j\hat{\nu}\big(\{n = j\}\big) = \int n\, d\hat{\nu}.$$

gives the last part of the lemma. □

Remark 3.5.2. If the inducing time $n(\cdot)$ satisfies a stopping rule

$$g(\{x : n(x) = j\}) \subset \{y : n(y) = j-1\}\quad \text{for any } j > 1, \tag{3.51}$$

then $\nu(\{n = k\})$ is finite for every $k \geq 1$, and so ν is σ-finite. Indeed, given $j > i \geq 0$, the set $\{n = j\} \cap g^{-i}(\{n = k\})$ is empty unless $k = j - i$, and so

$$\begin{aligned}\nu(\{n = k\}) &= \sum_{j=1}^{\infty}\sum_{i=0}^{j-1} \hat{\nu}\big(\{n = j\} \cap g^{-i}(\{n = k\})\big)\\ &= \sum_{j=k}^{\infty} \hat{\nu}\big(\{n = j\} \cap g^{k-j}(\{n = k\})\big) \leq \sum_{j=k}^{\infty} \hat{\nu}\big(\{n = j\}\big) \leq \hat{\nu}(M).\end{aligned}$$

Rule (3.51) holds in our setting, because $f(\eta_j) = \eta_{j-1}$ for all $j > 1$. Hence, the measure μ in (3.50) is always σ-finite. Moreover,

Corollary 3.5.1. *The measure μ is finite if and only if $d < 1$.*

Proof. We have

$$\int n\, d\hat{\mu} = \sum_{i=1}^{N} \hat{\mu}(\xi_i) + \sum_{j=1}^{\infty} j\hat{\mu}(\eta_j).$$

By Lemma 3.5.3, the density $\hat{\varphi}$ of $\hat{\mu}$ is bounded away from zero and infinity in a neighbourhood of zero. So $\hat{\mu}(\eta_j) \approx m(\eta_j)$, at least for all large j ($\approx$ means equality up to a positive factor independent of j). Moreover, $m(\eta_j) \approx j^{-1-1/d}$ for all j, by Lemma 3.5.1. This gives $j\hat{\mu}(\eta_j) \approx j^{-1/d}$, if j is large. So $n(\cdot)$ is integrable if and only if $1/d > 1$. □

Lemma 3.5.5. *The measure μ is absolutely continuous relative to Lebesgue measure, with density*

$$\varphi = \hat{\varphi} \sum_{i=1}^{N} \mathcal{X}_{\xi_i} + \mathcal{X}_{\xi_0} \sum_{i=0}^{\infty} \frac{\hat{\varphi}}{|Df^i|} \circ h_0^i$$

Proof. The definition (3.50) means that, for any measurable subset B,

$$\mu(B) = \sum_{i=1}^{N} \hat{\mu}(B \cap \xi_i) + \sum_{i=0}^{\infty} \hat{\mu}\big((0, c_{i+1}) \cap f^{-i}(B)\big).$$

For any $i \geq 0$, the map f^i sends $(0, c_{i+1})$ diffeomorphically onto ξ_0. The inverse of this diffeomorphism is the ith iterate h_0^i of the inverse branch h_0. So, by change of variables,

$$\hat{\mu}\big((0, c_{i+1}) \cap f^{-i}(B)\big) = \int_{(0,c_{i+1})\cap f^{-i}(B)} \hat{\varphi}\, dm = \int_{\xi_0 \cap B} \left(\frac{\hat{\varphi}}{|Df^i|} \circ h_0^i\right) dm.$$

This gives

$$\mu(B) = \sum_{i=1}^{N} \int_B \mathcal{X}_{\xi_i} \hat{\varphi}\, dm + \sum_{i=0}^{\infty} \int_B \mathcal{X}_{\xi_0} \left(\frac{\hat{\varphi}}{|Df^i|} \circ h_0^i \right) dm,$$

for any measurable set B, which was our claim. □

We use this expression of $\hat{\varphi}$ to show that any finite absolutely continuous invariant measure of f must be a multiple of μ.

Lemma 3.5.6. *If ν is a finite f-invariant measure absolutely continuous with respect to Lebesgue measure, then there exists $C > 0$ such that $\nu = C\mu$.*

Proof. We write $\nu = \psi m$, for some nonnegative integrable function *psi*. Given any function ϕ, we define

$$\mathcal{L}_0 \phi = \frac{\phi \mathcal{X}_{\xi_0}}{|Df|} \circ h_0 \quad \text{and} \quad \mathcal{L}_1 \phi = \sum_{i=1}^{N} \frac{\phi \mathcal{X}_{\xi_i}}{|Df|} \circ h_i\,,$$

where each term is understood to be identically zero outside the corresponding domain $f(\xi_i)$ of h_i. Then the transfer operator $\mathcal{L}$ of f is given by

$$\mathcal{L}\phi = \sum_{i=0}^{N} \frac{\phi \mathcal{X}_{\xi_i}}{|Df|} \circ h_i = \mathcal{L}_1 \phi + \mathcal{L}_0 \phi\,,$$

and the transfer operator $\widehat{\mathcal{L}}$ of $\hat{f}$ is given by

$$\widehat{\mathcal{L}}\phi = \sum_{i=1}^{N} \frac{\phi \mathcal{X}_{\xi_i}}{|Df|} \circ h_i + \sum_{j=1}^{\infty} \frac{\phi \mathcal{X}_{\eta_j}}{|Df^j|} \circ h_0^j = \mathcal{L}_1 \phi + \sum_{j=1}^{\infty} \mathcal{L}_0^j (\phi \mathcal{X}_{\eta_j}).$$

The conclusion of Lemma 3.5.5 may be restated

$$\varphi = \hat{\varphi} \sum_{i=1}^{N} \mathcal{X}_{\xi_i} + \sum_{j=1}^{\infty} (\mathcal{L}_0^i \hat{\varphi}) \mathcal{X}_{\xi_0}\,. \tag{3.52}$$

We are going to show that ψ is given by a similar expression, with $\hat{\varphi}$ replaced by some function $\hat{\psi}$. Moreover, $\hat{\psi}$ is a multiple of $\hat{\varphi}$.

Take $\hat{\psi} = \psi - (\mathcal{L}_0 \psi) \mathcal{X}_{\xi_0}$. Then $\hat{\psi}$ is an integrable function:

$$\int |\hat{\psi}|\, dm \le \int \psi\, dm + \int \mathcal{L}_0 \psi\, dm = \int \psi\, dm + \int \psi \mathcal{X}_{\xi_0}\, dm \le 2 \int \psi\, dm.$$

By recurrence (note that $\mathcal{L}_0(g\mathcal{X}_{\xi_0}) = \mathcal{L}_0(g)$ for any g),

$$\psi = \hat{\psi} + (\mathcal{L}_0 \psi)\, \mathcal{X}_{\xi_0} = \hat{\psi} + \sum_{i=1}^{n-1} (\mathcal{L}_0^i \hat{\psi})\, \mathcal{X}_{\xi_0} + (\mathcal{L}_0^n \psi)\, \mathcal{X}_{\xi_0}$$

for every $n \geq 1$. Moreover,

$$\int (\mathcal{L}_0^n \psi)\, \mathcal{X}_{\xi_0}\, dm = \int \psi(\mathcal{X}_{\xi_0} \circ f^n)\, dm = \int \psi \mathcal{X}_{(0,c_{n+1})}\, dm \to 0$$

as $n \to \infty$. Therefore,

$$\psi = \hat{\psi} + \sum_{i=1}^{\infty} (\mathcal{L}_0^i \hat{\psi})\, \mathcal{X}_{\xi_0} = \hat{\psi} \sum_{i=1}^{N} \mathcal{X}_{\xi_1} + \sum_{i=0}^{\infty} (\mathcal{L}_0^i \hat{\psi})\, \mathcal{X}_{\xi_0}. \tag{3.53}$$

We already know, from Lemma 3.5.3, that $\hat{f}$ has a unique absolutely continuous invariant probability $\hat{\mu} = \hat{\varphi} m$. This means that any fixed point of the operator $\widehat{\mathcal{L}}$ in $L^1(m)$ is a multiple of $\hat{\varphi}$. Therefore, to prove that $\hat{\psi} = C\hat{\varphi}$ for some constant $C > 0$ it suffices to check that $\widehat{\mathcal{L}}\hat{\psi} = \hat{\psi}$. The first step is just the definition of $\hat{\psi}$ and $\widehat{\mathcal{L}}$:

$$\widehat{\mathcal{L}}\hat{\psi} = \mathcal{L}_1 \psi - \mathcal{L}_1\big((\mathcal{L}_0\psi)\, \mathcal{X}_{\xi_0}\big) + \sum_{j=1}^{\infty} \mathcal{L}_0^j \big(\psi\, \mathcal{X}_{\eta_j}\big) - \sum_{j=1}^{\infty} \mathcal{L}_0^j \big((\mathcal{L}_0\psi)\, \mathcal{X}_{\xi_0}\, \mathcal{X}_{\eta_j}\big).$$

Observe that

$$\mathcal{L}_1\big((\mathcal{L}_0\psi)\, \mathcal{X}_{\xi_0}\big) = \sum_{i=1}^{N} \frac{(\mathcal{L}_0\psi)\, \mathcal{X}_{\xi_0}}{|Df|} \circ h_i = 0$$

because $(\mathcal{L}_0\psi)\, \mathcal{X}_{\xi_0}$ is zero on the image ξ_i of h_i. Moreover,

$$\mathcal{L}_0^j\big((\mathcal{L}_0\psi)\, \mathcal{X}_{\xi_0}\, \mathcal{X}_{\eta_j}\big) = \mathcal{L}_0^j\big((\mathcal{L}_0\psi)\, \mathcal{X}_{\eta_j}\big) = \mathcal{L}_0^{j+1}\big(\psi\, \mathcal{X}_{\eta_{j+1}}\big),$$

for every $j \geq 1$. This means that the two sums above cancel each other out, except for the very first term in the first sum. In this way we get

$$\widehat{\mathcal{L}}\hat{\psi} = \mathcal{L}_1\psi + \mathcal{L}_0\big(\psi\, \mathcal{X}_{\eta_1}\big) = \mathcal{L}\psi - \mathcal{L}_0\psi + \mathcal{L}_0\big(\psi\, \mathcal{X}_{\eta_1}\big).$$

Since $\nu = \psi\, m$ is an f-invariant measure, $\mathcal{L}\psi = \psi$. On the other hand,

$$\mathcal{L}_0\psi - \mathcal{L}_0\big(\psi\, \mathcal{X}_{\eta_1}\big) = (\mathcal{L}_0\psi)\mathcal{X}_{f(\xi_0)} - (\mathcal{L}_0\psi)\mathcal{X}_{f(\eta_1)} = (\mathcal{L}_0\psi)\mathcal{X}_{\xi_0}\,.$$

This shows that $\widehat{\mathcal{L}}\hat{\psi} = \hat{\psi}$. As already explained, it follows that $\hat{\psi} = \hat{\varphi}$ for some $C > 0$. From (3.52) and (3.53), it follows that $\psi = C\varphi$ or, equivalently, $\nu = C\mu$. □

Suppose $d < 1$. Corollary 3.5.1 and Lemma 3.5.6 imply that f has exactly one absolutely continuous invariant probability $\mu_0 = \mu/\mu(M)$. Its basin $B(\mu_0)$ has positive Lebesgue measure, and is an f-invariant set. Then $B(\mu_0)$ is also $\hat{f}$-invariant and, by Lemma 3.5.3, it has full Lebesgue measure in M. This completes the proof of the first part of Theorem 3.5.1.

When $d \geq 1$, Corollary 3.5.1 and Lemma 3.5.6 imply that f has no finite absolutely continuous invariant measures. On the other hand, μ is σ-finite, invariant, and absolutely continuous. In the next lemma we show that in this case all typical trajectories spend most of the time in the vicinity of the neutral fixed point 0.

Lemma 3.5.7. *Let $\tau_n(k,x)$ be the fraction of time, up to the nth iterate, spent by the orbit of x outside the interval $(0, c_k)$. In other words,*

$$\tau_n(k,x) = \frac{1}{n}\#\{0 \le s < n : f^s(x) \notin (0, c_k)\}.$$

If $d \ge 1$ then $\tau_n(k,x) \to 0$ as $n \to \infty$, for any $k \ge 1$ and Lebesgue almost any point $x \in M$.

Proof. First we observe that the average time the $\hat{f}$-orbit of x spends in η_j,

$$\theta_m(j,x) = \frac{1}{m}\#\{0 \le i < m : \hat{f}^i(x) \in \eta_j\} \to \hat{\mu}(\eta_j), \tag{3.54}$$

as $m \to \infty$, for each $j \ge 1$ and every point $x \in B(\hat{\mu})$. Analogously,

$$\theta_m(x) = \frac{1}{m}\#\{0 \le i < m : \hat{f}^i(x) \in M \setminus \xi_0\} \to \hat{\mu}(M \setminus \xi_0), \tag{3.55}$$

as $m \to \infty$, for every point $x \in B(\hat{\mu})$. This is because $m^{-1}\sum_{i=0}^{m} \delta_{\hat{f}^i(x)}$ converges to $\hat{\mu}$ in the weak* sense, and the boundary of η_j or $M \setminus \xi_0$ has zero $\hat{\mu}$-measure. In view of Lemma 3.5.3, this means that (3.54) and (3.55) are true for Lebesgue almost every $x \in M$.

We want to express this information in terms of iterates of the map f. Let $k \ge 1$ be fixed, and x be any point satisfying (3.54). To deal with the two different time scales corresponding to our two maps, we introduce the increasing sequence of functions $l(i)$, $i \ge 0$, with values in $\mathbb{N}$, given by $\hat{f}^i = f^{l(i)}$. By the definition of $\hat{f}$,

$$l(i+1) = \begin{cases} l(i) + 1 & \text{if } \hat{f}^i(x) \in M \setminus \xi_0 \\ l(i) + j & \text{if } \hat{f}^i(x) \in \eta_j\,, \quad j \ge 1. \end{cases}$$

Given any $n \ge 1$, let m be the unique integer such that $l(m) \le n < l(m+1)$. Up to time m, the $\hat{f}$-trajectory of x spends $m\theta(x)$ iterates in $M \setminus \xi_0$, and it hits each interval η_j exactly $m\theta_m(j,x)$ times. So,

$$n = (n - l(m)) + l(m) = (n - l(m)) + m\theta_m(x) + \sum_{j=1}^{\infty} jm\theta_m(j,x). \tag{3.56}$$

Suppose $i \le m$ is such that $\hat{f}^i(x) \in \eta_j$ for some $j > k$. Then there are exactly k iterates of the f-orbit of x in the time interval from $l(i)$ to $l(i+1)$ that are outside $(0, c_k)$. In particular, this is also true for the time interval from $l(m)$ to n. This shows that the f-orbit of x spends not more than

$$k + m\theta_m(x) + \sum_{j \le k} jm\theta_m(j,x) + \sum_{j > k} km\theta_m(j,x)$$

of its first n iterates outside $(0, c_k)$. Moreover, this is less than $2km$ because $\theta_m(x) + \sum_{j\geq 0} \theta_m(j,x) = 1$. So, using (3.56),

$$\tau_n(k,x) \leq \frac{2km}{n} \leq \frac{2k}{\theta_m(x) + \sum_{j=1}^{\infty} j\theta_m(j,x)}$$

By (3.54), the denominator converges to $\hat{\mu}(M \setminus \xi_0) + \sum_{j=1}^{\infty} j\hat{\mu}(\eta_j) = \mu(I) = \infty$ as n and m go to ∞. It follows that $\tau_n(k) \to 0$ when $n \to \infty$. □

As observed in Example 1.1.2, this means that the Dirac measure at the point $p = 0$ is the physical measure of f, with basin having full Lebesgue measure in M. Since $\log |Df|$ is continuous at zero, the same argument shows that

$$\lim_n \frac{1}{n} \log |Df^n(x)| = \log |Df(0)| = 0.$$

This completes the proof of Theorem 3.5.1.

3.5.2 Polynomial Convergence

One reason why maps with neutral fixed points attracted so much of attention is that they provide the simplest examples with decay of correlations that is slower than exponential.

Suppose $f : M \to M$ satisfies conditions (N1), (N2), (N3). In what follows we restrict ourselves to the case $0 < d < 1$, so that f has a unique f-invariant probability measure μ_0. $C_n(\cdot,\cdot)$ will denote correlations with respect to μ_0:

$$C_n(\varphi,\psi) = \int (\psi \circ f^n)\psi \, d\mu_0 - \int \psi \, d\mu_0 \int \varphi \, d\mu_0 .$$

As before, we consider $M = [0,1]$ and the neutral fixed point to be $p = 0$. In addition, we suppose that f has a unique singular point c, and maps both $[0,c)$ and $[c,1)$ monotonically onto $[0,1)$. The latter is not strictly necessary for the following theorem, but it makes the proof much more transparent.

Theorem 3.5.2. *For any Hölder continuous function $\varphi : M \to \mathbb{R}$ and any $\beta < (1/d) - 1$ there exists $K(\varphi) > 0$ such that*

$$|C_n(\varphi,\psi)| \leq K(\varphi) n^{-\beta} \|\psi\|_{1,\mu}$$

for any $\psi \in L^1(\mu)$.

As we shall see in Subsection 3.5.5, one may even take $\beta = (1/d) - 1$, and this bound is sharp: correlations decay not faster than $n^{1-1/d}$. The way we prove Theorem 3.5.2, is by reduction to an abstract model, a tower extension of the original map.

Let $(X_0, \mathcal{A}, m)$ be a probability space: $\mathcal{A}$ is a σ-algebra of subsets of X_0, and m is a probability measure in $\mathcal{A}$. Let $f : X_0 \to X_0$ be a measurable transformation, and $n : X_0 \to \mathbb{N}$ be a measurable integer function on X_0. Denote

$$X_j = \{x \in X_0 : n(x) > l\} \quad \text{and} \quad Y_j = \{x \in X_0 : n(x) = l\}$$

for each $j \geq 1$. Define $F : X_0 \to X_0$ by $(F \mid Y_l) = (f^l \mid Y_l)$ for each $l \geq 1$. In other words, $F(x) = f^{n(x)}(x)$ for every $x \in X_0$. We assume

(T1) For each $l \geq 1$, either Y_l is empty or $(F \mid Y_l)$ is a bijection from Y_l onto X_0, absolutely continuous with respect to the measure m.

The last conditions means that $F_*(m \mid Y_l)$ is absolutely continuous with respect to m, for every $l \geq 1$ (and so the same is true for $F_* m$). Then

$$JF(x) = \left(\frac{dF_*(m \mid Y_l)}{dm} \right)^{-1} (F(x))$$

defines a Jacobian for F with respect to m:

$$m(F(A)) = \int_A JF(x)\, dm(x)$$

for any measurable set $A \subset Y_l$. And

$$JF^n(x) = JF(x)\, JF(F(x)) \cdots JF(F^{n-1}(x))$$

defines a Jacobian for F^n, for any $n \geq 1$. Note that Jacobians are uniquely defined m-almost everywhere. In what follows, we assume that a representative of JF has been chosen that is defined at all points and satisfies

(T2) there exist $a_0 > 0$, $0 < \nu_0 \leq 1$, and a bounded distance $d(\cdot,\cdot)$ on X_0 such that

$$\log JF^m(x) - \log JF^m(y) \leq a_0\, d(F^m(x), F^m(y))^{\nu_0}$$

for every $m \geq 1$, and any $x, y \in X_0$ such that $F^j(x)$ and $F^j(y)$ are in the same set $Y_{l(j)}$ for each $0 \leq j \leq m$.

One particular case is when $\log JF$ is Hölder continuous on each Y_l, and there exists $\sigma > 1$ such that each $F \mid Y_l$ is a σ-expansion for $d(\cdot,\cdot)$.

Furhermore, we assume that

(T3) the function $n(\cdot)$ is m-integrable: $\displaystyle\sum_{j=1}^{\infty} j\, m(Y_j) = \sum_{j=0}^{\infty} m(X_j) < \infty.$

The extension $\tilde{f} : \widetilde{X} \to \widetilde{X}$ of $f : X_0 \to X_0$ that we are going to consider is defined as follows. Firstly, let

$$\widetilde{X} = \bigcup_{j=0}^{\infty} E_j\,,$$

where each E_j is the disjoint union of all $f^j(Y_l)\times\{j\}$ with $l > j$. For simplicity, we shall represent by $f^j(X_k)$ the disjoint union of all $f^j(Y_l)$ over all $l > k \geq j$, although this notation is strictly accurate only if f^j is one-to-one on X_k. So, up to this slight abuse of language, $E_j = f^j(X_j) \times \{j\}$. Next, define $\tilde{f} : \widetilde{X} \to \widetilde{X}$ by

$$\tilde{f}(x,j) = \begin{cases} (f(x), j+1) & \text{if } x \in f^j(X_{j+1}) \\ (f(x), 0) & \text{if } x \in f^j(Y_{j+1}) \end{cases}$$

By construction, $\pi \circ \tilde{f} = f \circ \pi$, where $\pi : \widetilde{X} \to X_0$ is the natural projection $\pi(x,j) = x$. Finally, let us fix some nondecreasing sequence $(\omega_j)_{j\geq 0}$ of real numbers with $\omega_0 = 1$, such that $\lambda_j = \omega_j/\omega_{j+1}$, is also nondecreasing, and

$$\sum_{j=0}^{\infty} \omega_j m(X_j) < \infty. \tag{3.57}$$

Under hypothesis (T3) such a sequence always exists. Then we endow $\widetilde{X}$ with the measure $\widetilde{m}$ defined by

$$\widetilde{m}(f^j(A) \times \{j\}) = \omega_j\, m(A) \tag{3.58}$$

for any mesurable set $A \subset Y_l$ and every $l > j$. In particular, this gives $\widetilde{m}(E_j) = \sum_{l>j} \omega_j\, m(Y_l) = \omega_j\, m(X_j)$ for each $j \geq 0$. So, (3.57) amounts to saying that $\widetilde{m}$ is a finite measure on $\widetilde{X}$.

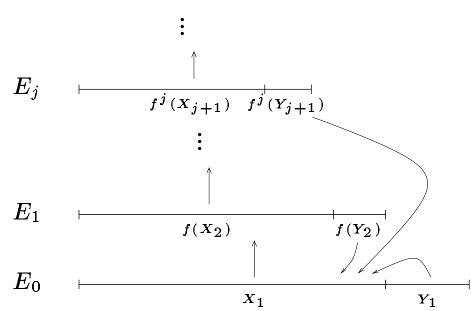

Fig. 3.4. A tower extension

We say that $\tilde{f} : \widetilde{X} \to \widetilde{X}$ is *aperiodic* if

$$\gcd\{l \geq 1 : Y_l \text{ is nonempty }\} = 1. \tag{3.59}$$

This terminology is inherited from the theory of Markov chains: as we shall see later, (3.59) is equivalent to saying that for each $i, j \geq 0$ there exists $k(i,j) \geq 1$ such that $E_i \cap \tilde{f}^{-k}(E_j) \neq \emptyset$ for all $k \geq k(i,j)$.

Theorem 3.5.3. *Let $f : X_0 \to X_0$ satisfy (T1), (T2), (T3), and $\widetilde{X}$, $\tilde{f}$, and $\widetilde{m}$ be as above. Then there exists some $\tilde{f}$-invariant probability $\tilde{\mu}_0$ that is absolutely continuous with respect to $\widetilde{m}$. Moreover, $\mu_0 = \pi_* \tilde{\mu}_0$ is an f-invariant probability absolutely continuous with respect to m.*

If $\tilde{f} : \widetilde{X} \to \widetilde{X}$ is aperiodic then $\tilde{\mu}_0$ is unique, and it is exact for $\tilde{f}$. Moreover, for any Hölder continuous function $\hat{\varphi} : \widetilde{X} \to \mathbb{R}$ there is $K(\hat{\varphi}) > 0$ such that

$$\left| \int (\Psi \circ \tilde{f}^n) \Phi \, d\widetilde{m} - \int \Psi \, d\tilde{\mu}_0 \int \Phi \, d\widetilde{m} \right| \le \frac{K(\Phi)}{\omega_n} \|\psi\|_1$$

for every $n \ge 1$ and $\psi \in L^1(\widetilde{m})$.

Before we start to prove Theorem 3.5.3, let us explain how Theorem 3.5.2 can be deduced from it.

Proof. Let $f : M \to M$ be as in the hypotheses of Theorem 3.5.2. Recall that $c_1 = c$ and c_{j+1} is the unique pre-image of c_j in the interval $(0, c)$, for each $j \ge 1$. We also write $c_0 = 1$. Let $X_0 = [0, 1)$, $X_j = [0, c_j)$ and $Y_j = [c_j, c_{j-1})$, for $j \ge 1$, and $n : X_0 \to \mathbb{N}$ be given by $n(x) = j$ for every $x \in Y_j$ and $j \ge 1$. Then f^j maps Y_j diffeomorphically onto $[0, 1)$, as we assumed that f sends $[0, c)$ and $[c, 1)$ onto $[0, 1)$. In particular, $F \mid Y_j = f^j \mid Y_j$ is absolutely continuous with respect to the Lebesgue measure m, with Jacobian

$$JF(x) = |Df^j(x)| \,.$$

Thus, condition (T1) is satisfied. Moreover, by Lemma 3.5.1, $m(X_j) \le \gamma_2 j^{-1/d}$. Then, given any $\beta < (1/d) - 1$, we may take $\omega_j = j^{-\beta}$ for each $j \ge 1$, and condition (T3) holds. □

3.5.3 Exact Invariant Measures

Here we are going to prove the first part of Theorem 3.5.3: we construct an $\tilde{f}$-invariant probability $\tilde{\mu}_0$ on $\widetilde{X}$, and we prove that $\tilde{\mu}_0$ is unique and exact if one has the aperiodicity condition.

Firstly, we endow each level E_j of $\widetilde{X}$ with the distance defined by

$$\tilde{d}\big((f^j(x), j), (f^j(y), j)\big) = d(x, y), \tag{3.60}$$

where $d(\cdot\,, \cdot)$ is as in hypothesis (T2). Let $\mathcal{X}_0 : \widetilde{X} \to \mathbb{R}$ be the characteristic function of the ground level E_0 . For every $n \ge 1$, define

$$\tilde{\mu}_n = \frac{1}{n} \sum_{j=0}^{n-1} \tilde{f}^j (\mathcal{X}_0 \widetilde{m}).$$

Observe that $\tilde{f}$ is absolutely continuous with respect to the measure $\widetilde{m}$. So, each $\tilde{\mu}_n$ may be written as $\Phi_n \widetilde{m}$ for some nonnegative function $\Phi_n : \widetilde{X} \to \mathbb{R}$. Moreover, every $\tilde{\mu}_n$ is a probability because $\widetilde{m}(E_0) = m(X_0) = 1$.

Lemma 3.5.8. *There exists a constant $K_1 > 0$ such that, for every $n \geq 1$ and $j \geq 0$, either Φ_n is identically zero on E_j, or else $\Phi_n > 0$ and $\log \Phi_n$ is (K_1, ν_0)-Hölder on E_j.*

Proof. Fix $K_1 > 0$ large enough so that $a_0 + K_1 *** \leq K_1$, where a_0 is a constant as in condition (T2). The proof of the lemma is by induction on n. The case $n = 0$ is trivial, due to the choice of the initial density $\mathcal{X}_0$. For the inductive step, consider first the case $j \geq 1$. In view of the definition of $\widetilde{m}$ in (3.57),

$$\Phi_{n+1}(f(x), j) = \frac{\omega_{j-1}}{\omega_j} \Phi(x, j-1) = \lambda_{j-1} \Phi(x, j-1) \tag{3.61}$$

for any $(x, j-1) \in f^{j-1}(X_j) \times \{j-1\} = E_{j-1} \cap \tilde{f}^{-1}(E_j)$. Thus Φ_{n+1} is identically zero on E_j if Φ_n is identically zero on E_{j-1}, and $\Phi_{n+1} > 0$ on E_j if $\Phi_n > 0$ on E_{j-1}. Moreover, in the latter case

$$\log \Phi_{n+1}(f(x), j) = \log \lambda_{j-1} + \log \Phi_n(x, j-1).$$

By induction, $\log \Phi_n$ is (K_1, ν_0)-Hölder on E_j. Since upwards iterates of $\tilde{f}$ preserve the distance $\tilde{d}$, by definition (3.60), it follows that $\log \Phi_{n+1}$ is (K_1, ν_0)-Hölder on E_j.

Finally, let us treat the case $j = 0$ of the inductive step. In view of our definitions, $\tilde{f}$ maps each $f^{i-1}(Y_i) \times \{i-1\} = E_{i-1} \cap \tilde{f}^{-1}(E_0)$ bijectively onto E_0, when Y_i is nonempty. Moreover, $\tilde{f}$ restricted to $f^{i-1}(Y_i) \times \{i-1\}$ has a Jacobian g_i, that is given by

$$g_j(\,\cdot\,, i-1) = \frac{1}{\omega_{i-1}} JF \circ (f^{i-1} \mid Y_i)^{-1}.$$

Hence, for any point $(y, 0) \in E_0$,

$$\Phi_{n+1}(y, 0) = \sum_i \frac{1}{\omega_{i-1}} JF(y_i) \Phi(f^{i-1}(y_i), i-1),$$

where $y_i = (f^i \mid Y_i)^{-1}(y)$ and the sum is over all the $i \geq 1$ for which Y_i is nonempty. Let $(x, 0)$ be another point of E_0 and $x_i = (f^i \mid Y_i)^{-1}(x)$. Using condition (T2) and the induction hypothesis, we get that

$$\log \frac{JF(x_i)\, \Phi(f^{i-1}(x_i), i-1)}{JF(y_i)\, \Phi(f^{i-1}(y_i), i-1)}$$

is less than

$$a_0\, d(x,y)^{\nu_0} + K_1\, \tilde{d}\left((f^{i-1}(x_i), i-1), (f^{i-1}(y_i), i-1)\right)^{\nu_0} \leq *** \leq K_1 d(x,y)^{\nu_0}$$

for each i. Then, comparing the expressions of $\Phi_{n+1}(x, 0)$ and $\Phi_{n+1}(y, 0)$ term-by-term, we find that

$$\log \frac{\Phi_{n+1}(x,0)}{\Phi_{n+1}(y,0)} \le K_1 d(x,y)^{\nu_0}.$$

This means that $\log \Phi$ is (K_1, ν_0)-Hölder. □

We would like to conclude that the sequence μ_n has some accumulation point that is an absolutely continuous invariant measure. Unlike in Section 1.2 for instance, the uniform control of Hölder constants given by the previous lemma is not enough for that. This has to do with the essential "noncompactness" of the tower $\widetilde{X}$: successive iterations under $\tilde{f}$ might push the initial mass distribution $\mathcal{X}_0 \widetilde{m}$ upwards to infinity. The next lemma will permit us to show that, actually, that does not happen.

Lemma 3.5.9. *There exists a constant $K_2 > 0$ such that $\Phi_n(x,j) \le K_2$ for any $n \ge 1$ and any $(x,j) \in \widetilde{X}$.*

Proof. Take $K_2 = 2\exp(K_1 D_0^{\nu_0})$, where D_0 is the $\tilde{d}$-diameter of E_0. The proof is by induction. The case $n = 1$ is immediate, because $\Phi_1 = \mathcal{X}_0$ is bounded by $1 \le K_1$. Next, if the conclusion is true for Φ_n then (3.61) gives that

$$\Phi_{n+1}(y,j) \le \frac{1}{\omega_{j-1}} \sup(\Phi_n E_{j-1}) \le \frac{1}{\omega_{j-1}} K_2 \le K_2\,,$$

for any $(y,j) \in E_j$ and $j \ge 1$. To complete the proof we only have to settle the case $j = 0$. We do that by contradiction. Suppose there was $(y,0) \in E_0$ such that $\Phi_{n+1}(y,0) > K_2$. Then

$$\Phi_{n+1}(y,0) > K_2 \exp(-K_1 D_0^{\nu_0}) = 2$$

for every $(x,0) \in E_0$, because $\log \Phi_n$ is (K_1, ν_0)-Hölder. Consequently,

$$\int \Phi_{n+1}\, d\widetilde{m} \ge \int_{E_0} \Phi_{n+1}\, d\widetilde{m} \ge 2\widetilde{m}(E_0) = 2.$$

On the other hand, $\int \Phi_{n+1}\, d\widetilde{m} = \tilde{\mu}_{n+1}(\widetilde{X}) = 1$ because $\hat{\mu}_{n+1}$ is a probability. This contradiction shows that $\Phi_{n+1} \le K_2$ in E_0 as well. □

Corollary 3.5.2. *The sequence $(\tilde{\mu}_n)_n$ has some convergent subsequence, and every weak*-limit point $\tilde{\mu}_0$ is absolutely continuous with respect to $\widetilde{m}$, with density $d\tilde{\mu}_0/d\widetilde{m}$ positive at every point.*
Proof. □

So we have proven

Proposition 3.5.1. *There exists some $\tilde{f}$-invariant probability $\tilde{\mu}_0$ on $\widetilde{X}$, with density positive at every point.*

Let us fix any $\tilde{\mu}_0$ as in the proposition. We are going to prove that if $\tilde{f} : \widetilde{X} \to \widetilde{X}$ is aperiodic then $\tilde{\mu}_0$ is an exact measure for $\tilde{f}$, and the unique $\tilde{f}$-invariant probability measure absolutely continuous with respect to $\widetilde{m}$. We need a few preliminary lemmas.

Lemma 3.5.10. *There exists $A_0 > 0$ such that distortion lemma*
Proof. □

Lemma 3.5.11. *partitions density lemma*
Proof. □

Lemma 3.5.12. *The following condidtions are equivalent:*

(a) $\gcd\{l \in \mathbb{N} : Y_l$ *is nonempty} is equal to* 1*;*
(b) there exists $\bar{k} \geq 1$ *such that* $E_0 \cap \tilde{f}^{-k}(E_0) \neq \emptyset$ *for all* $k \geq \bar{k}$*;*
(c) given any $i, j \geq 0$ *there exists* $k(i,j) \geq 1$ *such that*

$$E_i \cap \tilde{f}^{-k}(E_j) \neq \emptyset \text{ for all } k \geq k(i,j);$$

Proof. Suppose $\gcd\{l \in \mathbb{N} : Y_l$ is nonempty$\}$ is equal to 1. Then, as is well known, there exists $\bar{k} \geq 1$ such that every integer $k \geq \bar{k}$ may be written as

$$k = n_1 l_1 + \cdots + n_s l_s$$

for some $s, n_1, \ldots, n_s, l_1, \ldots, l_s \geq 1$ such that $Y_{l_1}, \ldots, Y_{l_s}$ are nonempty. Now, $\tilde{f}^l(E_0) \supset \tilde{f}^l(Y_l \times \{0\}) = E_0$ for every l such that Y_l is nonempty. It follows, by induction on $n_1 + \cdots + n_s$, that $\tilde{f}^k(E_0)$ contains E_0. In particular, $E_0 \cap \tilde{f}^{-k}(E_0)$ is nonempty for every $k \geq \bar{k}$. This proves that (a) implies (b).
(b) implies (c)
(c) implies (a) □

Proposition 3.5.2. *The probability $\tilde{\mu}_0$ is exact for $\tilde{f}$ if and only if $\tilde{f} : \widetilde{X} \to \widetilde{X}$ is aperiodic. In that case $\tilde{\mu}$ is the unique $\tilde{f}$-invariant probability absolutely continuous with respect to $\widetilde{m}$.*
Proof. □

Corollary 3.5.3. *Suppose $\tilde{f} : \widetilde{X} \to \widetilde{X}$ is aperiodic. Then, for any $i, j \geq 0$,*

1. $\tilde{\mu}(E_i \cap \tilde{f}^{-n}(E_j)) \to \tilde{\mu}(E_i)\tilde{\mu}(E_j)$ *as* $n \to \infty$*, and*
2. $\widetilde{m}(E_i \cap \tilde{f}^{-n}(E_j)) \to \widetilde{m}(E_i)\tilde{\mu}(E_j)$ *as* $n \to \infty$*.*

Proof. □

Finally, we deduce corresponding statements for the map f, through

Lemma 3.5.13. *Given a probability $\tilde{\nu}$ on $\widetilde{X}$, let $\nu = \pi_* \tilde{\nu}$.*

1. *If $\tilde{\nu}$ is $\tilde{f}$-invariant then ν is f-invariant.*
2. *If $\tilde{\nu}$ is absolutely continuous with respect to $\widetilde{m}$ then ν is absolutely continuous with respect to m.*
3. *If the density $d\tilde{\nu}/d\widetilde{m}$ is positive then $d\nu/dm$ is also positive.*
4. *If $\tilde{\nu}$ is exact for $\tilde{f}$ then nu is exact for f.*

Proof. □

Proposition 3.5.3. *same facts for f.*
Proof. □

3.5.4 Subexponential Decay via Invariant Cones

3.5.5 Subexponential Decay via Coupling

3.6 Hyperbolic Attractors

3.6.1 An Invariant Cone

Let m be Lebesgue measure on Q, normalized by $m(Q) = 1$. We introduce linear operators

$$(U\varphi)(x) = \varphi(f(x))$$

and

$$(\mathcal{L}\varphi)(y) = \begin{cases} \varphi(f^{-1}(y))|\det Df(f^{-1}(y))|^{-1}, & \text{if } y \in f(Q); \\ 0, & \text{otherwise.} \end{cases}$$

Observe that, changing variables $y = f(x)$,

$$\begin{aligned} \int_Q (\mathcal{L}\varphi)\psi\, dm &= \int_{f(Q)} \frac{\varphi(f^{-1}(y))}{|\det Df(f^{-1}(y))|}\psi(y)dm(y) \\ &= \int_Q \varphi(x)\psi(f(x))dm(x) = \int_Q \varphi(U\psi)dm. \end{aligned}$$

As mentioned before, we want to analyse the action of $\mathcal{L}$ on observable functions in terms of corresponding averages on local stable leaves. By a *local stable leaf* we mean any connected component γ of the intersection $W^s(\xi) \cap Q$ of the stable manifold of some point $\xi \in \Lambda$ with the open set Q; see Figure 2.1. Of course, there is no canonical choice of probabilities supported on stable leaves. Instead, we average with respect to a whole class of measures, namely, the cone of Hölder continuous densities $\mathcal{D}(\gamma) = \mathcal{D}(a, \mu, \gamma)$, defined by

$$\begin{aligned} \mathcal{D}(\gamma) = \{\rho : \gamma \to \mathbb{R} \text{ such that } \rho(x) > 0 \text{ for all } x \in \gamma \\ \text{and } \log \rho \text{ is } (a, \mu) - \text{Hölder}\}, \end{aligned}$$

for each local stable leaf $\gamma \subset Q$. It is easy to check that $\mathcal{D}(\gamma)$ is a convex cone, let $\theta = \theta_\gamma$ denote the corresponding projective metric. We also need a similar cone

$$\mathcal{D}_1(\gamma) = \mathcal{D}(a_1, \mu_1, \gamma),$$

corresponding to better Hölder constants $a_1 > 0$ and $0 < \mu_1 < 1$: we shall take $a \gg a_1 \gg 1$ and $0 < \mu < \mu_1 < 1$, cf. Lemma3.6.1, (3.67), (3.71) below. Finally, we also consider the projective metric $\theta_+ = \theta_{+,\gamma}$ associated to the cone of positive densities

$$\mathcal{D}_+(\gamma) = \{\rho : \gamma \to \mathbb{R} \text{ such that } \rho > 0\}.$$

Given φ: $M \to \mathbb{R}$ and $\rho \in \mathcal{D}(\gamma)$ we denote by $\int_\gamma \varphi\rho$ the integral of φ for the measure ρm_γ, where m_γ is the smooth measure induced on γ by the

Riemannian metric. We always suppose $\int_\gamma \rho = 1$, unless otherwise specified. Let $\varphi\colon M \to \mathbb{R}$ and $\mathcal{L}\varphi$ be as defined above. Given any stable leaf γ, let γ_1 and γ_2 be the stable leaves such that $f(\gamma_j) \subset \gamma$ for $j = 1, 2$. Then, for any $\rho \in \mathcal{D}(\gamma)$,

$$\begin{aligned}\int_\gamma (\mathcal{L}\varphi)\rho &= \sum_{j=1}^{2} \int_{f(\gamma_j)} \frac{\varphi(f^{-1}(y))}{|\det Df(f^{-1}(y))|}\rho(y) \\ &= \sum_{j=1}^{2} \int_{\gamma_j} \varphi(x) \frac{\det(Df \mid \gamma_j)(x)|}{|\det Df(x)|}\rho(f(x)) = \sum_{j=1}^{2} \int_{\gamma_j} \varphi\rho_j\end{aligned}$$

where $(Df \mid \gamma_j)$ denotes the restriction of Df to the tangent space of γ_j and

$$\rho_j = \frac{|\det(Df \mid \gamma_j)|}{|\det Df|}(\rho \circ f). \tag{3.62}$$

In the next lemma we use the fact that local stable leaves form a continuous C^2 foliation of Q and, thus, have have uniformly bounded curvature; see Remark 2.3.3.

Lemma 3.6.1. *There are $\lambda_1 < 1$ and $\Lambda_1 < 1$ such that, if a is large enough,*

(a) $\rho \in \mathcal{D}(\gamma) \Rightarrow \rho_j \in \mathcal{D}(\lambda_1 a, \mu, \gamma_j)$ for $j = 1, 2$
(b) $\rho', \rho'' \in \mathcal{D}(\gamma) \Rightarrow \theta_j(\rho_j', \rho_j'') \leq \Lambda_1 \theta(\rho', \rho'')$ for $j = 1, 2$,

where θ and θ_j are the projective metrics associated to $\mathcal{D}(\gamma)$ and to $\mathcal{D}(\gamma_j)$, respectively.

Proof. Clearly, $\rho > 0 \Rightarrow \rho_j > 0$. Let $K_1 > 0$, respectively $K_2 > 0$, be a Lipschitz constant for $\log|\det Df|$, respectively for $\log|\det(Df \mid \gamma_j)|$. Note that K_1 depends only on f, whereas K_2 depends also on some uniform bound for the curvature of stable leaves. Then

$$\begin{aligned}&|\log \rho_j(x) - \log \rho_j(y)| \\ &\qquad \leq |\log \rho(f\,x) - \log \rho(f\,y)| + |\log|\det Df(y)| - \log|\det Df(x)|| \\ &\qquad\qquad + |\log|\det(Df \mid \gamma_j)(x)| - \log|\det(Df \mid \gamma_j)(y)|| \\ &\qquad \leq a\,d(f(x), f(y))^\mu + K_1\,d(x,y) + K_2\,d(x,y) \\ &\qquad \leq (a\,\lambda_s^\mu + K_1 + K_2)\,d(x,y)^\mu \quad \leq \quad a\,\lambda_1\,d(x,y)^\mu\end{aligned}$$

where $\lambda_s < 1$ is a uniform bound for the contraction of f along stable leaves. We suppose

$$a > \frac{K_1 + K_2}{1 - \lambda_s^\mu},$$

and then we fix $\lambda_1 \in (\lambda_s^\mu, 1)$ so that $a \geq (K_1 + K_2)/(\lambda_1 - \lambda_s^\mu)$. This proves a).

Now, in view of Proposition 3.2.2, to prove b) it suffices to show that $\mathcal{D}(\lambda_1 a, \mu, \gamma_j)$ has finite diameter in $\mathcal{D}(\gamma_j)$. This is similar to the proof of Proposition 3.3.1. Let $\theta_{+,j} = \theta_{+,\gamma_j}$ denote the projective metric associated to the cone $\mathcal{D}_+(\gamma_j)$ of positive densities on γ_j. Observe that $\theta_{+,j}$ and θ_j are given by expressions analogous to Example 3.2.2 and Lemma 3.3.2. Given $\rho', \rho'' \in \mathcal{D}(\lambda_1 a, \mu, \gamma_j)$ and $x, y \in \gamma_j$, we have

$$\frac{\exp(a\,d(x,y)^\mu) - \rho''(y)/\rho''(x)}{\exp(a\,d(x,y)^\mu) - \rho'(y)/\rho'(x)} \geq \frac{\exp(a\,d(x,y)^\mu) - \exp(\lambda_1 a\,d(x,y)^\mu)}{\exp(a\,d(x,y)^\mu) - \exp(-\lambda_1 a\,d(x,y)^\mu)} \geq \tau_1$$

where $\tau_1 = \inf\{(z - z^{\lambda_1})/(z - z^{-\lambda_1}) : z > 1\} \in (0,1)$. Therefore,

$$\alpha_j(\rho', \rho'') \geq \tau_1\, \alpha_{+,j}(\rho', \rho'').$$

In just the same way, one finds $\tau_2 > 1$ such that $\beta_j(\rho', \rho'') \leq \tau_2\, \beta_{+,j}(\rho', \rho'')$. Thus,

$$\theta_j(\rho', \rho'') \leq \theta_{+,j}(\rho', \rho'') + \log(\tau_2/\tau_1)$$

for every $\rho', \rho'' \in \mathcal{D}(\lambda_1 a, \mu, \gamma_j)$.

Next we find a uniform upper bound for $\theta_{+,j}(\rho', \rho'')$ with $\rho', \rho'' \in \mathcal{D}(\gamma_j)$. We normalize $\int_{\gamma_j} \rho' = 1 = \int_{\gamma_j} \rho''$ and then the mean value theorem gives

$$\frac{\rho''}{\rho'}(x) \geq \frac{\exp(-a\,(\operatorname{diam}\gamma_j)^\mu)}{\exp(a\,(\operatorname{diam}\gamma_j)^\mu)} \geq \exp\left(-2a\Delta^\mu\right),$$

where $\Delta = \operatorname{diam}(Q)$ plays the role of an upper bound for the diameter of local stable leaves. It follows that

$$\alpha_{+,j}(\rho', \rho'') \geq \exp\left(-2a\Delta^\mu\right) \quad \text{and} \quad \beta_{+,j}(\rho', \rho'') \leq \exp\left(2a\Delta^\mu\right)$$

and so $\theta_{+,j}(\rho', \rho'') \leq 4a\Delta^\mu$, for all $\rho', \rho'' \in \mathcal{D}(\gamma_j)$. Altogether, we have shown that the θ_j-diameter of $\mathcal{D}(\lambda_1 a, \mu, \gamma_j)$ is bounded by $4a\Delta^\mu + \log(\tau_2/\tau_1)$. □

Let us recall a few facts concerning invariant foliations of hyperbolic attractors, cf. Subsection 2.2.2. As already mentioned, local stable leaves form a continuous family $\mathcal{F}^s_{loc}$ of C^2 embedded submanifolds of Q. In particular, given any pair of nearby stable leaves γ and $\tilde{\gamma}$, there is a C^2 diffeomorphism

$$\pi = \pi(\tilde{\gamma}, \gamma) : \tilde{\gamma} \to \gamma,$$

C^2-close to the inclusion map of $\tilde{\gamma}$ in M. Of course, such a diffeomorphism is not unique, but for what follows we only need to know that it has been chosen satisfying (p1), (p2), (p3) below, which is always possible. For instance, in the case of the solenoid it suffices to take π to be the projection along the leaves of the horizontal foliation $\{S^1 \times \{z\} : z \in B^2\}$ of $Q = S^1 \times B^2$. In other words, for each $z \in B^2$ the unique point $(\tilde{\theta}, z) \in \tilde{\gamma}$ is mapped by π to the unique point $(\theta, z) \in \gamma$.

Let γ_1, γ_2 be the connected components of $f^{-1}(\gamma) \cap Q$ and $\tilde{\gamma}_1$, $\tilde{\gamma}_2$ be the connected components of $f^{-1}(\tilde{\gamma}) \cap Q$ (numbered in such a way that $\tilde{\gamma}_1$ is closer to γ_1 than to γ_2), and let $\pi_j = \pi(\tilde{\gamma}_j, \gamma_j)$, for $j = 1, 2$. Let $d(y', y'')$ denote the distance between two points y', y'' belonging in a same horizontal leaf Γ, measured along Γ. Then there are constants $a_0 > 0$, $\nu_0 > 0$, $\lambda_u < 1$, depending only on f, such that

(p1) π and $\log|\det D\pi|$ are a_0-Lipschitz maps;
(p2) $\log|\det D\pi(y)| \leq a_0\, d(y, \pi(y))^{\nu_0}$ for every $y \in \tilde{\gamma}$;
(p3) $d(x, \pi_j(x)) \leq \lambda_u\, d(f(x), \pi f(x))$ for every $x \in \tilde{\gamma}_j$ and $j = 1, 2$.

Indeed, (p1) follows directly from γ and $\tilde{\gamma}$ being the graphs of C^2 maps $B^2 \to S^1$ with uniformly bounded C^2-norm. Property (p2) also uses the fact that the tangent spaces to the leaves of $\mathcal{F}^s_{loc}$ form a Hölder continuous subbundle of the tangent bundle TQ, in particular,

$$\text{angle}(T_y\gamma, T_{\pi(y)}\tilde{\gamma}) \leq A_0\, d(y, \pi(y))^{\nu_0}, \tag{3.63}$$

where $A_0 > 0$ and $\nu_0 \in (0, 1]$ depend only on the map f.

To prove (p3), begin by noting that if ξ_0 is some curve joining x to $\pi_j(x)$ inside the horizontal leaf Γ_0 that contains x, then ξ_0 is expanded by iteration under f. Even more, the horizontal projection of $f(\xi_0)$ has length larger than $\sigma_u\, \text{length}(\xi_0)$, for some uniform $\sigma_u > 1$. Let ξ_1 be a curve joining $f(x)$ to $\pi f(x)$ inside the corresponding horizontal leaf, with

$$d(f(x), \pi f(x)) = \text{length}(\xi_1).$$

Observe that the angle between $f(\Gamma_0)$ and the leaves of the horizontal foliation is bounded, at every point, by some constant $H > 0$. On the other hand, the angle between γ and the leaves of the vertical foliation $\{\{\theta\} \times B^2 : \theta \in S^1\}$ is also uniformly bounded, by some constant $\delta > 0$ which can be made arbitrarily small by taking f close enough to (2.4). It follows that we can take a curve ξ_2 joining $f(x)$ to $f\pi_j(x)$ inside $f(\Gamma_0)$, and such that the length of the horizontal projection of ξ_2 is smaller than

$$(1 + H\delta)\, \text{length}(\xi_1) = (1 + H\delta)\, d(f(x), \pi f(x)).$$

We suppose that $\delta > 0$ is so that $1 + H\delta < \sigma_u$, and then take $\sigma_u^{-1}(1 + H\delta) \leq \lambda_u < 1$. Then, denoting $\xi_0 = f^{-1}(\xi_2)$, we obtain (p2):

$$d(x, \pi_j(x)) \leq \text{length}(\xi_0) \leq \sigma_u^{-1}(1 + H\delta)\, d(f(x), \pi f(x)) \leq \lambda_u\, d(f(x), \pi f(x)).$$

For any γ, $\tilde{\gamma}$ as before, we define the *distance* between γ and $\tilde{\gamma}$ by

$$d(\gamma, \tilde{\gamma}) = \sup\{d(y, \pi(y)) : y \in \tilde{\gamma}\}. \tag{3.64}$$

As a direct consequence of (p3), the map induced by f in the space of local stable leaves is expanding for this distance:

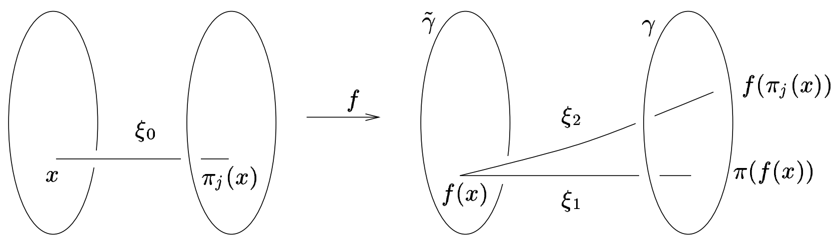

Fig. 3.5. Expansion in the space of stable leaves

$$d(\gamma,\tilde{\gamma}) \geq \lambda_u^{-1}\, d(\gamma_j,\tilde{\gamma}_j), \quad \text{for every } \gamma,\tilde{\gamma}, \text{ and } j=1,2. \tag{3.65}$$

Next, to every $\rho \in \mathcal{D}_1(\gamma)$ we associate the density $\tilde{\rho}\colon \tilde{\gamma} \to \mathbb{R}$ defined by

$$\tilde{\rho}(y) = \rho(\pi(y)) \cdot |\det D\pi(y)|. \tag{3.66}$$

Clearly, $\tilde{\rho} > 0$. Moreover, as a consequence of (p1), $\log\tilde{\rho} = \log\rho\circ\pi + \log|\det D\pi|$ is $(\tilde{a}_1,\mu_1)$-Hölder continuous, where $\tilde{a}_1 = a_1 a_0^{\mu_1} + a_0 \Delta^{1-\mu_1}$, recall that $\Delta = \operatorname{diam}(Q)$. This also implies that $\log\tilde{\rho}$ is $(\tilde{a},\mu)$-Hölder continuous, for $\tilde{a} = a_1 a_0^{\mu_1}\Delta^{\mu_1-\mu} + a_0\Delta^{1-\mu}$. We suppose

$$\frac{a}{2} \geq \tilde{a} = a_1 a_0^{\mu_1}\Delta^{\mu_1-\mu} + a_0\Delta^{1-\mu}, \tag{3.67}$$

so that, in particular, $\rho \in \mathcal{D}_1(\gamma) \Rightarrow \tilde{\rho} \in \mathcal{D}(\tilde{\gamma})$. (This step is one of the reasons why we need the auxiliary cone $\mathcal{D}_1(\gamma)$, the other one is in the proof of Lemma 3.6.3.) Note also that $\int_\gamma \rho = \int_{\tilde{\gamma}} \tilde{\rho}$, by change of variables.

At last, we are in a position to introduce the invariant cone of observables we are interested in. Given $b>0$, $c>0$, and $\nu \in (0,1]$, we let $C(b,c,\nu)$ be the cone of bounded functions $\varphi : Q \to \mathbb{R}$ satisfying conditions (A), (B), (C) below:

(A) $\int_\gamma \varphi\rho > 0$ for every γ and every $\rho \in \mathcal{D}(\gamma)$;
(B) the map $\rho \mapsto \log\int_\gamma \varphi\rho$ is b-Lipschitz as a function of $\rho \in \mathcal{D}(\gamma)$, that is,

$$|\log\int_\gamma \varphi\rho' - \log\int_\gamma \varphi\rho''| \leq b\,\theta(\rho',\rho'')$$

for every $\rho',\rho'' \in \mathcal{D}(\gamma)$ with $\int_\gamma \rho' = 1 = \int_\gamma \rho''$, and every $\gamma \in \mathcal{F}^s_{loc}$;
(C) the map $\gamma \mapsto \int_\gamma \varphi\rho$ is (c,ν)-Hölder as a function of γ, more precisely,

$$|\log\int_\gamma \varphi\rho - \log\int_{\tilde{\gamma}} \varphi\tilde{\rho}| \leq c\, d(\gamma,\tilde{\gamma})^\nu$$

for every $\rho \in \mathcal{D}_1(\gamma)$ and every pair $\gamma,\tilde{\gamma} \in \mathcal{F}^s_{loc}$.

As a matter of fact, we want to think of the elements of $C(b,c,\nu)$ as equivalence classes of bounded functions for the equivalence relation

$$\varphi_1 \sim \varphi_2 \quad \Leftrightarrow \quad \varphi_1|\gamma = \varphi_2|\gamma \quad m_\gamma\text{-almost everywhere, for every } \gamma.$$

However, replacing an equivalence class by any of its members never results in ambiguity, and so we ignore this formal distinction, in order not to overload the notations.

(A) and (C) are natural reformulations of the properties we required in the definition of the cone $C(a,\nu)$ in Subsection 3.3.1. Condition (B) is necessary to compensate for the fact that the functions $\varphi \in C(b,c,\nu)$ may take negative values. Observe, indeed, that (B) is automatically satisfied (with $b=1$) in the particular case when φ is nonnegative:

$$\frac{\int_\gamma \varphi\rho'}{\int_\gamma \varphi\rho''} \le \frac{\sup\rho'}{\inf\rho''} \le \frac{\sup\rho'/\inf\rho'}{\inf\rho''/\sup\rho''} = \exp(\theta_+(\rho',\rho'')) \le \exp(\theta(\rho',\rho'')). \tag{3.68}$$

In the sequel we take b and c large, and ν close to zero, cf. Proposition 3.6.1 and (3.71).

Lemma 3.6.2. *$C(b,c,\nu)$ is a proper convex cone, for any $b>0$, $c>0$, and $0<\nu\le 1$.*

Proof. The convexity of the cone is a direct consequence of the convexity of the logarithm function, so we only have to prove that the closure $\widetilde{C}(b,c,\nu)$ of $C(b,c,\nu)$ satisfies (3.19). In other words,

$$\int_\gamma \varphi\rho = 0 \;\text{ for all } \rho\in\mathcal{D}(\gamma) \text{ and all } \gamma \quad\Rightarrow\quad \varphi = 0.$$

Let γ be fixed. Given any μ-Hölder continuous function $\psi:\gamma\to\mathbb{R}$ and any $B>0$, we may write

$$\psi = (\psi^+ + B) - (\psi^- + B) \quad\text{where}\quad \psi^\pm(x) = \frac{1}{2}(|\psi(x)| \pm \psi(x)).$$

Now, it is easy to see that $(\psi^\pm + B) \in \mathcal{D}(\gamma)$ if B is large enough. Hence, by the linearity of the integral, $\int \varphi\psi = 0$ for every μ-Hölder ψ. Since any bounded function can be L^1-approximated by μ-Hölder continuous functions, it follows that $\int\varphi\psi = 0$ for every bounded $\psi:\gamma\to\mathbb{R}$. Taking $\psi = \varphi|\gamma$ we conclude that $\varphi|\gamma = 0$ at m_γ-almost every point, for arbitrary γ, and so $\varphi = 0$. $\square$

Let us calculate the projective metric $\Theta = \Theta_{b,c,\nu}$ associated to $C(b,c,\nu)$. Given $\rho'\in\mathcal{D}(\gamma)$ with $\int_\gamma \rho' = 1$,

$$\int_\gamma (\varphi_2 - t\varphi_1)\rho' > 0 \Leftrightarrow t < \frac{\int_\gamma \varphi_2\rho'}{\int_\gamma \varphi_1\rho'}.$$

Next, for $\rho', \rho'' \in \mathcal{D}(\gamma)$ with $\int_\gamma \rho' = 1 = \int_\gamma \rho''$,

$$\frac{\int_\gamma (\varphi_2 - t\varphi_1)\rho''}{\int_\gamma (\varphi_2 - t\varphi_1)\rho'} \le \exp(b\theta(\rho', \rho''))$$

or, equivalently,

$$t \le \frac{\int_\gamma \varphi_2\rho'}{\int_\gamma \varphi_1\rho'} \frac{\exp(b\theta(\rho', \rho'')) - (\int_\gamma \varphi_2\rho'' / \int_\gamma \varphi_2\rho')}{\exp(b\theta(\rho', \rho'')) - (\int_\gamma \varphi_1\rho'' / \int_\gamma \varphi_1\rho')}.$$

We denote the expression in the last fraction by $\xi(\rho', \rho'', \varphi_1, \varphi_2)$. Then we also have

$$\frac{\int_\gamma (\varphi_2 - t\varphi_1)\rho''}{\int_\gamma (\varphi_2 - t\varphi_1)\rho'} \ge \exp(-b\theta(\rho', \rho'')) \Leftrightarrow t \le \frac{\int_\gamma \varphi_2\rho''}{\int_\gamma \varphi_1\rho''} \xi(\rho'', \rho', \varphi_1, \varphi_2).$$

Finally, given $\rho \in \mathcal{D}_1(\gamma)$ with $\int_\gamma \rho = 1$, we have $\tilde{\rho} \in \mathcal{D}(\gamma)$ with $\int_{\tilde{\gamma}} \tilde{\rho} = 1$, and

$$\frac{\int_{\tilde{\gamma}} (\varphi_2 - t\varphi_1)\widetilde{\rho}}{\int_\gamma (\varphi_2 - t\varphi_1)\rho} \le \exp(c\, d(\gamma, \tilde{\gamma})^\nu)$$

which is the same as

$$t \le \frac{\int_\gamma \varphi_2\rho}{\int_\gamma \varphi_1\rho} \frac{\exp(c\, d(\gamma, \tilde{\gamma})^\nu) - (\int_{\tilde{\gamma}} \varphi_2\widetilde{\rho} / \int_\gamma \varphi_2\rho)}{\exp(c\, d(\gamma, \tilde{\gamma})^\nu) - (\int_{\tilde{\gamma}} \varphi_1\widetilde{\rho} / \int_\gamma \varphi_1\rho)}.$$

Let $\eta(\rho, \widetilde{\rho}, \varphi_1, \varphi_2)$ denote the expression in the last fraction. Then, analogously,

$$\frac{\int_{\tilde{\gamma}} (\varphi - t\varphi_1)\tilde{\rho}}{\int_\gamma (\varphi_2 - t\varphi_1)\rho} \ge \exp(-c\, d(\gamma, \tilde{\gamma})^\nu),$$

that is,

$$t \le \frac{\int_{\tilde{\gamma}} \varphi_2\tilde{\rho}}{\int_{\tilde{\gamma}} \varphi_2\tilde{\rho}} \eta(\tilde{\rho}, \rho, \varphi_1, \varphi_2).$$

Therefore, $\alpha(\varphi_1, \varphi_2)$ is given by

$$\inf \left\{ \frac{\int_\gamma \varphi_2\rho'}{\int_\gamma \varphi_1\rho'}, \frac{\int_\gamma \varphi_2\rho'}{\int_\gamma \varphi_1\rho'} \xi(\rho', \rho'', \varphi_1, \varphi_2), \right.$$
$$\left. \frac{\int_\gamma \varphi_2\rho}{\int_\gamma \varphi_1\rho} \eta(\rho, \tilde{\rho}, \varphi_1, \varphi_2), \frac{\int_{\tilde{\gamma}} \varphi_2\tilde{\rho}}{\int_{\tilde{\gamma}} \varphi_1\tilde{\rho}} \eta(\tilde{\rho}, \rho, \varphi_1, \varphi_2) \right\}$$

where the infimum runs over all $\rho' \in \mathcal{D}(\gamma)$, $\rho'' \in \mathcal{D}(\gamma)$, $\rho \in \mathcal{D}_1(\gamma)$, and every pair of local stable leaves γ and $\tilde{\gamma}$. Moreover, $\beta(\varphi_1, \varphi_2)$ is given by a similar expression, with inf replaced by sup.

Proposition 3.6.1. *There is* $\lambda_2 < 1$ *so that* $\mathcal{L}(C(b,c,\nu)) \subset C(\lambda_2 b, \lambda_2 c, \nu)$ *for every large enough* b *and* c.

Proof. Let $\rho \in \mathcal{D}(\gamma)$ and ρ_j be as defined above. Lemma3.6.1(a) ensures that $\rho_j \in \mathcal{D}(\gamma_j)$ and so $\int_{\gamma_j} \varphi\rho_j > 0$ for each $j = 1, 2$. As a consequence, $\int_\gamma (\mathcal{L}\varphi)\rho = \sum_j \int \varphi\rho_j > 0$, which proves the invariance of (A).

To prove the invariance of condition (B), let $\rho', \rho'' \in \mathcal{D}(\gamma)$ with $\int_\gamma \rho' = 1 = \int_\gamma \rho''$. Denote, cf. (3.62),

$$\rho'_j = \frac{|\det(Df \mid \gamma_j)|}{|\det Df|}(\rho' \circ f) \quad \text{and} \quad \rho''_j = \frac{|\det(Df \mid \gamma_j)|}{|\det Df|}(\rho'' \circ f).$$

Moreover, let $\rho_j^- = \rho'_j / \int_{\gamma_j} \rho'_j$ and $\rho_j^= = \rho''_j / \int_{\gamma_j} \rho''_j$. Then, using condition (B) for φ, followed by Lemma 3.6.1(b),

$$\begin{aligned}
\int_\gamma (\mathcal{L}\varphi)\rho'' &= \sum_j \int_{\gamma_j} \varphi\rho''_j = \sum_j \int_{\gamma_j} \rho''_j \int_{\gamma_j} \varphi\rho_j^= \\
&\leq \sum_j \int_{\gamma_j} \rho''_j \cdot \exp(b\theta(\rho_j^-, \rho_j^=)) \int_{\gamma_j} \varphi\rho_j^- \\
&\leq \sum_j \exp(b\theta(\rho'_j, \rho''_j)) \frac{\int_{\gamma_j} \rho''_j}{\int_{\gamma_j} \rho'_j} \int_{\gamma_j} \varphi\rho'_j \\
&\leq \exp(b\Lambda_1\, \theta(\rho', \rho'')) \sum_j \frac{\int_{\gamma_j} \rho''_j}{\int_{\gamma_j} \rho'_j} \int_{\gamma_j} \varphi\rho'_j .
\end{aligned}$$

By the same arguments as in (3.68),

$$\frac{\rho''_j}{\rho'_j}(x) = \frac{\rho''(f(x))}{\rho'(f(x))} \leq \exp\left(\theta(\rho', \rho'')\right) \leq \exp\left(\theta_+(\rho', \rho'')\right).$$

Thus, replacing above we conclude that

$$\begin{aligned}
\int_\gamma (\mathcal{L}\varphi)\rho'' &\leq \exp(b\Lambda_1\, \theta(\rho', \rho'') + \theta(\rho', \rho'')) \sum_j \int_{\gamma_j} \varphi\rho'_j \\
&\leq \exp(b\lambda_2\, \theta(\rho', \rho'')) \int_\gamma (\mathcal{L}\varphi)\rho',
\end{aligned}$$

as long as we fix $\lambda_2 \in (\Lambda_1, 1)$ and suppose $b \geq 1/(\lambda_2 - \Lambda_1)$.

Now we prove invariance of condition (C). Given two stable leaves γ and $\tilde{\gamma}$, let γ_j and $\tilde{\gamma}_j$, $j = 1, 2$, be the connected components of $f^{-1}(\gamma) \cap Q$ and $f^{-1}(\tilde{\gamma}) \cap Q$, respectively. Let $\rho \in \mathcal{D}_1(\gamma) \subset \mathcal{D}(\gamma)$. As we have seen before,

$$\int_\gamma (\mathcal{L}\varphi)\rho = \sum_j \int_{\gamma_j} \varphi\rho_j \qquad \text{and} \qquad \int_{\tilde{\gamma}} (\mathcal{L}\varphi)\tilde{\rho} = \sum_j \int_{\tilde{\gamma}_j} \varphi(\tilde{\rho})_j\, ,$$

where $(\tilde{\rho})_j : \tilde{\gamma}_j \to \mathbb{R}$ is defined by

$$
\begin{aligned}
(\tilde{\rho})_j(x) &= \tilde{\rho}(f(x)) \frac{|\det(Df \mid \tilde{\gamma}_j)(x)|}{|\det Df(x)|} \\
&= \rho(\pi f(x)) \, |\det D\pi(f(x))| \, \frac{|\det(Df \mid \tilde{\gamma}_j)(x)|}{|\det Df(x)|}.
\end{aligned}
$$

Since $\rho_j \in \mathcal{D}_1(\gamma_j)$, we may invoke property (C) for φ together with (3.65), to conclude that

$$
|\log \int_{\gamma_j} \varphi \rho_j - \log \int_{\tilde{\gamma}_j} \varphi \widetilde{\rho_j}| \le c \, d(\gamma_j, \tilde{\gamma}_j)^\nu \le c \, \lambda_u^\nu \, d(\gamma, \tilde{\gamma})^\nu,
$$

where

$$
\begin{aligned}
\widetilde{\rho_j}(x) &= \rho_j(\pi_j(x)) \, |\det D\pi_j(x)| \\
&= \rho(f \pi_j(x)) \frac{|\det(Df \mid \gamma_j)(\pi_j(x))|}{|\det Df(\pi_j(x))|} \, |\det D\pi_j(x)|
\end{aligned}
$$

with $\pi_j = \pi(\tilde{\gamma}_j, \gamma_j)$. Next, we use the following estimate, which is given by the auxiliary Lemma 3.6.3 below:

$$
|\log \int_{\tilde{\gamma}_j} \varphi \widetilde{\rho_j} - \log \int_{\tilde{\gamma}_j} \varphi (\tilde{\rho})_j| \le K_0 \, d(\gamma, \tilde{\gamma})^\nu,
$$

for some constant $K_0 > 0$ that does not depend on c. Altogether,

$$
|\log \int_{\gamma_j} \varphi \rho_j - \log \int_{\tilde{\gamma}_j} \varphi (\tilde{\rho})_j| \le (c \, \lambda_u^\nu + K_0) \, d(\gamma, \tilde{\gamma})^\nu,
$$

for $j = 1, 2$, and so

$$
|\log \int_{\gamma} (\mathcal{L}\varphi) \rho - \log \int_{\tilde{\gamma}} (\mathcal{L}\varphi) \tilde{\rho}| \le (c \, \lambda_u^\nu + K_0) \, d(\gamma, \tilde{\gamma})^\nu \le \lambda_2 c \, d(\gamma, \tilde{\gamma})^\nu,
$$

as long as we take $\lambda_2 \in (\lambda_u^\nu, 1)$ and suppose $c \ge K_0 / (\lambda_2 - \lambda_u^\nu)$. □

In this way we have reduced the proof of Proposition 3.6.1 to proving the following auxiliary statement.

Lemma 3.6.3. *There is $K_0 > 0$ depending only on f, a, a_1, b, such that*

$$
|\log \int_{\tilde{\gamma}_j} \varphi \widetilde{\rho_j} - \log \int_{\tilde{\gamma}_j} \varphi (\tilde{\rho})_j| \le K_0 \, d(\gamma, \tilde{\gamma})^\nu
$$

for every $\varphi \in C(b, c, \nu)$ and every $\gamma, \tilde{\gamma} \in \mathcal{F}^s_{loc}$, $\rho \in \mathcal{D}_1(\gamma)$, and $j = 1, 2$.

Proof. We use $K_1, \ldots, K_6$ to denote sufficiently large constants, depending only on f, a, a_1, b. The previous arguments prove that

(1) $\rho \in \mathcal{D}_1(\gamma) \Rightarrow \tilde{\rho} \in \mathcal{D}(\tilde{a}_1, \mu_1, \tilde{\gamma}) \Rightarrow \rho' = (\tilde{\rho})_j \in \mathcal{D}(\lambda_1 \tilde{a}_1, \mu_1, \tilde{\gamma}_j) \subset \mathcal{D}(\tilde{a}_1, \mu_1, \tilde{\gamma}_j)$;
(2) $\rho \in \mathcal{D}_1(\gamma) \Rightarrow \rho_j \in \mathcal{D}(\lambda_1 a_1, \mu_1, \gamma_j) \subset \mathcal{D}(a_1, \mu_1, \gamma_j) \Rightarrow \rho'' = \widetilde{\rho_j} \in \mathcal{D}(\tilde{a}_1, \mu_1, \tilde{\gamma}_j)$.

where $\tilde{a}_1 = a_1 a_0^{\mu_1} + a_0 \Delta^{1-\mu_1}$. In (1) we suppose that a_1 is large enough so that Lemma 3.6.1(a) holds with the cone $\mathcal{D}(\tilde{a}_1, \mu_1, \gamma)$ in the place of $\mathcal{D}(\gamma) = \mathcal{D}(a, \mu, \gamma)$.

It follows from (1) and (2) that both ρ' and ρ'' belong in $\mathcal{D}(\tilde{a}_1 \Delta^{\mu_1 - \mu}, \mu, \tilde{\gamma}_j)$ which, by (3.67), is contained in $\mathcal{D}(a/2, \mu, \tilde{\gamma}_j) \subset \mathcal{D}(\tilde{\gamma}_j)$. Hence, using condition (B) for the normalized densities $\rho' / \int_{\tilde{\gamma}_j} \rho'$ and $\rho'' / \int_{\tilde{\gamma}_j} \rho''$,

$$|\log \int_{\tilde{\gamma}_j} \varphi \rho' - \log \int_{\tilde{\gamma}_j} \varphi \rho''| \le b\, \theta_j(\rho', \rho'') + |\log \int_{\tilde{\gamma}_j} \rho' - \log \int_{\tilde{\gamma}_j} \rho''|. \quad (3.69)$$

In order to bound the two terms on the right hand side, let us take a look at the relation

$$\frac{\rho'(x)}{\rho''(x)} = \frac{\rho(\pi f(x))}{\rho(f \pi_j(x))} \frac{|\det D\pi(f(x))|}{|\det D\pi_j(x)|} \frac{|\det(Df \mid \tilde{\gamma}_j)(x)|}{|\det(Df \mid \gamma_j)(\pi_j(x))|} \frac{|\det Df(\pi_j(x))|}{|\det Df(x)|}. \quad (3.70)$$

First, in view of (p3) and the fact that f is Lipschitz continuous, the distance between $\pi f(x)$ and $f \pi_j(x)$ is from above bounded by

$$\begin{aligned} d(\pi f(x), f(x)) + K_1\, d(x, \pi_j(x)) &\le (1 + \lambda_u K_1)\, d(\pi f(x), f(x)) \\ &\le (1 + \lambda_u K_1)\, d(\gamma, \tilde{\gamma}). \end{aligned}$$

Combining with the assumption $\rho \in \mathcal{D}_1(\gamma)$, we find

$$\left| \log \rho(\pi f(x)) - \log \rho(f \pi_j(x)) \right| \le a_1 (1 + \lambda_u K_1)^{\mu_1}\, d(\gamma, \tilde{\gamma})^{\mu_1}.$$

On the other hand, by (p2),

$$\begin{aligned} \left| \log |\det D\pi(f(x))| - \log |\det D\pi_j(x)| \right| &\le a_0\, d(f(x), \pi f(x))^{\nu_0} + a_0\, d(x, \pi_j(x))^{\nu_0} \\ &\le a_0 (1 + \lambda_u^{\nu_0})\, d(\gamma, \tilde{\gamma})^{\nu_0}. \end{aligned}$$

Moreover, since $\log |\det Df|$ is Lipschitz continuous,

$$\left| \log |\det Df(\pi_j(x))| - \log |\det Df(x)| \right| \le K_2\, d(\pi_j(x), x) \le K_2 \lambda_u\, d(\gamma, \tilde{\gamma}).$$

Next, using also the Hölder property (3.63) of the tangent bundle to $\mathcal{F}^s_{loc}$,

$$\begin{aligned} \left| \log |\det(Df \mid \tilde{\gamma}_j)(x)| - \log |\det(Df \mid \gamma_j)(\pi_j(x))| \right| &\le K_3\, d(x, \pi_j(x))^{\nu_0} \\ &\le K_3 \lambda_u^{\nu_0}\, d(\gamma, \tilde{\gamma})^{\nu_0}. \end{aligned}$$

At this point we assume that

$$0 < \mu < \mu + \nu \le \mu_1 \le \nu_0. \tag{3.71}$$

Then, replacing the previous bounds in (3.70),

$$\big|\log \rho'(x) - \log \rho''(x)\big| \le K_4\, d(\gamma, \tilde\gamma)^{\mu_1} \tag{3.72}$$

for some sufficiently large $K_4 > 0$, and every $x \in \tilde\gamma_j$ and $j = 1, 2$.

In particular,

$$|\log \int_{\tilde\gamma_j} \rho' - \log \int_{\tilde\gamma_j} \rho''| \le K_4\, d(\gamma, \tilde\gamma)^{\mu_1} \tag{3.73}$$

and

$$\theta_{+,j}(\rho', \rho'') = \log \left(\frac{\sup_{\tilde\gamma_j}(\rho''/\rho')}{\inf_{\tilde\gamma_j}(\rho''/\rho')} \right) \le 2K_4\, d(\gamma, \tilde\gamma)^{\mu_1}. \tag{3.74}$$

Inequality (3.73) provides the kind of bound we want for the last term in (3.69).

To bound the term $b\,\theta_j(\rho', \rho'')$, we combine (3.74) with the relation (recall e.g. the proof of Lemma 3.6.1)

$$\theta_j(\rho', \rho'') \le \theta_{+,j}(\rho', \rho'') + \log(\hat\tau_2/\hat\tau_1), \tag{3.75}$$

where

$$\hat\tau_1 = \inf \left\{ \frac{\exp(a\,d(x,y)^\mu) - \rho''(y)/\rho''(x)}{\exp(a\,d(x,y)^\mu) - \rho'(y)/\rho'(x)} : x, y \in \tilde\gamma_j \text{ with } x \ne y \right\}$$

and $\hat\tau_2$ is given by a similar expression, with inf replaced by sup. All that is left to do is to bound $|\log \hat\tau_1|$ and $|\log \hat\tau_2|$. Let us denote

$$B' = (\rho'(y)/\rho'(x)) \exp(-a\,d(x,y)^\mu)$$

and

$$B'' = (\rho''(y)/\rho''(x)) \exp(-a\,d(x,y)^\mu).$$

Clearly, $\rho' \in \mathcal{D}(a/2, \mu, \tilde\gamma_j)$ implies $\log B' \le -(a/2)\,d(x,y)^\mu < 0$, and analogously for ρ'' and B''. In particular,

$$|B' - B''| \le |\log B' - \log B''| = |\log \rho'(y) - \log \rho'(x) - \log \rho''(y) + \log \rho''(x)|.$$

On the one hand, (3.72) implies

$$|B' - B''| \le |\log \rho'(y) - \log \rho''(y)| + |\log \rho'(x) - \log \rho''(x)| \le 2K_4\, d(\gamma, \tilde\gamma)^\mu. \tag{3.76}$$

On the other hand,

$$|B' - B''| \le |\log \rho'(y) - \log \rho'(x)| + |\log \rho''(y) - \log \rho''(x)| \le 2\tilde{a}_1\, d(x,y)^{\mu_1}, \tag{3.77}$$

because $\rho', \rho'' \in \mathcal{D}(\tilde{a}_1, \mu_1, \tilde{\gamma}_j)$. Since we are taking $\mu_1 \ge \mu + \nu$, it follows that

$$|B' - B''| \le K_5\, d(x,y)^{\mu}\, d(\gamma, \tilde{\gamma})^{\nu},$$

as long as $K_5 \ge \max\{2K_4, 2\tilde{a}_1\}$. Indeed, this last inequality is a direct consequence of ((3.76)) if $d(x,y) \ge d(\gamma,\tilde{\gamma})$, and of (3.77) in the case when $d(x,y) \le d(\gamma,\tilde{\gamma})$. Then,

$$\left|\log \frac{1-B''}{1-B'}\right| \le \frac{|B'-B''|}{1-\max\{B',B''\}} \le \frac{K_5\, d(x,y)^{\mu}\, d(\gamma,\tilde{\gamma})^{\nu}}{1-\exp(-(a/2)\, d(x,y)^{\mu})} \le K_6\, d(\gamma,\tilde{\gamma})^{\nu}.$$

Replacing in $\hat{\tau}_1$ and $\hat{\tau}_2$, we find

$$\log \hat{\tau}_1 \ge -K_6\, d(\gamma,\tilde{\gamma})^{\nu} \quad \text{and} \quad \log \hat{\tau}_2 \le K_6\, d(\gamma,\tilde{\gamma})^{\nu},$$

and so, in view of (3.74) and (3.75),

$$\theta_j(\rho', \rho'') \le 2K_4\, d(\gamma,\tilde{\gamma})^{\mu_1} + 2K_6\, d(\gamma,\tilde{\gamma})^{\nu}.$$

Finally, by (3.69) and (3.73),

$$|\log \int_{\tilde{g}_j} \varphi\rho' - \log \int_{\tilde{\gamma}_j} \varphi\rho''| \le (2b+1)\, K_4\, d(\gamma,\tilde{\gamma})^{\mu_1} + 2bK_6\, d(\gamma,\tilde{\gamma})^{\nu} \le K_0\, d(\gamma,\tilde{\gamma})^{\nu}$$

for some $K_0 > 0$, which concludes the proof of the lemma. □

In the proof of the next result we use the projective metric Θ_+ associated to the cone of bounded functions satisfying $\int_\gamma \varphi\rho > 0$ for every γ and every $\rho \in \mathcal{D}(\gamma)$. In the same way as we calculated Θ, one checks that $\Theta_+(\varphi_1,\varphi_2) = \log(\beta_+(\varphi_1,\varphi_2)/\alpha_+(\varphi_1,\varphi_2))$, with

$$\alpha_+(\varphi_1,\varphi_2) = \inf_{\rho,\gamma} \left\{ \frac{\int_\gamma \varphi_2\rho}{\int_\gamma \varphi_1\rho} \right\} \quad \text{and} \quad \beta_+(\varphi_1,\varphi_2) = \sup_{\rho,\gamma} \left\{ \frac{\int_\gamma \varphi_2\rho}{\int_\gamma \varphi_1\rho} \right\}$$

(taken over every $\rho \in \mathcal{D}(\gamma)$ and every stable leaf γ).

Proposition 3.6.2. *For $b > 0$, $c > 0$, $\nu \in (0,1]$, the Θ-diameter $D_2 = \sup\{\Theta(\mathcal{L}\varphi_1, \mathcal{L}\varphi_2) : \varphi_1,\varphi_2 \in C(b,c,\nu)\}$ of $\mathcal{L}(C(b,c,\nu))$ is finite.*

Proof. As in the proof of Proposition 3.3.2, the argument has two parts. In a first step, we bound D_2 in terms of the Θ_+-diameter of $\mathcal{L}(C(b,c,\nu))$. Let $\varphi_1,\varphi_2 \in C(\lambda_2 b, \lambda_2 c, \nu)$. Given $\rho', \rho'' \in \mathcal{D}(\gamma)$ and $\rho \in \mathcal{D}_1(\gamma)$, we have

$$\xi(\rho', \rho'', \varphi_1, \varphi_2) = \frac{\exp(b\theta(\rho', \rho'')) - \int_\gamma \varphi_2\rho'' / \int_\gamma \varphi_2\rho'}{\exp(b\theta(\rho', \rho'')) - \int_\gamma \varphi_1\rho'' / \int_\gamma \varphi_1\rho'}$$
$$\geq \frac{\exp(b\theta(\rho', \rho'')) - \exp(b\lambda_2\,\theta(\rho', \rho''))}{\exp(b\theta(\rho', \rho'')) - \exp(-b\lambda_2\,\theta(\rho', \rho''))} \geq \tau_1$$

where $\tau_1 = \inf\{(z - z^{\lambda_2})/(z - z^{-\lambda_2}) : z > 1\} \in (0,1)$. In just the same way,

$$\xi(\rho', \rho'', \varphi_1, \varphi_2) \leq \tau_2\,, \quad \eta(\rho', \rho'', \varphi_1, \varphi_2) \in [\tau_1, \tau_2]\,, \quad \eta(\tilde\rho, \rho, \varphi_1, \varphi_2) \in [\tau_1, \tau_2],$$

where $\tau_2 = \sup\{(z - z^{-\lambda_2})/(z - z^{-\lambda_2}) : z > 1\} \in (1, +\infty)$. As a direct consequence, $\alpha(\varphi_1, \varphi_2) \geq \tau_1\,\alpha_+(\varphi_1, \varphi_2)$ and $\beta(\varphi_1, \varphi_2) \leq \tau_2\,\beta_+(\varphi_1, \varphi_2)$, and so

$$\Theta(\varphi_1, \varphi_2) \leq \Theta_+(\varphi_1, \varphi_2) + \log(\tau_2/\tau_1)$$

for all $\varphi_1, \varphi_2 \in C(\lambda_2 b, \lambda_2 c, \nu)$.

Now we start the second and last step, where we show that the Θ_+-diameter of $\mathcal{L}(C(b,c,\nu))$ is finite, that is, there is a uniform upper bound for

$$\frac{\int_{\gamma''}(\mathcal{L}\varphi_2)\rho''}{\int_{\gamma''}(\mathcal{L}\varphi_1)\rho''} \Big/ \frac{\int_{\gamma'}(\mathcal{L}\varphi_2)\rho'}{\int_{\gamma'}(\mathcal{L}\varphi_1)\rho'}\,, \quad \varphi_1, \varphi_2 \in C(b,c,\nu),\ \rho' \in \mathcal{D}(\gamma'),\ \rho'' \in \mathcal{D}(\gamma'').$$

In fact, we prove a bit more:

$$\frac{\int_{\gamma''}(\mathcal{L}\varphi)\rho''}{\int_{\gamma'}(\mathcal{L}\varphi)\rho'}\,, \quad \varphi \in C(b,c,\nu),\ \rho' \in \mathcal{D}(\gamma'),\ \rho'' \in \mathcal{D}(\gamma''),$$

is uniformly bounded. We keep the notations of the proof of Proposition 3.6.1. First, we use condition (B) to get

$$\begin{aligned}\int_{\gamma'}(\mathcal{L}\varphi)\rho' &= \sum_{j=1}^{2}\int_{\gamma'_j}\varphi\rho'_j = \sum_{j=1}^{2}\left(\int_{\gamma'_j}\rho'_j\right)\left(\int_{\gamma_j}\varphi\rho_j^-\right)\\ &\geq \sum_{j=1}^{2}\left(\int_{\gamma'_j}\rho'_j\right)\exp(-b\theta_j(\rho_j^-, \mathbb{1}))\left(\int_{\gamma'_j}\varphi\mathbb{1}\right)\end{aligned} \tag{3.78}$$

where $\mathbb{1} = \mathbb{1}_{\gamma'_j}$ denotes the constant function on γ'_j satisfying $\int_{\gamma'_j}\mathbb{1} = 1$, and ρ_j^- is the multiple of ρ'_j such that $\int_{\gamma'_j}\rho_j^- = 1$. For the the last inequality recall that,

$$\rho_j^- \in \mathcal{D}(\lambda_1 a, \mu, \gamma'_j) \subset \mathcal{D}(\gamma'_j).$$

by Lemma 3.6.1. In just the same way,

$$\int_{\gamma''}(\mathcal{L}\varphi)\rho'' \leq \sum_{j=1}^{2}\left(\int_{\gamma''_j}\rho''_j\right)\exp(b\theta_j(\rho_j^=, \mathbb{1}))\left(\int_{\gamma''_j}\varphi\mathbb{1}\right). \tag{3.79}$$

Let us show that corresponding factors in (3.78) and (3.79) are comparable, up to uniform multiplicative constants. For the first factors we use change of variables:

$$\int_{\gamma_j'} \rho_j' = \int_{\gamma_j'} (\rho' \circ f)\,|\det(Df \mid \gamma_j')|\,|\det Df|^{-1}$$
$$= \int_{f(\gamma_j')} \rho' |\det(Df^{-1})| \geq \Gamma_1\,(\inf \rho')$$

and, analogously,

$$\int_{\gamma_j''} \rho_j'' \leq \Gamma_2\,(\sup \rho''),$$

where $\Gamma_1 > 0$, respectively $\Gamma_2 > 0$, depends only on uniform lower bounds, respectively upper bounds, for the Jacobian $|\det(Df^{-1})|$ and the Riemannian volume of images of local stable leaves. On the other hand, using the mean value theorem and $\int_{\gamma'} \rho' = 1 = \int_{\gamma''} \rho''$,

$$\inf \rho' \geq \exp(-a\,\Delta^{\mu}) \quad \text{and} \quad \sup \rho'' \leq \exp(a\,\Delta^{\mu})$$

This ensures that

$$\frac{\int_{\gamma_j''} \rho_j''}{\int_{\gamma_j'} \rho_j'} \leq \frac{\Gamma_2}{\Gamma_1} \exp(2a\Delta^{\mu}).$$

The second factors are dealt with quite easily. Let D_1 be some uniform upper bound for the θ-diameter of $\mathcal{D}(\lambda_1 a, \mu, \gamma) \subset \mathcal{D}(\gamma)$, cf. Lemma 3.6.1. Then $1 \leq \theta_j(\rho_j^-, \psi) \leq D_1$ and $1 \leq \theta_j(\rho_j^=, \psi) \leq D_1$, and so

$$\frac{\exp(b\theta_j(\rho_j^=, \psi))}{\exp(b\theta_j(\rho_j^-, \psi))} \leq \exp(bD_1)$$

Finally, we compare the third factors in (3.78), (3.79), respectively. Let $\tilde{\psi}\colon \gamma_j'' \to \mathbb{R}$ be given by $\tilde{\psi}(x) = \psi(\pi_j(x))\,|\det D\pi_j(x)|$, where $\pi_j = \pi(\gamma_j'', \gamma_j')$. By (p1), both ψ and $\tilde{\psi}$ belong in $\mathcal{D}(a_0, 1, \gamma_j'')$. On the other hand, recall (3.67),

$$\mathcal{D}(a_0, 1, \gamma_j'') \subset \mathcal{D}(a_0 \Delta^{1-\mu}, \mu, \gamma_j'') \subset \mathcal{D}(a/2, \mu, \gamma_j'').$$

Let D_0 be a uniform upper bound for the θ-diameter of $\mathcal{D}(a/2, \mu, \gamma_j'') \subset \mathcal{D}(\gamma_j'')$, cf. Lemma 3.6.1. Then, using conditions (B) and (C) for the function φ,

$$\frac{\int_{\gamma_j''} \varphi\psi}{\int_{\gamma_j'} \varphi\psi} \leq \frac{\int_{\gamma_j''} \varphi\psi}{\int_{\gamma_j''} \varphi\tilde{\psi}} \frac{\int_{\gamma_j''} \varphi\tilde{\psi}}{\int_{\gamma_j'} \varphi\psi}$$
$$\leq \exp(b\theta_j(\psi, \tilde{\psi}))\,\exp(c\,d(\gamma_j', \gamma_j'')^{\nu}) \leq \exp(b\,D_0 + c\Delta^{\nu}).$$

Replacing all this in (3.78), (3.79) we get,

$$\int_{\gamma''} (\mathcal{L}\varphi)\rho'' \leq \Gamma_0 \int_{\gamma'} (\mathcal{L}\varphi)\rho', \qquad \Gamma_0 = \frac{\Gamma_2}{\Gamma_1} \exp(2a\,\Delta^\mu + b\,D_1 + b\,D_0 + c\,\Delta^\mu),$$

for all φ, ρ', ρ''. It follows that $\Theta_+(\mathcal{L}\varphi_1, \mathcal{L}\varphi_2) \leq 2\log\Gamma_0$ for all $\varphi_1, \varphi_2 \in C(b, c, \nu)$. □

In what follows we let $\Lambda_2 = \tanh(D_2/4) < 1$. Then, in view of Propositions 3.6.2 and 3.2.2, the operator $\mathcal{L}$ is a Λ_2-contraction for the projective metric Θ.

Similarly to what we did in the expanding case, we now use the sequence $(\varphi_n = \mathcal{L}^n\,1)_n$ to construct an SRB-measure μ for f on Λ. It follows, from $\mathcal{L}$ being a contraction, that

$$\Theta_+(\varphi_k, \varphi_l) \leq \Theta(\varphi_k, \varphi_l) \to 0 \quad \text{(exponentially fast) as } k, l \to \infty.$$

Note, however, that in the present case $(\varphi_n)_n$ can not be expected to converge to a limit function, since μ and Lebesgue measure are mutually singular, in general. Instead, we have the following statement of weak-convergence.

Proposition 3.6.3. *Given any Θ_+-Cauchy sequence $(\varphi_n)_n$ in $C(b, c, \nu)$, normalized by $\int_Q \varphi_n\,dm = 1$ for all $n \geq 1$, and given any continuous function $\psi : Q \to \mathbb{R}$, the sequence $\left(\int \varphi_n\,\psi\,dm\right)_n$ is Cauchy in $\mathbb{R}$.*

In the proof of this proposition we make important use of the fact that the local stable foliation is absolutely continuous; see Theorem 2.2.2. In order to explain this in more precise terms, let $\tilde{m}$ be the quotient measure induced by Lebesgue measure m in the space of local stable leaves, that is,

$$\tilde{m}(\tilde{A}) = m\Bigg(\bigcup_{\gamma\in\tilde{A}} \gamma\Bigg).$$

By a disintegration of m with respect to the local stable foliation, one means a family $(p_\gamma)_{\gamma\in\mathcal{F}^s_{loc}}$ such that each p_γ is a probability measure on γ and

$$\int_Q \psi\,dm = \int \left(\int (\psi \mid \gamma)\,dp_\gamma\right) d\tilde{m}(\gamma)$$

for every m-integrable function ψ. We use the following fact, which is a consequence of the absolute continuity of $\mathcal{F}^s_{loc}$: there are constants $a_0 > 0$ and $0 < \nu_0 \leq 1$, and there is an (a_0, ν_0)-Hölder continuous function $H : Q \to (0, +\infty)$ such that $p_\gamma = (H|\gamma)m_\gamma$ defines a disintegration of m.

Proof. Consider first the case when $\psi > 0$ and $\log\psi$ is $(a/2, \mu)$-Hölder. We write

$$\int_Q \varphi_n \psi \, dm = \int \left(\int_\gamma \varphi_n \psi \, dp_\gamma \right) d\tilde{m}(\gamma) = \int \left(\int_\gamma \varphi_n \, \psi \, H_\gamma \right) d\tilde{m}(\gamma)$$

where $H_\gamma = H \mid \gamma$. Note that (ψH_γ) is strictly positive and $\log(\psi H_\gamma)$ is (a, μ)-Hölder, as long as we fix $a \geq 2a_0 \Delta^{\nu_0 - \mu}$ and $\mu \leq \nu_0$, cf. (3.67) and (3.71). Moreover,

$$\int_Q \varphi_n \, dm = \int \left(\int_\gamma \varphi_n \, H_\gamma \right) d\tilde{m}(\gamma)$$

and $H_\gamma > 0$ with $\log H_\gamma$ an (a, μ)-Hölder function. Therefore, given $k, l \geq 1$,

$$\frac{\int_\gamma \varphi_k H_\gamma}{\int_\gamma \varphi_l H_\gamma} \geq \alpha_+(\varphi_k, \varphi_l) \quad \text{and} \quad \frac{\int_\gamma \varphi_k \psi H_\gamma}{\int_\gamma \varphi_l \psi H_\gamma} \leq \beta_+(\varphi_k, \varphi_l) \quad \text{for all } \gamma.$$

On the other hand, $\int_Q \varphi_k \, dm = 1 = \int_Q \varphi_l \, dm$ implies that $\int_{\hat{\gamma}} \varphi_k H_{\hat{\gamma}} \leq \int_{\hat{\gamma}} \varphi_l H_{\hat{\gamma}}$ for some local leaf $\hat{\gamma}$. Thus,

$$\frac{\int_\gamma \varphi_k \psi H_\gamma}{\int_\gamma \varphi_l \psi H_\gamma} \leq \frac{\beta_+(\varphi_k, \varphi_l)}{\alpha_+(\varphi_k, \varphi_l)} \frac{\int_{\hat{\gamma}} \varphi_k H_{\hat{\gamma}}}{\int_{\hat{\gamma}} \varphi_l H_{\hat{\gamma}}} \leq e^{\Theta_+(\varphi_k, \varphi_l)} \quad \text{for all } \gamma,$$

implying that

$$\frac{\int_Q \varphi_k \psi \, dm}{\int_Q \varphi_l \psi \, dm} \leq e^{\Theta_+(\varphi_k, \varphi_l)} \quad \text{for all } k, l \geq 1.$$

As a consequence,

$$\begin{aligned} \left| \int_Q \varphi_k \psi \, dm - \int_Q \varphi_l \psi \, dm \right| &= \left| \int_Q \varphi_l \psi \, dm \right| \left| \frac{\int_Q \varphi_k \psi \, dm}{\int_Q \varphi_l \psi \, dm} - 1 \right| \\ &\leq \sup |\psi| \left(e^{\Theta_+(\varphi_k, \varphi_l)} - 1 \right), \end{aligned} \tag{3.80}$$

and the proposition is proved in this case.

Now, for an arbitrary μ-Hölder continuous function $\psi \colon Q \to \mathbb{R}$, we write

$$\psi = \psi_B^+ - \psi_B^-, \quad \text{where} \quad \psi_B^\pm = \frac{1}{2}(|\psi| \pm \psi) + B$$

and $B > 0$ is chosen large enough to ensure that $\log \psi_B^\pm$ is $(a/2, \mu)$-Hölder continuous. The previous argument applies to $\psi_B^\pm$ and so, by linearity, the proposition holds for ψ.

Finally, given any continuous function ψ and any $\varepsilon > 0$, we may take $\tilde{\psi}$ a μ-Hölder function such that $\sup |\psi - \tilde{\psi}| \leq \varepsilon$. Then, for every $k, l \geq 1$,

$$\left| \int_Q \varphi_k \psi \, dm - \int_Q \varphi_l \psi \, dm \right| \leq \left| \int_Q \varphi_k \tilde{\psi} \, dm - \int_Q \varphi_l \tilde{\psi} \, dm \right| + 2\varepsilon,$$

recall that we suppose $\int_Q \varphi_n \, dm = 1$ for all n. By the previous case, the right hand side is bounded by 3ε if k and l are large enough, and so we have proved that $\int_Q \varphi_n \psi \, dm$ is a Cauchy sequence also in this case. □

We are now in a position to introduce the SRB-measure μ_0 of the map f on Q. For that we consider $\varphi_n = \mathcal{L}^n 1$, for each $n \geq 1$. Then, by Proposition 3.6.3, $(\varphi_n)_n$ is a Θ-Cauchy sequence, and so it is also a Θ_+-Cauchy sequence. Moreover,

$$\int_Q \varphi_n \, dm = \int_Q (\mathcal{L}^n 1) \, 1 \, dm = \int_Q 1 \, (U^n 1) \, dm = \int_Q 1 \, dm = 1, \quad \text{for all } n \geq 1.$$

Then we define μ_0 to be the weak-limit of $(\mathcal{L}^n 1) \, m = (f^n)_* \, m$, that is,

$$\int \psi \, d\mu_0 = \lim \int_Q (\mathcal{L}^n 1) \psi \, dm = \lim \int_Q (\psi \circ f^n) \, dm,$$

for any continuous $\psi : Q \to \mathbb{R}$. We shall see in Section 3.6.2 that

$$\frac{1}{n} \sum_{j=0}^{n-1} \psi(f^j(x)) \to \int \psi \, d\mu_0 \tag{3.81}$$

for every $\psi \in C^0(Q, \mathbb{R})$ and m-almost all $x \in Q$. As a consequence, μ_0 is indeed the (unique) SRB-measure for the attractor Λ of f in Q.

Remark 3.6.1. Given any $\varphi_0 \in C(b, c, \nu)$ with $\int_Q \varphi_0 \, dm = 1$, consider the sequence $(\hat{\varphi}_n)_n$ defined by

$$\hat{\varphi}_{2k-1} = \mathcal{L}^k \varphi_0 \quad \text{and} \quad \hat{\varphi}_{2k} = \mathcal{L}^k 1, \quad \text{for } k \geq 1.$$

Then $(\hat{\varphi}_n)_n$ satisfies the hypothesis of Proposition 3.6.3: in particular,

$$\begin{aligned} \Theta_+(\hat{\varphi}_n, \hat{\varphi}_{n+1}) &\leq \Theta(\hat{\varphi}_n, \hat{\varphi}_{n+1}) \\ &\leq \Lambda_2^{[n/2]} \max \left\{ \Theta(1, \mathcal{L} \varphi_0), \Theta(\mathcal{L} \varphi_0, \mathcal{L} 1) \right\} \leq \Lambda_2^{[n/2]} \, D_2, \end{aligned}$$

and so the sequence is Θ_+-Cauchy. It follows that $(\hat{\varphi}_n)_n$ is weak-Cauchy and so

$$\int \psi \, d\mu_0 = \lim \int_Q \hat{\varphi}_{2k} \psi \, dm = \lim \int_Q \hat{\varphi}_{2k-1} \psi \, dm = \lim \int_Q (\mathcal{L}^k \varphi_0) \, \psi \, dm,$$

for every continuous ψ. Thus, μ_0 is also the weak-limit of $(\mathcal{L}^k \varphi_0) \, m = (f^k)_* (\varphi_0 \, m)$ for any $\varphi_0 \in C(b, c, \nu)$.

For the proof of (3.81) we shall need the following lemma which, in rough terms, asserts that μ_0 behaves as an absolutely continuous measure (with respect to Lebesgue measure) *if one quotients out local stable leaves*. Let $\mathcal{F}_0$ be the σ-algebra of Borel sets which are union of local stable leaves: $B \in \mathcal{F}_0$ if and only if B is a Borel subset of M and, given any local stable leaf γ, either $\gamma \cap B = \emptyset$ or $\gamma \subset B$. Clearly, $\tilde{m}$ is just the restriction of m to $\mathcal{F}_0$.

Lemma 3.6.4. *There is $K > 0$ such that, for every $\psi \in L^1(\mathcal{F}_0)$,*

$$\frac{1}{K}\int_Q \psi\, dm \le \int \psi\, d\mu_0 \le K \int_Q \psi\, dm.$$

Proof. Let γ, $\tilde\gamma$ be local stable leaves and $H_\gamma = H|\gamma$ and $H_{\tilde\gamma} = H|\tilde\gamma$ be as in the proof of Proposition 3.6.3. In addition, let $\widetilde{H}_\gamma = (H_\gamma \circ \pi)\,|\det D\pi|$, with $\pi = \pi(\tilde\gamma, \gamma)$. Recall that $\log H$ is (a_0, ν_0)-Hölder, with (a_0, ν_0) depending only on f. Therefore, up to choosing a, a_1 larger than a_0, and μ, μ_1 smaller than ν_0, as in (3.67), (3.71), we have $H_\gamma \in \mathcal{D}_1(\gamma)$ and $H_{\tilde\gamma}$, $\widetilde{H}_\gamma \in \mathcal{D}(a/2, \mu, \tilde\gamma)$. Then properties (B) and (C) give, for each $\varphi_k = \mathcal{L}^k\, 1$,

$$\begin{aligned}\frac{\int_\gamma \varphi_k\, H_\gamma}{\int_{\tilde\gamma} \varphi_k\, H_{\tilde\gamma}} &= \frac{\int_{\tilde\gamma} \varphi_k\, \widetilde{H}_\gamma}{\int_{\tilde\gamma} \varphi_k\, H_{\tilde\gamma}} \frac{\int_\gamma \varphi_k\, H_\gamma}{\int_{\tilde\gamma} \varphi_k\, \widetilde{H}_\gamma} \\ &\le \exp(b\theta_+(\widetilde{H}_\gamma, H_{\tilde\gamma}) + c\, d(\gamma, \tilde\gamma)^\nu) \le \exp(bD + c\Delta^\nu)\end{aligned}$$

where D is a uniform bound for the θ-diameter of $\mathcal{D}(a/2, \mu, \tilde\gamma) \subset \mathcal{D}(\tilde\gamma)$, see Lemma 3.6.1. For simplicity, we write $K = \exp(c\Delta^\nu + bD)$. Recalling that

$$\int \left(\int_{\tilde\gamma} \varphi_k\, H_{\tilde\gamma} \right) d\tilde{m}(\tilde\gamma) = \int \varphi_k\, dm = 1$$

we conclude that $\int_\gamma \varphi_k\, H_\gamma \le K$ for every local stable leaf γ. Then

$$\begin{aligned}\int_Q \psi(\mathcal{L}^k 1)\, dm &= \int \psi(\gamma)\left(\int_\gamma (\mathcal{L}^k 1)\, H_\gamma \right) d\tilde{m}(\gamma) \\ &\le K \int \psi(\gamma)\, d\tilde{m}(\gamma) = K \int_Q \psi\, dm,\end{aligned}$$

note that functions $\psi \in L^1(\mathcal{F}_0)$ are constant on each stable leaf γ. Passing to the limit as $k \to \infty$, $\int \psi\, d\mu_0 \le K \int_Q \psi\, dm$. The dual inequality $\int \psi\, d\mu_0 \ge K^{-1} \int_Q \psi\, dm$ may be derived in just the same way, and so the argument is complete. □

3.6.2 Exponential Convergence

Our next goal is to prove the following statement of exponential decay of correlations with respect to Hölder continuous functions.

Proposition 3.6.4. *Given any ν-Hölder continuous function φ and any μ-Hölder continuous function ψ, there is $C_2 = C_2(\varphi, \psi) > 0$ such that, for all $n \ge 0$,*

(a) $|\int_Q (\psi \circ f^n)\varphi\, dm - \int \psi\, d\mu_0 \int_Q \varphi\, dm| \le C_2 \Lambda_2^n$
(b) $|\int (\psi \circ f^n)\varphi\, d\mu_0 - \int \psi\, d\mu_0| \le C_2 \Lambda_2^n$

Proof. First we suppose that $\psi > 0$ and $\log \psi$ is $(a/2, \mu)$-Hölder. Then, as in (3.80),

$$\begin{aligned}
&\Big|\int_Q \psi(\mathcal{L}^n \varphi)dm - \int_Q \psi(\mathcal{L}^{n+k}\varphi)dm\Big| \\
&\qquad \le \sup \psi \int_Q (\mathcal{L}^{n+k}\varphi)\, dm \,\left(e^{\Theta_+(\mathcal{L}^n\varphi, \mathcal{L}^{n+k}\varphi)} - 1\right) \\
&\qquad \le \sup \psi \int_Q \varphi\, dm \,\left(e^{\Lambda_2^{n-1}\Theta(\mathcal{L}\varphi, \mathcal{L}^{k+1}\varphi)} - 1\right) \\
&\qquad \le \sup \psi \int_Q \varphi\, dm \,\left(e^{C_2'\Lambda_2^n} - 1\right) \\
&\qquad \le C_2''\, \Lambda_2^n \sup \psi \int_Q \varphi\, dm,
\end{aligned}$$

for every $\varphi \in C(b, c, \nu)$, where C_2', $C_2'' > 0$ are independent of ψ or φ. The third inequality uses Proposition 3.6.2. Since

$$\int_Q \psi(\mathcal{L}^n\varphi)dm = \int_Q (\psi \circ f^n)\varphi\, dm$$

and

$$\lim_{k\to\infty} \int_Q \psi(\mathcal{L}^{n+k}\varphi)dm = \int \psi\, d\mu_0 \int_Q \varphi\, dm$$

(by the definition of μ_0, together with Remark 3.6.1, we conclude that

$$\left|\int_Q (\psi \circ f^n)\varphi\, dm - \int \psi\, d\mu_0 \int_Q \varphi\, dm\right| \le C_2''\Lambda_2^n \sup \psi \int_Q \varphi\, dm \tag{3.82}$$

and this implies (a) for the class of functions φ and ψ as above.

Now we prove (b), under the further assumption that $\varphi > 0$ and $\log \psi$ is (c_1, ν)-Hölder continuous for some small $c_1 > 0$. We need the following statement, whose proof we postpone for a while (see Lemma 3.6.5 below):

$$\varphi\,(\mathcal{L}^l\, 1) \in C(b, c, \nu) \quad \text{for every } l \ge 0, \tag{3.83}$$

as long as $c_1 > 0$ is small enough (depending only on f). Indeed, (3.83) allows us to replace φ by $\varphi(\mathcal{L}^l\, 1)$ in 3.82, thus getting

$$\begin{aligned}
&\Big|\int_Q (\psi \circ f^n)\varphi(\mathcal{L}^l\, 1)dm - \int \psi\, d\mu_0 \int_Q \varphi(\mathcal{L}^l\, 1)dm\Big| \\
&\qquad\qquad \le C_2''\, \Lambda_2^n \sup \psi \int_Q \varphi(\mathcal{L}^l\, 1)dm.
\end{aligned}$$

Passing to the limit as $l \to \infty$,

$$\left| \int (\psi \circ f^n)\varphi \, d\mu_0 - \int \psi \, d\mu_0 \int \varphi \, d\mu_0 \right| \le C_2'' \Lambda_2^n \sup \psi \int \varphi \, d\mu_0$$

for all $n \ge 0$.

So far we have proved (a) and (b) for strictly positive φ, ψ such that $\log \varphi$ is (c_1, ν)-Hölder and $\log \psi$ is $(a/2, \mu)$-Hölder. The general case is a straightforward consequence. Just write $\varphi = \varphi^+ - \varphi^-$ with

$$\varphi^\pm = \frac{1}{2}(|\varphi| \pm \varphi) + B$$

and $B > 0$, and decompose ψ in the same way. Take B large enough so that $\log \varphi$ be (c_1, ν)-Hölder and $\log \psi_B^\pm$ be $(a/2, \mu)$-Hölder. Then use the previous particular case, together with linearity of the integral, to complete the argument. □

That is, we reduced Proposition 3.6.4 to checking the claim 3.83. We restate this claim in the next lemma. The proof uses the same kind of ideas as that of Lemma 3.6.3.

Lemma 3.6.5. *There is $c_1 > 0$, depending only on the map f, such that given any function $\varphi > 0$ such that $\log \varphi$ is (c_1, ν)-Hölder continuous, then*

$$\varphi \, (\mathcal{L}^l \, 1) \in C(b, c, \nu) \quad \textit{for every } l \ge 0.$$

Proof. Indeed, $1 \in C(b, c, \nu)$ and so, by Proposition 3.6.1, $\mathcal{L}^l \, 1 \in C(b, c, \nu)$. It follows that

$$\int_\gamma \varphi \, (\mathcal{L}^l \, 1) \, \rho \ge \inf \varphi \int_\gamma (\mathcal{L}^l \, 1)\rho > 0,$$

for every $\rho \in \mathcal{D}(\gamma)$, which proves property (A). We have already observed that (B) is automatic for nonnegative functions, as long as $b \ge 1$.

To prove property (C), let γ, $\tilde{\gamma}$ be arbitrary local stable leaves, and let $\rho \in \mathcal{D}_1(\gamma)$. Fix $l \ge 0$ and let γ_J and $\tilde{\gamma}_J$, $J = 1, \dots, 2^l$, the connected components of $f^{-l}(\gamma) \cap Q$ and $f^{-l}(\tilde{\gamma}) \cap Q$, respectively. Then,

$$\begin{aligned}
&\int_\gamma \varphi \, (\mathcal{L}^l \, 1) \, \rho \\
&\quad = \sum_{J=1}^{2^l} \int_{\gamma_J} (\varphi \circ f^l) \, (\rho \circ f^l) \, \frac{|\det (Df^l | \gamma_J)|}{|\det Df^l|} \\
&\quad = \sum_{J=1}^{2^l} \int_{\tilde{\gamma}_J} (\varphi \circ f^l \circ \pi_J) \, (\rho \circ f^l \circ \pi_J) \, \frac{|\det (Df^l | \gamma_J)| \circ \pi_J}{|\det Df^l| \circ \pi_J} \, |\det D\pi_J|,
\end{aligned}$$

where $\pi_J = \pi(\tilde{\gamma}_J, \gamma_J)$, and

$$\begin{aligned}\int_{\tilde{\gamma}} \varphi\,(\mathcal{L}^l\,1)\,\tilde{\rho} &= \sum_{J=1}^{2^l} \int_{\tilde{\gamma}_J} (\varphi\circ f^l)\,(\tilde{\rho}\circ f^l)\,\frac{|\det(Df^l|\tilde{\gamma}_J)|}{|\det Df^l|} \\ &= \sum_{J=1}^{2^l} \int_{\tilde{\gamma}_J} (\varphi\circ f^l)\,(\rho\circ\pi\circ f^l)\,(|\det D\pi|\circ f^l)\,\frac{|\det(Df^l|\tilde{\gamma}_J)|}{|\det Df^l|}\,.\end{aligned}$$

Since all the functions involved here are positive, property (C) will follow if we show that

$$\log\left(\frac{\varphi(f^l\pi_J(x))}{\varphi(f^l(x))}\;\frac{\rho(f^l\pi_J(x))}{\rho(\pi f^l(x))}\;\frac{|\det(Df^l|\gamma_J)|(\pi_J(x))}{|\det(Df^l|\tilde{\gamma}_J)|(x)}\right.\\ \left.\frac{|\det Df^l|(x)}{|\det Df^l|(\pi_J(x))}\;\frac{|\det D\pi_J|(x)}{|\det D\pi|(f^l(x))}\right) \tag{3.84}$$

is bounded in norm by $c\,d(\gamma,\tilde{\gamma})^\nu$, at every $x\in\tilde{\gamma}$ and for every $J=1,\dots,2^l$.

Let Γ_0 be the horizontal leaf containing x and $\xi_0\subset\Gamma_0$ be a curve joining x to $\pi_J(x)$ such that $\operatorname{length}(\xi_0)=\operatorname{dist}(x,\pi_J(x))$. The key remark is that the angle of each iterate $f^i(\Gamma_0)$, $i\geq 0$, to the horizontal direction is bounded, at every point, by some constant $H>0$ that depends only on f. It follows that

$$\operatorname{dist}(f^l\pi_J(x),\pi f^l(x))\leq H\,d(\gamma,\tilde{\gamma}), \tag{3.85}$$

and

$$\operatorname{dist}(f^l(x),f^l\pi_J(x))\leq \operatorname{length}(f^l(\xi_0))\leq(1+H)\,d(\gamma,\tilde{\gamma}). \tag{3.86}$$

More generally, the distance from $f^i(x)$ to $f^i\pi_J(x)$ is bounded by

$$\operatorname{length}(f^i(\xi_0))\leq(1+H)\,\lambda_u^{l-i}\,d(\gamma,\tilde{\gamma}) \tag{3.87}$$

for every $0\leq i\leq l$. By the assumption on φ together with (3.86),

$$|\log\varphi(f^l\pi_J(x))-\log\varphi(f^l(x))|\leq c_1(1+H)^\nu\,d(\gamma,\tilde{\gamma})^\nu\leq c_1(1+H)\,d(\gamma,\tilde{\gamma})^\nu.$$

Analogously, $\rho\in\mathcal{D}_1(\gamma)$ together with (3.85) give

$$|\log\rho(f^l\pi_J(x))-\log\rho(\pi f^l(x))|\leq a_1H^{\mu_1}\,d(\gamma,\tilde{\gamma})^{\mu_1}.$$

On the other hand, using (3.87) and the fact that $\log|\det Df|$ is Lipschitz continuous,

$$\Big|\log|\det Df^l|(x)-\log|\det Df^l|(\pi_J(x))\Big|\leq K_7\,d(\gamma,\tilde{\gamma})$$

for some large $K_7>0$. Recalling also (3.63),

$$\Big|\log|\det(Df^l|\gamma_J)|(\pi_J(x))-\log|\det(Df^l|\tilde{\gamma}_J)|(x)\Big|\leq K_8\,d(\gamma,\tilde{\gamma})^{\nu_0},$$

for some $K_8 > 0$, and

$$\begin{aligned}\big|\log|\det D\pi_J|(x) - \log|\det D\pi|(f^l(x))\big| &\leq A_0\, d(\gamma_J, \tilde{\gamma}_J)^{\nu_0} + A_0\, d(\gamma, \tilde{\gamma})^{\nu_0} \\ &\leq 2A_0\, d(\gamma, \tilde{\gamma})^{\nu_0}.\end{aligned}$$

In view of our choice of μ, μ_1, ν in (3.71), we conclude that (3.84) is bounded by

$$c_1(1+H)\, d(\gamma,\tilde{\gamma})^{\nu} + K_9\, d(\gamma,\tilde{\gamma})^{\nu}$$

where the constant $K_9 > 0$ depends only on f and a_1. At this point, we assume that $c > 0$ has been taken large enough so that $K_9 \leq c/2$, and then we choose any $c_1 \leq c/(2(1+H))$. □

Remark 3.6.2. The form of the constant $C_2(\varphi, \psi)$ in Proposition 3.6.4 is relevant for the sequel. By the mean value theorem, $\sup|\varphi| \leq \int_Q |\varphi|\, dm + \|\varphi\|_\nu (\operatorname{diam} Q)^\nu$ if φ is $(\|\varphi\|_\nu, \nu)$-Hölder continuous. Therefore, the argument in the proof of the proposition yields

$$C_2(\varphi,\psi) \approx \text{ const } \left(\int_Q |\varphi|\, dm + \|\varphi\|_\nu\right)\left(\int_Q |\psi|\, dm + \|\psi\|_\mu\right). \tag{3.88}$$

The Hölder term $\|\psi\|_\mu$ is essential, as shown by the following type of examples. Let $\psi\colon Q \to [0,1]$ be a C^1 function with

$$\int_Q \psi\, dm > 0 \quad \text{and} \quad \operatorname{supp}(\psi) \cap \Lambda = \emptyset.$$

Then $\psi_n = \psi \circ f^{-n}$ is a sequence of C^1 functions with $\int_Q (\psi_n \circ f^n)\, 1\, dm = \int_Q \psi\, dm > 0$, but $\int \psi_n\, d\mu_0 = 0$ for all $n \geq 0$. Note that $\|\psi_n\|_\mu$ is not bounded.

On the other hand, it is possible to improve the estimate of (3.88) in the following useful way. In the first step of the proof we took $\log\psi$ to be $(a/2,\mu)$-Hölder, to ensure that $\psi H_\gamma \in \mathcal{D}(\gamma)$ for every γ. Now, for this last conclusion it suffices that $\log\psi$ be $(a/2,\mu)$-Hölder *along stable leaves*. Thus, actually, the proof of Proposition 3.6.4 gives

$$C_2(\varphi,\psi) \approx \text{ const } \left(\int_Q |\varphi|\, dm + \|\varphi\|_\nu\right)\left(\int_Q |\psi|\, dm + \|\psi\|_\mu^s\right) \tag{3.89}$$

$\|\psi\|_\mu^s$ denoting any uniform Hölder constant for the restriction of ψ to each stable leaf.

Corollary 3.6.1. *Given any ν-Hölder continuous function φ, there is $C_3 = C_3(\varphi) > 0$ such that, for every $\psi \in L^1(\mathcal{F}_0)$ and all $n \geq 0$,*

(a) $|\int_Q (\psi\circ f^n)\,\varphi\, dm - \int \psi\, d\mu_0 \int_Q \varphi\, dm| \leq C_3 \Lambda_2^n \int_Q |\psi|\, dm$
(b) $|\int (\psi \circ f^n)\,\varphi\, d\mu_0 - \int \psi\, d\mu_0 \int \varphi\, d\mu_0| \leq C_3 \Lambda_2^n \int_Q |\psi|\, dm.$

Proof. This is a direct consequence of Proposition 3.6.4 and the last part of Remark 3.6.2. If ψ is $\mathcal{F}_0$-measurable then it is constant on each stable leaf, and so we may choose $\|\psi\|_\mu^s = 0$. Then take $C_3(\varphi) = \text{ const}\,(\int_Q |\varphi|\,dm + \|\varphi\|_\nu)$. □

Corollary 3.6.2. *The measure μ_0 is ergodic and satisfies*

$$\frac{1}{n}\sum_{j=0}^{n-1}\varphi(f^j(x)) \to \int \varphi\,d\mu_0 \quad \textit{as } n\to\infty,$$

for every continuous function $\varphi\colon Q\to\mathbb{R}$ and m-almost all $x\in Q$. In particular, μ_0 is the unique SRB-measure for f in Q.

Proof. Clearly, given any continuous φ and any pair of points x_1, x_2 belonging in a same stable leaf, then

$$\frac{1}{n}\sum_{j=0}^{n-1}\varphi(f^j(x_1)) \text{ converges} \quad\Leftrightarrow\quad \frac{1}{n}\sum_{j=0}^{n-1}\varphi(f^j(x_2)) \text{ converges,}$$

and in that case the two limits are the same. Let A be the set of points $x\in Q$ such that $\lim n^{-1}\sum_{j=0}^{n-1}\varphi(f^j(x))$ does not exist, for some continuous φ. Then A is a union of stable leaves and the ergodic theorem gives $\mu_0(A)=0$. Applying Lemma 3.6.4 to $\psi = \mathcal{X}_A$ we get $m(A)=0$.

Now, given any continuous φ, define $\tilde\varphi(x) = \lim n^{-1}\sum_{j=0}^{n-1}\varphi(f^j(x))$. We have just shown that $\tilde\varphi$ is defined almost everywhere, with respect to both measures μ_0 and m. Moreover, by the ergodic theorem, $\tilde\varphi\in L^1(\mathcal{F}_0)$ and $\tilde\varphi\circ f = \tilde\varphi$ at μ_0-almost every point. Lemma 3.6.4 for the characteristic function of $\{x\in Q : \tilde\varphi(f(x)) = \tilde\varphi(x)\}$ gives that $\tilde\varphi\circ f = \tilde\varphi$ at m-almost every point. Then part (a) of Corollary 3.6.1 implies

$$\begin{aligned}\left|\int_Q (\tilde\varphi - \int\tilde\varphi\,d\mu_0)\phi\,dm\right| &= \left|\int_Q (\tilde\varphi\circ f^n)\,\phi\,dm - \int\tilde\varphi\,d\mu_0\int\phi\,dm\right|\\ &\le C_3\Lambda_2^n\int_Q|\tilde\varphi|\,dm\end{aligned}$$

for every $n\ge 0$ and every ν-Hölder function ϕ. Therefore,

$$\tilde\varphi = \int\tilde\varphi\,d\mu_0 = \int\varphi\,d\mu_0$$

m-almost everywhere and μ_0-almost everywhere. This proves ergodicity and the SRB-property, simultaneously. □

4. Large Deviations and Central Limit Theorem

In Chapters 1 and 2 we dealt with the asymptotic averages

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x))$$

of functions $\varphi : M \to \mathbb{R}$ over typical orbits of a system $f : M \to M$. Here we investigate finite-time averages and their oscillations with respect to the limit value $\int \varphi \, d\mu$. The main conclusion is the following: if the function φ exhibits sufficiently fast mixing, meaning that its correlations to other L^2-functions decay sufficiently fast, then the sequence $\varphi \circ f^n$ shares many properties of sequences of independent random variables, including *large deviations* and *central limit* properties. The former means that the probability that finite-time averages be ε-deviated from the average decreases exponentially fast with time n, for any $\varepsilon > 0$. The latter states that the these oscillations converge in distribution to a Gaussian process.

4.1 Random Variables

Let us begin by reviewing some elementary topics from Probability Theory. See, for instance, [38, Section 1.4] for more information.

Let $X_1, \dots, X_n, \dots$ be a sequence of real-valued random variables in some probability space $(M, \mathcal{F}, \mu)$, that is, $\mathcal{F}$-measurable functions $X_n : M \to \mathbb{R}$. The X_n are *identically distributed* if, given any measurable set $A \subset \mathbb{R}$, the probability $\mu(X_k \in A)$ is the same for all $k \geq 1$. The X_n are *independent* if, given any $k \geq 1$ and measurable sets $A_1, \dots, A_k$,

$$\mu(X_1 \in A_1, \dots, X_k \in A_k) = \mu(X_1 \in A_1) \cdots \mu(X_k \in a_k).$$

The *mean* and the *variance* of a random variable Y are, respectively,

$$E(Y) = \int Y \, d\mu \quad \text{and} \quad V(Y) = E\left((Y - E(Y))^2\right) = E(Y^2) - E(Y)^2 .$$

The X_n are called *uncorrelated* if

$$E(X_{i_1}\cdots X_{i_k}) = E(X_{i_1})\cdots E(X_{i_k})$$

for any $k \geq 1$ and $i_1 < \cdots < i_k$. Independent random variables are uncorrelated, but independence is a far stronger condition. If the X_n are independent then, given any measurable real functions φ_n on $\mathbb{R}$, the random variables $\varphi_n(X_n)$, $n \geq 1$, are also independent, and so

$$E(\varphi_{i_1}(X_{i_1})\cdots\varphi_{i_k}(X_{i_k})) = E(\varphi_{i_1}(X_{i_1}))\cdots E(\varphi_{i_k}(X_{i_k})) \tag{4.1}$$

for all $k \geq 1$ and $i_1 < \cdots < i_k$.

The sequence of random variables $(X_n)_n$ is called *stationary* if

$$\mu\big(X_{1+j} \in A_1, \ldots, X_{k+j} \in A_k\big) = \mu\big(X_1 \in A_1, \ldots, X_k \in A_k\big),$$

for any $k \geq 1$, and any measurable subsets $A_1, \ldots A_k$ of $\mathbb{R}$. Clearly, in that case the X_n are identically distributed. Stationarity may also be expressed in the following useful way. Let ν be the probability measure on $\mathbb{R}^{\mathbb{N}}$ defined by

$$\nu\big(\{(\omega_n)_{n\geq 1} : \omega_1 \in A_1, \ldots, \omega_k \in A_k\}\big) = \mu\big(X_1 \in A_1, \ldots, X_k \in A_k\big)$$

for any $k \geq 1$, and any measurable subsets $A_1, \ldots A_k$ of $\mathbb{R}$. Then $(X_n)_n$ is stationary is and only if ν is invariant under the *shift map* $S : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$,

$$S : (\omega_1, \omega_2, \ldots, \omega_n, \ldots) \mapsto (\omega_2, \omega_3, \ldots, \omega_n, \ldots).$$

A stationary sequence $(X_n)_n$ is called *ergodic* if the measure ν is ergodic for the shift map S.

The sequences of random variables we are interested in are of the kind described in the following construction.

Example 4.1.1. Let $f : M \to M$ be some $\mathcal{F}$-measurable map, and ϕ be some $\mathcal{F}$-measurable real function on M. Let $X_n = \phi \circ f^{n-1}$, for $n \geq 1$. Suppose the measure μ is f-invariant. Then $(X_n)_n$ is a stationary sequence:

$$\mu\left(\bigcap_{i=1}^{k} X_{i+j} \in A_i\right) = \mu\left(\bigcap_{i=1}^{k} f^{-(i+j)}\big(\phi^{-1}(A_i)\big)\right)$$

does not depend on j. Moreover, if the measure μ is f-ergodic then $(X_n)_n$ is an ergodic sequence. That is because if B is an S-invariant subset of $\mathbb{R}^{\mathbb{N}}$ then

$$A = \{x \in M : \big(X_n(x)\big)_n \in B\} = \{x \in M : \big(\phi(f^n(x))\big)_n \in B\}$$

is f-invariant, and so $\nu(B) = \mu(A)$ is either zero or 1.

4.1.1 Large Deviations for Independent Random Variables

In its most basic form, the theorem of large deviations for independent identically distributed random variables is stated as follows:

Theorem 4.1.1. *Let X_n, $n \geq 1$, be a sequence of independent identically distributed random variables with finite mean $\bar{X} = E(X_n)$ and finite positive variance $\sigma^2 = V(X_n)$. Assume, moreover, that $E(e^{tX_n}) < \infty$ for every $t \in \mathbb{R}$. Then, given any $\varepsilon > 0$, the probability $\mu(n, \epsilon)$ of*

$$\Big|\frac{1}{n}\sum_{j=0}^{n-1}(X_j - \bar{X})\Big| \geq \varepsilon$$

converges to zero exponentially fast when n goes to infinity, in the sense that $\limsup_{n\to\infty} \frac{1}{n}\log\mu(n,\epsilon) < 0$.

We are going to prove Theorem 4.1.1 under a simplifying assumption: we suppose that $E(X_n\, e^{tX_n})$ and $E(X_n^2\, e^{tX_n})$ are finite for all $t \in \mathbb{R}$. This is always true if the random variables are bounded, which is the case in all the situations we are interested in.

Proof. Replacing each X_n by $X_n - \bar{X}$ if necessary, we may suppose that $\bar{X} = 0$. We do so in what follows. The *free-energy function* $\phi : \mathbb{R} \to \mathbb{R}$ is defined by $\phi(t) = \log E(e^{t\,X_n})$ (any $n \geq 1$). Note that $\phi(0) = 0$. Direct differentiation gives $\phi'(0) = \bar{X} = 0$, and

$$\begin{aligned}\phi''(t) &= \frac{E(e^{tX_n})\,E(X_n^2\,e^{tX_n}) - E(X_n\,e^{tX_n})^2}{E(e^{tX_n})^2} \\ &= E\left(\left(X_n - E\left(X_n\,\frac{e^{tX_n}}{E(e^{tX_n})}\right)\right)^2 \frac{e^{tX_n}}{E(e^{tX_n})}\right).\end{aligned}$$

The argument of $E(\cdot)$ in the last term is never negative, and it can not be zero at almost every point either: X_n would have to be constant equal to $E\left(X_n e^{tX_n}/E(e^{tX_n})\right)$ almost everywhere, and that is not possible because we assumed that its variance σ^2 is positive. This proves that $\phi''(t) > 0$ for every $t \in \mathbb{R}$, and so ϕ is strictly convex.

Now we introduce the *entropy function* $h : \mathbb{R} \to \mathbb{R}$, defined to be the Legendre transform of ϕ, that is,

$$h(z) = \sup\{tz - \phi(t) : t \in \mathbb{R}\}.$$

See Figure 4.1. Given any $t > 0$,

$$\mu\Big(\frac{1}{n}\sum_{i=1}^{n} X_i \geq \varepsilon\Big)\, e^{tn\varepsilon} = \mu\Big(t\sum_{i=1}^{n} X_i \geq tn\varepsilon\Big)\, e^{tn\varepsilon} \leq E(e^{t\sum_{i=1}^{n} X_i}) = e^{n\phi(t)}.$$

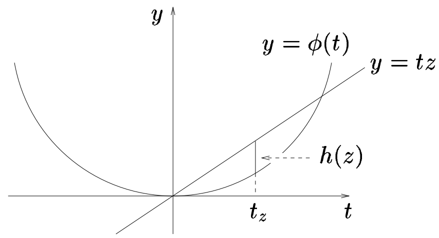

Fig. 4.1. Legendre transform of a convex function

The last equality uses (4.1) and the definition of ϕ. This shows that

$$\mu\big(\frac{1}{n}\sum_{i=1}^{n} X_i \geq \varepsilon\big) \leq e^{-n(t\varepsilon-\phi(t))} \tag{4.2}$$

for every $t > 0$. Actually, this inequality extends to every $t \in \mathbb{R}$: for $t \leq 0$, the right hand side of (4.2) is larger or equal than 1 (note that $\phi \geq 0$), while the left hand side is never larger than 1. It follows that

$$\mu\left(\frac{1}{n}\sum_{i=1}^{n} X_i \geq \varepsilon\right) \leq e^{-nh(\varepsilon)}.$$

A similar argument shows that the probability of $n^{-1}\sum_{i=1}^{n} X_i \leq -\varepsilon$ is bounded by $e^{-nh(-\varepsilon)}$. So,

$$\mu\left(\left|\frac{1}{n}\sum_{i=1}^{n} X_i\right| \geq \varepsilon\right) \leq e^{-nh(\varepsilon)} + e^{-nh(-\varepsilon)} \leq 2e^{-n\hat{h}(\varepsilon)},$$

where $\hat{h}(\varepsilon) = \min\{h(\varepsilon), h(-\varepsilon)\}$. Consequently,

$$\limsup_{n\to\infty} \frac{1}{n} \log \mu\left(\left|\frac{1}{n}\sum_{i=1}^{n} X_i\right| \geq \varepsilon\right) \leq -\hat{h}(\varepsilon).$$

To conclude the proof of the theorem, we only have to check that $\hat{h}(\varepsilon) > 0$, which is easy: since ϕ is strictly convex, with $\phi(0) = \phi'(0) = 0$, we have $h(z) > 0$ for every $z \neq 0$; in particular, $\hat{h}(\varepsilon)$ is positive for any $\varepsilon > 0$. □

Remark 4.1.1. It is easy to see that the entropy function h is monotone increasing on $\{z > 0\}$ and monotone decreasing on $\{z < 0\}$. This means that, $\hat{h}(\varepsilon) = \inf\{h(z) : |z| \geq \varepsilon\}$. Therefore, we have shown that

$$\limsup_{n\to\infty} \frac{1}{n} \log \mu\left(\frac{1}{n}\sum_{i=1}^{n} X_i \notin [-\varepsilon, \varepsilon]\right) \leq -\inf\{h(z) : z \notin [-\varepsilon, \varepsilon]\}. \tag{4.3}$$

There is also a lower bound: for any open set $O \subset \mathbb{R}$

$$\liminf_{n\to\infty} \frac{1}{n} \log \mu\big(\frac{1}{n}\sum_{i=1}^{n} X_i \in O\big) \geq \inf\{h(z) : z \in O\}.$$

See [41, Theorem II.4.1]. Taking $O = \mathbb{R}\setminus[-\varepsilon, \varepsilon]$, we get that the limit in (4.3) exists and is equal to the infimum.

4.1.2 Central Limit for Independent Random Variables

Let us state and prove the classical central limit theorem.

Theorem 4.1.2. *Let X_n, $n \geq 1$, be a sequence of independent identically distributed random variables in a probability space $(M, \mathcal{F}, \mu)$, with finite mean $\bar{X} = E(X_n)$ and finite positive variance $\sigma^2 = V(X_n)$. Then, given any $z \in \mathbb{R}$,*

$$\lim_{n\to\infty} \mu\left(\frac{1}{\sqrt{n}}\sum_{j=1}^{n}(X_j - \bar{X}) \leq z\right) \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{z} e^{\frac{-t^2}{2\sigma^2}}\, dt\,. \tag{4.4}$$

Proof. Replacing each X_n by $(X_n - \bar{X})/\sigma$ if necessary, we may suppose that $\bar{X} = 0$ and $\sigma = 1$. Let

$$Z_n = \frac{1}{\sqrt{n}}\,(X_1 + \cdots + X_n),$$

and $F(x) = \mu(X_n \leq x)$ and $H_n(x) = \mu(Z_n \leq x)$ be the *distribution functions* of X_n and Z_n. We also consider the normalized Gaussian distribution function,

$$G(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt$$

Then (4.4) is the claim that H_n converges everywhere to N, as n goes to infinity. To prove it, we introduce the *characteristic functions* (Fourier transforms) of X_n and Z_n, respectively,

$$f(t) = E\left(e^{itX_n}\right) = \int_{\mathbb{R}} e^{itx}\, dF(x) \quad \text{and} \quad h_n(t) = E\left(e^{itZ_n}\right) = \int_{\mathbb{R}} e^{itx}\, dH_n(x).$$

Using the Taylor series expansion of the exponential,

$$f(s) = \int_{\mathbb{R}} \left(1 + isx - \frac{1}{2}s^2x^2 + o(s^2)\right)\, dF(x) = 1 - \frac{1}{2}s^2 + o(s^2),$$

for every $s \in \mathbb{R}$. Here we also used that, by assumption,

$$\int_{\mathbb{R}} x\, dF(x) = E(X_n) = 0 \quad \text{and} \quad \int_{\mathbb{R}} x^2\, dF(x) = E(X_n^2) = 1.$$

Consequently, $\log f(s) = -(s^2/2) + o(s^2)$. On the other hand, since the X_n, $n \geq 1$, are independent,

$$h_n(t) = E\left(e^{it\sum_{j=1}^n X_j/\sqrt{n}}\right) = \prod_{j=1}^n E\left(e^{itX_j/\sqrt{n}}\right) = f\left(\frac{t}{\sqrt{n}}\right)^n$$
$$= \exp\left(n\log\left(\frac{t}{\sqrt{n}}\right)\right) = \exp\left(-\frac{t^2}{2} + n\,o\left(\frac{t^2}{n}\right)\right)$$

(apply (4.1) to the real and imaginary parts of $e^{it\sum_{j=1}^n X_j/\sqrt{n}}$). It follows that $h_n(t)$ converges to $g(t) = e^{-t^2/2}$ as $n \to \infty$, for all $t \in \mathbb{R}$. Now, $g(t)$ is the characteristic function of the Gaussian distribution G

$$\int_{\mathbb{R}} e^{itx}\,dG(x) = \int_{\mathbb{R}} e^{itx}\frac{e^{-x^2/2}}{\sqrt{2\pi}}\,dx = e^{-t^2/2}\int_{\mathbb{R}}\frac{e^{-(x-it)^2/2}}{\sqrt{2\pi}}\,dx = e^{-t^2/2},$$

and h_n was defined as the characteristic function of H_n. Then, since the Gaussian distribution G is continuous, we may use the inverse Fourier transform continuity theorem (see [16, Theorem 2.6] or [38, Theorem 2.3.4]), to conclude that H_n converges to G as n goes to infinity. □

Remark 4.1.2. The conclusion (4.4) may be rewritten, equivalently, as

$$\lim_{n\to\infty}\mu\left(\frac{1}{\sqrt{n}}\sum_{j=1}^n (X_j - \bar{X}) \in I\right) = \frac{1}{\sqrt{2\pi}\sigma}\int_I e^{\frac{-t^2}{2\sigma^2}}\,dt$$

for any interval $I \subset \mathbb{R}$. Clearly, (4.4) corresponds to the particular case $I = (-\infty, z]$. The converse is almost as easy, keep in mind that the Gauss distribution is continuous.

4.1.3 Martingale Differences

The sequences of random variables originating from a deterministic dynamical system, cf. Example 4.1.1, are not independent, and so the results in the previous subsections do not apply to them. However, under an assumption of fast mixing, it will be possible to deduce a central limit theorem for such sequences, from an extension of Theorem 4.1.2 for weakly dependent random variables. The precise notion is called *martingale differences*, and is defined in the sequel.

Let Y be a random variable on $(M, \mathcal{F}, \mu)$ and $\mathcal{G}$ be any sub-σ-algebra of the σ-algebra $\mathcal{F}$. Define

$$\mu_Y(B) = \int_B Y\,d\mu$$

for any $B \in \mathcal{G}$. Then μ_Y is a probability measure on $\mathcal{G}$ and, clearly, it is absolutely continuous with respect to μ. The *conditional expectation* of Y

with respect to $\mathcal{G}$, denoted $E(Y \mid \mathcal{G})$, is the Radon-Nikodym derivative of this measure μ_Y with respect to μ. That is, $E(Y \mid \mathcal{G})$ is a $\mathcal{G}$-measurable function such that

$$\int_B E(Y \mid \mathcal{G})\, d\mu = \int_B Y\, d\mu \tag{4.5}$$

for any $B \in \mathcal{G}$.

Remark 4.1.3. The conditional expectation admits the following geometric interpretation when Y is in the Hilbert space $L^2(\mu) = L^2(M, \mathcal{F}, \mu)$: it is the orthogonal projection of Y onto the subspace $L^2(M, \mathcal{G}, \mu)$. This can be seen as follows. First, (4.5) may be rewritten as $\int \big(Y - E(Y \mid \mathcal{G})\big)\, \mathcal{X}_B\, d\mu = 0$, for any $B \in \mathcal{G}$. Taking linear combinations and pointwise limits, we get that $Y - E(Y \mid \mathcal{G})$ is orthogonal to every $\mathcal{G}$-measurable function:

$$\int \big(Y - E(Y \mid \mathcal{G})\big)\, \psi\, d\mu = 0. \tag{4.6}$$

In particular, this holds for $\psi = E(Y \mid \mathcal{G})$. Then Y may be decomposed into two orthogonal terms $Y = E(Y \mid \mathcal{G}) + \big(Y - E(Y \mid \mathcal{G})\big)$, and so

$$\|E(Y \mid \mathcal{G})\|_2^2 + \|Y - E(Y \mid \mathcal{G})\|_2^2 = \|Y\|_2^2 < \infty.$$

This shows that the conditional expectation $E(Y \mid \mathcal{G})$ is in $L^2(M, \mathcal{G}, \mu)$ and, by (4.6), is the orthogonal projection of Y onto $L^2(M, \mathcal{G}, \mu)$.

Definition 4.1.1. *Let $\mathcal{F}_n$, $n \geq 1$, be a nonincreasing sequence of sub-σ-algebras of $\mathcal{F}$. A sequence of random variables X_n, $n \geq 1$, is a* reversed martingale difference *for $(\mathcal{F}_n)_n$ if*

1. *X_n is $\mathcal{F}_n$-measurable for every $n \geq 1$, and*
2. *$E(X_n \mid \mathcal{F}_{n+1}) = 0$ for every $n \geq 1$.*

In view of (4.6), the second condition means that

$$\int X_n \psi\, d\mu = \int (X_n - E(X_n \mid \mathcal{F}_{n+1}))\psi\, d\mu = 0,$$

for any $\mathcal{F}_{n+1}$-measurable function ψ. Observe that X_m is $\mathcal{F}_{n+1}$-measurable for every $m > n$, because it is $\mathcal{F}_m$-measurable and the sequence of sub-σ-algebras is nonincreasing. Taking $\psi = X_m$ gives that the random variables are two-by-two orthogonal:

$$\int X_n X_m\, d\mu = \quad \text{for every } 1 \leq n < m. \tag{4.7}$$

Moreover, taking $\psi = 1$ gives that every X_n has zero mean.

Definition 4.1.2. *Let $\mathcal{G}_n$, $n \geq 1$, be a nondecreasing sequence of sub-σ-algebras of $\mathcal{F}$. A sequence of random variables X_n, $n \geq 1$, is a* direct martingale difference *for $(\mathcal{G}_n)_n$ if*

1. *X_n is $\mathcal{G}_n$-measurable for every $n \geq 1$, and*
2. *$E(X_1) = 0$ and $E(X_n \mid \mathcal{G}_{n-1}) = 0$ for every $n \geq 2$.*

Clearly, the comments above about reversed martingale differences remain valid here: if X_n, $n \geq 1$, is a direct martingale difference then the X_n are two-by-two orthogonal and have zero mean.

We call a sequence of random variables X_n, $n \geq 1$, a reversed martingale difference if the conditions in Definition 4.1.1 hold for the following canonical choice of F_n, $n \geq 1$: each $\mathcal{F}_n = \sigma(X_m, m \geq n)$ is the σ-algebra generated by $\{X_m : m \geq n\}$ (the σ-algebra generated by a family of maps is the smallest σ-algebra with respect to which all those maps are measurable). Analogously, we call X_n, $n \geq 1$, a direct martingale difference if the conditions in Definition 4.1.2 hold for the canonical choice $\mathcal{G}_n = \sigma(X_m, m \leq n)$. Observe that $\mathcal{F}_n$ is nonincreasing and $\mathcal{G}_n$ is nondecreasing. Moreover, with these choices the first condition in Definitions 4.1.1 and 4.1.2 is automatic.

Example 4.1.2. A sequence of X_n, $n \geq 1$, independent random variables is always a reversed martingale difference and a direct martingale difference. Indeed, let us check that X_n satisfies the second condition in Definition 4.1.1:

$$\int_B X_n \, d\mu = 0 \quad \text{for every } B \in \mathcal{F}_{n+1}\,. \tag{4.8}$$

As a first step, suppose that $B = X_m^{-1}(A)$ for some $m > n$ and some measurable subset of $\mathbb{R}$ and $m > n$. By definition,

$$\int_B X_n \, d\mu = \lim_{k\to\infty} \sum_{j\in\mathbb{Z}} \frac{j}{k}\, \mu\left(B \cap \{X_n \in [\frac{j}{k}, \frac{j+1}{k})\}\right).$$

By independence, and the definition of B,

$$\begin{aligned}\mu\left(B \cap \{X_n \in [\frac{j}{k}, \frac{j+1}{k})\}\right) &= \mu\left(X_m \in A, X_n \in [\frac{j}{k}, \frac{j+1}{k})\right) \\ &= \mu\left(X_m \in A\right) \mu\left(X_n \in [\frac{j}{k}, \frac{j+1}{k})\right)\end{aligned}$$

Consequently, $\int_B X_n \, d\mu$ is equal to

$$\mu(X_m \in A) \lim_{k\to\infty} \sum_{j\in\mathbb{Z}} \frac{j}{k}\, \mu\left(X_n \in [\frac{j}{k}, \frac{j+1}{k})\right) = \mu(X_m \in A) E(X_n) = 0.$$

This proves (4.8) when B is of the form $X_m^{-1}(A)$. Next, it is easy to see that the family $\tilde{\mathcal{F}}$ of measurable sets B satisfying (4.8) is a σ-algebra: it is closed under complements (because $E(X_n) = 0$) and countable disjoint unions. It follows that, $\tilde{\mathcal{F}}$ contains the σ-algebra generated by the sets $X_m^{-1}(A)$, with $m > n$ and A a measurable subset of $\mathbb{R}$. In other words, $\tilde{\mathcal{F}}$ contains $\mathcal{F}_{n+1}$, as we wanted to prove.

Finally, we state the central limit theorem for martingale differences. A proof can be found in [81] and [38, Theorem 7.4].

Theorem 4.1.3. *Let $(X_n)_n$ be an ergodic, direct or reversed, martingale difference. Assume that $\sigma^2 = E(X_0^2)$ is positive and finite. Then, for every $z \in \mathbb{R}$,*

$$\lim_{n\to\infty} \mu\left(\frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} X_j \le z\right) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{z} e^{-t^2/2\sigma^2}\, dt\,. \tag{4.9}$$

4.2 Central Limit Theorems for Dynamical Systems

Using Theorem 4.1.3 we prove an abstract central limit theorem for dynamical systems, due to Gordin [49]. Related results had appeared previously in [15] and [57].

4.2.1 Noninvertible Maps

Let $(M, \mathcal{F}, \mu)$ be a probability space and $f\colon M \to M$ be a measurable map such that μ is f-invariant and f-ergodic. It is enough to state the results for functions with zero mean, because any L^2 function φ can be written $\varphi = \int \varphi\, d\mu + \phi$, with $\int \phi\, d\mu = 0$.

Theorem 4.2.1. *Let $\phi \in L^2(\mu)$ be such that $\int \phi\, d\mu = 0$, and $\mathcal{F}_n$ be the nonincreasing sequence of σ-algebras $\mathcal{F}_n = f^{-n}(\mathcal{F})$, $n \ge 0$. Assume that*

$$\sum_{n=0}^{\infty} \|E(\phi \mid \mathcal{F}_n)\|_2 < \infty. \tag{4.10}$$

Then $\sigma^2 = \int \phi^2\, d\mu + 2\sum_{n=1}^{\infty} \int \phi(\phi \circ f^n)\, d\mu$ is finite and nonnegative. Let σ be the nonnegative square root. If $\sigma > 0$ then, for any $z \in \mathbb{R}$,

$$\lim_{n\to\infty} \mu\left(\frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \phi \circ f^j \le z\right) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{t^2}{2\sigma^2}}\, dt\,.$$

Moreover, $\sigma = 0$ if and only if $\phi = u \circ f - u$ for some $u \in L^2(\mu)$.

It is not explicitly excluded that the map f be invertible, but the statement is of little use in the invertible case: the sequence $(\mathcal{F}_n)_n$ is constant, and so (4.10) can not be fulfilled (except for trivial circumstances). A version suitable for invertible maps is given in Theorem 4.2.3.

The main idea in the proof of Theorem 4.2.1 is contained in the following

Proposition 4.2.1. *Under the assumptions of Theorem 4.2.1, there exist functions η and ζ in $L^2(\mu)$ such that*

(i) $\phi = \eta + \zeta \circ f - \zeta$ and

(ii) $\eta \circ f^{n-1}$, $n \geq 1$, is a reversed martingale difference for $\mathcal{F}_{n-1}$, $n \geq 1$.

Moreover, if η is as above then $\|\eta\|_2^2 = \int \phi^2 \, d\mu + 2\sum_{n=1}^{\infty} \int \phi(\phi \circ f^n) \, d\mu$.

Proof. We write $L^2(\mu) = L^2(M, \mathcal{F}, \mu)$ and $L^2(\mathcal{F}_n) = L^2(M, \mathcal{F}_n, \mu)$ for each $n \geq 0$. Let $\mathcal{U}\colon L^2(\mu) \to L^2(\mu)$ be the linear operator given by $\mathcal{U}\varphi = \varphi \circ f$, and $\mathcal{U}_* : L^2(\mu) \to L^2(\mu)$ be its adjoint, defined by

$$\int (\mathcal{U}_*\varphi)\psi \, d\mu = \int \varphi(\mathcal{U}\psi) d\mu$$

for all φ and ψ in $L^2(\mu)$.

Lemma 4.2.1. *1. $\mathcal{U}$ is an isometry of $L^2(\mu)$ onto $L^2(\mathcal{F}_1)$.*
2. $\mathcal{U}_* \circ \mathcal{U} = \mathrm{id}$ and $(\mathcal{U} \circ \mathcal{U}_*) \mid L^2(\mathcal{F}_1)$ is the identity on $L^2(\mathcal{F}_1)$.
3. The restriction of $\mathcal{U}_*$ to $L^2(\mathcal{F}_1)$ is an isometry onto $L^2(\mu)$.
4. $\mathcal{U}$ maps $L^2(\mathcal{F}_n)$ onto $L^2(\mathcal{F}_{n+1})$, and $\mathcal{U}_*$ maps $L^2(\mathcal{F}_{n+1})$ onto $L^2(\mathcal{F}_n)$, for any $n \geq 1$.

Proof. A function φ is in $L^2(\mathcal{F}_1)$ if and only if $\varphi = \varphi_0 \circ f$ for $\varphi_0 \in L^2(\mu)$. Moreover, φ and φ_0 have the same L^2-norm, because the measure μ is invariant under f. This means that $\mathcal{U}$ maps $L^2(\mu)$ onto $L^2(\mathcal{F}_1)$, preserving the L^2-norm. In particular, $\mathcal{U}$ is one-to-one, hence it is an isometry. This gives the first statement. To get 2, consider arbitrary $\varphi \in L^2(\mathcal{F}_1)$ and $\varphi_0 \in L^2(\mu)$ with $\varphi = \varphi_0 \circ f$. Then

$$\int (\mathcal{U}_*\varphi)\psi \, d\mu = \int (\mathcal{U}\varphi_0)(\mathcal{U}\psi) d\mu = \int \varphi_0 \psi \, d\mu.$$

for any $\psi \in L^2(\mu)$, and so $\mathcal{U}_*\varphi = \varphi_0$. This proves that $\mathcal{U}_* \circ \mathcal{U}$ and $\mathcal{U} \circ \mathcal{U}_*$ are the identity maps, as stated. Claim 3 is a consequence of 1 and 2. Finally, a function φ is in $L^2(\mathcal{F}_{n+1})$ if and only if $\varphi = \varphi_0 \circ f^{n+1}$ for some $\varphi_0 \in L^2(\mu)$, that is, if and only if $\varphi \circ f$ is in $L^2(\mathcal{F}_n)$. This proves the fourth claim. □

Going back to the proof of Proposition 4.2.1, let

$$\zeta = -\sum_{j=1}^{\infty} \mathcal{U}_*^j \big(E(\phi \mid \mathcal{F}_j)\big) \quad \text{and} \quad \eta = \sum_{j=0}^{\infty} \mathcal{U}_*^j \big(E(\phi \mid \mathcal{F}_j) - E(\phi \mid \mathcal{F}_{j+1})\big).$$

According to the previous lemma, $\|\mathcal{U}_*^j(E(\phi \mid \mathcal{F}_j))\|_2 = \|E(\phi \mid \mathcal{F}_j)\|_2$ for all $j \geq 0$. So, the hypothesis (4.10) ensures that the series defining ζ converges in $L^2(\mu)$. Moreover, as we have seen in Remark 4.1.3, $E(\phi \mid \mathcal{F}_j) - E(\phi \mid \mathcal{F}_{j+1})$ is the orthogonal projection of $E(\phi \mid \mathcal{F}_j)$ onto the orthogonal complement $L^2(\mathcal{F}_{j+1})^\perp$ of $L^2(\mathcal{F}_{j+1})$. It follows that

$$\begin{aligned}\|\mathcal{U}_*^j(E(\phi \mid \mathcal{F}_j) - E(\phi \mid \mathcal{F}_{j+1}))\|_2 &= \|E(\phi \mid \mathcal{F}_j) - E(\phi \mid \mathcal{F}_{j+1})\|_2 \\ &\leq \|E(\phi \mid \mathcal{F}_j)\|_2\,,\end{aligned}$$

and so the series defining η also converges in $L^2(\mu)$.

To prove part (i) of the statement, write

$$\begin{aligned}\eta = E(\phi \mid \mathcal{F}_0) &+ \mathcal{U}_*(E(\phi \mid \mathcal{F}_1)) + \mathcal{U}_*^2(E(\phi \mid \mathcal{F}_2)) + \cdots \\ &- E(\phi \mid \mathcal{F}_1) - \mathcal{U}_*(E(\phi \mid \mathcal{F}_2)) - \mathcal{U}_*^2(E(\phi \mid \mathcal{F}_3)) - \cdots\end{aligned}$$

By definition, $E(\phi \mid \mathcal{F}_0) = \phi$. Using the previous lemma once again,

$$\mathcal{U}_*^{j-1}(E(\phi \mid \mathcal{F}_j)) = \mathcal{U}\mathcal{U}_*^j(E(\phi \mid \mathcal{F}_j)) = \mathcal{U}_*^j(E(\phi \mid \mathcal{F}_j)) \circ f.$$

for all $j \geq 1$. Therefore, we may rewrite

$$\eta = \phi + \sum_{j=1}^{\infty} \mathcal{U}_*^j(E(\phi \mid \mathcal{F}_j)) - \sum_{j=1}^{\infty} \mathcal{U}_*^j(E(\phi \mid \mathcal{F}_j)) \circ f = \phi - \zeta + \zeta \circ f.$$

Next, we show that $(\eta \circ f^{n-1})_n$ is a reversed martingale difference for $(\mathcal{F}_{n-1})_n$. The first condition in Definition 4.1.1 is clear in this case. The second one is checked in the following way. A while ago we observed that $E(\phi \mid \mathcal{F}_j) - E(\phi \mid \mathcal{F}_{j+1})$ is in the orthogonal complement of $L^2(\mathcal{F}_{j+1})$, for all $j \geq 0$. Since $\mathcal{U}_*$ is an isometry, it follows that

$$\mathcal{U}_*^j(E(\phi \mid \mathcal{F}_j) - E(\phi \mid \mathcal{F}_{j+1})) \in L^2(\mathcal{F}_1)^\perp,$$

for all $j \geq 0$. Consequently, $\eta \in L^2(\mathcal{F}_1)^\perp$ or, in other words, $E(\eta \mid \mathcal{F}_1) = 0$. The martingale difference property is an immediate consequence:

$$E(\eta \circ f^{n-1} \mid \mathcal{F}_n) = E(\eta \mid \mathcal{F}_1) \circ f^{n-1} = 0,$$

for all $n \geq 1$. This gives the second condition in Definition 4.1.1.

All that is left to do is prove the expression for the L^2-norm of η claimed in the statement. Since $\phi \circ f^n \in L^2(\mathcal{F}_n)$,

$$\Big|\int \phi(\phi \circ f^n)d\mu\Big| = \Big|\int E(\phi \mid \mathcal{F}_n)(\phi \circ f^n)d\mu\Big| \leq \|E(\phi \mid \mathcal{F}_n)\|_2 \, \|\phi\|_2 \quad (4.11)$$

for all $n \geq 0$. By (4.10), the series $\sum_n \int \phi\,(\phi \circ f^n)\,d\mu$ is absolutely convergent:

$$\sum_{n=1}^{\infty} \left| \int \phi\,(\phi \circ f^n)\,d\mu \right| \le \|\phi\|_2 \sum_{n=0}^{\infty} \|E(\phi \mid \mathcal{F}_n)\|_2 < \infty \tag{4.12}$$

Since $\eta \circ f^{n-1}$, $n \ge 1$, is a martingale difference, these random variables $\eta \circ f^{n-1}$ are two-by-two orthogonal. Therefore,

$$\|\eta\|_2^2 = \frac{1}{n} \sum_{j=0}^{n-1} \|\eta \circ f^j\|_2^2 = \frac{1}{n} \| \sum_{j=0}^{n-1} \eta \circ f^j \|_2^2 = \| \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \eta \circ f^j \|_2^2 .$$

Since

$$\| \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \phi \circ f^j - \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \eta \circ f^j \|_2 = \| \frac{1}{\sqrt{n}} (\zeta \circ f^n - \zeta) \|_2 \to 0,$$

as $n \to \infty$, we conclude that

$$\|\eta\|_2^2 = \lim_n \| \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \phi \circ f^j \|_2^2 .$$

On the other hand,

$$\begin{aligned} \| \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \phi \circ f^j \|_2^2 &= \frac{1}{n} \Big(\sum_{k=0}^{n-1} \int (\phi \circ f^k)^2\,d\mu + 2 \sum_{l>k\ge 0}^{n-1} \int (\phi \circ f^k)(\phi \circ f^l) d\mu \Big) \\ &= \int \phi^2\,d\mu + 2 \sum_{j=1}^{n-1} \Big(1 - \frac{j}{n}\Big) \int \phi(\phi \circ f^j) d\mu \end{aligned}$$

So, to conclude that

$$\|\eta\|_2^2 = \int \phi^2\,d\mu + 2 \sum_{n=1}^{\infty} \int \phi(\phi \circ f^n)\,d\mu$$

we only have to note that $\sum_{j=1}^{n-1} (j/n) \int \phi(\phi \circ f^j) d\mu$ goes to zero when $n \to \infty$. Indeed,

$$\left| \sum_{j=1}^{n} \frac{j}{n} \int \phi\,(\phi \circ f^j)\,d\mu \right| \le \varepsilon \sum_{j=1}^{\infty} \left| \int \phi\,(\phi \circ f^j)\,d\mu \right| + \sum_{j>n\varepsilon} \left| \int \phi\,(\phi \circ f^j)\,d\mu \right|,$$

for any $\varepsilon > 0$. Using (4.12), the right hand side can be made arbitrarily small, by first choosing $\varepsilon > 0$ close to zero and then taking n much larger than $1/\varepsilon$. The proof of the proposition is complete. □

Now we can prove Theorem 4.2.1:

Proof. The last statement in Proposition 4.2.1 means that $\sigma^2 = \|\eta\|_2^2$, which is finite and nonnegative. Of course, $\sigma^2 = 0$ implies that $\eta = 0$ and then $\phi = \zeta \circ f - \zeta$. Conversely, if ϕ has the form $\phi = u \circ f - u$ then, taking $\zeta = u$ and $\eta = 0$, the same arguments give $\sigma^2 = 0$.

From now on we suppose that $\sigma = \|\eta\|_2$ is positive. Then $\eta \circ f^j$ has positive variance for any $j \geq 0$. Let $\varepsilon > 0$ and $\delta > 0$ be fixed. Clearly,

$$\frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \phi \circ f^j \leq z \quad \Rightarrow \quad \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \eta \circ f^j \leq z + \varepsilon \text{ or } \frac{1}{\sqrt{n}} |\zeta \circ f^n - \zeta| > \varepsilon .$$

As a consequence of Proposition 4.2.1 and Theorem 4.2.1, there exists $n_0 \geq 1$ such that

$$\mu\left(\frac{1}{\sqrt{n}} |\zeta \circ f^n - \zeta| > \varepsilon\right) \leq \delta \quad \text{and} \quad \mu\left(\frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \eta \circ f^j \leq z + \varepsilon\right) \leq G_\sigma(z+\varepsilon) + \delta$$

for all $n \geq n_0$, where $G_\sigma(z) = (\sigma\sqrt{2\pi})^{-1} \int_{-\infty}^{z} e^{-\frac{t^2}{2\sigma^2}}\, dt$ is the Gaussian distribution with zero mean and variance σ^2. Then,

$$\mu\left(\frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \phi(f^j(x)) \leq z\right) \leq G_\sigma(z + \varepsilon) + 2\delta,$$

for any $n \geq n_0$. Since ε and δ are arbitrary, and G_σ is continuous, this proves that

$$\limsup_n \mu\left(\frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \phi \circ f^j \leq z\right) \leq G_\sigma(z).$$

A similar argument gives that the lim inf is at least $G_\sigma(z)$. So, the limit exists and is equal to $G_\sigma(z)$. □

The following useful extension of Theorem 4.2.1, due to Liverani [71], states that the same conclusion holds under weaker conditions than (4.10):

Theorem 4.2.2. *Let $\phi \in L^2(\mu)$ be such that $\int \phi \, d\mu = 0$, and $\mathcal{F}_n$ be the nonincreasing sequence of σ-algebras $\mathcal{F}_n = f^{-n}(\mathcal{F})$, $n \geq 0$. Assume that*

(a) $\sum_{j=0}^{\infty} \mathcal{U}_*^j \phi$ converges absolutely almost everywhere, and
(b) $\sum_{j=0}^{\infty} E\big(\phi\,(\phi \circ f^j)\big)$ converges absolutely.

Then all the conclusions of Theorem 4.2.1 remain valid.

The proof goes along the lines of that of Theorem 4.2.1, we just explain what the main additional difficulty is. First, let us comment on the hypotheses. Both conditions (a) and (b) were derived in the course of proving Theorem 4.2.1. Indeed, $\mathcal{U}_*^j \phi = \mathcal{U}_*^j\big(E(\phi \mid \mathcal{F}_j)\big)$, and so (a) is the same as the absolute convergence of the series that defines ζ. Condition (b) was obtained in (4.12). Thus, (4.10) is, indeed, a stronger hypothesis. On the other hand, these were the only places where the assumption (4.10) intervened. Under (a) and (b) the expression of η is also absolutely convergent, and we still have that $(\zeta \circ f^n - \zeta)/\sqrt{n}$ goes to zero in measure. The main missing step to conclude the proof of Theorem 4.2.2 is to show that η is in $L^2(\mu)$ if ϕ is, even though ζ may not be integrable. See [71] for an argument.

4.2.2 Invertible Maps

Next, we deduce a version of Theorem 4.2.1 for invertible maps. The proof is a variation of the proof of Theorem 4.2.1, inspired by [40] and [71].

Theorem 4.2.3. *Suppose the map $f\colon M \to M$ is invertible, and there exists some σ-algebra $\mathcal{G}_0 \subset \mathcal{F}$ such that $\mathcal{G}_n = f^{-n}(\mathcal{G}_0)$, $n \in \mathbb{Z}$, is a nonincreasing sequence, and*

$$\sum_{n=0}^{\infty} \|E(\phi \mid \mathcal{G}_n)\|_2 < \infty \quad \textit{and} \quad \sum_{n=0}^{\infty} \|\phi - E(\phi \mid \mathcal{G}_{-n})\|_2 < \infty. \tag{4.13}$$

Then $\sigma^2 = \int \phi^2 \, d\mu + 2\sum_{j=1}^{\infty} \int \phi(\phi \circ f^j)\, d\mu$ is finite and nonnegative. If $\sigma > 0$ then, for any $z \in \mathbb{R}$,

$$\lim_{n\to\infty} \mu\bigg(\frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \phi \circ f^j \le z\bigg) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{t^2}{2\sigma^2}}\, dt\,.$$

Moreover, $\sigma = 0$ if and only if $\phi = u \circ f - u$ for some $u \in L^2(\mu)$.

Recall that $E(\cdot \mid \mathcal{G}_j)$ is the orthogonal projection onto $L^2(\mathcal{G}_j)$, and so

$$\hat{E}(\varphi \mid \mathcal{G}_j) = \varphi - E(\varphi \mid \mathcal{G}_j).$$

is the orthogonal projection of $\varphi \in L^2(\mu)$ onto $L^2(\mathcal{G}_j)^{\perp}$, for any $j \in \mathbb{Z}$,. With this notation, (4.13) can be rewritten in a more symmetrical form

$$\sum_{n=0}^{\infty} \|E(\phi \mid \mathcal{G}_n)\|_2 < \infty \quad \text{and} \quad \sum_{n=0}^{\infty} \|\hat{E}(\phi \mid \mathcal{G}_{-n})\|_2 < \infty. \tag{4.14}$$

Proposition 4.2.2. *Under the assumptions of Theorem 4.2.3 there exist functions η and ζ in $L^2(\mu)$ such that*

(i) $\phi = \eta + \zeta \circ f - \zeta$ and

(ii) $\eta \circ f^{n-1}$, $n \geq 1$, *is a reversed martingale difference for* $\mathcal{G}_{n-1}$, $n \geq 1$.

Moreover, if η *is as above then* $\|\eta\|_2^2 = \int \phi^2 \, d\mu + 2 \sum_{n=1}^{\infty} \int \phi(\phi \circ f^n) \, d\mu$.

Proof. We begin with an outline, the detailed proof will follow. We decompose $\phi = \phi^+ + \phi^-$, with

$$\phi^+ = E(\phi \mid \mathcal{G}_0) \quad \text{and} \quad \phi^- = \hat{E}(\phi \mid \mathcal{G}_0).$$

Then we write each of the two terms ϕ^+ and ϕ^- as $\phi^\pm = \eta^\pm + \zeta^\pm \circ f - \zeta^\pm$, where $\eta^\pm$ and $\zeta^\pm$ are L^2 functions, and $\eta^\pm \circ f^{n-1}$, $n \geq 1$, are reversed martingale differences. For ϕ^+ this is similar to Theorem 4.2.1, based on the first part of (4.14). For ϕ^- we use a dual argument and the second condition in (4.14). Finally, we take $\eta = \eta^+ + \eta^-$ and $\zeta = \zeta^+ + \zeta^-$, and we prove that the L^2 norm of η is σ^2.

As before, we consider the linear operator $\mathcal{U} : L^2(\mu) \to L^2(\mu)$, $\mathcal{U}\varphi = \varphi \circ f$, and its adjoint $\mathcal{U}_* : L^2(\mu) \to L^2(\mu)$. In the present situation Lemma 4.2.1 asserts that $\mathcal{U}$ and $\mathcal{U}_*$ are isometries, with $\mathcal{U} \circ \mathcal{U}_* = \mathrm{id} = \mathcal{U}_* \circ \mathcal{U}$. Moreover, $\mathcal{U}$ maps $L^2(\mathcal{G}_j)$ onto $L^2(G_{j+1})$, and $\mathcal{U}_*$ maps $L^2(G_{j+1})$ onto $L^2(\mathcal{G}_j)$, for each $j \in \mathbb{Z}$. Define

$$\zeta^+ = -\sum_{j=1}^{\infty} \mathcal{U}_*^j \big(E(\phi \mid \mathcal{G}_j) \big) \quad \text{and} \quad \eta^+ = \sum_{j=0}^{\infty} \mathcal{U}_*^j \big(E(\phi \mid \mathcal{G}_j) - E(\phi \mid \mathcal{G}_{j+1}) \big),$$

$$\zeta^- = -\sum_{j=0}^{\infty} \mathcal{U}^j \big(\hat{E}(\phi \mid \mathcal{G}_{-j}) \big) \quad \text{and} \quad \eta^- = \sum_{j=1}^{\infty} \mathcal{U}^j \big(\hat{E}(\phi \mid \mathcal{G}_{-j+1}) - \hat{E}(\phi \mid \mathcal{G}_{-j}) \big).$$

All these expressions are well-defined. Indeed, from (4.14),

$$\|\zeta^+\|_2 \leq \sum_{j=1}^{\infty} \|E(\phi \mid \mathcal{G}_j)\|_2 < \infty \quad \text{and} \quad \|\zeta^-\|_2 \leq \sum_{j=0}^{\infty} \|\hat{E}(\phi \mid \mathcal{G}_{-j})\|_2 < \infty.$$

Recall that $(\mathcal{G}_n)_n$ is assumed nonincreasing. Then $E(\phi \mid \mathcal{G}_j) - E(\phi \mid \mathcal{G}_{j+1})$ is the orthogonal projection of $E(\phi \mid \mathcal{G}_j)$ onto $L^2(\mathcal{G}_{j+1})^\perp$, for every j. In particular,

$$\|\eta^+\|_2 \leq \sum_{j=0}^{\infty} \|E(\phi \mid \mathcal{G}_j) - E(\phi \mid \mathcal{G}_{j+1})\|_2 \leq \sum_{j=0}^{\infty} \|E(\phi \mid \mathcal{G}_j)\|_2 < \infty.$$

Similarly, $\hat{E}(\phi \mid \mathcal{G}_{-j+1}) - \hat{E}(\phi \mid \mathcal{G}_{-j}) = E(\phi \mid \mathcal{G}_{-j}) - E(\phi \mid \mathcal{G}_{-j+1})$ is the orthogonal projection of $\hat{E}(\phi \mid \mathcal{G}_{-j+1})$ onto $L^2(\mathcal{G}_{-j})$, and so

$$\|\eta^-\|_2 \leq \sum_{j=1}^{\infty} \|\hat{E}(\phi \mid \mathcal{G}_{-j+1}) - \hat{E}(\phi \mid \mathcal{G}_{-j})\|_2 \leq \sum_{j=1}^{\infty} \|\hat{E}(\phi \mid \mathcal{G}_{-j+1})\|_2 < \infty.$$

We have shown that ζ^+, η^+, ζ^-, η^- are all in $L^2(\mu)$.

The relation $\phi^+ = \eta^+ + \zeta^+ \circ f - \zeta^+$ is proved in the same way as the corresponding statement in Proposition 4.2.1. Similarly, adding $\hat{E}(\phi \mid \mathcal{G}_0)$ in the first row and subtracting it from the second row of

$$\begin{aligned}\eta^- = &\,\mathcal{U}(\hat{E}(\phi \mid \mathcal{G}_0)) + \mathcal{U}^2(\hat{E}(\phi \mid \mathcal{G}_{-1})) + \mathcal{U}^3(\hat{E}(\phi \mid \mathcal{G}_{-2})) + \cdots \\ &- \mathcal{U}(\hat{E}(\phi \mid \mathcal{G}_{-1})) - \mathcal{U}^2(\hat{E}(\phi \mid \mathcal{G}_{-2})) - \mathcal{U}^3(\hat{E}(\phi \mid \mathcal{G}_{-3})) - \cdots\end{aligned}$$

gives $\eta^- = \hat{E}(\phi \mid \mathcal{G}_0) - \mathcal{U}(\zeta^-) + \zeta^-$, that is, $\phi^- = \eta^- + \zeta^- \circ f - \zeta^-$. It follows that $\phi = \eta + \zeta \circ f - \zeta$, where $\eta = \eta^+ + \eta^-$ and $\zeta = \zeta^+ + \zeta^-$. This gives part (i) of the conclusion.

Now let us explain why $\eta \circ f^{n-1}$, $n \geq 1$, is a reversed martingale difference for the sequence $\mathcal{G}_{n-1}$, $n \geq 1$. Since $L^2(\mathcal{G}_j)$ contains $L^2(\mathcal{G}_{j+1})$, the difference $E(\phi \mid \mathcal{G}_j) - E(\phi \mid \mathcal{G}_{j+1})$ is in $L^2(\mathcal{G}_j)$. As observed before, it is also in $L^2(\mathcal{G}_{j+1})^\perp$. Then, each $E(\phi \mid \mathcal{G}_j) - E(\phi \mid \mathcal{G}_{j+1})$ is in the orthogonal complement $L^2(\mathcal{G}_j) \ominus L^2(\mathcal{G}_{j+1})$ of $L^2(\mathcal{G}_{j+1})$ inside $L^2(\mathcal{G}_j)$. Then

$$\mathcal{U}_*^j\big(E(\phi \mid \mathcal{G}_j) - E(\phi \mid \mathcal{G}_{j+1})\big) \in L^2(\mathcal{G}_0) \ominus L^2(\mathcal{G}_1) \quad \text{for all } j \in \mathbb{Z},$$

and so η^+ is in $L^2(\mathcal{G}_0) \ominus L^2(\mathcal{G}_1)$. Similarly, $\hat{E}(\phi \mid \mathcal{G}_{-j+1}) - \hat{E}(\phi \mid \mathcal{G}_{-j})$ is in $L^2(\mathcal{G}_{-j}) \ominus L^2(\mathcal{G}_{-j+1})$ for every $j \in \mathbb{Z}$, and so $\eta^- \in L^2(\mathcal{G}_0) \ominus L^2(\mathcal{G}_1)$. It follows that

$$\eta \in L^2(\mathcal{G}_0) \ominus L^2(\mathcal{G}_1)$$

Then $\eta \circ f^{n-1} \in L^2(\mathcal{G}_{n-1})$ is $\mathcal{G}_{n-1}$-measurable for all $n \geq 1$, corresponding to the first requirement in Definition 4.1.1. Moreover,

$$E(\eta \circ f^{n-1} \mid \mathcal{G}_n) = E(\eta \mid \mathcal{G}_1) \circ f^{n-1} = 0,$$

because η is in the orthogonal complement of $L^2(\mathcal{G}_1)$. We have completed the proof of part (ii).

Finally, we prove the expression for the L^2-norm of η claimed in the statement. Observe that (4.11) is no longer valid, because $\phi \circ f^j$ may not be in $L^2(\mathcal{G}_j)$. But we do have the analog of (4.12):

Lemma 4.2.2. *The series $\sum_{n=1}^\infty \int \phi\,(\phi \circ f^j)\,d\mu$ is absolutely convergent.*

Proof. Let us decompose

$$\phi = E(\phi \mid \mathcal{G}_0) + \sum_{k=1}^{j} \big(E(\phi \mid \mathcal{G}_{-k}) - E(\phi \mid \mathcal{G}_{-k+1})\big) + \big(\phi - E(\phi \mid \mathcal{G}_{-j})\big)$$

Since $E(\phi \mid \mathcal{G}_0) \circ f^j$ is $\mathcal{G}_j$-measurable,

$$\left|\int \phi\left(E(\phi \mid \mathcal{G}_0)\circ f^j\right) d\mu\right| = \left|\int E(\phi \mid \mathcal{G}_j)(E(\phi \mid \mathcal{G}_0)\circ f^j) d\mu\right| \le \|E(\phi \mid \mathcal{G}_j)\|_2 \,\|\phi\|_2 \,. \tag{4.15}$$

We have used $\|E(\phi \mid \mathcal{G}_0)\|_2 \le \|\phi\|_2$. Similarly, given $1 \le k \le j$,

$$\left|\int \phi\left(E(\phi \mid \mathcal{G}_{-k}) - E(\phi \mid \mathcal{G}_{-k+1})\right)\circ f^j\, d\mu\right| \le \le \|E(\phi \mid \mathcal{G}_{j-k})\|_2 \,\|\hat{E}(\phi \mid \mathcal{G}_{-k+1})\|_2 \,, \tag{4.16}$$

because $\|E(\phi \mid \mathcal{G}_{-k}) - E(\phi \mid \mathcal{G}_{-k+1})\|_2 \le \|\hat{E}(\phi \mid \mathcal{G}_{-k+1})\|_2$. Finally,

$$\left|\int \phi\,(\phi - E(\phi \mid \mathcal{G}_{-j}))\circ f^j\, d\mu\right| \le \|\phi\|_2 \,\|\phi - E(\phi \mid \mathcal{G}_{-j})\|_2 = \|\phi\|_2 \,\|\hat{E}(\phi \mid \mathcal{G}_{-j})\|_2 \,. \tag{4.17}$$

Putting (4.15), (4.16), (4.17), together, we obtain

$$\sum_{j=1}^{\infty} \left|\int \phi(\phi \circ f^j) d\mu\right| \le \|\phi\|_2 \sum_{j=1}^{\infty} \left(\|E(\phi \mid \mathcal{G}_j)\|_2 + \|\hat{E}(\phi \mid \mathcal{G}_{-j})\|_2\right) + + \sum_{i=0}^{\infty} \|E(\phi \mid \mathcal{G}_i)\|_2 \sum_{l=0}^{\infty} \|\hat{E}(\phi \mid \mathcal{G}_{-l})\|_2$$

(the last term comes from (4.16), with $l = k - 1$ and $i = j - k$). By (4.14), the right hand side is finite. This finishes the proof of the lemma. □

Now the proof of Proposition 4.2.2 proceeds in the precisely the same way as in Proposition 4.2.1. □

From this point on the proof of Theorem 4.2.3 is entirely analogous to that of Theorem 4.2.1.

4.2.3 The Fast Mixing Condition

Now, our goal is to express the assumptions (4.10) and 4.13 of these central limit theorems, in terms of properties of fast decay of correlations for the dynamical system.

According to Remark 4.1.3, the conditional expectation $E(\phi \mid \mathcal{F}_n)$ of a function $\phi \in L^2(\mu)$ is just the orthogonal projection of ϕ onto the subspace $L^2(\mathcal{F}_n)$ of $\mathcal{F}_n$-measurable functions. Thence,

$$\begin{aligned}\|E(\phi \mid \mathcal{F}_n)\|_2 &= \sup\{\int \xi\phi\, d\mu : \psi \in L^2(\mathcal{F}_n) \text{ with } \|\xi\|_{2,\mu} = 1\}\\ &= \sup\{\int (\psi \circ f^n)\, d\mu : \psi \in L^2(\mu) \text{ with } \|\psi\|_{2,\mu} = 1\}\\ &= \sup\{\int |C_n(\phi,\psi)| : \psi \in L^2(\mu) \text{ with } \|\psi\|_{2,\mu} = 1\}\end{aligned}$$

We need the following variation of Corollary 3.1.2:

Lemma 4.2.3. *Under the hypotheses of Proposition 3.1.2, there exists* $\tau < 1$*, and for every bounded function* $\phi \in F_0$ *there exists* $K(\phi) > 0$*, such that*

$$\|E(\phi \mid \mathcal{F}_n)\|_2^2 \le K(\phi)\tau_2^n \|\psi\|_{2,\mu}$$

for any , and for any $n \ge 1$ *and* $\psi \in L^q(\mu)$.

Proof. Using Lemma 4.2.1 and Proposition 3.1.2,

$$\begin{aligned}\|E(\phi \mid \mathcal{F}_n)\|_2^2 = \|\mathcal{U}^n \mathcal{U}_*^n\|_2^2 = \|\mathcal{U}_*^n\|_2\|_2^2 &= \int (\mathcal{U}_*^n \phi)\,(\mathcal{U}_*^n \phi)\,d\mu \\ &= \int (\mathcal{L}^n \phi)\,(\mathcal{U}_*^n \phi)\,dm \le C_1 \tau_1^n \|\phi\varphi_0\|_E \|\mathcal{U}_*^n \phi\|_q\,.\end{aligned}$$

Furthermore,

$$\|\mathcal{U}_*^n \phi\|_q \le \|\mathcal{U}_*^n \phi\|_\infty = \|\frac{1}{\varphi_0}\mathcal{L}^n\left(\varphi_0 \phi\right)\|_\infty \le \sup|\varphi_0|\,,$$

because $\mathcal{L}$ is a nonnegative operator and $\varphi_0 \ge 0$ is a fixed point of $\mathcal{L}$. This gives

$$\|E(\phi \mid \mathcal{F}_n)\|_2\|_2^2 \le C_1 \tau_1^n \|\phi\varphi_0\|_E \sup|\phi|\,.$$

So, we may take τ and $K(\phi)$ to be the square roots of τ_1 and $\|\phi\varphi_0\|_E \sup|\phi|$, respectively. □

Corollary 4.2.1. *Under the hypotheses of Proposition 3.1.2, every bounded function* $\phi \in F_0$ *satisfies condition 4.10. Consequently,* ϕ *has the central limit property.*
Proof. □

4.3 Large Deviations for Dynamical Systems

Now we prove a general version of this result Theorem 4.1.1 for random variables with rapid decay of correlations, and use it to show that observable functions in certain dynamical systems have the large deviations property.

4.4 Applications

Here we prove large deviations and central limit theorems various classes of dynamical systems, using the abstract results in the previous subsections. Several other results are available in the literature, see e.g. [36], [29] and bibliography therein.

Definition 4.4.1. *Let $f : M \to M$ be a measure-preserving transformation in a probability space $(M, \mathcal{F}, \mu)$, and $\varphi : M \to \mathbb{R}$ be a measurable function. We say that φ* has the central limit property for f *if there exists $\sigma > 0$ such that*

$$\lim_{n\to\infty} \mu\Big(\frac{1}{\sqrt{n}}\sum_{j=0}^{n-1}(\varphi - \int \varphi\, d\mu) \le z\Big) = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{z} e^{\frac{-t^2}{2\sigma^2}}\, dt$$

for all $z \in \mathbb{R}$.

Uniformly expanding maps. First we get a central limit theorem for uniformly expanding maps and Hölder functions.

Proposition 4.4.1. *Let $f : M \to M$ be a $C^{1+\nu_0}$ uniformly expanding map on a compact manifold M. Then every Hölder continuous function $\varphi : M \to \mathbb{R}$ has the central limit property for f, or else it can be written as $\varphi = u \circ f - u + \int \varphi\, d\mu$ for some $u \in L^2(\mu)$.*

Proof. **Corollary 4.4.1.** *For every ν-Hölder continuous function φ with $\int \varphi\, d\mu_0 = 0$ there is $R_0 = R_0(\varphi)$ such that $\|E(\varphi \mid \mathcal{F}_n)\|_2 \le R_0 \Lambda_1^n$ for all $n \ge 0$.*

Proof. It suffices to note that

$$\begin{aligned}\|E(\varphi \mid \mathcal{F}_n)\|_2 &= \sup\{\int \xi\varphi\, d\mu_0 : \xi \in L^2(\mathcal{F}_n) \text{ and } \|\xi\|_2 = 1\}\\ &= \sup\{\int (\psi \circ f^n)\varphi\,\varphi_0\, dm : \psi \in L^2(\mu_0) \text{ and } \|\psi\|_2 = 1\}\\ &\le K_0''(\varphi\,\varphi_0)\Lambda_1^n\end{aligned}$$

(since $\|\psi\|_1 \le \|\psi\|_2 = 1$ and we suppose $\int \varphi\, d\mu_0 = \int \varphi\,\varphi_0\, dm = 0$). □

□

Remark 4.4.1. The following description of the action of $\mathcal{L}$ in L^2 is contained in what we have done so far. Let φ_0 be as above and $H = \{\varphi \in L^2(m) : \int \varphi\, dm = 0\}$. We have $\mathcal{L}(\varphi_0) = \varphi_0$ and $\mathcal{L}(H) \subset H$, by Lemma 3.1.1. Consider the isometry $h : L^2(m) \to L_2(\mu_0)$, $h(\varphi) = \varphi/\varphi_0$, and introduce $\mathcal{U}_* : L^2(\mu_0) \to L^2(\mu_0)$, $\mathcal{U}_* = h \circ \mathcal{L} \circ h^{-1}$, and

$$N = h(H) = \{\psi \in L^2(m) : \int \psi\, d\mu_0 = 0\}.$$

It follows that $\mathcal{U}_*(1) = 1$ and $\mathcal{U}_*(N) \subset N$, and then Lemma 3.1.1 asserts that $\mathcal{U}_*$ is the adjoint operator of $U : L^2(\mu_0) \to L^2(\mu_0)$, $U(\psi) = \psi \circ f$. Let us denote $L_0^2(\mathcal{F}_n) = N \cap L^2(\mathcal{F}_n)$. Then U and $\mathcal{U}_*$ are unitary operators with $U(L_0^2(\mathcal{F}_n)) = L_0^2(\mathcal{F}_{n+1})$ and $\mathcal{U}_*(L_0^2(\mathcal{F}_{n+1})) = L_0^2(\mathcal{F}_n)$. Exactness means that $L^2(\mu_0)$ splits as an orthogonal sum

$$L^2(\mu_0) = \{\text{constants}\} \oplus N = \{\text{constants}\} \oplus \left(\oplus_{n=0}^{\infty} \left[L_0^2(\mathcal{F}_n) \ominus L_0^2(\mathcal{F}_{n+1})\right]\right),$$

where $\ominus$ denotes orthogonal complement. The last corollary implies that the components of any $\psi \in L^2(\mu_0)$ in this splitting decrease exponentially fast as $n \to +\infty$.

Piecewise expanding maps. The following central limit theorem for piecewise expanding maps of the interval and functions with bounded variation was first obtained by [64].

The next result follows, in the same way as Corollary 4.4.1 and Proposition 4.4.1.

Corollary 4.4.2. *Let φ be a function with bounded variation and $\sigma^2 = \int \phi^2 \, d\mu_0 + 2\sum_{j=1}^{\infty} \int \phi(\phi \circ f^j) d\mu_0$, where $\phi = \varphi - \int \varphi \, d\mu_0$. Then $\sigma < \infty$ and $\sigma = 0$ if and only if $\phi = u \circ f - u$ for some $u \in L^2(\mu_0)$. Moreover, if $\sigma > 0$ then for every interval $A \subset \mathbb{R}$*

$$\mu_0 \left\{ x \in M : \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \phi(f^j(x)) \in A \right\} \longrightarrow \int_A \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{t^2}{2\sigma^2}} \, dt \qquad \text{as } n \to +\infty.$$

Proposition 4.4.2. *Let $f : I \to I$ be a piecewise expanding map of the interval. Then function $\varphi : I \to \mathbb{R}$ with bounded variation has the central limit property for f, unless it can be written as $\varphi = u \circ f - u + \int \varphi \, d\mu$ for some $u \in L^2(\mu)$.*

Proof. □

Maps with neutral fixed points.

Hyperbolic attractors.

5. Stochastic Stability

5.1 Definitions

5.2 Expanding Maps

Next we prove that expanding maps $f\colon M \to M$ are stable under random perturbations. We consider parametrized families $f_t\colon M \to M$ of $C^{1+\nu_0}$-Hölder continuous maps, where t belongs in some metric space T. Suppose that there is $\tau \in T$ such that

$$f_\tau = f \quad \text{and} \quad T \ni t \mapsto f_t \in C^{1+\nu_0}(M, \mathbb{R}) \text{ is continuous at } \tau.$$

This means that if t is close to τ then f_t is uniformly close to f and Df_t is close to Df with respect to the ν_0-Hölder norm

$$|||G|||_{\nu_0} = \sup\{\|G(x)\| : x \in M\} + \sup\left\{\frac{\|G(x) - G(y)\|}{d(x,y)^{\nu_0}} : x, y \in M, 0 < d(x,y) \le \rho_0\right\}.$$

An important particular case is $T =$ some neighbourhood of f in $C^{1+\nu_0}$ and $f_t = t$. We also consider a family $(\theta_\varepsilon)_{\varepsilon>0}$ of regular probability measures in T such that supp $\theta_\varepsilon \to \{\tau\}$ as $\varepsilon \to 0$. Then we are interested in comparing the asymptotics of random trajectories

$$x_j = f_{t_j} \circ \cdots \circ f_{t_1}(x)$$

where $t_1, \ldots, t_j, \ldots$ are independent random variables with distribution θ_ε, with the asymptotics of deterministic trajectories $f^j(x)$.

For that we introduce perturbed versions of the linear operators U and $\mathcal{L}$ we used before for the map f:

$$(U_t\,\varphi)(x) = \varphi(f_t(x)) \qquad (\mathcal{L}_t\varphi)(y) = \sum_{f_t(x)=y} \varphi(x)\,|Df_t(x)|^{-1}$$

and also

$$(\widehat{U}_\varepsilon\,\varphi)(x) = \int (U_t\,\varphi)(x)d\theta_\varepsilon(t) \qquad (\widehat{\mathcal{L}}_\varepsilon\varphi)(y) = \int (\mathcal{L}_t\varphi)(y)d\theta_\varepsilon(t),$$

acting on $E = C^0(M, \mathbb{R})$. By Fubini's theorem and (**??**)

$$\begin{aligned}
\int (\widehat{\mathcal{L}}_\varepsilon \varphi)(y)\psi(y)dm(y) &= \int \left(\int (\mathcal{L}_t \varphi)(y) d\theta_\varepsilon(t) \right) \psi(y)dm(y) \\
&= \int \left(\int (\mathcal{L}_t \varphi)(y)\psi(y)dm(y) \right) d\theta_\varepsilon(t) \\
&= \int \left(\int \varphi(x)(U_t\psi)(x)dm(x) \right) d\theta_\varepsilon(t) \\
&= \int \varphi(x) \left(\int (U_t\psi)(x) d\theta_\varepsilon(t) \right) dm(x) \\
&= \int \varphi(x)(\widehat{U}_t\psi(x))dm(x).
\end{aligned}$$

Hence, if φ_ε is a nonnegative $L^1(m)$ function with $\widehat{\mathcal{L}}_\varepsilon \varphi_\varepsilon = \varphi_\varepsilon$ and $\int \varphi_\varepsilon \, dm = 1$ then $\mu_\varepsilon = \varphi_\varepsilon \, m$ is a stationary probability measure:

$$\int (\widehat{U}_\varepsilon \psi) d\mu_\varepsilon = \int \psi \, d\mu_\varepsilon \quad \text{for all } \psi \text{ continuous.} \tag{5.1}$$

We proceed to show that there exists such a function φ_ε. Moreover, it is unique and it is close to φ_0 if ε is small.

Our assumptions imply that if $\varepsilon > 0$ is small enough then every f_t with $t \in \operatorname{supp} \theta_\varepsilon$ is a $C^{1+\nu_0}$ expanding map, with uniform bounds $\sigma > 1$ and $a_0 > 0$ for the rate of expansion of f_t and the Hölder constant of $\log |\det Df_t|$, respectively. This means that the estimates in the proof of Proposition 3.3.1 apply uniformly

$$\mathcal{L}_t(C(a,\nu)) \subset C(\lambda_1 \, a, \nu) \quad \text{for every } t \in \operatorname{supp} \theta_\varepsilon,$$

as long as $\sigma^{-1} < \lambda_1 < 1$ and $a \geq a_0/(\lambda_1 - \sigma^{-1})$. Therefore, by convexity and closedness of the cone $C(\lambda_1 a, \nu)$,

$$\widehat{\mathcal{L}}_\varepsilon(C(a,\nu)) \subset C(\lambda_1 \, a, \nu) \quad \text{for every small } \varepsilon > 0.$$

Arguing as we did before for $\varphi_n = \mathcal{L}^n 1$, we conclude that $\varphi_{\varepsilon,n} = \widehat{\mathcal{L}}_\varepsilon^n 1$ converges uniformly to some $\varphi_\varepsilon \in C(\lambda_1 a, \nu)$, which is a fixed point of $\widehat{\mathcal{L}}_\varepsilon$. We take $\mu_\varepsilon = \varphi_\varepsilon \, m$. Note that this probability measure is equivalent to m.

Next, we show that μ_ε determines the asymptotics of almost all random trajectories $(x_j)_{j \geq 0}$, in the sense that

$$\frac{1}{n} \sum_{j=0}^{n-1} \varphi(x_j) \to \int \varphi \, d\mu_\varepsilon,$$

for m-almost all choices of the initial point x_0 and θ_ε-almost all choices of the perturbations $t_1, \ldots, t_j, \ldots$. First, we state a consequence of the previous arguments which is also interesting in itself.

Corollary 5.2.1. *Given φ a ν-Hölder function and ψ an $L^1(m)$-function on M there is $K_0 = K_0(\varphi, \psi) > 0$ such that*

$$\left| \int (\widehat{U}_\varepsilon^n \psi)\varphi \, dm - \int \psi \, d\mu_\varepsilon \int \varphi \, dm \right| \le K_0 \Lambda_1^n \quad \textit{for all } n \ge 0 \textit{ and } \varepsilon > 0 \textit{ small.}$$

Proof. Analogous to Proposition 3.3.3, just replace U, $\mathcal{L}$, μ_0, φ_0, by $\widehat{U}_\varepsilon$, $\widehat{\mathcal{L}}_\varepsilon$, μ_ε, φ_ε, respectively.

Now consider the probability measure $\nu_\varepsilon = \mu_\varepsilon \times \theta_\varepsilon^{\mathbb{N}}$ defined on $M \times T^{\mathbb{N}}$ by

$$\nu_\varepsilon(A \times B_1 \times \cdots \times B_k) = \mu_\varepsilon(A) \times \theta_\varepsilon(B_1) \times \cdots \times \theta_\varepsilon(B_k)$$

for Borel sets $A \subset M$, $B_1, \ldots, B_k \subset T$, and $k \ge 0$. We introduce the shift map

$$\sigma \colon M \times T^{\mathbb{N}} \to M \times T^{\mathbb{N}}, \quad \sigma(x, t_1, t_2, \ldots) = (f_{t_1}(x), t_2, \ldots).$$

It is easy to see that ν_ε is a σ-invariant measure, because μ_ε is stationary, recall (5.1). Hence, by the ergodic theorem,

$$\widetilde{\varphi}(x, t_1, t_2, \ldots) = \lim \frac{1}{n} \sum_{j=0}^{n-1} (\varphi \circ \pi_0)(\sigma^j(x, t_1, t_2, \ldots))$$

exists for ν_ε-almost every $(x, t_1, t_2, \ldots) \in M \times T^{\mathbb{N}}$. Here π_0 is the canonical projection $\pi_0 \colon M \times T^{\mathbb{N}} \to M$, $\pi_0(x, t_1, t_2, \ldots) = x$. Note that $\pi_0(\sigma^j(x, t_1, t_2, \ldots))$ is precisely what we have been denoting x_j. We are left to show that

$$\widetilde{\varphi}(x, t_1, t_2, \ldots) = \int (\varphi \circ \pi_0) d\nu_\varepsilon = \int \varphi \, d\mu_\varepsilon \quad \nu_\varepsilon\text{-almost everywhere.}$$

For each $k \ge 0$ we define

$$\widetilde{\varphi}_k(x, t_1, \ldots, t_k) = \int \widetilde{\varphi}(x, t_1, \ldots, t_k, t_{k+1}, \ldots) d\theta_\varepsilon(t_{k+1}) d\theta_\varepsilon(t_{k+2}) \ldots .$$

Since $\widetilde{\varphi}$ is σ-invariant, i.e., $\widetilde{\varphi} \circ \sigma = \widetilde{\varphi}$,

$$\begin{aligned} \widetilde{\varphi}_0(x) &= \int \widetilde{\varphi}(x, t_1, t_2, \ldots) d\theta_\varepsilon(t_1) d\theta_\varepsilon(t_2) \ldots \\ &= \int \widetilde{\varphi}(f_{t_1}(x), t_2, \ldots) d\theta_\varepsilon(t_2) d\theta_\varepsilon(t_3) \ldots d\theta_\varepsilon(t_1) \\ &= \int \widetilde{\varphi}_0(f_{t_1}(x)) d\theta_\varepsilon(t_1) = (\widehat{U}_\varepsilon \widetilde{\varphi}_0)(x) \end{aligned}$$

for μ_ε-almost all $x \in M$. Then Corollary 5.2.1 implies

$$\int (\widetilde{\varphi}_0 - \int \widetilde{\varphi}_0 \, d\mu_\varepsilon) \varphi \, dm = \int (\widehat{U}_\varepsilon^n \widetilde{\varphi}_0) \varphi dm - \int \widetilde{\varphi}_0 \, d\mu_\varepsilon \int \varphi dm \to 0$$

for all ν-Hölder φ. Therefore,

$$\tilde{\varphi}_0 = \int \tilde{\varphi}_0 \, d\mu_\varepsilon = \int \tilde{\varphi} \, d\nu_\varepsilon = \int (\varphi \circ \pi_0) d\nu_\varepsilon = \int \varphi \, d\mu_\varepsilon$$

m-almost everywhere, and so also μ_ε-almost everywhere. More generally, for $k \geq 1$,

$$\begin{aligned}\tilde{\varphi}_k(x, t_1, \ldots, t_k) &= \int \tilde{\varphi}(x, t_1, \ldots, t_k, t_{k+1}, \ldots) d\theta_\varepsilon(t_{k+1}) d\theta_\varepsilon(t_{k+2}) \ldots \\ &= \int \tilde{\varphi}(f_{t_1}(x), t_2, \ldots, t_k, t_{k+1}, \ldots) d\theta_\varepsilon(t_{k+1}) d\theta_\varepsilon(t_{k+2}) \ldots \\ &= \tilde{\varphi}_{k-1}(f_{t_1}(x), t_2, \ldots, t_k).\end{aligned}$$

It follows, by induction, that for every $k \geq 0$

$$\tilde{\varphi}_k = \int \varphi \, d\mu_\varepsilon, \qquad (\mu_\varepsilon \times \theta_\varepsilon^k)\text{-almost everywhere.}$$

This gives $\tilde{\varphi} = \int \varphi \, d\mu_\varepsilon$ at $(\mu_\varepsilon \times \theta_\varepsilon^{\mathbb{N}})$-almost every point, and so completes our argument.

Proposition 5.2.1. *φ_ε converges uniformly to φ_0 as $\varepsilon \to 0$. In particular, $\mu_\varepsilon \to \mu_0$ weakly* as $\varepsilon \to 0$.*

Proof. Proposition 3.3.3 gives, for some $R > 0$,

$$\left| \int \psi(\mathcal{L}^n 1) dm - \int \psi \, d\mu_0 \right| \leq R \|\psi\|_1 \Lambda_1^n$$

and a similar result for $\hat{\mathcal{L}}_\varepsilon$ is deduced in precisely the same way, using Corollary 5.2.1,

$$\left| \int \psi(\hat{\mathcal{L}}_\varepsilon^n 1) dm - \int \psi \, d\mu_\varepsilon \right| \leq R \|\psi\|_1 \Lambda_1^n .$$

Now, given $y \in M$ and $t \in \operatorname{supp} \theta_\varepsilon$ we write

$$f^{-1}(y) = \{x_1, \ldots, x_k\} \quad \text{and} \quad f_t^{-1}(y) = \{x_{1,t}, \ldots, x_{k,t}\},$$

with $\sup\{d(x_{i,t}, x_i) : 1 \leq i \leq k, \, t \in \operatorname{supp} \theta_\varepsilon, \, y \in M\} \to 0$ as $\varepsilon \to 0$. From

$$\mathcal{L}\varphi(y) = \sum_{i=1}^{k} \varphi(x_i) |\det Df(x_i)|^{-1} \quad \text{and} \quad \mathcal{L}_t\varphi(y) = \sum_{i=1}^{k} \varphi(x_{i,t}) |\det Df_t(x_{i,t})|^{-1}$$

one concludes easily that

$$1 - \xi(\varepsilon) \leq \frac{\mathcal{L}_t\varphi(y)}{\mathcal{L}\varphi(y)} \leq 1 + \xi(\varepsilon) \quad \text{for all } y \in M, \; t \in \operatorname{supp} \theta_\varepsilon, \; \varphi \in C(a, \nu),$$

where $\xi(\varepsilon)$ is independent of y, t, or φ, and converges to zero as $\varepsilon \to 0$. As a consequence, given any $\varphi \in C(a,\nu)$,

$$\left| \frac{\widehat{\mathcal{L}}_\varepsilon \varphi(y)}{\mathcal{L}\varphi(y)} - 1 \right| \le \xi(\varepsilon) \text{ for all } y \in M \text{ and so } \|\widehat{\mathcal{L}}_\varepsilon \varphi - \mathcal{L}\varphi\|_0 \le \xi(\varepsilon)\|\mathcal{L}\varphi\|_0 .$$

Applying this to each $\varphi = \mathcal{L}^i\, 1$, $0 \le i < n$, we get

$$\begin{aligned}
\left| \int (\widehat{\mathcal{L}}_\varepsilon^n\, 1 - \mathcal{L}^n\, 1)\psi\, dm \right| &= \left| \sum_{i=0}^{n-1} \int \widehat{\mathcal{L}}_\varepsilon^{n-i-1} (\widehat{\mathcal{L}}_\varepsilon - \mathcal{L})(\mathcal{L}^i\, 1) \cdot \psi\, dm \right| \\
&= \left| \sum_{i=0}^{n-1} \int (\widehat{\mathcal{L}}_\varepsilon - \mathcal{L})(\mathcal{L}^i\, 1)(\widehat{U}_\varepsilon^{n-i-1}\, \psi) dm \right| \\
&\le \sum_{i=0}^{n-1} \|(\widehat{\mathcal{L}}_\varepsilon - \mathcal{L})(\mathcal{L}^i\, 1)\|_0\ \|\widehat{U}_\varepsilon^{n-i-1}\, \psi\|_1\ \sup|\varphi_\varepsilon|^{-1} \\
&\le \sum_{i=0}^{n-1} \xi(\varepsilon) \cdot \sup|\mathcal{L}^{i+1}\, 1| \cdot \|\psi\|_1 \cdot \sup|\varphi_\varepsilon|^{-1} \\
&\le Kn\, \xi(\varepsilon)\|\psi\|_1, \quad \text{where } K \text{ depends only on } f.
\end{aligned}$$

In the last inequality we use the fact that $\mathcal{L}^i\, 1 \in C(\lambda_1 a, \nu)$ and $\int \mathcal{L}^i\, 1\, dm = 1$, hence $\sup|\mathcal{L}^{i+1}\, 1|$ admits an upper bound independent of i, see the proof of Proposition 3.2.3. Altogether, the previous estimates imply

$$\left| \int (\varphi_\varepsilon - \varphi_0)\psi\, dm \right| = \left| \int \psi\, d\mu_\varepsilon - \int \psi\, d\mu_0 \right| \le (Kn\, \xi(\varepsilon) + 2R\, \Lambda_1^n)\|\psi\|_1,$$

for every $n \ge 0$, $\varepsilon > 0$ small, and $\psi \in L^1(m)$. Therefore

$$\|\varphi_\varepsilon - \varphi_0\|_0 \le (Kn\, \xi(\varepsilon) + 2R\, \Lambda_1^n)$$

for every $n \ge 0$. We fix $n \ge 0$ such that $\lambda_1^n \ge \xi(\varepsilon) > \Lambda_1^{n+1}$ and then we get

$$\|\varphi_\varepsilon - \varphi_0\|_0 \le K'\, \xi(\varepsilon) \log \xi(\varepsilon)$$

for some $K' > 0$ depending only on f. This proves the proposition.

A special feature of these uniformly expanding systems is that all the previous arguments could be carried out with no assumption on the class of regular probability distributions θ_ε, (apart from $\operatorname{supp}\theta_\varepsilon \to \{\tau\}$). In particular, one may easily extract from Proposition 5.2.1 the following statement of stability of the absolutely continuous invariant measure under *deterministic* perturbations of the map.

Let f be an expanding map and $(g_n)_n$ be any sequence converging to f in $C^{1+\nu_0}(M)$. Define θ_ε to be the Dirac measure supported on g_n for all

$\varepsilon \in (1/n{+}1, 1/n]$. Then, as a particular case of Proposition 5.2.1, the densities $\varphi_{1/n} = \varphi_{0,g_n}$ of the stationary measures $\mu_{1/n} = \mu_{0,g_n}$ converge uniformly to φ_0 as $n \to +\infty$. This proves that the absolutely continuous invariant measure varies continuously with the expanding map. In more precise terms,

Corollary 5.2.2. *Let f be as before and g be another expanding map, close to f in $C^{1+\nu_0}(M)$. Let $\varphi_{0,g}$ be the fixed point of the corresponding transfer operator and $\mu_{0,g} = \varphi_{0,g} m$. Then $\varphi_{0,g}$ is uniformly close to φ_0, in particular $\mu_{0,g}$ is close to μ_0 in the weak*-sense.*

5.3 Piecewise Expanding Maps

Example **??** shows that *not all* piecewise expanding maps are stochastically stable. However, as we shall now show, stochastic stability does hold under a mild "generic" condition (E4) to be stated below. Observe that, unlike the smooth expanding maps treated in Chapter **??**, piecewise expanding maps are, generally, not structurally stable.

The present statement of stochastic stability is analogous to the one we obtained before in the smooth case. We let T be any metric space and $T \ni t \longmapsto f_t$ be any parametrized family of piecewise expanding maps satisfying

$$f_t \to f_\tau = f \quad \text{as } t \to \tau$$

for some $\tau \in T$. (We shall explain below which topology we want to consider in the space of piecewise expanding maps). Moreover, let $(\theta_\varepsilon)_{\varepsilon>0}$ be an arbitrary family of probabilities on T, such that

$$\operatorname{supp} \theta_\varepsilon \to \{\tau\} \quad \text{as } \varepsilon \to 0.$$

We show that for each small $\varepsilon > 0$ there is a probability measure μ_ε on M such that

$$\frac{1}{n} \sum_{j=0}^{n-1} \varphi(x_j) \to \int \varphi \, d\mu_\varepsilon \quad \text{as} \quad n \to +\infty,$$

for every continuous function $\varphi : M \to \mathbb{R}$ and $m \times \theta_\varepsilon^{\mathbb{N}}$-almost every random trajectory $x_j = f_{t_j} \circ \cdots \circ f_{t_1}(x)$. Furthemore, μ_ε converges to μ_0 as $\varepsilon \to 0$.

The precise content of the condition that $f_t \to f$ as $t \to \tau$ requires some words of explanation. Let us start by the circle case. We require that, for every t in a neighbourhood of τ, there exist

$$a_0(t) < a_1(t) < \cdots < a_{l-1}(t) < a_l(t) = a_0(t)$$

such that

(i) the restriction f_t to $(a_{i-1}(t), a_i(t))$ is a C^1 map, for each $i = 1, \ldots, l$;

(ii) $|Df_t| > 0$ on the complement of $\{a_1(t), \dots, a_l(t)\}$.

For $i = 1, \dots, l$, we denote $\eta_{i,t} = (a_{i-1}(t), a_i(t))$, and let $\phi_{i,t} \colon \eta_i \to \eta_{i,t}$ be the affine bijection mapping a_i to $a_i(t)$. Then we ask that there be some $\widetilde{C}_1 > 0$ close to C_1 do that, for every $i = 1, \dots, l$,

(1) $a_i(t) \to a_i(\tau) = a_i$ as $t \to \tau$;
(2) $(1/|Df_t|\eta_{i,t}|) \circ \phi_{i,t}$ converges uniformly to $(1/|Df|\eta_i|)$ as $t \to \tau$;
(3) $\operatorname{var}(1/|Df_t|\eta_{i,t}|) \le \widetilde{C}_1$ if t is close to τ.

We shall denote $g_{\eta_i} = (1/|Df|\eta_i|)$, for each $i = 1, \dots, l$.

Example 5.3.1. Let $a_0(t) < a_1(t) < \cdots < a_{l-1}(t) < a_l(t) = a_0(t)$ be continuous functions, defined for t in some metric space T and taking values in S^1. Let $(f_t)_{t \in T}$ be a family of transformations on M such that

(a) each $f_t|(a_{i-1}(t), a_i(t))$ has a C^2 extension to $[a_{i-1}(t), a_i(t)]$ with $|Df_t| \ge 1/\tilde{\lambda}_1 > 1$;
(b) each $(f_t \circ \phi_{i,t}) : [a_{i-1}, a_i] \to S^1$ varies continuously with t in the C^2-topology.

Then, clearly, $(f_t)_t$ is continuous at every $\tau \in T$, in the sense we have defined.

The definition of continuity for families of piecewise expanding maps of the interval is similar, but it is convenient to avoid the phenomenon of (random) trajectories escaping I_* and never returning. With that in mind, we always take the unperturbed map $f = f_\tau$ to extend to some larger interval $[\tilde{a}_0, \tilde{a}_l] \supset (\tilde{a}_0, \tilde{a}_l) \supset I_*$, in such a way that

(i) the restrictions of f to both $(\tilde{a}_0, a_1)$ and $(a_{l-1}, \tilde{a}_l)$ are of class C^1;
(ii) $|Df|\eta| > 0$ and $\operatorname{var}(1/|Df|\eta|) \le C_1$ for both $\eta = (\tilde{a}_0, a_1)$ and $\eta = (a_{l-1}, \tilde{a}_l)$;
(iii) $f([\tilde{a}_0, \tilde{a}_l]) \subset (\tilde{a}_0, \tilde{a}_l)$, and $f^q([\tilde{a}_0, \tilde{a}_l]) = I_*$ for some $q \ge 1$.

Then we require that, for each t close to τ, there exist

$$\tilde{a}_0 = \tilde{a}_0(t) < a_1(t) < \cdots < a_{l-1}(t) < \tilde{a}_l(t) = \tilde{a}_l$$

satisfying the analogs of conditions (1), (2), (3). In particular, $f_t([\tilde{a}_0, \tilde{a}_l]) \subset (\tilde{a}_0, \tilde{a}_l)$ for every t close enough to τ.

Example 5.3.2. Let $f \colon I \to I$ be a tent map and c be its turning point, recall Figure 1.3. As we have seen in Example **??**, f is topologically mixing on $I_* = [f^2(c), f(c)]$ if $|Df| > \sqrt{2}$. In this case one may take $l = 2$, and it suffices to choose $0 < \tilde{a}_0 < f^2(c)$, $a_1 = c$, and $f(c) < \tilde{a}_2 < 1$, such that $f(\tilde{a}_0) = f(\tilde{a}_2) > \tilde{a}_0$.

Another important class of continuous families of piecewise expanding maps is provided by the following construction.

Example 5.3.3. Let $f_t : [0,1] \to [0,1]$, $t \in T$, be Lorenz-like maps given by

$$f_t(x) = \begin{cases} h_t^-(|x-1/2|^{\alpha_t}) \text{ if } x < 1/2 \\ h_t^+(|x-1/2|^{\alpha_t}) \text{ if } x > 1/2 \end{cases}$$

where $\alpha_t \in (0,1)$ varies continuously with t, and the $h_t^\pm$ are C^2 strictly monotone maps depending continuously on t in the C^2-topology. For the sake of definiteness, we let h_t^- be decreasing with $h_t^-(0) = 1$ and $h_t^-(1/2^{\alpha_t}) > 0$, and h_t^+ be increasing with $h_t^+(0) = 0$ and $h_t^+(1/2^{\alpha_t}) < 1$. We also take $h_t^\pm$ to be such that $|Df_t(x)| \geq 1/\tilde{\lambda}_1$ for some $\tilde{\lambda}_1 < 1$ and every $t \in T$ and $x \neq 1/2$, which ensures that the f_t are piecewise expanding maps. Then $(f_t)_t$ is continuous at every $\tau \in T$: it is clear that f may be C^2 extended to some $[\tilde{a}_0, \tilde{a}_2]$ as above, and C^2-closeness of $h_t^\pm$ to $h_\tau^\pm$ is easily seen to imply uniform closeness of $1/|Df_t|$ to $1/|Df|$, together with uniformly bounded variation for $1/|Df_t|$ on each of the intervals $(\tilde{a}_0, 1/2)$ and $(1/2, \tilde{a}_2)$.

Observe that if $(X_t)_{t\in T}$ is a continuous family of flows in three dimensions exhibiting geometric Lorenz attractors, cf. [51], then the corresponding family $(f_t)_{t\in T}$ of one-dimensional Lorenz-like maps as in this example. It follows that such a family is continuous in the sense defined above, which means that the topology we introduced above is natural also for these applications in the setting of flows.

We use $\underline{t} = (t_1, \ldots, t_n, \ldots)$ to denote the generic element of $T^{\mathbb{N}}$, with $\underline{\tau}$ representing $(\tau, \ldots, \tau, \ldots)$. Since modifying a map over a finite set of points does not affect its statistical properties, it is no restriction to suppose that every f_t is left-continuous at $a_i(t)$ or else every f_t is right-continuous at $a_i(t)$, for all $0 \leq i \leq l$ and t close to τ. Let $\mathcal{P}_t$ be any partition into subintervals η_t such that

(a) $\eta_{i,t} \subset \eta_t \subset \operatorname{clos}(\eta_{t,i})$ for some $i \in \{1, \ldots, l\}$;
(b) and $(f_t|\eta_t)$ is continuous, for every $\eta_t \in \mathcal{P}_t$.

For $n \geq 1$, we write $f_{\underline{t}}^n = f_{t_n} \circ \cdots f_{t_1}$ and let $\mathcal{P}_{\underline{t}}^{(n)}$ be the partition given by

$$\mathcal{P}_{\underline{t}}^{(n)}(x) = \mathcal{P}_{\underline{t}}^{(n)}(y) \quad \Leftrightarrow \quad \mathcal{P}_{t_{j+1}}(f_{\underline{t}}^j(x)) = \mathcal{P}_{t_{j+1}}(f_{\underline{t}}^j(y)) \text{ for every } 0 \leq j < n.$$

It follows from (2) that we may choose $\tilde{\lambda}_1 < 1$ close to λ_1, so that

$$\sup(1/|Df_{\underline{t}}^n|\eta_{\underline{t}}|) \leq \widetilde{C}_1 \tilde{\lambda}_1^n \quad \text{for all } \eta_{\underline{t}} \in \mathcal{P}_{\underline{t}}^{(n)} \text{ and } \underline{t} \text{ close to } \underline{\tau}. \tag{5.2}$$

Moreover, (2) and (3) imply, cf. (**??**),

$$\operatorname{var}(1/|Df_{\underline{t}}^n|\eta_{\underline{t}}|) \leq \widetilde{C}_2 \tilde{\lambda}_2^n \quad \text{for all } \eta_{\underline{t}} \in \mathcal{P}_{\underline{t}}^{(n)} \text{ and } \underline{t} \text{ close to } \underline{\tau}, \tag{5.3}$$

where $\widetilde{C}_2 > 0$ and $\tilde{\lambda}_2 < 1$ may be taken close to C_2 and λ_2 , respectively.

To state our final hypothesis on the expanding map f, we introduce the notations

$$b_i^- = \lim_{x \to a_i^-} f(x) \quad \text{and} \quad b_i^+ = \lim_{x \to a_i^+} f(x),$$

for each $0 \le i \le l$ (except that, by convention, $b_0^- = b_0^+$ and $b_l^+ = b_l^-$ in the case $M = I$). Then we suppose

(E4) for every $n \ge 0$ and $0 \le i \le l$, we have $f^n(b_i^\pm) \notin \{a_1, \ldots, a_{l-1}, a_l\}$, if $M = S^1$, respectively, $f^n(b_i^\pm) \notin \{\tilde{a}_0, a_1, \ldots, a_{l-1}, \tilde{a}_l\}$, if $M = I$.

In fact a slightly weaker condition suffices for stochastic stability, namely, that no $b_i^\pm$ be a periodic point for f. Observe that, in either form, this is not an open condition, as it concerns the behaviour over an infinite time-scale.

Condition (E4) will used through the following direct consequences. The length $m(\eta)$ of each interval $\eta \in \mathcal{P}^{(n)}$ is positive, and so $\delta_n = \inf_{\eta \in \mathcal{P}^{(n)}} m(\eta) > 0$ for every $n \ge 1$. Then, for each fixed $n \ge 1$, there is a neighbourhood V_n of τ in T such that given any $t_1, \ldots, t_n \in V_n$ there is a bijection

$$\mathcal{P}^{(n)} \ni \eta \to \eta_{\underline{t}} \in \mathcal{P}_{\underline{t}}^{(n)}, \qquad \text{with} \;\; \eta_{\underline{t}} \to \eta = \eta_{\underline{\tau}} \;\; \text{as} \;\; \underline{t} \to \underline{\tau}.$$

In particular, if V_n is chosen small enough, $m(\eta_{\underline{t}}) \ge \tilde{\delta}_n$ for some $\tilde{\delta}_n > 0$ close to δ_n and every $\eta \in \mathcal{P}^{(n)}$. Moreover, up to further restricting V_n, we may suppose that the family of intervals $\{\eta \cup \eta_{\underline{t}} : \eta \in \mathcal{P}^{(n)}\}$ has overlap index 2: every point belongs in no more than 2 of its elements.

After these preliminaries, we may start proving that piecewise expanding maps satisfying (E1)-(E4) are stochastically stable. First, we introduce transfer operators corresponding to the perturbed maps f_t,

$$U_t \varphi(x) = \varphi(f_t(x)) \qquad\qquad \mathcal{L}_t \varphi(y) = \sum_\eta \frac{\varphi}{|Df_t|}\big((f_t|\eta_t)^{-1}(y)\big)$$

where the sum is over all the $\eta \in \mathcal{P}^{(1)}$ for which $y \in f_t(\eta_t)$. We also introduce their iterates

$$U_{\underline{t}} \varphi(x) = \varphi(f_{\underline{t}}^n(x)) \qquad\qquad \mathcal{L}_{\underline{t}}^n \varphi = \sum_\eta \frac{\varphi}{|Df_{\underline{t}}^n|}\big((f_{\underline{t}}^n|\eta_{\underline{t}})^{-1}(y)\big)$$

where the sum is over all $\eta \in \mathcal{P}^{(n)}$ such that $y \in f_{\underline{t}}^n(\eta_{\underline{t}})$. The same arguments as in the proof of Proposition 1.3.1 yield $\widetilde{C}_0 > 0$ and $\tilde{\lambda}_0 < 1$, close to C_0 and λ_0, respectively, such that

$$\operatorname{var}(\mathcal{L}_{\underline{t}}^n \varphi) \le \widetilde{C}_0 \tilde{\lambda}_0^n \operatorname{var} \varphi + \widetilde{C}_0 \int |\varphi| \, dm \tag{5.4}$$

for every $t_1, \ldots, t_n$ in V_n and φ with bounded variation.

Lemma 5.3.1. *There are* $\widetilde{C} > 0$ *and* $\tilde{\lambda} \in (0,1)$, *and for each* $n \geq 1$ *there is a neighbourhood* Y_n *of* τ, *such that*

$$\|\mathcal{L}^n_{\underline{t}}\,\varphi - \mathcal{L}^n\,\varphi\|_1 \leq \widetilde{C}\tilde{\lambda}^n(\mathrm{var}\,\varphi + \int |\varphi| dm)$$

for every $t_1, \ldots, t_n \in Y_n$ *and every function* φ *with bounded variation.*

Proof. For notational simplicity, we write f^{-n} and $f^{-n}_{\underline{t}}$ to mean $(f^n\eta)^{-1}$ and $(f^n_{\underline{t}}|\eta_{\underline{t}})^{-1}$, respectively. Then

$$\begin{aligned}\int |\mathcal{L}^n_{\underline{t}}\,\varphi - \mathcal{L}^n\,\varphi|\,dm & \\ \leq \sum_{\eta} \int_{f^n(\eta)\cap f^n_{\underline{t}}(\eta_{\underline{t}})} & \big|(\varphi\circ f^{-n}_{\underline{t}})\,|Df^{-n}_{\underline{t}}| - (\varphi \circ f^{-n})\,|Df^{-n}|\,\big|\,dm \\ + \sum_{\eta}\Bigg(\int_{f^n(\eta)\setminus f^n_{\underline{t}}(\eta_{\underline{t}})} & |\varphi\circ f^{-n}|\,|Df^{-n}|\,dm \\ & + \int_{f^n_{\underline{t}}(\eta_{\underline{t}})\setminus f^n(\eta)} |\varphi \circ f^{-n}_{\underline{t}}|\,|Df^{-n}_{\underline{t}}| dm\Bigg),\end{aligned} \tag{5.5}$$

the sums being over all $\eta \in \mathcal{P}^{(n)}$. The first sum may be estimated as follows. If $t_1, \ldots, t_n$ are taken in some neighbourhood Y of τ then, at every point in $f^n(\eta) \cap f^n_{\underline{t}}(\eta_{\underline{t}})$,

$$\begin{aligned} a & \big|(\varphi\circ f^{-n}_{\underline{t}})\,|Df^{-n}_{\underline{t}}| - (\varphi\circ f^{-n})\,|Df^{-n}|\,\big| \\ a & \leq |(\varphi\circ f^{-n}_{\underline{t}}) - (\varphi \circ f^{-n})|\,|Df^{-n}_{\underline{t}}| + |\varphi\circ f^{-n}|\,|Df^{-n}_{\underline{t}} - Df^{-n}| \\ & \leq \mathrm{var}(\varphi|\eta\cup\eta_{\underline{t}})\,\widetilde{C}_1\tilde{\lambda}^n_1 + \sup|\varphi|\,\xi_n(Y)\end{aligned}$$

where $\xi_n(Y) \to 0$ when $Y \to \{\tau\}$. Recalling that $\{\eta \cup \eta_{\underline{t}} : \eta \in \mathcal{P}^{(n)}\}$ has overlap index 2, and choosing Y small enough so that

$$\#\mathcal{P}^{(n)}\,\xi_n(Y) \leq \widetilde{C}_1\tilde{\lambda}^n_1,$$

we conclude that the first sum in (5.5) is bounded by

$$\begin{aligned}\sum_{\eta}\Big(\widetilde{C}_1\tilde{\lambda}^n_1\,\mathrm{var}(\varphi|\eta\cup\eta_{\underline{t}}) + \sup|\varphi|\,\xi_n(Y)\Big) &\leq 2\widetilde{C}_1\tilde{\lambda}^n_1\ \mathrm{var}\,\varphi + \widetilde{C}_1\tilde{\lambda}^n_1\ \sup|\varphi| \\ &\leq 3\widetilde{C}_1\tilde{\lambda}^n_1\ \mathrm{var}\,\varphi + \widetilde{C}_1\tilde{\lambda}^n_1\int|\varphi|dm.\end{aligned}$$

Moreover, using once again the continuity of f_t at $t = \tau$,

$$m(f^n(\eta) \setminus f_{\underline{t}}^n(\eta_{\underline{t}})) \le \xi_n(Y) \quad \text{and} \quad m(f_{\underline{t}}^n(\eta_{\underline{t}}) \setminus f^n(\eta)) \le \xi_n(Y),$$

where $\xi_n(Y)$ has the same meaning as before. It follows that the second sum in (5.5) is bounded by

$$\sum_\eta 2 \sup |\varphi| \widetilde{C}_1 \lambda_1^n \, \xi_n(Y) \le 2 \sup |\varphi| \widetilde{C}_1 \tilde{\lambda}_1^n \le 2\widetilde{C}_1 \tilde{\lambda}_1^n \left(\operatorname{var} \varphi + \int |\varphi| dm \right),$$

as long as Y is small enough so that $\#\mathcal{P}^{(n)} \, \xi_n(Y) \le 1$. To conclude the proof it suffices to take $\widetilde{C} = 5\widetilde{C}_1$ and $\tilde{\lambda} = \tilde{\lambda}_1$.

Now we introduce the linear operators

$$\widehat{U}_\varepsilon \varphi(x) = \int U_t \varphi(x) \, d\theta_\varepsilon(t) \qquad \widehat{\mathcal{L}}_\varepsilon \varphi(y) = \int \mathcal{L}_t \varphi(y) \, d\theta_\varepsilon(t)$$

and note that

$$\int (\widehat{U}_\varepsilon \psi) \, \varphi \, dm = \int \psi \, (\widehat{\mathcal{L}}_\varepsilon \varphi)$$

for every $\varphi \in L^\infty(m)$ and every $\psi \in L^1(m)$. This is, simply, because the analogous duality property holds for U_t and $\mathcal{L}_t$ for every t (use Fubini's theorem in the same way as in Section **??**). Moreover, (5.4) implies (recall property (v6) of the variation)

$$\operatorname{var}(\widehat{\mathcal{L}}_\varepsilon^n \varphi) \le \widetilde{C}_0 \tilde{\lambda}_0^n \operatorname{var} \varphi + \widetilde{C}_0 \int |\varphi| \, dm \tag{5.6}$$

for every $n \ge 1$ and every function φ with bounded variation. Then, cf. Corollary **??**, there is a nonnegative function φ_ε which is fixed under the operator $\widehat{\mathcal{L}}_\varepsilon$, that is, $\widehat{\mathcal{L}}_\varepsilon \, \varphi_\varepsilon = \varphi_\varepsilon$. We normalize φ_ε by asking that $\int \varphi_\varepsilon \, dm = 1$, and denote $\mu_\varepsilon = \varphi_\varepsilon \, m$. Then μ_ε is a probability measure and it is a stationary measure:

$$\int (\widehat{U}_\varepsilon \psi) \, d\mu_\varepsilon = \int \psi \, (\widehat{\mathcal{L}}_\varepsilon \varphi_\varepsilon) \, dm = \int \psi \, \varphi_\varepsilon \, dm = \int \psi \, d\mu_\varepsilon$$

for every integrable function ψ.

Proposition 5.3.1. *There are $\widetilde{\Lambda} < 1$ and $\widetilde{K} > 0$ such that, given φ with bounded variation and $\psi \in L^1(m)$,*

$$\left| \int (\widehat{U}_\varepsilon^n \, \psi) \varphi \, dm - \int \psi \, d\mu_\varepsilon \int \varphi \, dm \right| \le \widetilde{K} \widetilde{\Lambda}^n \, \|\varphi\|_{\mathrm{BV}} \int |\psi| \, dm$$

for all $n \ge 0$ and every small enough $\varepsilon > 0$.

The approach we take for proving this proposition is to think of $\mathcal{L}_\varepsilon$ as a kind of perturbation of $\mathcal{L}$, to deduce that $\mathcal{L}_\varepsilon$ is quasi-compact from the fact that $\mathcal{L}$ is quasi-compact. The following perturbation lemma is what we need in this setting.

Lemma 5.3.2. *Let $C > 0$, $\lambda < 1$, $\Lambda < 1$, and $\mathcal{P}_\varepsilon : \mathrm{BV} \to \mathrm{BV}$, $\varepsilon \geq 0$, be a family of linear operators satisfying*

1. $\int \mathcal{P}_\varepsilon \varphi \, dm = \int \varphi \, dm$ *and* $\varphi \geq 0 \Rightarrow \mathcal{P}\varphi \geq 0$;
2. $\|\mathcal{P}_\varepsilon^n \varphi\|_{\mathrm{BV}} \leq C\lambda^n \|\varphi\|_{\mathrm{BV}} + C\|\varphi\|_1$;

for every $n \geq 1$, $\varepsilon \geq 0$, and $\varphi \in \mathrm{BV}$. Suppose that

(a) *given $n \geq 1$ there is $\varepsilon(n) > 0$ so that, for all $\varphi \in \mathrm{BV}$ and $0 \leq \varepsilon \leq \varepsilon(n)$,*

$$\|\mathcal{P}_\varepsilon^n \varphi - \mathcal{P}_0^n \varphi\|_1 \leq C\lambda^n \|\varphi\|_{\mathrm{BV}};$$

(b) $\mathrm{spec}(\mathcal{P}_0) = \{1\} \cup \Sigma_0$ *where 1 is a simple eigenvalue and $\Sigma_0 \subset \{z \in \mathbb{C} : |z| \leq \Lambda\}$.*

Fix $\sigma \in (\max\{\sqrt{\Lambda}, \sqrt{\lambda}\}, 1)$. Then, for any small enough $\varepsilon > 0$, $\mathrm{spec}(\mathcal{P}_\varepsilon) = \{1\} \cup \Sigma_\varepsilon$, where 1 is a simple eigenvalue and $\Sigma_\varepsilon \subset \{z \in \mathbb{C} : |z| \leq \sigma\}$.

Proof. As a first step, we claim that if n is sufficiently large and $0 < \varepsilon \leq \varepsilon(n)$ then $R(\mathcal{P}_\varepsilon^n, z^n) = (z^n I - \mathcal{P}_\varepsilon^n)^{-1}$ is a bounded operator on BV for any z with $\sigma \leq |z| < 1$. To prove this, we write

$$\begin{aligned} R(\mathcal{P}_\varepsilon^n, z^n) &= \big((z^n I - \mathcal{P}_0^n) + (\mathcal{P}_0^n - \mathcal{P}_\varepsilon^n)\big)^{-1} \\ &= \big((z^n I - \mathcal{P}_0^n) \cdot (I + R(\mathcal{P}_0^n, z^n)\,(\mathcal{P}_0^n - \mathcal{P}_\varepsilon^n))\big)^{-1} \\ &= \left(I + \sum_{k=1}^{\infty} \big(R(\mathcal{P}_0^n, z^n)(\mathcal{P}_\varepsilon^n - \mathcal{P}_0^n)\big)^k\right) R(\mathcal{P}_0^n, z^n). \end{aligned} \tag{5.7}$$

Of course, the operator $R(\mathcal{P}_0^n, z^n)$ is bounded, since z does not belong in the spectrum of $\mathcal{P}_0$. A main tool is the sequence $|||\cdot|||_n$ of norms on BV defined by

$$|||\varphi|||_n = \theta^n \|\varphi\|_{\mathrm{BV}} + \|\varphi\|_1,$$

where $\max\{\sqrt{\Lambda}, \sqrt{\lambda}\} < \theta < \sigma$. Note that $\theta^n\|\varphi\|_{\mathrm{BV}} \leq |||\varphi|||_n \leq 2\|\varphi\|_{\mathrm{BV}}$, and so each $|||\cdot|||_n$ is equivalent to $\|\cdot\|_{\mathrm{BV}}$. Then, in order to prove that the sum in (5.7) converges in the space of bounded operators on BV, it suffices to show that (for all large n)

$$|||R(\mathcal{P}_0^n, z^n)\,(\mathcal{P}_\varepsilon^n - \mathcal{P}_0^n)|||_n < 1. \tag{5.8}$$

We start with a few simple observations. Assumption (1) implies that $X_0 = \{\varphi \in \mathrm{BV} : \int \varphi \, dm = 0\}$ is invariant under $\mathcal{P}_0$. Any accumulation point φ_0 of $\frac{1}{n}\sum_{j=0}^{n-1} \mathcal{P}_0^j 1$ is a fixed point of $\mathcal{P}_0$ with $\int \varphi_0 \, dm = 1$, and such accumulation points do exist, by Helly's theorem. In view of assumption (b), the accumulation point φ_0 is unique, and the spectral splitting corresponding to the decomposition $\mathrm{spec}(\mathcal{P}_0) = \{1\} \cup \Sigma_0$ must be $\mathrm{BV} = \mathbb{R}\varphi_0 \oplus X_0$. Condition (1) also implies $\int (\mathcal{P}_\varepsilon^n - \mathcal{P}_0^n)\varphi \, dm = 0$ for every $n \geq 1$, $\varepsilon > 0$, $\varphi \in \mathrm{BV}$.

In other words, the image of $\mathcal{P}_\varepsilon^n - \mathcal{P}_0^n$ is contained in the eigenspace X_0 of $\mathcal{P}_0$. Now, for some $K > 0$ and any $\psi \in X_0$,

$$|||\mathcal{P}_0^n \psi|||_n \le 2\|\mathcal{P}_0^n \psi\|_{\mathrm{BV}} \le 2K\Lambda^n \|\psi\|_{\mathrm{BV}} \le 2K\Lambda^n \theta^{-n} |||\psi|||_n \le \frac{1}{2}|z|^n |||\psi|||_n$$

if n is large, recall that $|z| \ge \sigma > \theta > \sqrt{\Lambda}$. Then $|||(z^n I - \mathcal{P}_0^n)\psi|||_n \ge (1/2)|z|^n |||\psi|||_n$ for every $\psi \in X_0$, which gives

$$|||R(\mathcal{P}_0^n, z^n)|X_0|||_n \le 2|z|^{-n} \le 2\sigma^{-n}.$$

Next, from assumptions (2) and (a) we get

$$\begin{aligned} |||(\mathcal{P}_\varepsilon^n - \mathcal{P}_0^n)\varphi|||_n &\le \theta^n \left(2C\lambda^n \|\varphi\|_{\mathrm{BV}} + 2C\|\varphi\|_1\right) + C\lambda^n \|\varphi\|_{\mathrm{BV}} \\ &\le 2C\theta^n \left((\lambda^n + \lambda^n/2\theta^n)\|\varphi\|_{\mathrm{BV}} + \|\varphi\|_1\right) \\ &\le 2C\theta^n |||\varphi|||_n, \end{aligned}$$

for all large n and $0 < \varepsilon \le \varepsilon(n)$, recall that $\theta > \sqrt{\lambda}$. Combining these estimates we find

$$|||R(\mathcal{P}_0^n, z^n)\,(\mathcal{P}_\varepsilon^n - \mathcal{P}_0^n)|||_n \le 4C\theta^n |z|^{-n} \le 4C\theta^n \sigma^{-n}. \tag{5.9}$$

Since $\theta < \sigma$, this implies (5.8) and completes the proof of our claim.

As an immediate consequence, we get that the spectrum of $\mathcal{P}_\varepsilon$ contains no point z with $\sigma < |z| < 1$, whenever $0 \le \varepsilon \le \varepsilon(n)$ for some sufficiently large n. Let us denote $U_\varepsilon = \{z \in \mathrm{spec}(\mathcal{P}_\varepsilon) : |z| \ge 1\}$ and $\Sigma_\varepsilon = \{z \in \mathrm{spec}(\mathcal{P}_\varepsilon) : |z| \le \sigma\}$. Note that U_ε is contained in the unit circle (and nonempty), as assumption (2) implies that the spectral radius of $\mathcal{P}_\varepsilon$ is equal to 1. For each $\varepsilon \ge 0$, let Y_ε be the eigenspace of $\mathcal{P}_\varepsilon$ associated to U_ε, and $\pi^\varepsilon : \mathrm{BV} \to Y_\varepsilon$ denote the corresponding spectral projection. We are left to prove that $U_\varepsilon = \{1\}$ and 1 is a simple eigenvalue or, in other words, that Y_ε has dimension 1. With that in mind, we show that $|||\pi^\varepsilon - \pi^0|||_n$ is small if $0 < \varepsilon \le \varepsilon(n)$ and n is large. Fix $\sigma < \sigma_1 < 1 < \sigma_2$ and let C_l^n denote the circle of radius σ_l^n around the origin, for $l = 1, 2$ and $n \ge 1$. Noting that π^ε is also the spectral projection for $\mathcal{P}_\varepsilon^n$ associated to $\mathrm{spec}(\mathcal{P}_\varepsilon^n) \cap \{w \in \mathbb{C} : \sigma_1^n < |w| < \sigma_2^n\}$,

$$\pi^\varepsilon = \frac{1}{2\pi i} \int_{C_1^n \cup C_2^n} R(\mathcal{P}_\varepsilon^n, w)\, dw.$$

for each $\varepsilon \ge 0$. Then, recall also (5.7), (5.9),

$$
\begin{aligned}
|||\pi^\varepsilon - \pi^0|||_n &\le \frac{1}{2\pi}\sum_{l=1}^{2} \text{length}(C_l^n)\cdot \sup_{w\in C_l^n} |||R(\mathcal{P}_\varepsilon^n, w) - R(\mathcal{P}_0^n, w)|||_n \\
&\le \sum_{l=1}^{2} \sigma_l^n \sup_{|z|=\sigma_l} |||\sum_{k=1}^{\infty}\big(R(\mathcal{P}_0^n, z^n)(\mathcal{P}_\varepsilon^n - \mathcal{P}_0^n)\big)^k|||_n \\
&\le \sum_{l=1}^{2} \sigma_l^n \sum_{k=1}^{\infty} (4C\theta^n\sigma_l^{-n})^k \le \sum_{l=1}^{2} \frac{4C\theta^n}{1-(\theta/\sigma_l)^n} \le 16C\theta^n \le \frac{1}{2}
\end{aligned}
$$

if n is large and $0 < \varepsilon \le \varepsilon(n)$. It follows that, given any $\varphi \in Y_\varepsilon \cap X_0$,

$$
|||\varphi|||_n = |||\pi^\varepsilon\varphi||| = |||(\pi^\varepsilon - \pi^0)\varphi|||_n \le \frac{1}{2}|||\varphi|||_n
$$

and so $\varphi = 0$. Since X_0 is a hyperplane, this proves that Y_ε is 1-dimensional.

We apply this statement to $\mathcal{P}_0 = \mathcal{L}$ and $\mathcal{P}_\varepsilon = \widehat{\mathcal{L}}_\varepsilon$, any small $\varepsilon > 0$. Assumption (1) is clear from the definitions, (2) was proved in Proposition 1.3.1 and in (5.6), (a) is a direct consequence of Lemma 5.3.1 and Fubini's theorem, and (b) was obtained in Proposition **??**. Then, Proposition 5.3.1, with $\widetilde{\Lambda} = \sigma$, follows from this lemma in just the same way as we derived Corollary **??** from Proposition **??**.

Observe that, through the arguments following Corollary 5.2.1, the previous proposition implies

$$
\lim \frac{1}{n}\sum_{j=0}^{n-1} \varphi(x_j) = \int \varphi \, d\mu_\varepsilon
$$

for every continuous function φ and $m \times \theta_\varepsilon^{\mathbb{N}}$-almost all random trajectories $x_j = f_{\underline{t}}^j(x)$. At last, we are in a position to prove the main result in this section.

Proposition 5.3.2. *φ_ε converges to φ_0 in $L^1(m)$ as $\varepsilon \to 0$.*

Proof. Let $\widetilde{C} > 0$, $\tilde{\lambda} < 1$, and Y_n, $n \ge 1$, be as in Lemma 5.3.1, that is,

$$
\|\mathcal{L}_{\underline{t}}^n 1 - \mathcal{L}^n 1\|_1 \le \widetilde{C}\tilde{\lambda}^n
$$

for all $t_1, \dots, t_n \in Y_n$ and each $n \ge 1$. Given any $n \ge 1$, take $\varepsilon > 0$ to be small enough so that supp $\theta_\varepsilon \subset Y_n$. Then,

$$
\|\widehat{\mathcal{L}}_\varepsilon^n 1 - \mathcal{L}^n 1\|_1 \le \widetilde{C}\tilde{\lambda}^n .
$$

On the other hand, Proposition 5.3.1 implies that

$$
\left|\int (\widehat{\mathcal{L}}_\varepsilon^n 1 - \varphi_\varepsilon)\,\psi\, dm\right| = \left|\int (\widehat{U}_\varepsilon^n \psi) dm - \int \psi \, d\mu_\varepsilon\right| \le \widetilde{K}\widetilde{\Lambda}^n \int |\psi|\, dm \le \widetilde{K}\widetilde{\Lambda}^n \sup|\psi|
$$

for every $n \geq 0$ and ε sufficiently small, and every bounded function ψ. Hence

$$\|\widehat{\mathcal{L}}_\varepsilon^n 1 - \varphi_\varepsilon\|_1 \leq \widetilde{K}\widetilde{\Lambda}^n \quad \text{for all } n \geq 0 \text{ and } \varepsilon > 0 \text{ small.}$$

In a similar way, using Corollary **??**, $\|\mathcal{L}^n 1 - \varphi_0\|_1 \leq \widehat{K}\Lambda^n$ for all $n \geq 0$ and some $\widehat{K} > 0$. Altogether, this gives

$$\|\varphi_\varepsilon - \varphi_0\|_1 \leq \widehat{K}\Lambda^n + \widetilde{K}\widetilde{\Lambda}^n + \widetilde{C}\tilde{\lambda}^n$$

if ε is small enough, depending on n. The right hand side can be made arbitrarily small by choosing n large and so the proof is complete.

Remark 5.3.1. Since every $\widehat{\mathcal{L}}_\varepsilon$ preserves $X_0 = \{\varphi \in \mathrm{BV} : \int \varphi\, dm = 0\}$, the spectral splitting corresponding to $\mathrm{spec}(\widehat{\mathcal{L}}_\varepsilon) = \{1\} \cup \Sigma_\varepsilon$ must be $\mathrm{BV} = \mathbb{R}\varphi_\varepsilon \oplus X_0$. It follows that the spectral projection π^ε is given by $\pi^\varepsilon \varphi = \varphi_\varepsilon \int \varphi\, dm$, and so Proposition 1.3.4 implies that $\|\pi_\varepsilon - \pi_0\|_1 = \|\varphi_\varepsilon - \varphi_0\|_1 \to 0$ as $\varepsilon \to 0$.

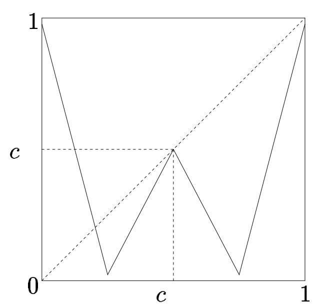

Fig. 5.1. A stochastically unstable expanding map

A counterexample.

5.4 Hyperbolic Attractors

5.4.1 Invariant Foliations for Random Maps

We close by giving precise statements of versions of the previous results for skew-product maps

$$F\colon Q \times T^{\mathbb{N}} \to Q \times T^{\mathbb{N}}, \quad F(x, t_1, t_2, \dots) = (f_{t_1}(x), t_2, \dots)$$

which are used in Section **??**. Here T is just a metric space. We continue to suppose that $\Lambda = \cap_{n \geq 0} f^n(Q)$ is a hyperbolic attractor for f, and we assume that all the maps f_t, with $t \in T$, belong in a sufficiently small neighborhood of f.

We define the *tangent space* at a point $(x, \underline{t}) \in Q \times T^{\mathbb{N}}$ to be $T_x Q \times \{\underline{t}\}$. We say that $N \subset Q \times T^{\mathbb{N}}$ is a *submanifold* of $Q \times T^{\mathbb{N}}$ if for every $(x, \mathbf{t}) \in N$ the set $N_{\mathbf{t}} = N \cap (Q \times \{\mathbf{t}\})$ is a submanifold of $Q \times \{\mathbf{t}\} \approx Q$. Moreover, by definition, the tangent space to N at $(x, \mathbf{t}) \in N$ is $T_x N_{\mathbf{t}} \times \{\mathbf{t}\}$.

It is easy to check that

$$\hat{\Lambda} = \bigcap_{n \geq 0} F^n \left(Q \times T^{\mathbb{N}}\right)$$

is compact and F-invariant. We define the tangent bundle to $Q \times T^{\mathbb{N}}$ over $\hat{\Lambda}$ as

$$T_{\hat{\Lambda}}\big(Q \times T^{\mathbb{N}}\big) = \bigcup_{(x, \underline{t}) \in \hat{\Lambda}} \big(T_x Q \times \{\underline{t}\}\big)$$

and endow it with the metric $\|\cdot\|$ given by the riemannian metric induced on each $T_x Q \times \{\mathbf{t}\}$. We also define the derivative of F over $\hat{\Lambda}$ by

$$DF\colon T_{\hat{\Lambda}}\big(Q \times T^{\mathbb{N}}\big) \to T_{\hat{\Lambda}}\big(Q \times T^{\mathbb{N}}\big), \qquad DF(x, \underline{t})(v, \underline{t}) = (Df_{t_1}(x)v, \sigma(\underline{t})).$$

Proposition 5.4.1. *There is a continuous splitting $T_{\hat{\Lambda}}\big(Q \times T^{\mathbb{N}}\big) = \hat{E}^s \oplus \hat{E}^u$ invariant under DF, and there is a constant $0 < \lambda < 1$ for which*

$$\|DF|\hat{E}^s\| \leq \lambda \quad \textit{and} \quad \|DF|\hat{E}^u\| \geq \lambda^{-1}.$$

This proposition may be proved by using invariant cone fields for f, defined in a neighbourhood of Λ. Note that such cone fields area also invariant under every small perturbation of f.

Then, using this hyperbolic structure for $\hat{\Lambda}$ one can prove the existence of local stable manifolds for points in $\hat{\Lambda}$, in much the same way as one proves Theorem **??**.

Theorem 5.4.1. *There is $\lambda_s < 1$ and for each $(x, \mathbf{t}) \in \hat{\Lambda}$ there is a C^k disk $W^s_{loc}(x, \mathbf{t})$ embedded in $Q \times \{\mathbf{t}\}$, such that*

1. $T_{(x,\mathbf{t})} W^s_{loc}(x, \mathbf{t}) = \hat{E}^s_{(x,\mathbf{t})}$,
2. $F\left(W^s_{loc}(x, \mathbf{t})\right) \subset W^s_{loc}(F(x, \mathbf{t}))$,
3. $F\colon W^s_{loc}(x, \mathbf{t}) \to W^s_{loc}(F(x, \mathbf{t}))$ *is a λ_s-contraction.*

Moreover, the disk $W^s_{loc}(x, \mathbf{t})$ varies continuously with the point $(x, \mathbf{t}) \in \hat{\Lambda}$ in the C^k topology.

We call $W^s_{loc}(x, \mathbf{t})$ the *local stable manifold* of F at the point $(x, \mathbf{t}) \in \hat{\Lambda}$. Then we let $\widehat{\mathcal{F}}^s_{loc}$ be the foliation of a neighbourhood $\widehat{W}^s_{loc}$ of $\hat{\Lambda}$ whose leaves are these local stable manifolds. We want to state that the foliation $\widehat{\mathcal{F}}^s_{loc}$ is absolutely continuous, Theorem 5.4.2 below, but this requires some words of explanation.

A submanifold N of $Q \times T^{\mathbb{N}}$ is said to be transverse to $\widehat{\mathcal{F}}^s_{loc}$ if $N \subset \widehat{W}^s_{loc}$ and

$$T_{(x,\mathbf{t})}N \oplus T_{(x,\mathbf{t})}\widehat{W}^s_{loc} = T_x Q \times \{\mathbf{t}\}.$$

for every $(x, \mathbf{t}) \in N \cap \widehat{W}^s_{loc}$. Given submanifolds N_1 and N_2 of $Q \times T^{\mathbb{N}}$ transverse to $\widehat{\mathcal{F}}^s_{loc}$, we say that $P\colon N_1 \to N_2$ is a *Poincaré map* for $\widehat{\mathcal{F}}^s_{loc}$ if it is injective and continuous, and

$$P(x, \mathbf{t}) \in W^s_{loc}(x, \mathbf{t}) \cap N_2 \quad \text{for every } (x, \mathbf{t}) \in N_1.$$

A Poincaré map $P\colon N_1 \to N_2$ is *absolutely continuous* if there is a continuous map $J\colon N_1 \to \mathbb{R}$, the *jacobian* of P, such that for every $\mathbf{t} \in T^{\mathbb{N}}$ and every Borel subset $A \subset N_1 \cap (Q \times \{\mathbf{t}\})$ we have

$$m_2(P(A)) = \int_A J(\cdot, \mathbf{t})\, dm_1$$

where m_i is the riemannian measure on $N_i \cap (Q \times \{\mathbf{t}\})$ for $i = 1, 2$. We say that P is Hölder continuous if for fixed $\mathbf{t} \in T^{\mathbb{N}}$ the map $P(\cdot, \mathbf{t})$ is Hölder continuous, with uniform Hölder constants. Analogously, we define Hölder continuity of the jacobian J. Finally, we say that $\widehat{\mathcal{F}}^s_{loc}$ is absolutely (resp. Hölder) continuous if every Poincaré map for $\widehat{\mathcal{F}}^s_{loc}$ is absolutely (resp. Hölder) continuous.

Theorem 5.4.2. *Let f and $F\colon Q \times T^{\mathbb{N}} \to Q \times T^{\mathbb{N}}$ be as before. The stable foliation $\widehat{\mathcal{F}}^s_{loc}$ associated to F is absolutely continuous and Hölder continuous, and the jacobian of any Poincaré map for $\widehat{\mathcal{F}}^s_{loc}$ is also Hölder continuous. Moreover, all the Hölder constants involved may be taken uniform (i.e., independent of the map) in a neighbourhood of f.*

Let θ_ε be a probability measure on T and $\widehat{m}_\varepsilon$ be the measure induced by $(m \times \theta_\varepsilon^{\mathbb{N}})$ on the quotient space (the space of leaves) of $\widehat{\mathcal{F}}^s_{loc}$:

$$\widehat{m}_\varepsilon(\tilde{A}) = (m \times \theta_\varepsilon^{\mathbb{N}})\Big(\bigcup \{\gamma : \gamma \in \tilde{A}\}\Big)$$

for every measurable subset $\tilde{A}$ of the quotient space. Recall that given any leaf $\gamma \in \widehat{\mathcal{F}}^s_{loc}$, there is $\mathbf{t} \in T^{\mathbb{N}}$ such that $\gamma \subset Q \times \{\mathbf{t}\}$. Let m_γ be the smooth measure induced on γ by the riemannian metric of $Q \approx Q \times \{\mathbf{t}\}$. Using the previous theorem and the same arguments as in the deterministic case, one proves that the measure $(m \times \theta_\varepsilon^{\mathbb{N}})$ admits a disintegration $(p_{\varepsilon,\gamma})_\gamma$ along the leaves γ of $\widehat{\mathcal{F}}^s_{loc}$ which have Hölder continuous densities with respect to the corresponding riemannian measure m_γ. More precisely, there are constants $a_0 > 0$ and $0 < \nu_0 < 1$, depending only on the map f, and there exists a continuous function

$$H_\varepsilon\colon Q \times T^{\mathbb{N}} \to (0, +\infty)$$

bounded away from zero and infinity, such that $\log H$ is (a_0, ν_0)-Hölder continuous on every $Q \times \{\mathbf{t}\}$, and $p_{\varepsilon,\gamma} = (H_\varepsilon \mid \gamma)\, m_\gamma$, $\gamma \in \widehat{\mathcal{F}}^s_{loc}$, defines a disintegration of $m \times \theta_\varepsilon^{\mathbb{N}}$:

$$\int \Psi d(m \times \theta_\varepsilon^{\mathbb{N}}) = \int \left(\int (\Psi \mid \gamma)\, dp_{\varepsilon,\gamma} \right) d\widehat{m}_\varepsilon(\gamma).$$

for any $\Psi \in L^1(m \times \theta_\varepsilon^{\mathbb{N}})$.

5.4.2 Stochastic Stability

In this last section we prove that the maps $f\colon Q \to Q$ we have been considering are stable under small random perturbations. The setting is formally the same as in previous sections. We consider an arbitrary parametrized family $(f_t)_{t\in T}$ of C^2 maps from Q to Q, where T is any metric space. We suppose that, for some $\tau \in T$,

$$f_\tau = f \quad \text{and} \quad T \ni t \mapsto f_t \text{ is continuous at } \tau \ \text{(with respect to the } C^2\text{-topology)}.$$

The basic example corresponds to T being some neighbourhood of f in the space of C^2 embeddings of Q into itself, with $f_t = t$ for each $t \in T$. We also consider an arbitrary family $(\theta_\varepsilon)_{\varepsilon>0}$ of regular probability measures on T such that

$$\operatorname{supp} \theta_\varepsilon \to \{\tau\} \quad \text{as} \quad \varepsilon \to 0.$$

Then we construct, for each small $\varepsilon > 0$, a probability measure μ_ε which is stationary under the random process associated to $((f_t)_{t\in T}\,,\ \theta_\varepsilon)$:

$$\int \left(\int (\psi \circ f_t)\, d\mu_\varepsilon \right) d\theta_\varepsilon(t) = \int \psi\, d\mu_\varepsilon \tag{5.10}$$

for every continuous function $\psi : Q \to \mathbb{R}$. Moreover, this measure μ_ε determines the asymptotic time-averages of continuous functions over almost all random trajectories $x_j = f_{t_j} \circ \cdots f_{t_1}(x)$:

$$\frac{1}{n} \sum_{j=0}^{n-1} \psi(x_j) \to \int \psi\, d\mu_\varepsilon$$

for every continuous ψ and $(m \times \theta_\varepsilon^{\mathbb{N}})$-almost every $(x, t_1, \ldots, t_j, \ldots)$. Finally, we show that μ_ε converges weakly to the SRB-measure μ_0 of f when $\varepsilon \to 0$.

Our strategy for proving these statements is somewhat different from the one we used in the expanding case. In Section **??** we took advantage of the fact that the cone $C(a, \mu)$, that had been constructed for studying the unperturbed map f, is also invariant under transfer operators $\mathcal{L}_t$ associated to the perturbations f_t. As it turns out, the corresponding statement no longer holds in the present setting. The reason is that the definition of our cone $C(b,c,\nu)$ involves the stable foliation $\mathcal{F}^s_{loc}$, and the proof that $C(b,c,\nu)$ is invariant under the operator $\mathcal{L}$ makes use of the invariance of $\mathcal{F}^s_{loc}$ under the map f (more precisely, we needed the fact that the preimage of a local stable leaf is a union of local stable leaves).

One way to bypass this difficulty is to replace $C(b,c,\nu)$ by some other convex cone, invariant under the transfer operator of every f_t with t close to τ. Such a cone may be obtained, for instance, substituting in the definition of $C(b,c,\nu)$ the foliation $\mathcal{F}^s_{loc}$ by a larger class of submanifolds of Q with an invariance property with respect to every f_t.

An alternative approach, that we take here, relies on considering the skew-product map

$$F : Q \times T^{\mathbb{N}} \to Q \times T^{\mathbb{N}}, \qquad F(x,t_1,t_2,t_3,\dots) = (f_{t_1}(x),t_2,t_3,\dots), \quad (5.11)$$

and developping for this F, and every small $\varepsilon > 0$, a theory similar to the one presented in the previous sections for f, with $m \times \theta_\varepsilon^{\mathbb{N}}$ in the role of Lebesgue measure m. The statements of stability made above are then easily deduced from this theory. This approach also provides information on the individual behaviour (e.g. correlation functions) of typical random iterates, although we do not pursue this aspect here.

A good part of this treatment of the "random" system $(F, m \times \theta_\varepsilon^{\mathbb{N}})$ consists in adapting arguments we used previously for f, profitting from their robustness under small perturbations: most of what we have done so far remains valid, with uniform estimates, when f^n is replaced by $f_{t_n} \circ \dots \circ f_{t_1}$, for any $n \ge 1$ and any $f_{t_1}, \dots, f_{t_n}$ in a sufficiently small C^2 neighbourhood $\mathcal{V}$ of f. We give the guidelines of each step but, as a rule, do not reproduce in detail those arguments which appeared already in the unperturbed case and which can be translated in straightforward ways to the present setting. The reader who has gone through the previous sections should find no difficulty in providing those details, and may find it a good exercise to do so.

The constants λ_1, λ_2, Λ_1, Λ_2, λ_u, λ_s, μ, $\nu \in (0,1)$, and a, b, $c > 0$, have the same meaning as before, but may take slightly different values. More precisely, all these constants are *uniform*, meaning that they may be chosen depending only on f. We use $\mathbf{t} = (t_1,t_2,t_3,\dots)$ to represent a generic element of $T^{\mathbb{N}}$, and we also denote $\sigma(\mathbf{t}) = (t_2,t_3,\dots)$. Up to replacing, right from the start, the metric space T by a sufficiently small neighbourhood of τ, we may suppose that every f_t, $t \in T$, belongs in a neighbourhood $\mathcal{V}$ of f as above (where the estimates of the unperturbed case remain valid). The precise conditions we need on $\mathcal{V}$ are stated at a few places along the way.

We start by introducing linear operators $\widehat{U}$ and $\mathcal{L}_{t_1}$, for $t_1 \in T$, given by

$$(\widehat{U}\Phi)(x,\mathbf{t}) = (\Phi \circ F)(x,\mathbf{t}) = \Phi(f_{t_1}(x),\sigma(\mathbf{t}))$$

and

$$(\mathcal{L}_{t_1}\Phi)(y,\sigma(\mathbf{t})) = \begin{cases} \Phi(f_{t_1}^{-1}(y),\mathbf{t}) \,|\det Df_{t_1}(f_{t_1}^{-1}(y))|^{-1}, & \text{if } y \in f_{t_1}(Q) \\ 0, & \text{otherwise.} \end{cases}$$

for every function $\Phi : Q \times T^{\mathbb{N}} \to \mathbb{R}$. We also define, for $\varepsilon > 0$ and Φ as before,

$$(\widehat{\mathcal{L}}_\varepsilon \Phi)(y, \sigma(\mathbf{t})) = \int (\mathcal{L}_{t_1} \Phi)(y, \sigma(\mathbf{t}))\, d\theta_\varepsilon(t_1).$$

Let $\pi_0 : Q \times T^{\mathbb{N}} \to \mathbb{R}$ be the canonical projection $\pi_0(x, \mathbf{t}) = x$. We say that a function $\Phi : Q \times T^{\mathbb{N}} \to \mathbb{R}$ *depends only on* x if it can be written $\Phi = \varphi \circ \pi_0$ for some $\varphi : Q \to \mathbb{R}$. Observe that if Φ depends only on x then so do $\mathcal{L}_{t_1}\Phi$ and $\widehat{\mathcal{L}}_\varepsilon \Phi$, for every $t_1 \in T$ and $\varepsilon > 0$. Thus, we may also think of these operators as acting on the space of functions defined on Q, and sometimes we do so.

A main property of $\widehat{U}$ and $\widehat{\mathcal{L}}_\varepsilon$ is the following duality relation, which follows directly from the definitions and Fubini's theorem, using the change of variables $y = f_{t_1}^{-1}(x)$:

$$\int \Phi(\widehat{U}\Psi)\, d(m \times \theta_\varepsilon^{\mathbb{N}}) = \int (\widehat{\mathcal{L}}_\varepsilon \Phi)\Psi\, d(m \times \theta_\varepsilon^{\mathbb{N}}). \tag{5.12}$$

whenever the integrals make sense. This estalishes a close link between the operator $\widehat{\mathcal{L}}_\varepsilon$ and stationary measures of our random process, as illustrated by the following remarks.

Let $\hat{\mu}_\varepsilon$ be a probability measure on $Q \times T^{\mathbb{N}}$ given by $\hat{\mu} = \Phi\,(m \times \theta_\varepsilon^{\mathbb{N}})$, and suppose that $\Phi : Q \times T^{\mathbb{N}} \to \mathbb{R}$ is a fixed point of $\widehat{\mathcal{L}}_\varepsilon$. Then (5.12) gives, for any Ψ,

$$\int (\Psi \circ F)\, d\hat{\mu} = \int (\widehat{U}\Psi)\Phi\, d(m \times \theta_\varepsilon^{\mathbb{N}}) = \int \Psi\Phi\, d(m \times \theta_\varepsilon^{\mathbb{N}}) = \int \Psi\, d\hat{\mu}.$$

In other words, $\hat{\mu}$ is an invariant measure for F. The converse is proved in the same way: the density of an F-invariant measure absolutely continuous with respect to $m \times \theta_\varepsilon^{\mathbb{N}}$ is necessarily a fixed point of $\widehat{\mathcal{L}}_\varepsilon$. Now suppose that this fixed point Φ depends only on x, and write $\Phi = \varphi \circ \pi_0$. Then $\mu = \varphi m$ is a stationary probability measure on Q. To see this, let $\psi : Q \to \mathbb{R}$ be continuous, and define $\Psi : Q \times T^{\mathbb{N}} \to \mathbb{R}$ by $\Psi(x, \mathbf{t}) = \psi(x)$. Then, since $\Phi(x, \sigma(\mathbf{t})) = \varphi(x) = \Phi(x, \mathbf{t})$, and $d(m \times \theta_\varepsilon^{\mathbb{N}})(x, \sigma(\mathbf{t}))\, d\theta_\varepsilon(t_1)$ represents just the same as $d(m \times \theta_\varepsilon^{\mathbb{N}})(x, \mathbf{t})$,

$$\begin{aligned}
\int \Big(\int \psi(f_{t_1}(x))\, d\mu_\varepsilon(x) \Big) d\theta_\varepsilon(t_1) &= \int \Psi(f_{t_1}(x), \sigma(\mathbf{t}))\Phi(x, \sigma(\mathbf{t}))\, d(m \times \theta_\varepsilon^{\mathbb{N}})(x, \sigma(\mathbf{t})) d\theta_\varepsilon(t_1) \\
&= \int (\Psi \circ F)(x, \mathbf{t})\, \Phi(x, \mathbf{t})\, d(m \times \theta_\varepsilon^{\mathbb{N}})(x, \mathbf{t}) \\
&= \int (\Psi \circ F)\, d\hat{\mu}_\varepsilon = \int \Psi\, d\hat{\mu}_\varepsilon = \int \psi\, d\mu_\varepsilon .
\end{aligned}$$

In general, $\widehat{\mathcal{L}}_\varepsilon$ need not have a fixed point and the stationary measure μ_ε need not be absolutely continous with respect to Lebesgue measure. However, the considerations we have just made motivate our approach for constructing the stationary measure. Similarly to what we did in the unperturbed case, cf. Proposition **??**, we shall prove that successive push-forwards of $(m \times \theta_\varepsilon^{\mathbb{N}})$ form a sequence of probability measures

$$F_*^n(m \times \theta_\varepsilon^{\mathbb{N}}) = (\widehat{\mathcal{L}}_\varepsilon^n 1)\,(m \times \theta_\varepsilon^{\mathbb{N}})$$

that converges weakly to some measure $\hat{\mu}_\varepsilon$ on $Q \times T^{\mathbb{N}}$. Let μ_ε be the measure on Q defined by

$$\mu_\varepsilon = (\pi_0)_* \hat{\mu}_\varepsilon .$$

We shall seduce that μ_ε is the stationary measure we are looking for, from the fact that every $\widehat{\mathcal{L}}_\varepsilon^n 1$ depends only on x.

An important tool in this construction is the local stable foliation of the map F. The same kind of arguments as one uses for constructing the local stable foliation $\mathcal{F}_{loc}^s$ of the map f, see Appendix A, shows that there exists a (unique) foliation $\widehat{\mathcal{F}}_{loc}^s$ of $Q \times T^{\mathbb{N}}$ satisfying properties (1), (2), (3), (4) below. At this point we suppose that all the maps f_t are in a suffciently small neighbourhood $\mathcal{V}$ of f, cf. previous comments.

1. Each leaf $\widehat{\mathcal{F}}_{loc}^s(x, \mathbf{t})$ through a point $(x, \mathbf{t})$ is a C^2 submanifold of $Q \times \{\mathbf{t}\}$, with uniformly bounded curvature.
2. $F(\widehat{\mathcal{F}}_{loc}^s(x, \mathbf{t}))$ is contained in $\widehat{\mathcal{F}}_{loc}^s(F(x, \mathbf{t}))$, for every $(x, \mathbf{t}) \in Q \times T^{\mathbb{N}}$, and
$$F : \widehat{\mathcal{F}}_{loc}^s(x, \mathbf{t}) \to \widehat{\mathcal{F}}_{loc}^s(F(x, \mathbf{t}))$$
is a λ_s-contraction, for some uniform constant $\lambda_s \in (0, 1)$.
3. Given $(y, \sigma(\mathbf{t})) \in Q \times T^{\mathbb{N}}$ and any $t_1 \in T$, the intersection
$$F^{-1}(\widehat{\mathcal{F}}_{loc}^s(y, \sigma(\mathbf{t}))) \cap (Q \times \{\mathbf{t}\})$$
has exactly two connected components, and they are also leaves of $\widehat{\mathcal{F}}_{loc}^s$.
4. The foliation $\widehat{\mathcal{F}}_{loc}^s$ is absolutely continuous with respect to $(m \times \theta_\varepsilon^{\mathbb{N}})$.

Let us explain property (4) in more precise terms, before proceeding. Let $\widetilde{m}_\varepsilon$ be the measure induced by $(m \times \theta_\varepsilon^{\mathbb{N}})$ in the quotient space (the space of leaves) of $\widehat{\mathcal{F}}_{loc}^s$, that is,

$$\widetilde{m}_\varepsilon(\tilde{A}) = (m \times \theta_\varepsilon^{\mathbb{N}}) \left(\bigcup_{\gamma \imath \tilde{A}} \gamma \right)$$

for every measurable subset $\tilde{A}$ of the quotient space. Given any $\gamma \in \widehat{\mathcal{F}}_{loc}^s$, let m_γ be the smooth measure induced on γ by the riemannian metric of M. In (4) we mean that there exists a continuous function

$$H_\varepsilon : Q \times T^{\mathbb{N}} \to (0, +\infty),$$

bounded away from zero and infinity, such that $\{p_{\varepsilon,\gamma} = (H_\varepsilon \mid \gamma)\, m_\gamma : \gamma \in \widehat{\mathcal{F}}^s_{loc}\}$ defines a disintegration of $(m \times \theta_\varepsilon^{\mathbb{N}})$ along the leaves of the foliation: given any $\Psi \in L^1(m \times \theta_\varepsilon^{\mathbb{N}})$,

$$\int \Psi \, d(m \times \theta_\varepsilon^{\mathbb{N}}) = \int \left(\int_\gamma (\Psi \mid \gamma) \, dp_{\varepsilon,\gamma} \right) d\tilde{m}_\varepsilon(\gamma).$$

Moreover, H_ε may be taken such that $\log H_\varepsilon$ is (a_0, ν_0)-Hölder continuous on every $Q \times \{\mathbf{t}\}$, with $a_0 > 0$ and $\nu_0 \in (0,1)$ depending only on the initial map f.

Given a leaf $\gamma \in \widehat{\mathcal{F}}^s_{loc}$ we define cones of Hölder continuous densities $\mathcal{D}(\gamma) = \mathcal{D}(a, \mu, \gamma)$ and $\mathcal{D}_1(\gamma) = \mathcal{D}(a_1, \mu_1, \gamma)$, in just the same way as before:

$$\mathcal{D}(\gamma) = \{\rho : \gamma \to \mathbb{R} \text{ such that } \rho(x) > 0 \text{ for all } x \in \gamma \text{ and } \log \rho \text{ is } (a,\mu)\text{– Hölder}\},$$

and similarly for $\mathcal{D}_1(\gamma)$. The constants a, μ, a_1, μ_1 are chosen as in (3.67), (3.71).

For any bounded function Φ defined on γ and any $\rho \in \mathcal{D}(\gamma)$ we let $\int_\gamma \Phi\rho$ denote the integral of Φ with respect to the measure $\rho\, m_\gamma$ supported on γ. The following simple consequence of Fubini's theorem will be useful later:

$$\int_\gamma (\widehat{\mathcal{L}}_\varepsilon \Phi)\rho = \int \left(\int_\gamma (\mathcal{L}_{t_1} \Phi)\rho \right) d\theta_\varepsilon(t_1). \tag{5.13}$$

For each $\mathbf{t} \in T^{\mathbb{N}}$ we denote $\widehat{\mathcal{F}}^s_{loc}(\mathbf{t})$ the restriction of the foliation $\widehat{\mathcal{F}}^s_{loc}$ to $Q \times \{\mathbf{t}\}$. We often identify $Q \times \{\mathbf{t}\}$ with Q, through the canonical bijection

$$\phi_{\mathbf{t}} : (Q \times \{\mathbf{t}\}) \ni (x, \mathbf{t}) \mapsto x \in Q,$$

thus thinking of each $\widehat{\mathcal{F}}^s_{loc}(\mathbf{t})$ also as a foliation of Q. Observe that after identification the action of F on the leaves of this foliation is described by the map f_{t_1}.

For $\gamma \in \widehat{\mathcal{F}}^s_{loc}(\sigma(\mathbf{t}))$ and each $t_1 \in T$, let $\gamma_{1,t_1}, \gamma_{2,t_1} \in \widehat{\mathcal{F}}^s_{loc}(\mathbf{t})$ be the connected components of

$$F^{-1}(\gamma) \cap (Q \times \{\mathbf{t}\}),$$

cf. property (3) above. Then, for any $\rho \in \mathcal{D}(\gamma)$ and any bounded function Φ,

$$\int_\gamma (\mathcal{L}_{t_1} \Phi)\rho = \sum_{j=1}^{2} \int_{\gamma_{j,t_1}} \Phi \rho_{j,t_1}\,, \qquad \rho_{j,t_1} = \frac{|\det(Df_{t_1} \mid \gamma_{j,t_1})|}{|\det Df_{t_1}|} (\rho \circ f_{t_1}), \tag{5.14}$$

compare (3.62). As in Lemma **??**, there is some uniform constant $\lambda_1 < 1$ so that

$$\rho_{j,t_1} \in \mathcal{D}(\lambda_1 a, \mu, \gamma_{j,t_1}) \quad \text{for every } j = 1, 2.$$

Moreover, there is some other uniform constant $\Lambda_1 < 1$ so that

$$\theta_{j,t_1}(\rho'_{j,t_1}, \rho''_{j,t_1}) \le \Lambda_1 \theta(\rho', \rho''), \quad \text{for all } \rho', \rho'' \in \mathcal{D}(\gamma) \text{ and } j = 1, 2,$$

where θ and θ_{j,t_1} are the projective metrics associated to the cones $\mathcal{D}(\gamma)$ and to $\mathcal{D}(\gamma_{j,t_1})$, respectively.

The next step is to define a projection map $\pi = \pi(\tilde{\gamma}, \gamma)$ from a leaf $\tilde{\gamma}$ to another leaf γ of the local stable foliation of F, as well as a notion of distance $d(\gamma, \tilde{\gamma})$ between the two leaves. Let $\gamma \in \mathcal{F}^s_{loc}(\sigma(\mathbf{t}))$ and $\tilde{\gamma} \in \mathcal{F}^s_{loc}(\sigma(\mathbf{s}))$ for some $\sigma(\mathbf{t}) = (t_2, t_3, \dots)$ and $\sigma(\mathbf{s}) = (s_2, s_3, \dots)$ in $T^{\mathbb{N}}$. Identifying both $Q \times \{\sigma(\mathbf{t})\}$ and $Q \times \{\sigma(\mathbf{s})\}$ with Q, we may consider γ and $\tilde{\gamma}$ as submanifolds of Q, and then define π in just the same way as in Section **??**. In more precise terms, we set

$$\pi = \pi(\tilde{\gamma}, \gamma) = \phi^{-1}_{\sigma(\mathbf{t})} \circ \pi(\phi_{\sigma(\mathbf{s})}(\tilde{\gamma}), \phi_{\sigma(\mathbf{t})}(\gamma)) \circ \phi_{\sigma(\mathbf{s})},$$

where $\pi(\phi_{\sigma(\mathbf{s})}(\tilde{\gamma}), \phi_{\sigma(\mathbf{t})}(\gamma))$ is as defined in Section **??**. Analogously, given any $y \in \tilde{\gamma}$,

$$d(y, \pi(y)) = d(\phi_{\sigma(\mathbf{s})}(y), \phi_{\sigma(\mathbf{t})}(\pi(y))),$$

where the right hand side is meant as in Section **??**.

For $t_1 \in T$, let $\mathbf{s} = (t_1, s_2, s_3, \dots)$, recall that $\mathbf{t} = (t_1, t_2, t_3, \dots)$. Let $\gamma_{j,t_1} \in \widehat{\mathcal{F}}^s_{loc}(\mathbf{t})$ and $\tilde{\gamma}_{j,t_1} \in \widehat{\mathcal{F}}^s_{loc}(\mathbf{s})$, $j = 1, 2$, be the connected components of

$$F^{-1}(\gamma) \cap (Q \times \{\mathbf{t}\}) \quad \text{and} \quad F^{-1}(\tilde{\gamma}) \cap (Q \times \{\mathbf{s}\}),$$

respectively, and denote $\pi_{j,t_1} = \pi(\tilde{\gamma}_{j,t_1}, \gamma_{j,t_1})$. Then we have the following analogs of properties (p1) and (p3):

(q1) π and $\log|\det D\pi|$ are a_0-Lipschitz maps;

(q3) $d(x, \pi_{j,t_1}(x)) \le \lambda_0\, d(f_{t_1}(x), \pi f_{t_1}(x))$ for all $x \in \tilde{\gamma}_{j,t_1}$, $j = 1, 2$, and $t_1 \in T$.

where $\lambda_u < 1$, $a_0 > 0$, and $\nu_0 \in (0, 1)$, depend only on the unperturbed map f. Indeed, (q1) is a consequence of the fact that stable leaves have uniformly bounded curvature, and (q3) is proved in the same way as (p3), with f_{t_1} in the place of f (and further restricting the neighbourhoof $\mathcal{V}$ of f, if necessary).

We also need an analog of property (p2), but this is more subtle. Indeed, (p2) relied on the Hölder property (3.63) of the tangent spaces to stable leaves, which has no straightforward analog in the present situation. To see this observe that, although the canonical identifications $\phi_{\sigma(\mathbf{t})}$ and $\phi_{\sigma(\mathbf{s})}$ allow us to think of both $\widehat{\mathcal{F}}^s_{loc}(\sigma(\mathbf{t}))$ and $\widehat{\mathcal{F}}^s_{loc}(\sigma(\mathbf{s}))$ as foliations of Q, in general these foliations do not coincide. For instance, a leaf γ of one foliation may intersect a leaf $\tilde{\gamma}$ of the other foliation transversely at some point y, in which case

$$d(y, \pi(y)) = 0 \quad \text{but} \quad \text{angle}(T_y\tilde{\gamma}, T_{\pi(y)}\gamma) > 0. \tag{5.15}$$

An exception occurs in the particular case $\sigma(\mathbf{t}) = \sigma(\mathbf{s})$, since a Hölder property similar to (3.63) can be proved for leaves γ_1, γ_2 within a same $Q \times \{\sigma(\mathbf{t})\}$, any $\sigma(\mathbf{t}) \in T^{\mathbb{N}}$: denoting $\pi = \pi(\gamma_1, \gamma_2)$,

$$\text{angle}(T_z\gamma_1, T_{\pi(z)}\gamma_2) \leq a_0\, d(z, \pi(z))^{\nu_0} \quad \text{for all } z \in \gamma_1 . \tag{5.16}$$

This property follows from the same methods as (3.63), and will be useful below. However, this particular case is not sufficient for our purposes (the statement of the analog of condition (C) in the definition of the new cone of observable functions in $Q \times T^{\mathbb{N}}$ must involve all the pairs of local stable leaves γ and $\tilde{\gamma}$ in $\widehat{\mathcal{F}}^s_{loc}$) and so we must deal with the difficulty expressed by (5.15).

The way we overcome this is by defining the distance $d(\gamma, \tilde{\gamma})$ between two general stable leaves in a careful way. Similarly to what we did before, we consider

$$d_1(\gamma, \tilde{\gamma}) = \sup\{d(y, \pi(y)) : y \in \tilde{\gamma}\}.$$

But a key point is to take angles in consideration too, when determining how far apart two leaves are from each other. Given any point $z \in \gamma \in \widehat{\mathcal{F}}^s_{loc}(\sigma(\mathbf{t}))$, let γ_z be the leaf of $\widehat{\mathcal{F}}^s_{loc}(\sigma(\mathbf{s}))$ that contains z. We define

$$\partial(\gamma, \tilde{\gamma}) = \sup\{\text{angle}(T_z\gamma, T_z\gamma_z)^{1/\nu_0} : z \in \gamma\},$$

and define $\partial(\tilde{\gamma}, \gamma)$ in the same way, just reversing the roles of γ, $\tilde{\gamma}$ and $\sigma(\mathbf{t})$, $\sigma(\mathbf{s})$. Our definition also involves explicitly the distance between $\sigma(\mathbf{t})$ and $\sigma(\mathbf{s})$ (this will be useful in the proof of Lemma 5.4.1): we let

$$d_2(\gamma, \tilde{\gamma}) = \|\sigma(\mathbf{t}) - \sigma(\mathbf{s})\| = \sum_{i=1}^{\infty} 2^{-i}\, d(t_{i+1}, s_{i+1})$$

Finally, we define

$$d(\gamma, \tilde{\gamma}) = \max\{d_1(\gamma, \tilde{\gamma}), d_2(\gamma, \tilde{\gamma}), \partial(\gamma, \tilde{\gamma}), \partial(\tilde{\gamma}, \gamma)\}. \tag{5.17}$$

Of course, $\partial(\gamma, \tilde{\gamma}) = \partial(\tilde{\gamma}, \gamma) = d_2(\gamma, \tilde{\gamma}) = 0$ when $\sigma(\mathbf{t}) = \sigma(\mathbf{s})$, and so in that case $d(\gamma, \tilde{\gamma})$ coincides with the "usual" distance $d_1(\gamma, \tilde{\gamma})$.

Now, observe that $\pi = \pi(\tilde{\gamma}, \gamma) = \pi(\gamma_{\pi(y)}, \gamma) \circ \pi(\tilde{\gamma}, \gamma_{\pi(y)})$ and $\pi(\tilde{\gamma}, \gamma_{\pi(y)})(y) = \pi(y)$, for any $y \in \tilde{\gamma}$. So,

$$\log|\det D\pi|(y) = \log|\det D\pi(\gamma_{\pi(y)}, \gamma)|(\pi(y)) + \log|\det D\pi(\tilde{\gamma}, \gamma_{\pi(y)})|(y).$$

By definition, $\pi(y)$ belongs in $\gamma_{\pi(y)} \cap \gamma$, and so the first term on the right hand side is bounded by

$$\alpha_0\, \text{angle}(T_{\pi(y)}\gamma, T_{\pi(y)}\gamma_{\pi(y)}) \leq \alpha_0 \partial(\gamma, \tilde{\gamma})^{\nu_0} \leq \alpha_0 d(\gamma, \tilde{\gamma})^{\nu_0}$$

for some universal constant $\alpha_0 > 0$. Also by definition, $\tilde{\gamma}$ and $\gamma_{\pi(y)}$ are both leaves of the foliation $\widehat{\mathcal{F}}^s_{loc}(\sigma(\mathbf{s}))$. Hence, by (**??**), the second term on the right hand is bounded by

$$a_0 d(y, \pi(y))^{\nu_0} \leq a_0 d_1(\gamma, \tilde{\gamma})^{\nu_0} \leq a_0 d(\gamma, \tilde{\gamma})^{\nu_0}.$$

In this way we conclude the analog of (p2) we were looking for:

(q2) $\log|\det D\pi|(y) \leq (a_0 + \alpha_0) d(\gamma, \tilde{\gamma})^{\nu_0}$ for every $y \in \tilde{\gamma}$.

Now we also need to generalize the expansion property (3.65) to arbitrary pairs γ and $\tilde{\gamma}$ of leaves in the local stable foliation of F. This easily done, in the following way. Up to further restricting the metric space T, the leaves of the foliation $\widehat{\mathcal{F}}^s_{loc}$ (viewed as submanifolds of Q, via canonical identification) are uniformly close to the leaves of the stable foliation $\mathcal{F}^s_{loc}$ of f. Then, let $\gamma_1 \in \widehat{\mathcal{F}}^s_{loc}(\mathbf{t})$ and $\gamma_2 \in \widehat{\mathcal{F}}^s_{loc}(\mathbf{s})$ be any two leaves intersecting at some point z, with $\mathbf{t} = (t_1, t_2, t_3, \dots)$ and $\mathbf{s} = (t_1, s_2, s_3, \dots)$. Since the action of F on γ_1, γ_2, is described by a same map f_{t_1}, a small perturbation of f, we have

$$\operatorname{angle}(T_{f_{t_1}(z)} F(\gamma_1), T_{f_{t_1}(z)} F(\gamma_2)) \geq \sigma \operatorname{angle}(T_z \gamma_1, T_z \gamma_2) \tag{5.18}$$

for some constant $\sigma > 1$ depending only on f. Let $\gamma \in \widehat{\mathcal{F}}^s_{loc}(\sigma(\mathbf{t}))$ and $\tilde{\gamma} \in \widehat{\mathcal{F}}^s_{loc}(\sigma(\mathbf{s}))$ be arbitrary leaves, and let $j = 1, 2$ and $t_1 \in T$. Then,

$$\partial(\gamma, \tilde{\gamma}) \geq \sigma \partial(\gamma_{j,t_1}, \tilde{\gamma}_{j,t_1})$$

follows from applying the previous relation to $\gamma_1 = \gamma_{j,t_1}$, any $z \in \gamma_1$, and $\gamma_2 = \gamma_{j,t_1,z}$ (the leaf of $\widehat{\mathcal{F}}^s_{loc}(\mathbf{s})$ through z). Analogously, $\partial(\tilde{\gamma}, \gamma) \geq \sigma \partial(\tilde{\gamma}_{j,t_1}, \gamma_{j,t_1})$. On the other hand, (q3) gives

$$d_1(\gamma, \tilde{\gamma}) \geq \lambda_u^{-1} d_1(\gamma_{j,t_1}, \tilde{\gamma}_{j,t_1})$$

for every $j = 1, 2$ and $t_1 \in T$. Finally,

$$\begin{aligned} d_2(\gamma, \tilde{\gamma}) = \|\sigma(\mathbf{t}) - \sigma(\mathbf{s})\| &= \sum_{i=1}^{\infty} 2^{-i} \, d(t_{i+1}, s_{i+1}) \\ &= 2 \sum_{i=2}^{\infty} 2^{-i} \, d(t_i, s_i) = 2\|\mathbf{t} - \mathbf{s}\| = 2 d_2(\gamma_{j,t_1}, \tilde{\gamma}_{j,t_1}). \end{aligned}$$

It is no restriction to suppose $2 \geq \sigma \geq \lambda_u^{-1} > 1$ (decreasing σ in (5.18) and increasing λ_u in (q3), if necessary), and then these remarks give

$$d(\gamma, \tilde{\gamma}) \geq \lambda_u^{-1} d(\gamma_{j,t_1}, \tilde{\gamma}_{j,t_1}) \tag{5.19}$$

for every $j = 1, 2$ and $t_1 \in T$. This is the analog of (3.65) that we wanted.

Finally, given any leaves γ, $\tilde{\gamma}$ of $\widehat{\mathcal{F}}^s_{loc}$ and given any $\rho \in \mathcal{D}_1(\gamma)$, we let $\tilde{\rho}\colon \tilde{\gamma} \to \mathbb{R}$ be defined by

$$\tilde{\rho}(y) = \rho(\pi(y)) \cdot |\det D\pi(y)|.$$

In the same way as before, our choice of a, μ, a_1, μ_1 in (3.67) ensures that $\tilde{\rho}$ is in $\mathcal{D}(\tilde{\gamma})$.

Now we have all we need to give the definition of our cone $\widehat{C}(b, c, \nu)$ of observable functions in $Q \times T^{\mathbb{N}}$. At this point this is a direct translation of the definition of the cone $C(b, c, \nu)$ in Section **??**. We let $\widehat{C}(b, c, \nu)$ consist of all the bounded functions $\Phi : Q \times T^{\mathbb{N}} \to \mathbb{R}$ that satisfy

(AA) $\int_\gamma \Phi\rho > 0$ for every $\gamma \in \widehat{\mathcal{F}}^s_{loc}$ and every $\rho \in \mathcal{D}(\gamma)$;

(BB) $\log \int_\gamma \Phi\rho$ is b-Lipschitz as a function of $\rho \in \mathcal{D}(\gamma)$:

$$|\log \int_\gamma \Phi\rho' - \log \int_\gamma \Phi\rho''| \leq b\,\theta(\rho', \rho'')$$

for every $\rho', \rho'' \in \mathcal{D}(\gamma)$ with $\int_\gamma \rho' = 1 = \int_\gamma \rho''$, and every $\gamma \in \widehat{\mathcal{F}}^s_{loc}$;

(CC) $\int_\gamma \Phi\rho$ is (c, ν)-Hölder as a function of γ:

$$|\log \int_\gamma \Phi\rho - \log \int_{\tilde{\gamma}} \Phi\tilde{\rho}| \leq c\,d(\gamma, \tilde{\gamma})^\nu$$

for every $\rho \in \mathcal{D}_1(\gamma)$ and every pair $\gamma, \tilde{\gamma} \in \widehat{\mathcal{F}}^s_{loc}$.

The argument of Lemma **??** applies to $\widehat{C}(b, c, \nu)$, so that this is indeed a convex cone satisfying condition (3.19). We denote $\widehat{\Theta}$ the corresponding projective metric, and we also let $\widehat{\Theta}_+$ be the projective metric associated to the cone of strictly positive functions on $Q \times T^{\mathbb{N}}$. They can be calculated in the same way as Θ and Θ_+ in Section **??**, in fact, one obtains similar expressions (up to replacing $\gamma \in \mathcal{F}^s_{loc}$ by $\gamma \in \widehat{\mathcal{F}}^s_{loc}$).

Let $\Phi \in \widehat{C}(b, c, \nu)$. The same argument as in the proof of Proposition **??** shows that there exists a uniform $\lambda_2 < 1$ such that, given any $t_1 \in T$ and any $\gamma, \tilde{\gamma} \in \widehat{\mathcal{F}}^s_{loc}$,

1. $\int_\gamma (\mathcal{L}_{t_1}\Phi)\rho > 0$ for every $\rho \in \mathcal{D}(\gamma)$,
2. for every $\rho', \rho'' \in \mathcal{D}(\gamma)$,

$$\exp(-b\lambda_2\,\theta(\rho', \rho'')) \int_\gamma (\mathcal{L}_{t_1}\Phi)\rho'' \leq \int_\gamma (\mathcal{L}_{t_1}\Phi)\rho' \leq \exp(b\lambda_2\,\theta(\rho', \rho'')) \int_\gamma (\mathcal{L}_{t_1}\Phi)\rho''$$

3. for every $\rho \in \mathcal{D}_1(\gamma)$,

$$\exp(-c\lambda_2\,d(\gamma, \tilde{\gamma})) \int_\gamma (\mathcal{L}_{t_1}\Phi)\rho \leq \int_{\tilde{\gamma}} (\mathcal{L}_{t_1}\Phi)\tilde{\rho} \leq \exp(c\lambda_2\,d(\gamma, \tilde{\gamma})) \int_\gamma (\mathcal{L}_{t_1}\Phi)\rho,$$

as long as $b > 0$ and $c > 0$ are fixed large enough. This means, in other words, that

$$\mathcal{L}_{t_1}(\widehat{C}(b,c,\nu)) \subset \widehat{C}(\lambda_2 b, \lambda_2 c, \nu)$$

for every $t_1 \in T$. Moreover, integrating with respect to $d\theta_\varepsilon(t_1)$, cf. (5.13), we conclude that the same relations (1), (2), (3) hold with $\widehat{\mathcal{L}}_\varepsilon$ in the place of $\mathcal{L}_{t_1}$. Therefore,

$$\widehat{\mathcal{L}}_\varepsilon(\widehat{C}(b,c,\nu)) \subset \widehat{C}(\lambda_2 b, \lambda_2 c, \nu).$$

A straightforward translation of the first part of the proof of Proposition **??** shows that there is $T > 0$, depending only on λ_2, so that

$$\widehat{\Theta}\text{-diameter}\,(\mathcal{C}) \le \widehat{\Theta}_+\text{-diameter}\,(\mathcal{C}) + T$$

for every subset $\mathcal{C}$ of $C(\lambda_2 b, \lambda_2 c, \nu)$. In particular, we may take $\mathcal{C} = \mathcal{L}_{t_1}(C(b,c,\nu))$, any $t_1 \in T$, or $\mathcal{C} = \widehat{\mathcal{L}}_\varepsilon(C(b,c,\nu))$.

Besides, the same arguments as in the second part of the proof of Proposition **??** show that there is $\Gamma_0 > 0$ such that

$$\int_{\gamma''} (\mathcal{L}_{t_1}\Phi)\rho'' \le \Gamma_0 \int_{\gamma'} (\mathcal{L}_{t_1}\Phi)\rho'$$

for all $t_1 \in T$, $\Phi \in \widehat{C}(b,c,\nu)$, $\gamma', \gamma'' \in \widehat{\mathcal{F}}^s_{loc}$, and normalized densities $\rho' \in \mathcal{D}(\gamma')$, $\rho'' \in \mathcal{D}(\gamma'')$. Integrating with respect to $d\theta_\varepsilon(t_1)$ we get

$$\int_{\gamma''} (\widehat{\mathcal{L}}_\varepsilon\Phi)\rho'' \le \Gamma_0 \int_{\gamma'} (\widehat{\mathcal{L}}_\varepsilon\Phi)\rho'$$

for every Φ, γ', γ'', ρ', ρ'' as above. Consequently, the $\widehat{\Theta}_+$-diameter of $\mathcal{L}_\varepsilon(C(b,c,\nu))$ is bounded by $2\log\Gamma_0$. We conclude that, denoting $D_2 = 2\log\Gamma_0 + T$,

$$\widehat{\Theta}\text{-diameter}\,(\,\widehat{\mathcal{L}}_\varepsilon(\widehat{C}(b,c,\nu))\,) \le D_2 < \infty,$$

and so $\widehat{\mathcal{L}}_\varepsilon$ is a Λ_2-contraction for the projective metric $\widehat{\Theta}$, with $\Lambda_2 = 1 - e^{-D_2}$.

This implies that $(\widehat{\mathcal{L}}_\varepsilon^n 1)$ is a Cauchy sequence for $\widehat{\Theta}$. Then it is also Cauchy for $\widehat{\Theta}_+$, since $\widehat{\Theta}_+ \le \widehat{\Theta}$. It follows that the sequence of probability measures $(\widehat{\mathcal{L}}_\varepsilon^n 1)(m \times \theta_\varepsilon^{\mathbb{N}})$ is weak*-Cauchy, meaning that for every continuous function $\Psi : Q \times T^{\mathbb{N}} \to \mathbb{R}$,

$$\int \Psi\,(\widehat{\mathcal{L}}_\varepsilon^n 1)\,d(m \times \theta_\varepsilon^{\mathbb{N}}), \quad n \ge 1,$$

is a Cauchy sequence in $\mathbb{R}$. This last claim is proved in the same way as Proposition **??**, using the absolute continuity of the foliation $\widehat{\mathcal{F}}^s_{loc}$ stated in property (4) above, including the fact that H may be taken uniformly Hölder continuous on leaves of the foliation.

On the other hand, it enables us to define a probability measure $\hat{\mu}_\varepsilon$ on $Q \times T^{\mathbb{N}}$ by letting

$$\int \Psi \, d\hat{\mu}_\varepsilon = \lim \int \Psi \, (\widehat{\mathcal{L}}_\varepsilon^n 1) \, d(m \times \theta_\varepsilon^{\mathbb{N}})$$

for each continuous function $\Psi : Q \times T^{\mathbb{N}} \to \mathbb{R}$. It is easy to deduce from (5.12) that this $\hat{\mu}_\varepsilon$ is F-invariant: for every continuous Ψ,

$$\int (\Psi \circ F) \, d\hat{\mu}_\varepsilon = \lim \int (\Psi \circ F) \, (\widehat{\mathcal{L}}_\varepsilon^n 1) \, d(m \times \theta_\varepsilon^{\mathbb{N}}) = \lim \int \Psi \, (\widehat{\mathcal{L}}_\varepsilon^{n+1} 1) \, d(m \times \theta_\varepsilon^{\mathbb{N}}) = \int \Psi \, d\hat{\mu}_\varepsilon.$$

Moreover, the probability measure μ_ε on Q defined by

$$\int \psi \, d\mu_\varepsilon = \int (\psi \circ \pi_0) \, d\hat{\mu}_\varepsilon \quad \text{for each continuous } \psi : Q \to \mathbb{R},$$

is a stationary measure, recall (5.10). Indeed, let $\psi : Q \to \mathbb{R}$ be any continuous function, and $\Psi : Q \times T^{\mathbb{N}} \to \mathbb{R}$ be given by $\Psi(x, \mathbf{t}) = (\psi \circ \pi_0)(x, \mathbf{t}) = \psi(x)$. By definition and Fubini's theorem,

$$\begin{aligned}
\int \left(\int \psi(f_{t_1}(x)) \, d\theta_\varepsilon(t_1) \right) d\mu_\varepsilon(x) &= \int \left(\int \Psi(f_{t_1}(x), \sigma(\mathbf{t})) \, d\theta_\varepsilon(t_1) \right) d\hat{\mu}_\varepsilon(x, \sigma(\mathbf{t})) \\
&= \lim \int \left(\int \Psi(f_{t_1}(x), \sigma(\mathbf{t})) \, d\theta_\varepsilon(t_1) \right) (\widehat{\mathcal{L}}_\varepsilon^n 1)(x, \sigma(\mathbf{t})) \, d(m \times \theta_\varepsilon^{\mathbb{N}})(x, \sigma(\mathbf{t})) \\
&= \lim \int (\Psi \circ F)(x, \mathbf{t}) \, (\widehat{\mathcal{L}}_\varepsilon^n 1)(x, \sigma(\mathbf{t})) \, d\theta_\varepsilon(t_1) \, d(m \times \theta_\varepsilon^{\mathbb{N}})(x, \sigma(\mathbf{t})).
\end{aligned}$$

Since $\widehat{\mathcal{L}}_\varepsilon^n 1$ depends only on x, we may write

$$\begin{aligned}
\int \left(\int \psi(f_{t_1}(x)) \, d\theta_\varepsilon(t_1) \right) d\mu_\varepsilon(x) &= \lim \int (\Psi \circ F)(x, \mathbf{t}) (\widehat{\mathcal{L}}_\varepsilon^n 1)(x, \mathbf{t}) \, d(m \times \theta_\varepsilon^{\mathbb{N}})(x, \mathbf{t}) \\
&= \lim \int \Psi(x, \mathbf{t}) (\widehat{\mathcal{L}}_\varepsilon^{n+1} 1)(x, \mathbf{t}) \, d(m \times \theta_\varepsilon^{\mathbb{N}})(x, \mathbf{t}) \\
&= \int \Psi(x, \mathbf{t}) \, d\hat{\mu}_\varepsilon(x, \mathbf{t}) = \int \psi(x) \, d\mu_\varepsilon(x),
\end{aligned}$$

which is precisely what stationarity means.

Next, we want to prove that μ_ε describes the asymptotic Birkhoff averages of every continuous function $\varphi \colon Q \to \mathbb{R}$ over $(m \times \theta_\varepsilon^{\mathbb{N}})$-almost every random trajectory. This is stated in a precise form in Proposition 5.4.2 below. For the proof we need the following lemma.

Lemma 5.4.1. *Let $\Psi : Q \times T^{\mathbb{N}} \to \mathbb{R}$ be any bounded function such that $\Psi > 0$ and $\log(\Psi|\gamma)$ is $(a/2, \mu)$-Hölder continuous along every leaf γ of the stable foliation $\widehat{\mathcal{F}}_{loc}^s$.*

1. *For every* $\Phi \in \widehat{C}(b,c,\nu)$ *there exists* $C(\Phi) > 0$ *such that*

$$\left| \int \Psi\, (\widehat{\mathcal{L}}_\varepsilon^n\, \Phi)\, d(m \times \theta_\varepsilon^{\mathbb{N}}) - \int \Psi\, d\hat{\mu}_\varepsilon \int \Phi\, d(m \times \theta_\varepsilon^{\mathbb{N}}) \right| \le C(\phi) \Lambda_2^n \,\sup \Psi,$$

for every $n \ge 1$.

2. *If* Ψ *satisfies* $\Psi = \Psi \circ F$ *at* $(m \times \theta_\varepsilon^{\mathbb{N}})$*-almost every point, then* Ψ *is almost everywhere constant:*

$$\Psi(x, \mathbf{t}) = \int \Psi\, d\hat{\mu}_\varepsilon \quad \textit{for } (m \times \theta_\varepsilon^{\mathbb{N}})\textit{-almost every } (x, \mathbf{t}) \in Q \times T^{\mathbb{N}}.$$

Proof. The proof of part (1) is based on the arguments leading to (3.80) and to the first part of Proposition **??**. For $1 \le n < k$, denote $\Phi_n = \widehat{\mathcal{L}}_\varepsilon^n\, \Phi$ and $\Phi_k = \widehat{\mathcal{L}}_\varepsilon^k\, \Phi$. We write

$$\int \Phi_n \Psi\, d(m \times \theta_\varepsilon^{\mathbb{N}}) = \int \left(\int_\gamma \Phi_n\, (\Psi\, H_\varepsilon \mid \gamma) \right) d\widetilde{m}_\varepsilon(\gamma),$$

and analogously for $\int \Phi_k \Psi\, d(m \times \theta_\varepsilon^{\mathbb{N}})$. Moreover,

$$\int \Phi_n\, d(m \times \theta_\varepsilon^{\mathbb{N}}) = \int \left(\int_\gamma \Phi_n\, (H_\varepsilon \mid \gamma) \right) d\widetilde{m}_\varepsilon(\gamma),$$

and analogously for $\int \Phi_k\, d(m \times \theta_\varepsilon^{\mathbb{N}})$. Recall that $\log H_\varepsilon$ is (a_0, ν_0)-Hölder continuous along leaves of the local stable foliation of F. In particular, $\log (H_\varepsilon | \gamma)$ is (a, μ)-Hölder along every stable leaf γ, recal (3.67), (3.71). Moreover, our assumptions on Ψ imply that $\log (\Psi H_\varepsilon | \gamma)$ is also (a, μ)-Hölder continuous along every leaf γ. Then, from the expression of $\widehat{\Theta}_+ = \log(\hat{\beta}_+ / \hat{\alpha}_+)$,

$$\frac{\int_\gamma \Phi_k\, (H_\varepsilon \mid \gamma)}{\int_\gamma \Phi_n\, (H_\varepsilon \mid \gamma)} \ge \hat{\alpha}_+(\Phi_k, \Phi_n) \quad \text{and} \quad \frac{\int_\gamma \Phi_k\, (\Psi H_\varepsilon \mid \gamma)}{\int_\gamma \Phi_n\, (\Psi H_\varepsilon \mid \gamma)} \le \hat{\beta}_+(\Phi_k, \Phi_n) \quad \text{for all } \gamma.$$

Moreover, using (5.12),

$$\int \Phi_n\, d(m \times \theta_\varepsilon^{\mathbb{N}}) = \int \Phi(\widehat{U}^n 1)\, d(m \times \theta_\varepsilon^{\mathbb{N}}) = \int \Phi\, d(m \times \theta_\varepsilon^{\mathbb{N}}) \tag{5.20}$$

and, analogously, $\int \Phi_k\, d(m \times \theta_\varepsilon^{\mathbb{N}}) = \int \Phi\, d(m \times \theta_\varepsilon^{\mathbb{N}})$. As consequence, there must be some local stable leaf $\hat{\gamma}$ so that $\int_{\hat{\gamma}} \Phi_k\, (H_\varepsilon \mid \hat{\gamma}) \le \int_{\hat{\gamma}} \Phi_n\, (H_\varepsilon \mid \hat{\gamma})$, and then

$$\frac{\int_\gamma \Phi_k\, (\Psi H_\varepsilon \mid \gamma)}{\int_\gamma \Phi_n\, (\Psi H_\varepsilon \mid \gamma)} \le \frac{\beta_+(\Phi_k, \Phi_n)}{\alpha_+(\Phi_k, \Phi_n)} \frac{\int_{\hat{\gamma}} \Phi_k\, (H_\varepsilon \mid \hat{\gamma})}{\int_{\hat{\gamma}} \Phi_n\, (H_\varepsilon \mid \hat{\gamma})} \le \exp(\widehat{\Theta}_+(\Phi_k, \Phi_n))$$

for every γ. This implies

$$\frac{\int \Phi_k \Psi \, d(m \times \theta_\varepsilon^{\mathbb{N}})}{\int \Phi_n \Psi \, d(m \times \theta_\varepsilon^{\mathbb{N}})} \leq \exp(\widehat{\Theta}_+(\Phi_k, \Phi_n)),$$

and so

$$\begin{aligned}\left| \int \Phi_k \Psi \, d(m \times \theta_\varepsilon^{\mathbb{N}}) - \int \Phi_n \Psi \, d(m \times \theta_\varepsilon^{\mathbb{N}}) \right| &= \\ = \left| \int \Phi_n \Psi \, d(m \times \theta_\varepsilon^{\mathbb{N}}) \right| &\left| \frac{\int \Phi_k \Psi \, d(m \times \theta_\varepsilon^{\mathbb{N}})}{\int \Phi_n \Psi \, d(m \times \theta_\varepsilon^{\mathbb{N}})} - 1 \right| \\ \leq \sup \Psi \int |\Phi_n| \, d(m \times \theta_\varepsilon^{\mathbb{N}}) &\big(\exp(\widehat{\Theta}_+(\Phi_k, \Phi_n)) - 1 \big).\end{aligned} \tag{5.21}$$

It is easy to see from the definition of $\widehat{\mathcal{L}}_\varepsilon$ that $|\widehat{\mathcal{L}}_\varepsilon \Upsilon| \leq \widehat{\mathcal{L}}_\varepsilon |\Upsilon|$ for every function Υ on $Q \times T^{\mathbb{N}}$. Then, using also the analog of (5.20) for $|\Phi|$,

$$\int |\Phi_n| \, d(m \times \theta_\varepsilon^{\mathbb{N}}) \leq \int \widehat{\mathcal{L}}_\varepsilon^n |\Phi| \, d(m \times \theta_\varepsilon^{\mathbb{N}}) = \int |\Phi| \, d(m \times \theta_\varepsilon^{\mathbb{N}})$$

The fact that $\widehat{\mathcal{L}}_\varepsilon$ is a Λ_2-contraction for the projective metric $\widehat{\Theta}$ implies that

$$\widehat{\Theta}_+(\Phi_k, \Phi_n) \leq \widehat{\Theta}(\Phi_k, \Phi_n) \leq \Lambda_2^{n-1} \widehat{\Theta}(\Phi_{k-n+1}, \Phi_1) \leq D_2 \Lambda_2^{n-1},$$

where $D_2 > 0$ is an upper bound for the $\widehat{\Theta}$-diameter of $\widehat{\mathcal{L}}_\varepsilon(\widehat{C}(b, c, \nu))$. It follows that

$$\exp(\widehat{\Theta}_+(\Phi_k, \Phi_n)) - 1 \leq C_2 \Lambda_2^n$$

for some $C_2 > 0$ depending only on D_2 and Λ_2. Replacing in (5.21) and passing to the limit as $k \to \infty$ we obtain the conclusion of part (1) of the lemma, with $C(\Phi) = C_2 \int \Phi \, d(m \times \theta_\varepsilon^{\mathbb{N}})$.

In particular, if Ψ is such that $\Psi \circ = \Psi$ at $(m \times \theta_\varepsilon^{\mathbb{N}})$-almost every point,

$$\int \Psi \, \Phi \, d(m \times \theta_\varepsilon^{\mathbb{N}}) = \int (\Psi \circ F^n) \, \Phi \, d(m \times \theta_\varepsilon^{\mathbb{N}}) = \int \Psi \, (\widehat{\mathcal{L}}_\varepsilon^n \, \Phi) \, d(m \times \theta_\varepsilon^{\mathbb{N}})$$

for every $n \geq 1$. the last term converges to $\int \Psi \, d\hat{\mu}_\varepsilon \int \Phi \, d(m \times \theta_\varepsilon^{\mathbb{N}})$ as n goes to infinity, cf. Remark 3.6.1. Therefore,

$$\int \left(\Psi - \int \Psi \, d\hat{\mu}_\varepsilon \right) \Phi \, d(m \times \theta_\varepsilon^{\mathbb{N}}) = 0, \tag{5.22}$$

for every $\Phi \in \widehat{C}(b, c, \nu)$. We are left to explain why this implies

$$\Psi - \int \Psi \, d\hat{\mu}_\varepsilon = 0 \quad (m \times \theta_\varepsilon^{\mathbb{N}})\text{-almost everywhere.} \tag{5.23}$$

First, note that every bounded function on $Q \times T^{\mathbb{N}}$ can be approximated in $L^1(m \times \theta_\varepsilon^{\mathbb{N}})$ by a ν-Hölder function. Next, every ν-Hölder function may be written as the difference of two strictly positive functions whose logarithm is (c,ν)-Hölder continuous. Next, every function $\Upsilon > 0$ such that $\log \Upsilon$ is (c,ν)-Hölder belongs in the cone $\widehat{C}(b,c,\nu)$. Indeed, conditions (AA) and (BB) are automatic, since Υ is positive. To check (CC), observe that given $\gamma \in \mathcal{F}^s_{loc}(\mathbf{t})$ and $\tilde{\gamma} \in \mathcal{F}^s_{loc}(\mathbf{s})$, and given $\rho \in \mathcal{D}(\gamma)$,

$$\begin{aligned}\Big|\log \int_\gamma \Upsilon\rho - \log \int_{\tilde{\gamma}} \Upsilon\tilde{\rho}\Big| &= \Big|\log \int_\gamma \Upsilon\rho - \log \int_{\tilde{\gamma}} \Upsilon(\rho\circ\pi)|\det D\pi|\Big| \\ &= \Big|\log \int_\gamma \Upsilon\rho - \log \int_\gamma (\Upsilon\circ\pi^{-1})\rho\Big| \\ &\le \sup\{|\log \Upsilon(z,\mathbf{t}) - \log \Upsilon(\pi^{-1}(z),\mathbf{s})| : z \in \gamma\}.\end{aligned}$$

Hölder continuity of $\log \Upsilon$ implies that the last term is bounded by

$$c\left(\max\left\{d(z,\pi^{-1}(z)), \|\mathbf{t}-\mathbf{s}\|\right\}\right)^\nu \le c\left(\max\left\{d_1(\gamma,\tilde{\gamma}), d_2(\gamma,\tilde{\gamma})\right\}\right)^\nu \le c d(\gamma,\tilde{\gamma})^\nu,$$

as we wanted to prove. In view of these remarks, (5.22) implies that

$$\int \left(\Psi - \int \Psi \, d\hat{\mu}_\varepsilon\right) \Phi \, d(m \times \theta_\varepsilon^{\mathbb{N}}) = 0$$

for every bounded function Φ. Taking $\Phi = \Psi - \int \Psi \, d\hat{\mu}_\varepsilon$ we obtain (5.23), thus completing the proof.

Proposition 5.4.2. *Given any continuous function $\varphi \colon Q \to \mathbb{R}$, we have*

$$\lim \frac{1}{n} \sum_{j=0}^{n-1} \varphi(x_j) = \int \varphi \, d\mu_\varepsilon \tag{5.24}$$

for $(m \times \theta_\varepsilon^{\mathbb{N}})$-almost every $(x,\mathbf{t}) \in Q \times T^{\mathbb{N}}$, where $x_j = f_{t_j} \circ \cdots \circ f_{t_1}(x)$.

Proof. Let $\Phi : Q \times T^{\mathbb{N}} \to \mathbb{R}$ be defined by $\Phi(x,\mathbf{t}) = \varphi(x)$. Observe that $\varphi(x_j)$ is precisely the same as $\Phi(F^j(x,\mathbf{t}))$. Then, by the ergodic theorem, the set $B = B(\varphi)$ of points in $Q \times T^{\mathbb{N}}$ such that the limit in (5.24) exists has $\hat{\mu}_\varepsilon(B) = 1$. It is easy to see that $(x,\mathbf{t})$ belongs in B if and only if $F(x,\mathbf{t})$ does. That is, the characteristic function χ_B satisfies $\chi_B = \chi_B \circ F$. Moreover, B is a union of entire local stable leaves of F, which means precisely that χ_B is constant on every stable leaf. Then the (positive) function $\chi_B + 1$ satisfies the assumptions of Lemma **??**, and so it is constant

$$\chi_B + 1 = \int (\chi_B + 1) \, d\hat{\mu}_\varepsilon = \hat{\mu}_\varepsilon(B) + 1$$

$(m \times \theta_\varepsilon^{\mathbb{N}})$-almost everywhere. Then either $\chi_B = 0$ almost everywhere or $\chi_B = 1$ almost everywhere, with respect to the measure $(m \times \theta_\varepsilon^{\mathbb{N}})$. The first alternative would lead to

$$\hat{\mu}_\varepsilon(B) = \int \chi_B \, d\hat{\mu}_\varepsilon = \lim \int \chi_B(\widehat{\mathcal{L}}_\varepsilon^n 1) \, d(m \times \theta_\varepsilon^{\mathbb{N}}) = 0,$$

(the last integral is zero for every n), contradicting the ergodic theorem. Therefore, we must have $(m \times \theta_\varepsilon^{\mathbb{N}})(B) = 1$. Now let $\beta : Q \times T^{\mathbb{N}} \to \mathbb{R}$ be the Birkhoff average of Φ. More precisely,

$$\beta(x, \mathbf{t}) = \lim_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \Phi(F^j(x, \mathbf{t})) = \lim_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(x_j), \qquad x_j = f_{t_j} \circ \cdots \circ f_{t_1},$$

if $(x, \mathbf{t}) \in B$, and $\beta(x, \mathbf{t}) = 0$ otherwise. Once again, it is easy to see that β is constant on stable leaves and satisfies $\beta \circ F = \beta$. Moreover, $\beta \geq \inf \varphi$ at every point in B. Then $\beta + |\inf \varphi| + 1$ is a strictly positive satisfying the assumptions of Lemma **??**, and so it is constant $(m \times \theta_\varepsilon^{\mathbb{N}})$-almost everywhere. Then β is also constant $(m \times \theta_\varepsilon^{\mathbb{N}})$-almost everywhere

$$\beta(x, \mathbf{t}) = \int \beta \, d\hat{\mu}_\varepsilon = \int \Phi \, d\hat{\mu}_\varepsilon = \int \varphi \, d\mu_\varepsilon$$

for $(m \times \theta_\varepsilon^{\mathbb{N}})$-almost every point $(x, \mathbf{t}) \in Q \times T^{\mathbb{N}}$ (the second equality is part of the ergodic theorem).

Proposition 5.4.3. *As $\varepsilon \to 0$, the stationary measure μ_ε converges to the SRB measure μ_0 of f in the weak* topology:*

$$\int \psi \, d\mu_\varepsilon \to \int \psi \, d\mu_0 \,, \quad \textit{for any continuous function } \psi : Q \to \mathbb{R}.$$

Proof. As every continuous function is uniformly approximated by μ-Hölder continuous functions, it suffices to consider the case when ψ is μ-Hölder continuous. Moreover, every μ-Hölder function ψ may be written $\psi = \psi^+ - \psi^-$ where $\psi^\pm > 0$ and $\log \psi^\pm$ are $(a/2, \mu)$-Hölder continuous. Therefore, we may assume right from the start that

$$\psi > 0 \quad \text{and} \quad \log \psi \text{ is } \left(\frac{a}{2}, \mu\right) - \text{Hölder.}$$

Then, as in (3.80), there is some uniform constant $C > 0$, so that

$$\begin{aligned}\left| \int_Q \psi(\mathcal{L}^n 1) dm - \int_Q \psi(\mathcal{L}^{n+j} 1) dm \right| &\leq \sup |\psi| \left(\exp(\Theta_+(\mathcal{L}^n 1, \mathcal{L}^{n+j} 1)) - 1 \right) \\ &\leq \sup |\psi| \, C \Lambda_2^n\end{aligned}$$

for every $j \geq 1$ ($\mathcal{L}$ is the transfer operator of the unperturbed map f). Passing to the limit as $j \to \infty$,

$$\left| \int_Q \psi(\mathcal{L}^n 1) dm - \int \psi \, d\mu_0 \right| \leq C \Lambda_2^n \sup \psi.$$

Similarly, taking $\Phi = 1$ and $\Psi = \psi \circ \pi_0$ in Lemma **??**,

$$\left|\int_Q \psi(\widehat{\mathcal{L}}_\varepsilon^n 1)\, dm - \int \psi\, d\mu_\varepsilon\right| = \left|\int_Q \Psi(\widehat{\mathcal{L}}_\varepsilon^n 1)\, d(m \times \theta_\varepsilon^{\mathbb{N}}) - \int \Psi\, d\hat{\mu}_\varepsilon\right| \le C\,\Lambda_2^n \sup \psi.$$

On the other hand,

$$\begin{aligned}\left|\int_Q \psi\,(\widehat{\mathcal{L}}_\varepsilon^n 1 - \mathcal{L}^n 1)\, dm\right| &= \left|\sum_{i=0}^{n-1} \int_Q \psi\, \widehat{\mathcal{L}}_\varepsilon^{n-i-1}(\widehat{\mathcal{L}}_\varepsilon - \mathcal{L})(\mathcal{L}^i 1)\, dm\right| \\ &\le \sum_{i=0}^{n-1} \left|\int_Q (\widehat{U}^{n-i-1}\psi)(\widehat{\mathcal{L}}_\varepsilon - \mathcal{L})(\mathcal{L}^i 1)\, dm\right| \\ &\le \sum_{i=0}^{n-1} \sup \psi \int_Q |(\widehat{\mathcal{L}}_\varepsilon - \mathcal{L})(\mathcal{L}^i 1)|\, dm.\end{aligned}$$

The first inequality uses (5.12), for functions that depend only on x. In the second one note that $\sup|\widehat{U}^{n-i}\psi| \le \sup \psi$. We claim that, for each fixed i,

$$\int_Q |(\widehat{\mathcal{L}}_\varepsilon - \mathcal{L})(\mathcal{L}^i 1)|\, dm \to 0 \quad \text{as } \varepsilon \to 0. \tag{5.25}$$

Let us assume this statement for a while, and explain why the proposition is now an easy consequence. Indeed, we find that

$$\left|\int_Q \psi(\widehat{\mathcal{L}}_\varepsilon^n 1 - \mathcal{L}^n 1)\, dm\right| \le \sum_{i=0}^{n-1} \sup \psi \cdot \xi_i(\varepsilon) \le \sup \psi \cdot \xi_n(\varepsilon)$$

where $\xi_j(\varepsilon)$ is a generic notation for a function of (j, ε) such that $\xi_j(\varepsilon) \to 0$ when $\varepsilon \to 0$, keeping j fixed, Together with the previous bounds this gives

$$\left|\int \psi\, d\mu_\varepsilon - \int \psi\, d\mu_0\right| \le (2C\Lambda_2^n + \xi_n(\varepsilon)) \sup \psi \text{ for every } n \ge 1.$$

Given any $\delta > 0$, fix n large enough so that $2C\Lambda_2^n \le \delta/2$. Then

$$\left|\int \psi\, d\mu_\varepsilon - \int \psi\, d\mu_0\right| \le \delta,$$

if $\varepsilon > 0$ is small enough so that $\xi_n(\varepsilon) \le \delta/2$. This completes the proof of the proposition, up to justifying claim (5.25).

Let $i \ge 0$ be fixed and denote $\varphi = \mathcal{L}^i 1$. We start by noting that

$$\begin{aligned}\int_Q |(\widehat{\mathcal{L}}_\varepsilon - \mathcal{L})\varphi|\, dm &= \int \left|\int_Q (\mathcal{L}_t - \mathcal{L})\varphi\, d\theta_\varepsilon(t)\right|\, dm \\ &\le \int \left(\int_Q |(\mathcal{L}_t - \mathcal{L})\varphi|\, dm\right)\, d\theta_\varepsilon(t).\end{aligned}$$

as a consequence of Fubini's theorem. Thus, in order to prove the claim, it suffices to show that

$$\int_Q |(\mathcal{L}_t - \mathcal{L})\varphi|\, dm \to 0 \quad \text{as } t \to \tau.$$

To do that, we disintegrate the integral with respect to the partition of Q into local stable manifolds $\gamma \in \mathcal{F}^s_{loc}$ of f:

$$\int_Q |(\mathcal{L}_t - \mathcal{L})\varphi|\, dm = \int \left(\int_\gamma |(\mathcal{L}_t - \mathcal{L})\varphi|\, H_\gamma \right) d\tilde{m}(\gamma).$$

Moreover, for each $\gamma \in \mathcal{F}^s_{loc}$, we write

$$\int_\gamma |(\mathcal{L}_t - \mathcal{L})\varphi|\, H_\gamma = \int_{\gamma \cap f_t(Q) \cap f(Q)} |\tilde{\phi}_t - \tilde{\phi}|\, H_\gamma + \\ + \int_{\gamma \cap f_t(Q) \setminus f(Q)} |\tilde{\phi}_t|\, H_\gamma + \int_{\gamma \cap f(Q) \setminus f_t(Q)} |\tilde{\phi}|\, H_\gamma$$

where $\tilde{\phi} = (\phi \circ f^{-1})|/\det Df \circ f^{-1}|$ and $\tilde{\phi}_t = (\phi \circ f_t^{-1})/|\det Df_t \circ f_t^{-1}|$. Observe that $\tilde{\phi}_t$ converges uniformly to ϕ as $t \to \tau$, because we suppose that $f_t \to f$ in the C^2 topology (C^1 would be sufficient here). This ensures that the first term on the right hand side goes to zero as $t \to \tau$. Moreover, the riemannian volume of $\gamma \cap f_t(Q) \setminus f(Q)$ and of $\gamma \cap f(Q) \setminus f_t(Q)$ in γ converges uniformly to zero as $t \to \tau$. Thus, the other two terms also converge to zero. It follows that

$$\int_\gamma |(\mathcal{L}_t - \mathcal{L})\varphi| H_\gamma \to 0 \quad \text{as } t \to \tau,$$

uniformly in $\gamma \in \mathcal{F}^s_{loc}$, and so $\int_Q |(\mathcal{L}_t - \mathcal{L})\varphi|\, dm$ converges to zero as $t \to \tau$. As we already pointed out that this implies our claim (5.12).

We conclude this section by deducing that the SRB measure μ_0 varies continuously with the difeeomorphim f. To explain this, let g be any map C^2 close to f. Then $g(Q) \subset Q$ and the maximal invariant set

$$\Lambda_g = \bigcap_{n \geq 0} g^n(Q)$$

is a hyperbolic attractor for g, see Appendix A. Everything we did here applies to g, if it is close enough to f, and so g has a unique SRB measure $\mu_{0,g}$ supported on Λ_g. Moreover,

Corollary 5.4.1. *The measure $\mu_{0,g}$ is close to $\mu_0 = \mu_{0,f}$ in the weak* topology, if g is C^2 close to f: given any continuous function $\psi : Q \to \mathbb{R}$,*

$$\int \psi\, d\mu_{0,g} \to \int \psi\, d\mu_0 \quad \text{as } g \to f.$$

Proof. This uses precisely the same argument as Corollary 5.2.2. Let $(g_n)_n$ be any sequence converging to f in $C^2(M)$. Define θ_ε to be the Dirac measure supported on g_n, for all $\varepsilon \in (1/n+1, 1/n]$. Then the stationary measure μ_ε, $\varepsilon \in (1/n+1, 1/n]$, coincides with the SRB measure μ_{0,g_n} of g_n and, as a particular case of Proposition 5.4.3, these stationary measures converge weakly to μ_0 as $n \to +\infty$.

6. Smooth Interval Maps

In this chapter we study the statistical properties of a large class of nonuniformly hyperbolic maps of the interval. For simplicity, we state the results for the quadratic family

$$f : \mathbb{R} \to \mathbb{R}, \qquad f(x) = a - x^2, \qquad a \in \mathbb{R},$$

but all the arguments can be extended to general smooth unimodal maps with negative schwarzian derivative and nondegenerate critical point. The dynamics of these quadratic maps depends in a crucial way on the value of the parameter a. We begin by listing some main facts, referring the reader to [35] for definitions and more information.

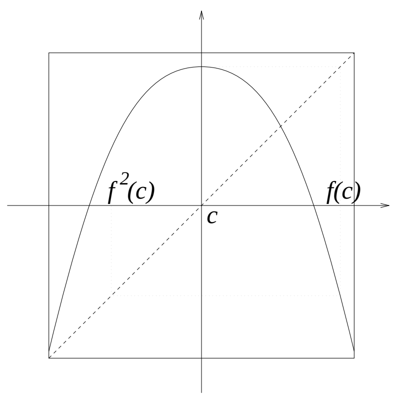

Fig. 6.1. A quadratic map $f(x) = a - x^2$

It is easy to see that if $a < -1/4$ then every trajectory $f^n(x)$ goes to $-\infty$ as $n \to +\infty$. The same happens, for all typical trajectories, when $a > 2$: the exceptions form a Cantor set K with zero Lebesgue measure. Moreover, K is a uniformly hyperbolic set for f: there are $\sigma > 1$ and $N \geq 1$ such that $|(f^N)'(x)| \geq \sigma$ for every $x \in K$. The asymptotic dynamics of f is much richer if a is in between $-1/4$ and 2, which we always suppose from now on. Denoting $q = (-1 - \sqrt{1+4a})/2$, the fixed point with largest absolute value, the interval $I = [q, -q]$ is invariant under f, that is, $f(I) \subset I$. All the trajectories starting outside I go to $-\infty$ as $n \to +\infty$, but the behaviour of $f|I$ may take very different forms.

For an open and dense subset H of values of $a \in (-1/4, 2)$ the map admits a unique attracting periodic orbit $\mathcal{O}$, which attracts every typical trajectory in the invariant interval: the basin $B(\mathcal{O}) = \{x \in I : f^n(x) \to \mathcal{O}\}$ is an open, full Lebesgue measure subset of I. Moreover, $B(\mathcal{O})$ contains the critical point $c = 0$, and $K = I \backslash B(\mathcal{O})$ is a uniformly hyperbolic set for f. From a statistical point of view the situation is still very simple: the Dirac probability measure uniformly distributed along the periodic attractor is the SRB-measure of $f|I$.

Another interesting case corresponds to the orbit of the critical point being non-recurrent, that is,

$$\inf_{n \geq 1} |f^n(c) - c| > 0. \tag{6.1}$$

Then the map is expanding over its critical orbit, in the sense that there are $b > 0$ and $\lambda > 1$ such that

$$|(f^n)'(f(c))| \geq b\lambda^n \quad \text{for every } n \geq 1. \tag{6.2}$$

Moreover, a similar property holds for Lebesgue almost every point $x \in I$, if one allows b to depend on x. In this case, Lebesgue almost every orbit starting in I has a limit distribution, which is described by an absolutely continuous f-invariant probability measure μ_0. This measure μ_0 is unique and ergodic, and it is supported on a finite union of subintervals of I. Then, in particular, the trajectory of every typical point $x \in I$ is dense in those subintervals. Observe, however, that condition (5.1) holds only for an (uncountable) zero Lebesgue measure set of values of a.

Here we study another, much larger, set of parameter values $a \in (-1/4, 2) \backslash H$ for which the map exhibits complex asymptotic behaviour. As we shall see below, it suffices to assume (5.2) together with a much weaker form of (5.1), to ensure the existence of an SRB-measure μ_0 which is absolutely continuous with respect to Lebesgue measure and which has very rich statistical properties. A crucial point is that these weaker conditions are satisfied by a set of parameter values with positive Lebesgue measure. In the sequel we give a precise definition of the systems we shall be dealing with, as well as the statement of the main result.

Before that, let us mention that the two types of behaviour we have been discussing are typical for quadratic maps: a recent result of [73] asserts that for Lebesgue almost every value of the parameter $a \in (-1/4, 2)$ either f has an attracting periodic orbit or it admits an absolutely continuous invariant measure.

We make the following assumptions on the orbit of the critical point $c = 0$: there are constants $\lambda_c > 1 \gg \alpha > 0$ (the precise condition is at the beginning of Section 6.1) such that

(U1) $|(f^n)'(f(c))| \geq \lambda_c^n$ for every $n \geq 1$;
(U2) $|f^n(c) - c| \geq e^{-\alpha n}$ for every $n \geq 1$;
(U3) f is topologically mixing on the interval $I_* = [f^2(c), f(c)]$.

A few words of motivation are in order on these hypotheses. The Collet-Eckmann condition (U1) means that f is expanding on its critical orbit, and is our main hyperbolicity assumption on the map. Property (U2) should be compared to (5.2): we allow the critical orbit to be recurrent, but we impose a bound on the speed of the recurrence. Altogether, these two conditions ensure a certain amount of expanding behaviour for the map f, as we shall see. Both of them can be further weakened, for instance multiplying the right hand side by a positive constant, but we keep this formulation for the sake of simplicity. The topological mixing condition (U3) plays essentially the same role as in previous cases, and we just add that in the present setting it is equivalent to asking that f be non-renormalizable. Recall that f is called *renormalizable* if there exists a subinterval $J \subset I$ and an integer $k \geq 2$ such that

$$c \in \text{interior}\,(J), \quad c \notin f^i(J) \text{ for } 0 < i < k, \quad f^k(J) \subset J.$$

As already mentioned, (U1), (U2), (U3) hold, simultaneously, for a large (positive Lebesgue measure) set S of values of the parameter a. Observe that S is disjoint from H, since no $a \in H$ satisfies (U1). Indeed, if f has an attracting periodic orbit $\mathcal{O}$ then $c \in B(\mathcal{O})$ implies that $(f^n)'(f(c))$ converges to zero (exponentially fast).

Our goal is to prove

Theorem 6.0.3. *Under assumptions (U1), (U2), (U3),*

1. *f admits a unique absolutely continuous invariant probability measure μ_0; moreover, μ_0 is ergodic and so it is an SRB-measure for f;*
2. *(f, μ_0) has exponential decay of correlations and satisfies the central limit theorem in the space of functions with bounded variation;*

Moreover, (f, μ_0) is stochastically stable, in a strong sense, under certain random perturbations. The proof of this result uses a "perturbed" version of the arguments we shall develop in Sections 6.1 through 6.4 to prove Theorem6.0.3, and we do not present it here. We just give the precise content of this stability statement, and refer the reader to [10] for the proof and further information.

The class of random perturbations one considers in this setting is necessarily more restricted than in the situations we treated before. For instance, statements of determistic stability analog to Corollary 5.2.2 or Corollary5.4.1, are known to be false for general quadratic maps [56]. We consider perturbations within the quadratic family,

$$\mathbb{R} \ni t \mapsto f_t(x) = f(x) + t = (a + t) - x^2,$$

and we also impose certain conditions on the probability distributions $(\theta_\varepsilon)_{\theta>0}$. A main one is that θ_ε be absolutely continuous with respect to Lebesgue measure m, and supported on some subsinterval J_ε of $[-\varepsilon, \varepsilon]$. The two other conditions, of a somewhat more technical kind, are

$$\sup_{\varepsilon>0} \left(\varepsilon \sup \frac{d\theta_\varepsilon}{dm}\right) < \infty \qquad \text{and} \qquad \log \frac{d\theta_\varepsilon}{dm} \text{ concave on } J_\varepsilon.$$

Then we conclude that the random scheme admits a unique stationary measure μ_ε, that describes the asymptotic time averages of almost every random trajectory. Moreover, as the noise level ε goes to 0 the density $d\mu_\varepsilon/dm$ of the stationary measure converges in $L^1(m)$ to the density $d\mu_0/dm$ of the SRB measure μ_0.

It is worth pointing out that the dynamics of these systems is very fragile under deterministic perturbations: the fact that H is dense in the parameter interval $(-1/4, 2)$ implies that the maps we are dealing with may be approximated by other quadratic maps having a periodic attractor and, thus, simple statistical features. This makes the stochastic stability statement all the more striking in the present setting.

In the sequel we sketch our approach to proving the ergodic properties in the statement of Theorem 5.1. It is, once more, based on studying the spectrum of convenient transfer operators. However, this time the "natural" operator

$$\mathcal{L}\varphi(y) = \sum_{f(x)=y} \frac{\varphi(x)}{|f'(x)|}$$

is not a right object to look at. To begin with, this is not well defined at $y = f(c)$. Moreover, the expansion properties of the dynamical system played a crucial role in all the situations we studied so far. This may lead one to suspect that for the present class of maps, which combine some amount of expansion with strong contraction (near c), the operator defined above may have poor spectral properties, and this is indeed so.

Instead, the basic strategy is to try and reduce this setting of nonuniformly hyperbolic dynamics to that of piecewise uniformly expanding maps treated in Sections 1.3 and 3.4. More precisely, in Section 6.1 we describe a procedure associating to each quadratic map $f : I \to I$ satisfying (U1) and (U2), an expanding map $\hat{f} : \hat{I} \to \hat{I}$ defined on a countable union $\hat{I} = \cup_{k\geq 0}\big(B_k \times \{k\}\big)$ of disjoint intervals. To make $\hat{f}$ expanding we have to consider an adapted riemannian metric on $\hat{I}$, of the form

$$\|\cdot\|_{(x,k)} = w_0(x,k)|\cdot|,$$

where $|\cdot|$ is the usual length (along each interval B_k) and w_0 is a convenient nonnegative function. Correspondingly, instead of the usual transfer operator, we consider its conjugate under multiplication by the *cocycle* w_0,

$$\mathcal{L}_0\varphi(\zeta) = \sum_{\hat{f}(\xi)=\zeta} \frac{1}{|\hat{f}'(\xi)|} \frac{w_0(\xi)}{w_0(\zeta)} \varphi(\xi), \tag{6.3}$$

where $\hat{f}'(\xi) = f'(x)$ for each $\xi = (x, k) \in \hat{I}$. This *tower extension* $\hat{f}$ is constructed in such a way that $\pi \circ \hat{f} = f \circ \pi$, where $\pi : \hat{I} \to I$ is the canonical projection given by $\pi(x, k) = x$. In other words, given any $(x, k) \in \hat{I}$ there is $l \geq 0$ so that $\hat{f}(x, k) = (f(x), l)$. Then

$$|\hat{f}'(\xi)| \frac{w_0(\hat{f}(\xi))}{w_0(\xi)}$$

is just the jacobian of $\hat{f}$ at ξ, with respect to the new metric. We introduce the measure $m_0 = w_0 m$ and then, by change of variables,

$$\int (\mathcal{L}_0\varphi)\psi \, dm_0 = \int \varphi(\psi \circ \hat{f}) \, dm_0, \tag{6.4}$$

whenever the integrals make sense. We take $\mathcal{L}_0$ to act on the space $\mathrm{BV}(\hat{I})$ of functions with bounded variation on $\hat{I}$, whose precise definition will be shortly given. We prove that $\mathcal{L}_0$ is a quasi-compact operator, from which we deduce ergodic properties of $\hat{f}$. Their analogs for f follow, easily, using $\pi \circ \hat{f} = f \circ \pi$.

6.1 Transfer Operators on Towers

Here we give the precise definitions of the objects introduced in the paragraphs above. Throughout, we suppose that the constant α in (U2) has been taken small enough so that $e^{2\alpha} < \sqrt{\lambda_c}$. Then, starting the construction of the tower extension $\hat{f} : \hat{I} \to \hat{I}$, we fix $\beta \in (3\alpha/2, 2\alpha)$. We also fix $\rho > \lambda > 1$ such that $\rho > e^{\alpha}$ and $\lambda\rho e^{\alpha} < \sqrt{\lambda_c}$. We use two more constants $1 < \sigma \leq \sigma_0$ and $0 < \delta \leq \delta_0$, where $\sigma_0 \in (1, \lambda)$ and $0 < \delta_0 \ll \alpha$ are given by Lemma 6.2.1 below. We denote by C various large positive constants depending only on α and λ_c, and use $C(\cdots)$ for large positive constants depending also on other parameters involved in our constructions. For simplicity, we write $c_j = f^j(c)$ for each $j \geq 0$.

We define $B_k = [c_k - e^{-\beta k}, c_k + e^{\beta k}]$, for each $k \geq 1$, and $B_0 = I = [q, -q]$. Then we let $\hat{I} = \cup_{k \geq 0} E_k$, where $E_k = B_k \times \{k\}$. Observe that the critical point c is not contained in B_k, for any $k \geq 1$, since (U2) implies $|c_k| \geq e^{-\alpha k} > e^{-\beta k}$. This simple fact will be useful in a number of occasions. As we already said, we want $\hat{f}$ to be such that $\hat{f}(x, k) = (f(x), l)$ for some $l \geq 0$ depending on (x, k). The definition of $\hat{f}$ is given by the following pair of rules:

1. whenever possible, $\hat{f}$ maps (x, k) one level higher in the tower, i.e., $l = k + 1$;

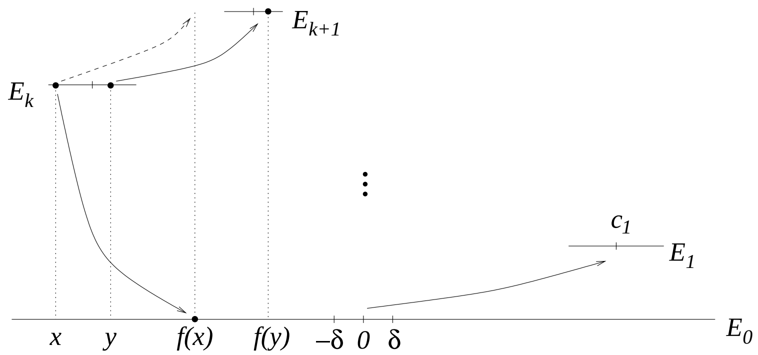

Fig. 6.2. The tower extension $\hat{f} : \hat{I} :\to \hat{I}$

2. if this is not allowed, that is, if $f(x) \notin B_{k+1}$, then $\hat{f}$ sends (x,k) directly to the ground level: $l = 0$.

Rule (1) admits an exception when $k = 0$: a point $(x,0)$ goes up to level 1 only if x is close to zero, otherwise it remains in level 0. The precise expression is

$$\hat{f}(x,k) = \begin{cases} (f(x),0) & \text{if either } f(x) \notin B_{k+1} \text{ or else } k = 0 \text{ and } |x| \geq \delta; \\ (f(x),k+1) & \text{otherwise.} \end{cases}$$

Typically, a point $(x,0)$ moves around in the zeroth level E_0 for a while, until it hits $(-\delta,\delta) \times \{0\}$ at some time $m \geq 0$. Then it starts climbing the tower

$$\hat{f}^{m+j}(x,0) = (f^{m+j}(x),j) \quad \text{for } 0 \leq j \leq h.$$

Unless $f^m(x)$ coincides with the critical point c, the integer h is finite and at the next iterate the orbit falls back to the ground level: $\hat{f}^{m+h+1}(x,0) = (f^{m+h+1}(x),0)$. Observe also that we must have $h \geq H(\delta)$, for some integer $H(\delta) \geq 1$ which can be made arbitrarily large by choosing δ small enough.

Next, we define our cocycle w_0. First, we set $w_0(x,0) = 1$ for every $x \in B_0$. Given any point $(x,k) \in E_k$, $k \geq 1$, there are two possibilities. If there is $z \in (0,\delta)$ such that $\hat{f}^k(z,0) = (x,k)$ then we define

$$w_0(x,k) = \lambda^k |(f^k)'(z)|^{-1}$$

(it is easy to see that z is unique, when it exists). If there is no such z then we set simply $w_0(x,k) = 0$. For each $k \geq 0$ we shall denote $W_k = \{x \in B_k : w_0(x,k) > 0\}$. Note that every W_k is an interval, whose closure contains c_k. We also write

$$W_* = \bigcup_{k \geq 0} (W_k \times \{k\}).$$

As mentioned before, we associate to w_0 the riemannian metric $\|\cdot\|_{(x,k)} = w_0(x,k)|\cdot|$ and the Borel measure $m_0 = w_0\, m$.

Remark 6.1.1. By definition w_0 and m_0 are supported on W_*, reflecting the fact that points in $\hat{I} \setminus W_*$ are transient for $\hat{f}$, and so play no role as far as asymptotic behaviour is concerned. Let us note that certain points in the ground level E_0 are also transient: $\hat{f}(W_*)$ does not intersect $(f(\delta), f(c)] \times \{0\}$, and $\hat{f}^2(W_*) \subset \hat{f}(W_*)$ does not intersect $[f^2(c), f^2(\delta)) \times \{0\}$. To see this, suppose there exists $(x,k) \in W_*$ such that

$$\text{either} \quad \text{(i)} \quad \hat{f}(x,k) \in (f(\delta), f(c)] \times \{0\} \quad \text{or} \quad \text{(ii)} \quad \hat{f}^2(x,k) \in (f(\delta), f(c)] \times \{0\}.$$

Then either $f(x) \in (f(\delta), f(c)]$ or $f^2(x) \in [f^2(c), f^2(\delta))$, respectively. In both cases we must have $x \in (-\delta, \delta)$: this is immediate in (i), and easy to obtain in (ii), if one assumes that $\delta > 0$ is small enough so that $f^2(c) < f^2(\delta) < f^3(c)$. Now, in order that the ith iterate of (x,k) be in E_0, for $i = 1$ or $i = 2$, we must have $k + i > H(\delta)$. Assume that $0 < \delta < 1/64$ is small enough so that this implies $\text{length}(B_k) \leq 2e^{-\beta k} \leq 1/64$. Then, since $x \in B_k \cap (-\delta, \delta)$, the interval B_k must be contained in $(-1/32, 1/32)$. It follows that $|f'(y)| \leq 1/16$ for every $y \in B_k$, and so

$$\begin{aligned} |x - c_k| \leq e^{-\beta k} \;\Rightarrow\; & |f(x) - c_{k+1}| \leq \frac{1}{16} e^{-\beta k} \leq \frac{1}{8} e^{-\beta(k+1)} \\ \Rightarrow\; & |f^2(x) - c_{k+2}| \leq \frac{1}{2} e^{-\beta(k+1)} \leq e^{-\beta(k+2)} \end{aligned}$$

(because $|f'| \leq 4$ and $e^{\beta} < e^{2\alpha} < \sqrt{\lambda_c} \leq 2$). This means that $\hat{f}^i(x,k) \in E_{k+i}$ for $i = 1, 2$, contradicting the choice of (x,k).

For any point (y,l) such that $\hat{f}(y,l) \in W_*$, we denote

$$g(y,l) = \frac{1}{|f'(y)|} \frac{w_0(y,l)}{w_0(\hat{f}(y,l))}. \tag{6.5}$$

Clearly, $g(y,l) > 0$ if and only if (y,l) is in W_*. Moreover, as we already pointed out, in that case $1/g(y,l)$ is the jacobian of $\hat{f}$ at (y,l), with respect to the metric $\|\cdot\|$ (or, equivalently, with respect to the measure m_0).

Given a measurable function $\varphi : \hat{I} \to \mathbb{R}$ we define

$$\operatorname{var} \varphi = \sum_{k \geq 0} \operatorname{var}(\varphi | E_k) \qquad \sup \varphi = \sup_{k \geq 0} \sup(\varphi | E_k) \qquad \int \varphi \, dm_0 = \sum_{k \geq 0} \int_{E_k} \varphi w_0 \, dm$$

where m denotes Lebesgue measure on $B_k \approx E_k$. Then we define the BV-norm of φ

$$\|\varphi\|_{\mathrm{BV}} = \operatorname{var} \varphi + \sup |\varphi| + \int |\varphi| \, dm_0,$$

and take $\mathrm{BV}(\hat{I})$ to be the (Banach) space of functions $\varphi : \hat{I} \to \mathbb{R}$ such that $\|\varphi\|_{\mathrm{BV}} < \infty$.

Finally, we describe the transfer operator $\mathcal{L}_0$ associated to $\hat{f}$. Given $\varphi \in \mathrm{BV}(\hat{I})$ and $(x,k) \in W_*$, we let

$$\mathcal{L}_0\varphi(x,k) = \sum_{\hat{f}(y,l)=(x,k)} \frac{\varphi(y,l)}{|f'(y)|}\frac{w_0(y,l)}{w_0(x,k)} = \sum_{\hat{f}(y,l)=(x,k)} (g\varphi)(y,l) \tag{6.6}$$

Observe that the sum involves exactly one term if $x \in W_k$, with $k \geq 2$, and exactly two terms if $x \in W_1$. For $k = 0$ there may be infinitely many terms: at most one for each value of $l \geq 1$, and at most two for $l = 0$. Then we extend $\mathcal{L}_0\varphi$ to $\hat{I} \setminus W_*$ by asking that it be constant on each connected component of $B_k \setminus W_k$, $k \geq 1$. More precisely, we let $a_k < b_k$ be the endpoints of the interval W_k, then we define

$$\mathcal{L}_0\varphi(x,k) = \begin{cases} \limsup_{y\to a_k^+} \mathcal{L}_0\varphi(y,k) & \text{if } x \leq a_k \\ \limsup_{y\to b_k^-} \mathcal{L}_0\varphi(y,k) & \text{if } x \geq b_k. \end{cases} \tag{6.7}$$

This seemingly unnatural definition is designed so that $\mathrm{var}(\mathcal{L}_0\varphi)$ and $\sup|\mathcal{L}_0\varphi|$ are left unchanged: the variation of $\mathcal{L}_0\varphi$ over B_k coincides with the variation of $\mathcal{L}_0\varphi$ over W_k, and a similar fact is true for the supremum. Of course, the same holds for $\int \mathcal{L}_0\varphi\, dm_0$, because m_0 is supported on W_*. In particular, the duality relation (5.4) is not affected by this convention.

Clearly, $\mathcal{L}_0$ is a nonnegative operator, in the sense that it maps nonnegative functions to nonnegative functions. Then, (5.4) also implies that $\mathcal{L}_0$ is nonincreasing with respect to the L^1-norm:

$$\int |\mathcal{L}_0\psi|\, dm_0 \leq \int \mathcal{L}_0|\psi|\, dm_0 = \int |\psi|\, dm_0 \quad \text{for every } \psi. \tag{6.8}$$

6.2 Uniform Expansion for the Tower Map

We shall see in the next sections that $\mathcal{L}_0$ is a bounded, and even quasi-compact, operator from $\mathrm{BV}(\hat{I})$ to itself. More precisely, 1 is a simple eigenvalue and the rest of the spectrum is contained in a disk of radius strictly smaller than 1. The proof of this relies on properties of uniform expansion of the map $\hat{f} : \hat{I} \to \hat{I}$ that we obtain in Lemma 5.4 below. First, we state and prove two key lemmas on the expanding behaviour of certain iterates of the map f.

Lemma 6.2.1. *There are $\sigma_0 > 1$, $b > 0$, and $\delta_0 > 0$ such that for any $0 < \delta \leq \delta_0$ there is $c(\delta) > 0$ such that, given any $x \in I$ and $n \geq 1$,*

1. *if $x, f(x), \ldots, f^{n-1}(x) \notin (-\delta,\delta)$ then $|(f^n)'(x)| \geq c(\delta)\sigma_0^n$;*
2. *(2) if, in addition, $f^n(x) \in (-\delta,\delta)$ then $|(f^n)'(x)| \geq b\sigma_0^n$.*

Proof. The arguments in the proof are now standard in one-dimensional dynamics. First, given $\delta_1 > 0$ there are $m \geq 1$ and $\sigma_1 > 1$ such that

$$|(f^m)'(y)| \geq \sigma_1^m \quad \text{whenever} \quad y, f(y), \ldots, f^{m-1}(y) \notin (-\delta_1, \delta_1). \tag{6.9}$$

Indeed, (U1) and [118] imply that all the periodic points of f are repelling, and then (5.9) is a consequence of [35, Section III.3]. In the sequel we fix $\delta_1 > 0$ small, depending only on α and λ_c, cf. (5.13)–(5.14). By (U1) and [85], there are $\sigma_2 > 1$, $\delta_2 > 0$, $K_2 \geq 1$ such that, given any $1 \leq l < m$,

$$|(f^l)'(y)| \geq \frac{1}{K_2}\sigma_2^l \quad \text{whenever} \quad f^l(y) \in (-\delta_2, \delta_2). \tag{6.10}$$

We take $\sigma_0 = \min\{\sigma_1, \sigma_2, \lambda\}$ and $\delta_0 = \min\{\delta_1, \delta_2\}$ and, for each $0 < \delta \leq \delta_0$, we define $c(\delta) = \inf\{(|f'(x)|/\sigma_0)^m : x \in I \setminus (-\delta, \delta)\}$. The constant $b > 0$ will be defined below, with $b \leq (1/K_2)$. Clearly, for all $0 \leq l < m$,

$$|(f^l)'(y)| \geq c(\delta)\sigma_0^l \quad \text{if} \quad y, f(y), \ldots, f^{l-1}(y) \notin (-\delta, \delta). \tag{6.11}$$

Given n and x as in the statement, we denote $x_j = f^j(x)$, for $0 \leq j \leq n$. If $x_j \notin (-\delta_1, \delta_1)$ for all $0 \leq j < n$ then the lemma follows immediately from (5.9), (5.10), (5.11). Indeed, writing $n = qm + l$, with $0 \leq l < m$, we get

$$|(f^{qm+l})'(x)| = |(f^l)'(f^{qm}(x))|\,|(f^{qm})'(x)| \geq c(\delta)\sigma_0^l\,(\sigma_1^m)^q,$$

which gives (1). Moreover, if $f^n(x) \in (-\delta, \delta)$ then we may replace $c(\delta)\sigma_0^l$ by $K_2^{-1}\sigma_2^l$ in this estimate, thus proving (2).

From now on we suppose that the trajectory of x up to time n does intersect $(-\delta_1, \delta_1)$, and we define $0 \leq \nu_1 < \cdots < \nu_s < n$ as follows. Let ν_1 be the smallest $j \geq 0$ with $x_j \in (-\delta_1, \delta_1)$. For each ν_i, $i \geq 1$, define

$$p_i = \max\{k \geq 1\colon |x_{\nu_i+j} - c_j| < e^{-\beta j} \text{ for every } 1 \leq j \leq k\}.$$

Then let ν_{i+1} be the smallest $n > r > \nu_i + p_i$ for which $x_r \in (-\delta_1, \delta_1)$. For the time being we fix $1 \leq i \leq s$, and write $p = p_i$ and $\nu = \nu_i$. The previous definition and (U2) yield $|x_{\nu+j} - c_j| \leq e^{-(\beta-\alpha)j}|c_j|$ and so

$$(1 - e^{-j(\beta-\alpha)})|f'(c_j)| \leq |f'(x_{\nu+j})| \leq (1 + e^{-j(\beta-\alpha)})|f'(c_j)|,$$

for all $1 \leq j \leq p$. Then

$$\frac{1}{C}|(f^p)'(c_1)| \leq |(f^p)'(x_{\nu+1})| \leq C|(f^p)'(c_1)|\,, \tag{6.12}$$

with $C^{-1} = \prod_{j\geq 1}(1 - e^{-j\alpha/2})$, recall that we take $\beta - \alpha > \alpha/2$. Moreover,

$$e^{-\beta(p+1)} \leq |x_{\nu+p+1} - c_{p+1}| \leq |x_{\nu+p} - c_p|(1 + e^{-p(\beta-\alpha)})|f'(c_p)|\,,$$

and so, by recurrence,

$$e^{-\beta(p+1)} \le |x_{\nu+1} - c_1| \prod_{j=1}^{p}(1+e^{-j(\beta-\alpha)})|(f^p)'(c_1)| \le C|(f^p)'(c_1)||x_\nu|^2 .$$

Combining this with (5.12) and (U1), we conclude that

$$|(f^{p+1})'(x_\nu)|^2 \ge \frac{1}{C}|(f^p)'(c_1)|^2|x_\nu|^2 \ge \frac{1}{C}(\lambda_c e^{-\beta})^{p+1}. \qquad (6.13)$$

Up to taking δ_1 small enough with respect to α and λ_c, we may suppose the p_i (uniformly) sufficiently large so that (5.13) implies

$$|(f^{p_i+1})'(x_{\nu_i})| \ge \frac{1}{C}(\lambda_c e^{-\beta})^{(p_i+1)/2} \ge \frac{1}{C}(\lambda\rho)^{p_i+1} \ge K_2\lambda^{p_i+1}, \qquad (6.14)$$

for each $1 \le i \le s$, recall that $e^{-\beta}\lambda_c > e^{-2\alpha}\lambda_c > \lambda^2\rho^2$. At this point we write $|(f^n)'(x)| = \prod_{j=0}^{n-1}|f'(x_j)|$ and partition the time range $[0,n)$ into subintervals $J \subset [0,n)$ as follows. Let $|J|$ denote the number of elements of each given J. First, we suppose $\nu_s + p_s < n$. For $J = [0,\nu_1)$ and for each $J = (\nu_i + p_i, \nu_{i+1})$, $1 \le i < s$, we have

$$\prod_{j\in J}|f'(x_j)| \ge K_2^{-1}\sigma_0^{|J|},$$

as a consequence of (5.9) and (5.10). The same holds for $J = (\nu_s + p_s, n)$ if $|x_n| < \delta$. In general, $J = (\nu_s + p_s, n)$ has

$$\prod_{j\in J}|f'(x_j)| \ge c(\delta)\sigma_0^{|J|},$$

by (5.9) and (5.11). Moreover,

$$\prod_{j\in J}|f'(x_j)| \ge K_2\sigma_0^{|J|}$$

for each $J = [\nu_i, \nu_i + p_i]$, $1 \le i \le s$, recall (5.14). Altogether, this proves both parts (1) and (2) of the lemma when $\nu_s + p_s < n$, since we take $b \le (1/K_2)$. Now we treat the case $\nu_s + p_s \ge n$. We only have to consider $J = [\nu_s, n)$, as the previous estimates remain valid for all other subintervals involved. In general, (5.9) and (5.11) give

$$\prod_{j\in J}|f'(x_j)| \ge c(\delta)\sigma_0^{|J|}.$$

Part (1) follows, in the same way as before. In order to prove (2), we let $q = n - \nu_s - 1$. Then $0 \le q < p_s$ and so, recall also (U2),

$$|x_{\nu_s}| \ge \delta > |x_n| \ge |c_{q+1}| - |x_n - c_{q+1}| \ge e^{-\alpha(q+1)} - e^{-\beta(q+1)} \ge \frac{1}{C}e^{-\alpha(q+1)}.$$

Moreover, (5.12) holds for $\nu = \nu_s$ and $p = q$. Hence,

$$|(f^{q+1})'(x_{\nu_s})| \geq \frac{1}{C}|(f^q)'(c_1)|\,|x_{\nu_s}| \geq \frac{1}{C}(\lambda_c e^{-\alpha})^{q+1} \geq \frac{1}{C}\lambda^{q+1}.$$

We take $b = (CK_2)^{-1}$, for $C > 0$ as in the last term. □

As we already said at the beginning of this section, we take the constant δ in the definition of our tower satisfying $0 < \delta \leq \delta_0$, and we also fix $\sigma \in (1, \sigma_0]$.

Lemma 6.2.2. *There is $C > 0$ such that, given any $z \in (-\delta, \delta)$ and $k \geq 1$,*

1. *if $|f^j(z) - c_j| \leq e^{-\beta j}$ for every $1 \leq j \leq k$, then*

$$\frac{1}{C} \leq \frac{|(f^k)'(f(z))|}{|(f^k)'(c_1)|} \leq C;$$

2. *if, in addition, $|f^{k+1}(z) - c_{k+1}| \geq e^{-\beta(k+1)}$, then*

$$|(f^k)'(f(z))| \geq \frac{1}{C}\lambda_c^k \qquad \textit{and} \qquad |(f^{k+1})'(z)| \geq \frac{1}{C}\rho^k\lambda^k.$$

Proof. This follows from the same arguments as (5.12)–(5.14). Indeed, the assumption together with (U2) imply

$$(1 - e^{j(\alpha-\beta)})|f'(c_j)| \leq |f'(f^j(z))| \leq (1 + e^{j(\alpha-\beta)})|f'(c_j)|\,,$$

for every $1 \leq j \leq k$. Then, multiplying over $1 \leq j \leq k$ and taking C as in (5.12), we get the conclusion of (1):

$$\frac{1}{C}|(f^k)'(c_1)| \leq |(f^k)'(f(z))| \leq C|(f^k)'(c_1)|\,.$$

The first claim in (2) is a direct consequence of (1) and (U1), and the second one can be derived as follows. Given z and k as in the statement,

$$e^{-\beta(k+1)} \leq |f^{k+1}(z) - c_{k+1}| = |(f^k)'(y)|\,|f(z) - c_1|$$

for some y in the interval bounded by c_1 and $f(z)$. Then $f^j(y)$ is in between $f^{j+1}(z)$ and c_{j+1} for each $0 \leq j < k$ and so (1) remains valid with y in the place of $f(z)$. In particular, $|(f^k)'(y)| \leq C|(f^k)'(c_1)|$ and so

$$e^{-\beta(k+1)} \leq C|(f^k)'(c_1)|\,|f(z) - c_1| \leq C|(f^k)'(c_1)|\,|z|^2\,. \tag{6.15}$$

This implies

$$|f'(z)|^2 \geq \frac{1}{C}|z|^2 \geq \frac{1}{C}e^{-\beta k}|(f^k)'(c_1)|^{-1}$$

and so, using the conclusion of (1) once again,

$$|(f^{k+1})'(z)| \geq \frac{1}{C}|(f^k)'(c_1)|\,|f'(z)| \geq \frac{1}{C}|(f^k)'(c_1)|^{1/2}e^{-\beta k/2} \geq \frac{1}{C}\lambda_c^{k/2}e^{-\beta k/2} \geq \frac{1}{C}\rho^k\lambda^k$$

(because $\beta < 2\alpha$ and $\lambda\rho e^{\alpha} < \sqrt{\lambda_c}$). □

We denote by $\mathcal{P}^{(n)}$ the partition of $\hat{I}$ into monotonicity intervals of $\hat{f}^n$, $n \geq 1$, characterized in the following way. For every $k \geq 1$, let

$$U_k = \{(x,k) \in E_k : \hat{f}(x,k) = (f(x), k+1)\}$$

and D_k^-, D_k^+ be the connected components of $E_k \setminus U_k$. That is, points in U_k are sent by $\hat{f}$ to an upper level of the tower, whereas points in $D_k^+ \cup D_k^-$ are mapped down to the ground level E_0. Note that U_k is a neighbourhood of c_k, but $D_k^{\pm}$ may be empty. For $k = 0$, we set

$$U_0^- = (-\delta, 0] \times \{0\}, \quad U_0^+ = (0,\delta) \times \{0\}, \quad D_0^- = [q, -\delta] \times \{0\}, \quad D_0^+ = [\delta, -q] \times \{0\}.$$

Then, we take

$$\mathcal{P}^{(1)} = \{U_k, D_k^-, D_k^+ : k \geq 1\} \cup \{U_0^-, U_0^+, D_0^-, D_0^+\}.$$

Then, for any $n > 1$, we take $\mathcal{P}^{(n)}$ to be the nth iterate of $\mathcal{P}^{(1)}$. That is, by definition, $\mathcal{P}^{(n)}(\xi_1) = \mathcal{P}^{(n)}(\xi_2)$ if and only if $\mathcal{P}^{(1)}(\hat{f}^i(\xi_1)) = \mathcal{P}^{(1)}(\hat{f}^i(\xi_2))$ for each $0 \leq i < n$.

In what follows we always assume that every $\eta \in \mathcal{P}^{(n)}$ has strictly positive length, moreover, the intersection of η with W_* is either empty or an interval with positive length. Note that in order to have this it suffices that the orbits of

$$(c, 0), \qquad (\pm\delta, 0), \quad \text{and} \quad (c_k \pm e^{-\beta k}) \quad k \geq 1,$$

be two-by-two disjoint injective sequences on $\hat{I}$, which can always be obtained by slightly modifying β and δ if necessary (so as to avoid a countable set of relations involving these two constants).

We shall also need the iterated versions $g^{(n)}$ of g, defined by

$$g^{(n)}(\xi) = g(\xi)\, g(\hat{f}(\xi)) \cdots g(\hat{f}^{n-1}(\xi)) = \frac{1}{|(f^n)'(x)|} \frac{w_0(\xi)}{w_0(\hat{f}^n(\xi))}$$

for every $\xi = (x,k)$ such that $\hat{f}^i(\xi) \in W_*$ for $1 \leq i \leq n$. For use in Lemma 5.5, let us observe that, cf. (5.5),

$$\mathcal{L}_0^n \varphi(\xi) = \sum_{\eta \in \mathcal{P}^{(n)}} \Big((g^{(n)}\varphi) \circ (\hat{f}^n|\eta)^{-1} \chi_{\hat{f}^n(\eta)} \Big)(\xi) \tag{6.16}$$

for every $n \geq 1$ and ξ with $\hat{f}^i(\xi) \in W_*$ for $1 \leq i \leq n$.

Remark 6.2.1. It follows from our definitions that if (x,k) belongs in $U_k \cap W_k$, $k \geq 1$, and $z \in (0,\delta)$ is such that $\hat{f}^k(z,0) = (x,k)$, then

$$g(x,k) = \frac{1}{|f'(x)|} \frac{w_0(x,k)}{w_0(\hat{f}(x,k))} = \frac{1}{|f'(x)|} \frac{\lambda^k |(f^k)'(z)|^{-1}}{\lambda^{k+1}|(f^{k+1})'(z)|^{-1}} = \frac{1}{\lambda}.$$

The same remains true for $(x, 0) \in (-\delta, \delta) = U_0^+ \cup U_0^-$, On the other hand, if (x, k) is in $D_k \cap W_k$, $k \geq 1$, and z is as before,

$$g(x,k) = \frac{1}{|f'(x)|} \frac{w_0(x,k)}{w_0(\hat{f}(x,k))} = \frac{1}{|f'(x)|} \frac{\lambda^k |(f^k)'(z)|^{-1}}{1} = \frac{\lambda^k}{|(f^{k+1})'(z)|} \leq C\rho^{-k}$$

as a consequence of Lemma 6.2.2. Observe that $k \geq H(\delta)$, where $H(\delta)$ is the minimum height from which orbits starting in $(-\delta, \delta) \times \{0\}$ can fall down back to E_0, cf. Section 6.1. We suppose that $\delta > 0$ is small (and so $H(\delta)$ is large enough) so that this implies $C\rho^{-k} < 1/\lambda$. Moreover, Lemma 6.2.1 gives $g^{(n)}(x, 0) \leq 1/(c(\delta)\sigma_0^n)$ for every point $(x, 0)$ whose trajectory remains in E_0 up to time $n \geq 1$. These remarks express the uniformly expanding character of $\hat{f}$.

Lemma 6.2.3. *1. Let $\gamma \subset \eta \in \mathcal{P}^{(n)}$ be such that $\hat{f}^j(\gamma) \subset E_0$ for every $0 \leq j \leq n$. Then*

$$\sup_\gamma g^{(n)} \leq \begin{cases} C\sigma^{-n} & \textit{if } \hat{f}^n(\gamma) \subset (-\delta, \delta) \times \{0\} \\ C(\delta)\sigma^{-n} & \textit{in general.} \end{cases}$$

Moreover, $\operatorname{var}_\gamma g^{(n)} \leq 2 \sup_\gamma g^{(n)}$.

2. Let $\gamma \subset \eta \cap W_*$ for some $\eta \in \mathcal{P}^{(n)}$ and let $0 \leq l \leq \min\{k, n-1\}$ be such that $\hat{f}^i(\gamma) \in E_{k-l+i}$ for $0 \leq i \leq l$ and $\hat{f}^i(\gamma) \in E_0$ for $l < i \leq n$. Then

$$\sup_\gamma g^{(n)} \leq \begin{cases} C\lambda^{-l}\rho^{-k}\sigma^{-n+l+1} & \textit{if } \hat{f}^n(\gamma) \subset (-\delta, \delta) \times \{0\} \\ C(\delta)\lambda^{-l}\rho^{-k}\sigma^{-n+l+1} & \textit{in general.} \end{cases}$$

Moreover, $\operatorname{var}_\gamma g^{(n)} \leq 2 \sup_\gamma g^{(n)}$.

3. Let $\gamma \subset \eta \cap W_*$ for some $\eta \in \mathcal{P}^{(n)}$ and let $l \geq 0$ be such that $\hat{f}^i(\gamma) \in E_{l+i}$ for $0 \leq i \leq n$. Then $g^{(n)} = \lambda^{-n}$ on γ.

Proof. The first statement in (1) follows immediately from Lemma 5.2 and the observation that

$$g^{(n)}(x,k) = \frac{1}{|(f^n)'(x)|}$$

for every $(x, k) \in \gamma$. For the second statement we use the fact that f has negative schwarzian derivative:

$$Sf(x) = \frac{f'''}{f'}(x) - \frac{3}{2}\left(\frac{f''}{f'}\right)^2(x) = -\frac{3}{2x^2} < 0.$$

Indeed, since the class of maps with negative schwarzian derivative is closed under composition, we have $Sf^n < 0$. Then $(f^n)'''$ and $(f^n)'$ must have opposite signs at every local extremum of the first derivative, in other words, every local minimum of $(f^n)'$ is negative, and every local maximum of $(f^n)'$ is positive. Since the derivative is nonzero on the interior of each monotonicity interval, we conclude that $1/|(f^n)'(x)|$ has no local maximum, and so it has at most one local minimum on γ. As a consequence, $\operatorname{var}_\gamma g \le 2\sup_\gamma g$, as claimed.

For the first claim in (2), let $(x, k-l) \in \gamma$ and $z \in (0,\delta)$ with $\hat{f}^{k-l}(z,0) = (x, k-l)$. Then, by Lemma 5.3(2) and Lemma 5.2(1),

$$\begin{aligned} g^{(n)}(x, k-l) &= \frac{1}{|(f^n)'(x)|}\frac{\lambda^{k-l}\,|(f^{k-l})'(z)|^{-1}}{1} \\ &= \lambda^{k-l}\,|(f^{k+1})'(z)|^{-1}|(f^{n-l-1})'(f^{k+1}(z))|^{-1} \\ &\le \lambda^{k-l}\cdot C\rho^{-k}\lambda^{-k}\cdot C(\delta)\sigma^{-(n-l-1)}\,. \end{aligned}$$

Moreover, $C(\delta)$ may be replaced by C if

$$f^{n-l-1}(f^{k+1}(z)) = f^n(x) \in (-\delta,\delta).$$

The last statement in (2) follows from applying to $1/|(f^{n+k-l})'(z)|$ the same argument as we used in the previous case (1) for $1/|(f^n)'(x)|$.

Finally, (3) follows immediately from the first observation in Remark 6.2.1. □

6.3 Absolutely Continuous Invariant Measures

The main result in this section is Proposition 6.3.1, a version of Proposition 1.3.1 for the tower map $\hat{f}$. From it we deduce that $\hat{f}$ and f admit absolutely continuous invariant measures $\hat{\mu_0}$ and μ_0, respectively.

We begin by proving a partial statement, Lemma 6.3.1, which contains most of the technical difficulty of Proposition 6.3.1. The proof is fairly long, but the reader should find it useful to bear in mind that it is closely related to the calculation leading to Proposition **??**. Indeed, the key fact which is implicit in the argument is that the first-return map of $\hat{f}$ to the ground floor E_0 is uniformly expanding, with properties akin to (a), (b2), (c2) of Section **??**; see Remark 6.2.1.

Lemma 6.3.1. *There is $C > 0$ and, for each $n \ge 1$, there is $C(n) > 0$ such that for every $\varphi \in \mathrm{BV}(\hat{I})$ and every interval $A \subset E_0$,*

$$\operatorname*{var}_A(\mathcal{L}_0^n\varphi) \le \operatorname{var}(\chi_A\mathcal{L}_0^n\varphi) \le C\sigma^{-n}(\operatorname{var}\varphi + \sup|\varphi|) + C(n)\int|\varphi|\,dm_0.$$

Proof. Fix $n \ge 1$ and any interval $A \subset E_0$. Whenever our estimates involve some constant $C(\delta)$, it is implicitly stated that the dependence on δ may be removed (that is, $C(\delta)$ may be replaced by C) if A is contained in $(-\delta,\delta)\times\{0\}$. For the sake of readability, we split the proof into several steps.

Step 1: We find a useful expression (5.18) for $\chi_A \mathcal{L}_0^n \varphi$, by decomposing backward orbits of points in A according to the instant when they have fallen down to the zeroth level E_0.

Let $\underline{\Gamma}(0)$ be the set of all nonempty intervals $\underline{\gamma}$ of the form $\underline{\gamma} = \eta \cap \hat{f}^{-1}(A)$, with $\eta \in \mathcal{P}^{(1)}$ and $\eta \subset E_0$, and let $\overline{\Gamma}(0)$ be defined in the same way, except that $\eta \subset E_k$ for some $k \geq 1$. Then, recalling (5.16) and the fact that $A \subset W_*$,

$$\chi_A \mathcal{L}_0^n \varphi = \sum_{\underline{\gamma} \in \underline{\Gamma}(0)} \big(g \mathcal{L}_0^{n-1} \varphi\big) \circ (\hat{f}|\eta)^{-1} \chi_{\hat{f}(\underline{\gamma})} + \sum_{\overline{\gamma} \in \overline{\Gamma}(0)} \big(g \mathcal{L}_0^{n-1} \varphi\big) \circ (\hat{f}|\eta)^{-1} \chi_{\hat{f}(\overline{\gamma})} . \tag{6.17}$$

Since g is zero outside W_*, we may just as well replace each $\overline{\gamma} \in \overline{\Gamma}(0)$ by $\gamma = \overline{\gamma} \cap W_*$. We call $\mathcal{G}(0)$ the set of all intervals γ obtained in this way. Now we repeat the same procedure for each $\underline{\gamma} \in \underline{\Gamma}(0)$ in the place of A. In this way we find sets of intervals $\underline{\Gamma}(1)$ and $\mathcal{G}(1)$ as follows: each $\underline{\gamma} \in \underline{\Gamma}(1)$ is given by

$$\underline{\gamma} = \eta \cap \hat{f}^{-2}(A)$$

for some $\eta \in \mathcal{P}^{(2)}$ such that $\eta \subset E_0$ and $\hat{f}(\eta) \subset E_0$, and each $\gamma \in \mathcal{G}(1)$ has the form

$$\gamma = \eta \cap \hat{f}^{-2}(A) \cap W_*$$

with $\eta \in \mathcal{P}^{(2)}$ contained in E_k for some $k \geq 1$ and $\hat{f}(\eta) \subset E_0$. Replacing in (5.17), we get that $\chi_A \mathcal{L}_0^n \varphi$ is given by

$$\sum_{\underline{\gamma} \in \underline{\Gamma}(1)} \big(g^{(2)} \mathcal{L}_0^{n-2} \varphi\big) \circ (\hat{f}^2|\eta)^{-1} \chi_{\hat{f}^2(\underline{\gamma})} + \sum_{j=0}^{1} \sum_{\gamma \in \mathcal{G}(j)} \big(g^{(j+1)} \mathcal{L}_0^{n-j-1} \varphi\big) \circ (\hat{f}^{j+1}|\eta)^{-1} \chi_{\hat{f}^{j+1}(\gamma)} .$$

Repeating this operation n times, we obtain

$$\begin{aligned} \chi_A \mathcal{L}_0^n \varphi = & \sum_{\gamma \in \Gamma} \big(g^{(n)} \varphi\big) \circ (\hat{f}^n|\eta)^{-1} \chi_{\hat{f}^n(\gamma)} + \\ & + \sum_{j=0}^{n-1} \sum_{\gamma \in \mathcal{G}(j)} \big(g^{(j+1)} \mathcal{L}_0^{n-j-1} \varphi\big) \circ (\hat{f}^{j+1}|\eta)^{-1} \chi_{\hat{f}^{j+1}(\gamma)} . \end{aligned} \tag{6.18}$$

Here $\Gamma = \underline{\Gamma}(n)$ is set of all intervals $\gamma = \eta \cap \big(\hat{f}^n\big)^{-1}(A)$ with $\eta \in \mathcal{P}^{(n)}$ and $\hat{f}^i(\eta) \subset E_0$ for $0 \leq i \leq n$, and $\mathcal{G}(j)$ is the set of all

$$\gamma = \eta \cap \big(\hat{f}^{j+1}\big)^{-1}(A) \cap W_* ,$$

where $\eta \in \mathcal{P}^{(j+1)}$ is contained in some E_k with $k \geq 1$, and $\hat{f}^i(\eta) \subset E_0$ for every $1 \leq i \leq j+1$. Then, using Lemma 1.3.1, we find

$$\operatorname{var}(\chi_A \mathcal{L}_0^n \varphi) \leq \sum_{\gamma \in \Gamma} \operatorname{var}(\chi_\gamma g^{(n)} \varphi) + \sum_{j=0}^{n-1} \sum_{\gamma \in \mathcal{G}(j)} \operatorname{var}(\chi_\gamma g^{(j+1)} \mathcal{L}_0^{n-j-1} \varphi) \qquad (6.19)$$
$$=: S_1 + S_2 .$$

Step 2: We bound the first term S_1 on the right hand side of (5.19), see (5.21). Suprema are bound in terms of variations and L^1-norms, through the mean value theorem. We use the fact that the lengths of the monotonicity intervals involved in S_1 are bounded from below and their number is bounded from above, by constants depending only on n.

Using Lemma 1.3.1,

$$S_1 \leq \sum_{\gamma \in \Gamma} \big(\operatorname*{var}_\gamma g^{(n)} + 2 \sup_\gamma g^{(n)} \big) \cdot \sup_\gamma |\varphi| + \sup_\gamma g^{(n)} \cdot \operatorname*{var}_\gamma \varphi. \qquad (6.20)$$

For each $\gamma \in \Gamma$, we also denote $\hat{\gamma}$ the monotonicity interval η that contains γ (this is for notational coherence with the sequel of the calculations). Moreover, we let $\mathcal{G}(S_1)$ be the set of all these $\hat{\gamma} = \eta$. Since

$$\mathcal{G}(S_1) \subset \mathcal{P}^{(n,0)} = \{\eta \in \mathcal{P}^{(n)} \text{ such that } \hat{f}^i(\eta) \subset E_0 \text{ for all } 0 \leq i \leq n\},$$

and $\mathcal{P}^{(n,0)}$ is a finite set of nonempty intervals depending only on n, there is some large constant $C(n) > 0$ such that

$$\#\mathcal{G}(S_1) \leq C(n) \qquad \text{and} \qquad \frac{1}{m_0(\hat{\gamma})} = \frac{1}{|\hat{\gamma}|} \leq C(n) \text{ for all } \hat{\gamma} \in \mathcal{G}(S_1).$$

Then, by the mean value theorem,

$$\sup_\gamma \varphi \leq \sup_{\hat{\gamma}} \varphi \leq \operatorname*{var}_{\hat{\gamma}} \varphi + \frac{1}{m_0(\hat{\gamma})} \int_{\hat{\gamma}} |\varphi| \, dm_0 \leq \operatorname*{var}_{\hat{\gamma}} \varphi + C(n) \int |\varphi| \, dm_0 .$$

Of course, we also have $\operatorname{var}_\gamma \varphi \leq \operatorname{var}_{\hat{\gamma}} \varphi$. Replacing this and Lemma 5.4(1) in (5.19),

$$S_1 \leq \sum_{\hat{\gamma} \in \mathcal{G}(S_1)} \big(C(\delta) \sigma^{-n} \operatorname*{var}_{\hat{\gamma}} \varphi + C(\delta) C(n) \int |\varphi| \, dm_0 \big)$$
$$\leq C(\delta) \sigma^{-n} \big(\sum_{\hat{\gamma} \in \mathcal{G}(S_1)} \operatorname*{var}_{\hat{\gamma}} \varphi \big) + C(\delta) C(n) \int |\varphi| \, dm_0 . \qquad (6.21)$$

Step 3: We decompose the last term S_2 in (5.19) into three parts, according to the height of the tower level containing the interval $\gamma \in \mathcal{G}(j)$.

For each $\gamma \in \mathcal{G}(j)$ we define $j(\gamma) = j$, and also $k(\gamma) = k$ if $\gamma \subset E_k$. Observe that $k(\gamma)$ is at least $H(\delta)$ since, by construction, all these intervals γ are contained in W_*. Then we split

$$
\begin{aligned}
S_2 &= \sum_{j=0}^{n-1} \sum_{\gamma \in \mathcal{G}(j)} \operatorname{var}(\chi_\gamma g^{(j+1)} \mathcal{L}_0^{n-j-1} \varphi) \\
&= \sum_{j=0}^{n-1} \Bigg(\sum_{k(\gamma) \geq N} + \sum_{n-j-1 \leq k(\gamma) < N} + \sum_{k(\gamma) < n-j-1} \Bigg) =: s_1 + s_2 + s_3 \,,
\end{aligned}
$$

where $N \geq n$ is to be fixed below, as a function of n only.

Step 4: We bound the term s_1 in S_2, see (5.22). The main point is that use of Lemma 5.3 introduces in the estimates a factor ρ^{-N}, which can be made small by choosing N large.

For each γ with $k = k(\gamma) \geq N$ let $\tilde{\gamma} = (\hat{f}^{n-j-1})^{-1}(\gamma) \subset E_{k-(n-j-1)}$. Since $\hat{f}$ is monotone on each level E_k, $k \geq 1$, we have that $\tilde{\gamma}$ is an interval and $\hat{f}^n$ is monotone on $\tilde{\gamma}$. Let $\tilde{\eta}$ the atom of $\mathcal{P}^{(n)}$ containing $\tilde{\gamma}$. Then

$$
\mathcal{L}_0^{n-j-1} \varphi = \big(g^{(n-j-1)} \varphi\big) \circ (\hat{f}^{n-j-1} | \tilde{\eta})^{-1}
$$

on γ, and so

$$
\chi_\gamma g^{(j+1)} \mathcal{L}_0^{n-j-1} \varphi = (\chi_{\tilde{\gamma}} g^{(n)} \varphi) \circ (\hat{f}^{n-j-1} | \tilde{\eta})^{-1}.
$$

Then, using (v2), (v5), and applying Lemma 5.4(2) to $g^{(n)}$ (with $l = n-j-1$),

$$
s_1 \leq \sum_{j=0}^{n-1} \sum_{k(\gamma) \geq N} C(\delta) \lambda^{-(n-j-1)} \rho^{-k} \sigma^{-j} (\operatorname*{var}_{\tilde{\gamma}} \varphi + \sup_{\tilde{\gamma}} |\varphi|) \,.
$$

Since $\hat{f}^j | E_0$ is at most 2^j-to-1, and E_k contains at most two intervals of monotonicity mapped to E_0 by $\hat{f}$, the sum above ranges over at most 2^{j+1} intervals γ for each given value of $j = j(\gamma)$ and $k = k(\gamma)$. Therefore, the previous inequality gives

$$
\begin{aligned}
s_1 &\leq \sum_{j=0}^{n-1} \sum_{k \geq N} 2^{j+1} C(\delta) \lambda^{-n+j+1} \rho^{-k} \sigma^{-j} (\operatorname{var} \varphi + \sup |\varphi|) \\
&\leq C(\delta) 2^n \sigma^{-n} \rho^{-N} (\operatorname{var} \varphi + \sup |\varphi|)
\end{aligned} \tag{6.22}
$$

(note that $\rho > 1$ and $\sigma < 2\lambda$).

Step 5: We bound the second term s_2 in S_2, see (5.23). The mean value theorem is invoked to bound suprema in terms of variations and L^1-norms. This uses the fact that the number of monotonicity intervals involved in s_2 is bounded from above, and the lengths of their intersections with W_* are bounded from below, by constants depending only on n and N.

Using Lemma 6.2.3(2) as in Step 4, we find

$$s_2 \le \sum_{j=0}^{n-1} \sum_{n-j-1 \le k(\gamma) < N} C(\delta)\lambda^{-(n-j-1)}\rho^{-k}\sigma^{-j}(\operatorname*{var}_{\tilde{\gamma}} \varphi + \sup_{\tilde{\gamma}} |\varphi|)$$

where $\tilde{\gamma} \subset E_{k(\gamma)-(n-j-1)}$ is defined as before. A slight difference with respect to the previous case is that $\tilde{\gamma}$ may not be an interval, if $k(\gamma) = n - j - 1$. However, this is not important, since in this exceptional case $\tilde{\gamma}$ is just the union of two (symmetric) subintervals of $U_0^- \times \{0\}$ and $U_0^- \times \{0\}$, respectively, and then it suffices to consider each of these subintervals separately. From this point on the argument is essentially the same as in Step 3. We let $\hat{\gamma} = \tilde{\eta} \cap W_*$, where $\tilde{\eta}$ is the element of $\mathcal{P}^{(n)}$ containing $\tilde{\gamma}$, and we call $\mathcal{G}(s_2)$ the set of all $\tilde{\eta}$ constructed in this way. By definition,

$$\mathcal{G}(s_2) \subset \mathcal{P}^{(n,N)} = \{\eta \cap W_* : \eta \in \mathcal{P}^{(n)} \text{ and } \hat{f}^i(\eta) \subset \cup_{l \le N} E_l \text{ for all } 0 \le i \le n\}.$$

Since $\mathcal{P}^{(n,N)}$ is a finite set of nonempty intervals contained in W_* which depends only on n and N, we may find $C(n, N) > 0$ large enough so that

$$\#\mathcal{G}(s_2) \le C(n, N) \qquad \text{and} \qquad \frac{1}{m_0(\hat{\gamma})} \le C(n, N)$$

for all $\hat{\gamma} \in \mathcal{G}(s_2)$. Then, the same calculations as we used to deduce (5.21), give

$$s_2 \le C(\delta)\sigma^{-n}(\sum_{\hat{\gamma} \in \mathcal{G}(s_2)} \operatorname*{var}_{\hat{\gamma}} \varphi) + C(\delta)C(n, N) \int |\varphi| \, dm_0 \,. \tag{6.23}$$

Step 6: We bound the last term s_3 in S_2, see (5.24). The reasonings are the same as in Step 5. Combined with the previous estimates, this gives the preliminary bounds for the variation of $\chi_A \mathcal{L}_0^n \varphi$ in (5.25), (5.26).

Using similar arguments and objects $\hat{\gamma} \in \mathcal{G}(s_3)$ for those intervals $\gamma \in \mathcal{G}(j)$ with $k(\gamma) < n - j - 1$, we find

$$s_3 \le \sum_{\hat{\gamma} \in \mathcal{G}(s_3)} C(\delta)\lambda^{-k}\rho^{-k}\sigma^{-l} \operatorname*{var}_{\hat{\gamma}}(\mathcal{L}_0^{n-l}\varphi) + C(\delta)C(n) \int |\varphi| dm_0 \,, \tag{6.24}$$

where $j = j(\gamma)$, $k = k(\gamma)$, and $l = j + k + 1 < n$. Note that we used once more the fact that $\mathcal{L}_0$ does not increase the L^1-norm, cf. (5.8).

Replacing (5.21) – (5.24) in (5.19) we conclude that

$$\begin{aligned} \operatorname*{var}_{A}(\mathcal{L}_0^n\varphi) \le C(\delta)\sigma^{-n} & \sum_{\hat{\gamma} \in \mathcal{G}(S_1) \cup \mathcal{G}(s_2)} \operatorname*{var}_{\hat{\gamma}} \varphi + \sum_{\hat{\gamma} \in \mathcal{G}(s_3)} C(\delta)\sigma^{-l} \operatorname*{var}_{\hat{\gamma}}(\mathcal{L}_0^{n-l}\varphi) \\ & + C(\delta)\left(2^n \rho^{-N}\right)\sigma^{-n}(\operatorname{var} \varphi + \sup |\varphi|) + C(\delta)C(n, N) \int |\varphi| \, dm_0 \,, \end{aligned} \tag{6.25}$$

for general $A \subset E_0$, and

$$\begin{aligned}\operatorname*{var}_A(\mathcal{L}_0^n\varphi) \le C\sigma^{-n} \sum_{\hat\gamma \in \mathcal{G}(S_1)\cup\mathcal{G}(s_2)} \operatorname*{var}_{\hat\gamma}\varphi + \sum_{\hat\gamma\in\mathcal{G}(s_3)} 1\cdot\sigma^{-l}\operatorname*{var}_{\hat\gamma}(\mathcal{L}_0^{n-l}\varphi) \\ + C\left(2^n\rho^{-N}\right)\sigma^{-n}(\operatorname{var}\varphi + \sup|\varphi|) + C(n,N)\int|\varphi|\,dm_0\,,\end{aligned} \tag{6.26}$$

if $A \subset (-\delta,\delta)\times\{0\}$. The factor in the second term of (5.26) is important for the last part of the argument. Note that the calculations we have just presented yield a factor $C\lambda^{-k}\rho^{-k}$, which one may replace by 1 as we did, since $k \ge H(\delta) \gg 1$.

Step 7: If the sum over $\hat\gamma \in \mathcal{G}(s_3)$ in (5.25) is not void, we proceed by recurrence: we apply the previous estimates to each such $\hat\gamma$ in the place of A.

A few notational arrangements are necessary at this point. We change the name of the $\mathcal{G}(\cdot)$, of the $\hat\gamma$, and of their indices k and j, to $\mathcal{G}_1$, $\hat\gamma_1$, k_1, and j_1, respectively. Corresponding objects appearing at the ith step (for each $\hat\gamma_{i-1}$) will be denoted $\mathcal{G}_i$, $\hat\gamma_i$, k_i, and j_i, and we also let $l_i = j_i + 1 + k_i$. By construction, every of these $\hat\gamma_i$ is a subset of $(-\delta,\delta)\times\{0\}$, and so we may apply (5.26) to it (rather than (5.25)). After one recurrence step,

$$\begin{aligned}\operatorname*{var}_A \mathcal{L}_0^n\varphi \le C(\delta)\sigma^{-n}\Big(\sum_{\hat\gamma_1}\operatorname*{var}_{\hat\gamma_1}\varphi + C\sum_{\hat\gamma_1,\hat\gamma_2}\operatorname*{var}_{\hat\gamma_2}\varphi\Big) + \sum_{\hat\gamma_1,\hat\gamma_2} C(\delta)\sigma^{-l_1-l_2}\operatorname*{var}_{\hat\gamma_2}(\mathcal{L}_0^{n-l_1-l_2}\varphi) \\ + C(\delta)(1+\#\mathcal{G}_1(s_3))\left[(2^n\rho^{-N})\sigma^{-n}(\operatorname{var}\varphi+\sup|\varphi|) + C(n,N)\int|\varphi|\,dm_0\right],\end{aligned}$$

the sums running over $\hat\gamma_1 \in \mathcal{G}_1(S_1)\cup\mathcal{G}_1(s_2)$, over $\hat\gamma_1\in\mathcal{G}_1(s_3)$, $\hat\gamma_2\in\mathcal{G}_2(S_1)\cup\mathcal{G}_2(s_2)$, and over $\hat\gamma_1\in\mathcal{G}_1(s_3)$, $\hat\gamma_2\in\mathcal{G}_2(s_3)$, respectively. By construction, each $\hat\gamma_1\in\mathcal{G}_1(s_3)$ is contained in some monotonicity interval $\eta_1\in\mathcal{P}^{(l_1)}$ such that $\hat f^i(\eta_1)\subset\cup_{k\le k_1}E_k$ for all $0\le i\le l_1$. Since the correspondence $\hat\gamma_1\mapsto\eta_1$ is one-to-one, and $k_1 < l_1 \le n$, we conclude that $\#\mathcal{G}(s_3)\le C(l_1,k_1)\le C(n)$ for large enough $C(l_1,k_1)$ and $C(n)$. In fact, this same argument shows that $\#\mathcal{G}_j(s_3)\le C(n)$ for all $1\le j\le n$. Hence, after at most n steps,

$$\begin{aligned}\operatorname*{var}_A \mathcal{L}_0^n\varphi \le C(\delta)\sigma^{-n}\sum_{i=1}^n\Bigg(C\sum_{\hat\gamma_1,\dots,\hat\gamma_{i-1},\hat\gamma_i}\operatorname*{var}_{\hat\gamma_i}\varphi\Bigg) \\ + C(\delta)\;C(n)\Bigg((2^n\rho^{-N})\sigma^{-n}(\operatorname{var}\varphi+\sup|\varphi|) + C(n,N)\int|\varphi|\,dm_0\Bigg),\end{aligned} \tag{6.27}$$

the second sum being over $\hat\gamma_1\in\mathcal{G}_1(s_3),\dots,\hat\gamma_{i-1}\in\mathcal{G}_{i-1}(s_3)$, $\hat\gamma_i\in\mathcal{G}_i(S_1)\cup\mathcal{G}_i(s_2)$. Observe that the intervals $\hat\gamma_i$ occurring in (5.27) are all contained in distinct atoms of the partition $\mathcal{P}^{(n)}$, and so they are two-by-two disjoint. Therefore, the variations of φ over such intervals add up, so that the first term on the right hand side of (5.27) is bounded by $C(\delta)\sigma^{-n}\operatorname{var}\varphi$. Now we fix $N \gg n$ in such a way that

$$C(n) \cdot 2^n \rho^{-N} \leq 1,$$

and then the second term is also bounded by $C(\delta)\sigma^{-n}(\operatorname{var}\varphi + \sup|\varphi|)$. Moreover, once we have chosen N in this way, depending only on n, we may replace $C(n, N)$ by $C(n)$ in the last term. Finally, since δ is also fixed at this point, we may omit the reference to δ, replacing $C(\delta)$ by C in (5.27). □

Proposition 6.3.1. *Given any $\bar{\sigma} \in (1, \sigma)$ there is $C > 0$ such that*

1. $\operatorname{var}(\mathcal{L}_0^n \varphi) \leq C\bar{\sigma}^{-n}(\operatorname{var}\varphi + \sup|\varphi|) + C \int |\varphi|\, dm_0$*;*
2. $\sup(\mathcal{L}_0^n \varphi) \leq C\bar{\sigma}^{-n}(\operatorname{var}\varphi + \sup|\varphi|) + C \int |\varphi|\, dm_0$

for any function $\varphi \in \mathrm{BV}(\hat{I})$ and any $n \geq 1$.

Proof. Let $1 < \bar{\sigma} < \tilde{\sigma} < \sigma$. First, we fix $n = n_0$ and decompose

$$\operatorname{var} \mathcal{L}_0^{n_0} \varphi = \sum_{k=0}^{\infty} \operatorname*{var}_{E_k} \mathcal{L}_0^{n_0} \varphi.$$

For $k > n_0$ and $(x, k) \in W_k \times \{k\}$ there exists a unique $(y, k - n_0) \in E_{k-n_0}$ such that $\hat{f}^{n_0}(y, k - n_0) = (x, k)$. Then, by Lemma 5.4(3),

$$\mathcal{L}_0^{n_0} \varphi(x, k) = g^{(n_0)}(y, k - n_0)\varphi(y, k - n_0) = \lambda^{-n_0}\varphi(y, k - n_0),$$

and so

$$\operatorname*{var}_{E_k} \mathcal{L}_0^{n_0} \varphi = \operatorname*{var}_{W_k \times \{k\}} \mathcal{L}_0^{n_0} \varphi \leq \lambda^{-n_0} \operatorname*{var}_{E_{k-n_0}} \varphi.$$

If $k \leq n_0$ then the same argument gives

$$\operatorname*{var}_{E_k} \mathcal{L}_0^{n_0} \varphi \leq 2\lambda^{-k} \operatorname*{var}_{E_0} \mathcal{L}_0^{n_0 - k} \varphi$$

(the factor 2 accounts for the fact that f is 2-to-1 on E_0). Combining with Lemma 5.5, for $A = E_0$ and $n = n_0 - k < n_0$,

$$\operatorname*{var}_{E_k} \mathcal{L}_0^{n_0} \varphi \leq 2\lambda^{-k}\left[C\sigma^{-(n_0-k)}(\operatorname{var}\varphi + \sup|\varphi|) + C(n_0)\int |\varphi|\, dm_0\right].$$

Recall that we are taking $\tilde{\sigma} < \sigma$ and that we have chosen $\sigma \leq \lambda$. Thus, adding the previous estimates over all $k \geq 0$,

$$\begin{aligned} \operatorname{var} \mathcal{L}_0^{n_0} \varphi &\leq \lambda^{-n_0} \operatorname{var}\varphi + n_0 C\sigma^{-n_0}(\operatorname{var}\varphi + \sup|\varphi|) + n_0 C(n_0) \int |\varphi|\, dm_0 \\ &\leq C\tilde{\sigma}^{-n_0}(\operatorname{var}\varphi + \sup|\varphi|) + C(n_0)\int |\varphi| dm_0\,, \end{aligned} \tag{6.28}$$

as long as the constants C and $C(n_0)$ in the last term be fixed large enough with respect to the ones in the second term.

In order to prove part (1) of the proposition one must now remove the dependence on n_0 of the factor in the last term in (5.28). We start by proving a similar inequality for the supremum:

$$\sup|\mathcal{L}_0^{n_0}\varphi| \le C\tilde{\sigma}^{-n_0}(\operatorname{var}\varphi + \sup|\varphi|) + C(n_0)\int|\varphi|dm_0\,. \tag{6.29}$$

In doing this it is convenient to consider separately the suprema over the the upper and over the lower part of the tower. On the one hand, we note that

$$\sup_{\cup_{k>n_0}E_k}|\mathcal{L}_0^{n_0}\varphi| = \sup_{k>n_0}\big(\sup_{W_k\times\{k\}}|\mathcal{L}_0^{n_0}\varphi|\big) \le \sup_{k>n_0}\big(\lambda^{-n_0}\sup_{E_{k-n_0}}|\varphi|\big) \le \lambda^{-n_0}\sup|\varphi|,$$

as a direct consequence of Lemma 5.4(3) applied n_0 times. Since $\sigma \le \lambda$, this implies (5.29) when the supremum of $|\mathcal{L}_0^{n_0}\varphi|$ is attained over the union of the E_k with $k > n_0$. From now on we suppose otherwise, that is, we suppose that $\sup|\mathcal{L}_0^{n_0}\varphi|$ is attained on some level E_k with $k \le n_0$. Using the mean value theorem,

$$\sup|\mathcal{L}_0^{n_0}\varphi| = \sup_{E_k}|\mathcal{L}_0^{n_0}\varphi| \le \operatorname*{var}_{E_k}\mathcal{L}_0^{n_0}\varphi + \frac{1}{m_0(E_k)}\int_{E_k}|\mathcal{L}_0^{n_0}\varphi|\,dm_0.$$

Clearly, $m_0(E_l) > 0$ for every $l \ge 0$: it suffices to note that each W_l contains a neighbourhood of c_l. As a consequence, $\sup_{l\le n_0} 1/m_0(E_l)$ is less than some constant $C(n_0) > 0$ depending only on n_0. Combining this with (5.8), we obtain

$$\sup|\mathcal{L}_0^{n_0}\varphi| \le \operatorname{var}\mathcal{L}_0^{n_0}\varphi + C(n_0)\int|\varphi|\,dm_0\,,$$

so that now (5.29) follows from (5.28).

At this point we choose an integer q large enough so that $2C\bar{\sigma}^{-q} < \bar{\sigma}^{-q} < 1/2$, where C is the constant multiplying $\tilde{\sigma}^{-n_0}(\operatorname{var}\varphi + \sup\varphi)$ in (5.28), (5.29). Then, given any $n \ge 1$, we write $n = pq + r$ with $0 \le r < q$. Using the inequalities above p times with $n_0 = q$, and once more with $n_0 = r$,

$$\begin{aligned}
\operatorname{var}(\mathcal{L}_0^{n}\varphi) &\le \frac{1}{2}\bar{\sigma}^{-q}\big(\operatorname{var}\mathcal{L}_0^{n-q}\varphi + \sup|\mathcal{L}_0^{n-q}\varphi|\big) + C(q)\int|\varphi|\,dm_0\\
&\le \frac{1}{2}\bar{\sigma}^{-pq}\big(\operatorname{var}\mathcal{L}_0^{r}\varphi + \sup|\mathcal{L}_0^{r}\varphi|\big) + C(q)\big(1 + 2^{-1} + \cdots + 2^{-p+1}\big)\int|\varphi|\,dm_0\\
&\le C\bar{\sigma}^{-pq-r}\big(\operatorname{var}\varphi + \sup|\varphi|\big) + \big(2C(q) + C(r)2^{-pq}\big)\int|\varphi|\,dm_0.
\end{aligned}$$

Finally, as $r < q$ and q has already been fixed, we may bound $2C(q) + C(r)2^{-pq}$ by some constant C independent of n. □

For constructing absolutely continuous invariant probability measures for the maps f and $\hat{f}$ we also need

Lemma 6.3.2. *$m_0 = w_0\, m$ is a finite measure on $\hat{I}$.*

Proof. Of course, $m_0(E_0) = m(E_0) = m(I)$ is finite. Moreover, for each $k \geq 1$,

$$m_0(E_k) = \int_{B_k} w_0(x,k)\, dm(x) = \int_{W_k} \frac{\lambda^k}{|(f^k)'(z)|}\, dm(x),$$

where $z \in (0,\delta)$ is uniquely defined by $\hat{f}^k(z,0) = (x,k)$. We change variables $x = f^k(z)$, and then we get

$$m_0(E_k) = \int_{Y_k} \lambda^k\, dm(z) = \lambda^k\, m(Y_k),$$

where $Y_k = \{z \in (0,\delta) : f^k(z) \in W_k\}$. Next, we observe that

$$2e^{-\beta k} \geq m(B_k) \geq m(W_k) \geq \frac{1}{C} |(f^{k-1})'(c_1)|\, m(f(Y_k)) \geq \frac{1}{C} \lambda_c^{k-1}\, m(Y_k)^2,$$

where the third inequality is a consequence of (5.16) and the mean value theorem. Replacing above, and recalling that we have chosen $\sqrt{\lambda_c} > \lambda\rho$ and $\beta > 0$,

$$m_0(E_k) \leq \sqrt{2C\lambda_c}\lambda^k \lambda_c^{-k/2} e^{-\beta k/2} \leq \sqrt{2C\lambda_c}\, \rho^{-k} \tag{6.30}$$

for every $k \geq 1$. Since $\rho > 1$, the claim follows immediately. □

Proposition 6.3.2. *1. $\hat{f}$ has some invariant probability measure $\hat{\mu}_0$ that is absolutely continuous with respect to m_0;*
2. f has a unique invariant probability measure μ_0 absolutely continuous with respect to Lebesgue measure m on I, and μ_0 is ergodic.

Proof. Part (1) is quite similar to Corollary **??**. Recall from Lemma 6.3.2 that $m_0(\hat{I})$ is finite. Proposition 6.3.1 implies that the sequence $\varphi_n = n^{-1} \sum_{j=0}^{n-1} \mathcal{L}_0^j\big(1/m_0(\hat{I})\big)$ is uniformly bounded and has uniformly bounded variation. By Lemma **??** there exists a subsequence φ_{n_k} converging in $L^1(m_0)$ to some function φ_0 in $\mathrm{BV}(\hat{I})$. The operator $\mathcal{L}_0$ being continuous with respect to the norm of $L^1(m_0)$, recall (5.8), it follows that φ_0 is a fixed point of $\mathcal{L}_0$. Then, using (5.4), the absolutely continuous measure $\hat{\mu}_0 = \varphi_0 m_0$ is $\hat{f}$-invariant, and

$$\int \varphi_n\, dm_0 = \frac{1}{n} \sum_{j=0}^{n-1} \int \mathcal{L}_0^j\big(1/m_0(\hat{I})\big)\, dm_0 = \frac{1}{n} \sum_{j=0}^{n-1} \int (1/m_0(\hat{I}))\, dm_0 = 1$$

for every $n \geq 1$. Therefore, $\int \varphi_0\, dm_0 = 1$ and so $\hat{\mu}_0$ is a probability measure.

To prove (2), we take $\mu_0 = \pi_* \hat{\mu}_0$, where $\pi : \hat{I} \to I$ continues to denote the projection $\pi(x,k) = x$. In other words, for each Borel subset A of I,

$$\mu_0(A) = \hat{\mu}_0(\pi^{-1}(A)) = \sum_{k=0}^{\infty} \hat{\mu}_0\big((A \cap B_k) \times \{k\}\big).$$

Then μ_0 is a probability, and the relation $\pi \circ \hat{f} = f \circ \pi$ ensures that μ_0 is f-invariant:

$$\mu_0(f^{-1}(A)) = \hat{\mu}_0(\pi^{-1}(f^{-1}(A))) = \hat{\mu}_0(\hat{f}^{-1}(\pi^{-1}(A))) = \hat{\mu}_0(\pi^{-1}(A)) = \mu_0(A).$$

Moreover, μ_0 is easily seen to be absolutely continuous with respect to Lebesgue measure. In fact, if A has zero Lebesgue measure then the same is true for every $A \cap B_k$, for every $k \geq 0$. This implies $m_0((A \cap B_k) \times \{k\}) = 0$ for all $k \geq 0$, and so $\mu_0(A) = 0$.

A main ingredient to prove that μ_0 is unique and ergodic is the result of [**?**] asserting that *any unimodal map with negative schwarzian derivative, non-degenerate critical points, and no periodic attractors is ergodic with respect to Lebesgue measure: if $A \subset I$ satisfies $f^{-1}(A) = A$ then either $m(A) = 0$ or $m(A^c) = 0$*. Then such an A must have $\mu_0(A) = 0$ or $\mu_0(A^c) = 0$, which proves ergodicity of μ_0. Now, we claim that the measure μ_0 is equivalent to Lebesgue measure m on the interval $I_* = [f^2(c), f(c)]$. This can be seen as follows. Since φ_0 has bounded variation, and $\int \varphi_0 \, dm_0 = 1$, there is some interval $\gamma \subset W_*$ such that $\inf_\gamma \varphi_0 > 0$. Then the density of $\hat{\mu}_0$ with respect to the usual length is bounded away from zero on γ. As a consequence, $\inf_{\pi(\gamma)}(d\mu_0/dm) > 0$. On the other hand, the assumption of topological mixing (U3) ensures that $f^N(\pi(\gamma)) = I_*$ for some $N \geq 1$. It follows that

$$\inf_{I_*} \left(\frac{d\mu_0}{dm}\right) \geq \inf_{\pi(\gamma)} \left(\frac{d\mu_0}{dm}\right) \cdot \frac{1}{\sup |(f^N)'|} > 0, \tag{6.31}$$

which implies our claim. Finally, let ν be any f-invariant probability measure which is absolutely continuous with respect to Lebesgue measure. It is easy to see that the support of ν must be contained in I_*, and so ν is absolutely continuous with respect to μ_0. Since μ_0 is ergodic, it follows that $\nu = \mu_0$ (because ergodic measures are minimal for the relation of absolute continuity), proving uniqueness. □

Closing this section, we prove that the support of φ_0 contains

$$W_\delta = W_* \setminus \big([f^2(c), f^2(\delta)) \cup (f(\delta), f(c)]\big) \times \{0\}.$$

Then $\operatorname{supp} \varphi_0$ must coincide with W_δ: $\varphi_0 = \mathcal{L}_0^n \varphi_0$ implies that φ_0 is identically zero on $\hat{I} \setminus \hat{f}^n(\hat{I})$, for every $n \geq 1$, and we have seen in Remark 5.1 that $\cap_{n \geq 1} \hat{f}^n(\hat{I})$ is contained in W_δ.

Lemma 6.3.3. *1.* $\inf \big(\varphi_0 | [f^2(\delta), f(\delta)] \times \{0\}\big) > 0$

2. $\inf\left(\varphi_0|W_k\right)>0$ *for every* $k\geq 1$.

Proof. Let $\gamma_1\subset W_*$ be some open interval such that $\inf_{\gamma_1}\varphi_0>0$. By the topological mixing assumption (U3), there exists some $n_1\geq 0$ such that

$$\pi(\hat{f}^{n_1}(\gamma_1))=f^{n_1}(\pi(\gamma_1))=I_*\,.$$

In particular, $\pi(\hat{f}^{n_1}(\gamma_1))$ contains the fixed point $p=(-1+\sqrt{1+4a})/2>0$ of f. Moreover, up to slightly modifying β if necessary, we may suppose that no endpoint $(c_k\pm e^{-\beta k})\times\{k\}$ of a level E_k, $k\geq 1$, projects down to p. Then there exists some open interval $\gamma_2\subset\hat{f}^{n_1}(\gamma_1)$ such that $\pi(\gamma_2)$ contains p. Clearly, $\pi(\hat{f}^n(\gamma_2))$ must contain p for every $n\geq 1$. Now we suppose that $p\neq c_k$ for every $k\geq 1$ (if this happens to be false, we simply replace p by any other periodic orbit of f whose orbit does not intersect $(-\delta,\delta)$, and the argument proceeds along the same lines). Then, there exists some finite time $n_2\geq 0$ at which the point $\xi\in\gamma_2$ satisfying $\pi(\xi)=p$ falls down to E_0: $\hat{f}^{n_2}(\xi)=(p,0)$. Up to another arbitrarily small modification of β, we may suppose that the orbit of ξ does not pass through any of the boundary points of the monotonicity intervals in $\mathcal{P}^{(1)}$. Then $\hat{f}^{n_2}(\xi)$ contains some open neighbourhood γ_3 of $(p,0)$ in E_0. Let $n_3\geq 0$ be minimum such that $f^{n_3}(\pi(\gamma_3))$ intersects $(-\delta,\delta)$. Then $\hat{f}^{n_3}(\gamma_3)=f^{n_3}(\pi(\gamma_3))\times\{0\}$ contains $[\delta,p]\times\{0\}$. Let us denote

$$\sigma_1=\hat{f}([\delta,p]\times\{0\})=[p,f(\delta)]\times\{0\}\quad\text{and}\quad\sigma_1=\hat{f}^2([\delta,p]\times\{0\})=[f^2(\delta),p]\times\{0\}.$$

Then $\sigma_1\cup\sigma_2=[f^2(\delta),f(\delta)]\times\{0\}$. We use the following property

$$\inf_{\gamma}\varphi_0>0\Rightarrow\inf_{\hat{f}(\gamma)}\varphi_0>0,$$

which is a direct consequence of the fact that φ_0 is a fixed point for the transfer operator of $\hat{f}$. Since $\sigma_i\subset\hat{f}^{n+i}(\gamma_1)$, with $n=n_1+n_2+n_3$, we get that $\inf_{\sigma_i}\varphi_0>0$ for $i=1,2$. Part (1) of the lemma follows immediately.

Moreover, given $(y,k)\in W_k$, $k\geq 1$, and $z\in(0,\delta)$ such that $f^k(z)=y$,

$$\varphi_0(y,k)=(\mathcal{L}_0^k\varphi_0)(y,k)=\frac{\varphi_0(z,0)}{\lambda^k}+\frac{\varphi_0(-z,0)}{\lambda^k}\geq\frac{2}{\lambda^k}\inf\left(\varphi_0|[f^2(\delta),f(\delta)]\times\{0\}\right).\tag{6.32}$$

This proves part (2). □

Remark 6.3.1. This last relation also yields another useful conclusion:

$$\varphi_0(y,k)\leq\frac{2}{\lambda^k}\sup\left(\varphi_0|[f^2(\delta),f(\delta)]\times\{0\}\right)\leq\frac{2}{\lambda^k}\sup\varphi_0\,,$$

and so

$$\sum_{k=0}^{\infty}\sup(\varphi_0|E_k)\leq\sum_{k=0}^{\infty}2\lambda^{-k}\sup\varphi_0<\infty.\tag{6.33}$$

6.4 Quasi-compacity and Decay of Correlations

In this section we prove that the measures $\hat{\mu}_0$ and μ_0 we have just constructed are exact, and so also mixing, for the corresponding dynamical systems $\hat{f}$ and f (Proposition 5.13). As a consequence, the transfer operator $\mathcal{L}_0$ is quasi-compact and both systems $(\hat{f}, \hat{\mu}_0)$ and (f, μ_0) have exponential decay of correlations in corresponding spaces of functions with bounded variation. Proposition 5.13 also provides another proof of the ergodicity of μ_0 (besides implying that $\hat{\mu}_0$ is also ergodic). For the proof we need a few preparatory lemmas.

Lemma 6.4.1. *Given $\varepsilon > 0$ there exists $N \geq 0$, and for each $n \geq 1$ there exists a subset $\mathcal{Q}(n, N)$ of $\mathcal{P}^{(n+N)}$, such that*

1. *$f^n(\eta) \in \mathcal{P}^{(N)}$ and $f^n(\eta) \subset \bigcup_{k=0}^{N} E_k$ for every $\eta \in \mathcal{Q}(n, N)$;*
2. *the $\hat{\mu}_0$-measure of the union of the intervals $\eta \notin \mathcal{Q}(n, N)$ is at most ε.*

Proof. Let us denote ∂_k, $k \geq 0$, the set of boundary points of the elements of the partition $\mathcal{P}^{(1)}$ contained in E_k. That is,

$$\partial_0 = \{q, -\delta, 0, \delta, -q\} = \partial D_0^- \cup \partial U_0^- \cup \partial U_0^+ \cup \partial D_0^+$$

and, for each $k \geq 1$,

$$\partial_k = \partial D_k^- \cup \partial U_k \cup \partial D_k^+.$$

Observe that each ∂_k, $k \geq 1$, contains at most 4 points. For $n \geq 1$, $N \geq 1$, and $\eta \in \mathcal{P}^{(N+n)}$, let $(k(i))_i$, be the sequence given by

$$\hat{f}^i(\eta) \subset E_{k(i)}, \quad \text{for each } i \geq 0.$$

Let $\tau > 0$ be fixed in the following way: for what concerns the present lemma τ is arbitrary, but for the proof of Lemma 5.11 it is convenient to choose $\tau = \log(\lambda\rho)/\log 8$. Then define $\mathcal{Q}(n, N)$ to be the subset of intervals $\eta \in \mathcal{P}^{(N+n)}$ such that

(i) $k(i) \leq N + (n - i)\tau$ for $0 \leq i \leq n$.
(ii) $\hat{f}^i(\partial\eta)$ is disjoint from $\partial_{k(i)}$ for every $0 \leq i < n$;

Condition (ii) is an analog of (3.7): it implies that $\hat{f}^n(\eta)$ belongs in $\mathcal{P}^{(N)}$. The case $i = n$ in condition (i) means that $\hat{f}^n(\eta) \subset E_k$ for some $k \leq N$. Thus, property (1) in the statement is satisfied by every element η of $\mathcal{Q}(n, N)$. Now we only have to show that the total measure of those intervals η for which either of (i) or (ii) fails can be made arbitrarily small by increasing N.

Let $0 \leq i \leq n$ be fixed. Then (i) fails for a given $\eta \in \mathcal{P}^{(N+n)}$ if and only if

$$\eta \subset \hat{f}^{-i}\Bigg(\bigcup_{k > N + (n-i)\tau} E_k\Bigg).$$

We have shown in Lemma 6.3.2 that the m_0-measure of the tower levels E_k decreases exponentially fast with k, recall (5.30). Then the same is true for the $\hat{\mu}_0$-measure, since $\hat{\mu}_0 = \varphi_0 \, m_0$ and φ_0 is a bounded function. So

$$\hat{\mu}_0\Big(\bigcup_{k>N+(n-i)\tau} E_k\Big) \le K_1 \rho^{-N-(n-i)\tau},$$

for some $K_1 > 0$. Since $\hat{\mu}_0$ is $\hat{f}$-invariant, it follows that the $\hat{\mu}_0$-measure of the union of all the $\eta \in \mathcal{P}^{(N+n)}$ such that (i) fails is also bounded by $K_1 \rho^{-N-(n-i)\tau}$.

Keeping $0 \le i \le n$ fixed, let us also estimate the total measure of the elements of $\mathcal{P}^{(N+n)}$ that satisfy (i) but not (ii). Let $\zeta = \zeta_i$ be the element of $\mathcal{P}^{(N+n-i)}$ containing $\hat{f}^i(\eta)$. Since (ii) breaks down, some boundary point of ζ must be in $\partial_{k(i)}$. On the other hand, in view of (i), there are at most $8+6\big(N+(n-i)\tau\big)$ such intervals ζ: 8 inside E_0 and not more than 6 inside each E_k, $1 \le k \le N+(n-i)\tau$. Moreover, the m_0-measure of each one of them is bounded by $\lambda^{-N-(n-i)}$, because the jacobian of $\hat{f}^{N+(n-i)}$ with respect to m_0 is everywhere larger than $\lambda^{N+(n-i)}$, cf. Remark 5.2. Using once more the fact that φ_0 is bounded, we conclude that the $\hat{\mu}_0$-measure of the union of these intervals $\zeta \in \mathcal{P}^{(N+n-i)}$ is at most

$$K_2\big(8+6\big(N+(n-i)\tau\big)\big)\lambda^{-N-(n-i)} \le 14K_2(N+n-i)\lambda^{-N-(n-i)}$$

for some $K_2 > 0$. Then, because $\hat{\mu}_0$ is $\hat{f}$-invariant, the same bound applies to the $\hat{\mu}_0$ measure of the union of all the intervals $\eta \in \mathcal{P}^{(N+n)}$ for which (i) holds but (ii) fails.

We conclude that the total measure of the monotonicity intervals η of $\hat{f}^{N+n}$ for which either (i) or (ii) fails for some $0 \le i \le n$ is bounded by

$$\begin{aligned} &\sum_{i=0}^{n}\big(K_1\rho^{-N-(n-i)\tau} + 14K_2(N+n-i)\lambda^{-N-(n-i)}\big) \le \\ &\qquad \le \rho^{-N}\big(\sum_{j=0}^{n} K_1\rho^{-j\tau}\big) + \lambda^{-N/2}\big(\sum_{j=0}^{n} 14K_2(N+j)\lambda^{-(N+j)/2}\big) \\ &\qquad \le K_3\rho^{-N} + K_3\lambda^{-N/2}, \end{aligned}$$

for some $K_3 > 0$. This can be made smaller than ε by choosing N sufficiently large, and so the proof is complete. □

Lemma 6.4.2. *Given $N \ge 1$ and $\varepsilon_2 > 0$ there exists $\varepsilon_1 > 0$ such that for any $n \ge 1$, any interval $\eta \in \mathcal{Q}(n,N)$, and any Borel subset $\xi \subset \eta$*

$$\frac{m(\xi)}{m(\eta)} \le \varepsilon_1 \quad \Rightarrow \quad m(\hat{f}^n(\xi)) \le \varepsilon_2\,.$$

Proof. Most of the proof is based on the same ideas as Lemmas 5.2 and 5.3. The main new ingredient is to use condition (i) $k(i) \leq N + (n-i)\tau$ in the definition of $\mathcal{Q}(n,N)$, cf. proof of Lemma 5.10, taking τ sufficiently small, e.g., $\tau = \log(\lambda\rho)/\log 8$.

Suppose first that $\eta \subset E_0$ and $\hat{f}^n(\eta) \subset E_0$. In this case we prove that $\hat{f}^n$ has uniformly bounded distortion on η (depending on N, but not on n nor on η). Let us consider the sequence of iterates $0 \leq \nu_1 < \nu_1 + p_1 < \nu_2 < \cdots < \nu_s < \nu_s + p_s < n$ defined by

(a) $\hat{f}^j(\eta) \subset E_0$ for $0 \leq j \leq \nu_1$, for $\nu_i + p_i < j \leq \nu_{i+1}$ and $1 \leq i \leq s-1$, and for $\nu_s + p_s < j \leq n$
(b) $\hat{f}^j(\eta) \subset E_{\nu_i - j}$ for $\nu_i \leq j \leq \nu_i + p_i$ and $1 \leq i \leq s$;

Let $\gamma = \pi(\eta) \subset I$ and $x, y \in \gamma$. First we consider $0 \leq j < \nu_1$. Using $f^j(\gamma) \cap (-\delta, \delta) = \emptyset$, and Lemma 5.2 together with the mean value theorem,

$$\sum_{j=0}^{\nu_1 - 1} \big| \log|f'(f^j(x))| - \log|f'(f^j(y))| \big| \leq \sum_{j=0}^{\nu_1 - 1} \frac{1}{\delta} |f^j(\gamma)| \leq \\ \leq \sum_{j=0}^{\nu_1 - 1} \text{const}\, \sigma^{j-\nu_1} |f^{\nu_1}(\gamma)| \leq \text{const}\, |f^{\nu_1}(\gamma)|. \tag{6.34}$$

For the same reasons,

$$\sum_{j=\nu_i + p_i + 1}^{\nu_{i+1} - 1} \big| \log|f'(f^j(x))| - \log|f'(f^j(y))| \big| \leq \text{const}\, |f^{\nu_{i+1}}(\gamma)| \tag{6.35}$$

for every $1 \leq i \leq s-1$, and

$$\sum_{j=\nu_s + p_s + 1}^{n-1} \big| \log|f'(f^j(x))| - \log|f'(f^j(y))| \big| \leq \text{const}\, |f^n(\gamma)|. \tag{6.36}$$

Now let $j = \nu_i$, and denote $\Delta_i = d(f^{\nu_i}(\gamma), c)$. Then

$$\big| \log|f'(f^j(x))| - \log|f'(f^j(y))| \big| \leq \text{const}\, \frac{|f^{\nu_i}(\gamma)|}{\Delta_i}$$

Next we consider $\nu_i < j \leq \nu_i + p_i$. Lemma 6.2.2(1) implies

$$\frac{|f^{\nu_i + 1}(\gamma)|}{\Delta_i^2} \leq \text{const}\, \frac{|D_{p_i}^{\pm}|}{|U_{p_i}|} \leq \text{const}\, \frac{e^{-\beta p_i}}{e^{-\beta(p_i+1)}/4} \leq \text{const}\ .$$

As a consequence,

$$|f^{\nu_i}(\gamma)| \leq \text{const}\, \Delta_i\,, \qquad \text{which implies} \qquad \frac{|f^{\nu_i+1}(\gamma)|}{\Delta_i^2} \leq \text{const}\, \frac{|f^{\nu_i}(\gamma)|}{\Delta_i}\,.$$

Using Lemma 5.3(1) again

$$\frac{|f^j(\gamma)|}{e^{-\beta(j-\nu_i)}} \le \text{const}\,\frac{|f^{\nu_i+1}(\gamma)|}{\Delta_i^2} \le \text{const}\,\frac{|f^{\nu_i}(\gamma)|}{\Delta_i}$$

On the other hand, by definition the of B_k, E_k, and assumption (U2), $f^j(\gamma)$ does not intersect $(-\gamma e^{-\alpha(j-\nu_i)}, \gamma e^{-\alpha(j-\nu_i)})$, where $\gamma = 1 - e^{\alpha-\beta}$. It follows that

$$\begin{aligned}\sum_{j=\nu_i+1}^{\nu_i+p_i} \big|\log|f'(f^j(x))| - \log|f'(f^j(y))|\big| &\le \sum_{j=\nu_i+1}^{\nu_i+p_i} \text{const}\, e^{\alpha(j-\nu_i)}|f^j(\gamma)| \\ &\le \sum_{j=\nu_i+1}^{\nu_i+p_i} \text{const}\, e^{(\alpha-\beta)\,(j-\nu_i)}\frac{|f^{\nu_i}(\gamma)|}{\Delta_i} \le \text{const}\,\frac{|f^{\nu_i}(\gamma)|}{\Delta_i}.\end{aligned} \tag{6.37}$$

Putting (5.34), (5.35), (5.36). (5.37) together, and noting that $\Delta_i \le 1$, we find

$$\sum_{j=0}^{n-1} \big|\log|f'(f^j(x))| - \log|f'(f^j(y))|\big| \le \text{const}\sum_{i=1}^{s}\frac{|f^{\nu_i}(\gamma)|}{\Delta_i} + \text{const}\,|f^n(\gamma)| \tag{6.38}$$

Of course, $|f^n(\gamma)| \le \text{const}$. Lemmas **??** and 6.2.2(2) imply

$$|f^{\nu_i}(\gamma)| \le \text{const}(\lambda\rho)^{\nu_i-n}|f^n(\gamma)| \le \text{const}(\lambda\rho)^{\nu_i-n}$$

for each $1 \le i \le s$, and (5.15) gives

$$\Delta_i^2 \ge \text{const}\, e^{-\beta(p_i+1)}|(f^{p_i})'(c_1)|^{-1} \ge \text{const}\, e^{-2\beta p_i}4^{-p_i}.$$

Now, condition (i) in the definition of $\mathcal{Q}(n,N)$ implies

$$p_i = k(\nu_i + p_i) \le N + (n - \nu_i - p_i)\tau \le N + (n-\nu_i)\tau$$

and so, recall that $\tau = \log(\lambda\rho)/\log 8$ and $e^\beta \le \sqrt{\lambda_c} \le 2$

$$\begin{aligned}\sum_{i=1}^{s}\frac{|f^{\nu_i}(\gamma)|}{\Delta_i} &\le \sum_{i=1}^{s}(\lambda\rho)^{\nu_i-n}(2e^\beta)^{p_i} \le \sum_{i=1}^{s}4^N(\lambda\rho 4^{-\tau})^{\nu_i-n} \\ &\le 4^N\sum_{i=1}^{s}2^{(\nu_i-n)\tau} \le \text{const}\,4^N.\end{aligned}$$

Replacing in (5.38), we conclude that f^n has bounded distortion on γ

$$\sum_{j=0}^{n-1} \big|\log|f'(f^j(x))| - \log|f'(f^j(y))|\big| \le \text{const}\,4^N. \tag{6.39}$$

In equivalent terms, $\hat{f}^n$ has bounded distortion on η, as we had claimed. In particular, in this case we may take $\varepsilon_1 = \big(\varepsilon_2/m(I)\big)\exp(-K_1)$, where $K_1 > 0$ denotes the right hand term in (5.39).

Now the remaining cases can be treated easily. If η is not contained in E_0 then we define $p_0 + 1 \geq 1$ to be the first iterate for which $\hat{f}^{p_0+1} \subset E_0$. Then we modify the first condition in (a) to $\hat{f}^j(\eta) \subset E_0$ for $p_0 + 1 \leq j \leq \nu_1$. The sum

$$\sum_{j=p_0+1}^{\nu_1-1} \big| \log|f'(f^j(x))| - \log|f'(f^j(y))| \big|$$

is estimated in just the same way as (5.34). For the sum over $0 \leq j \leq p_0$ we use a simpler version of (5.37): since $\hat{f}^j(\eta) \subset E_{k(0)+j}$,

$$\begin{aligned}\sum_{j=0}^{p_0} \big| \log|f'(f^j(x))| - \log|f'(f^j(y))| \big| &\leq \sum_{j=0}^{p_0} \text{const}\, e^{\alpha(j+k(0))} e^{-\beta(j+k(0))} \\ &\leq \sum_{k=1}^{\infty} \text{const}\, e^{(\alpha-\beta)k} \leq \text{const}\,. \end{aligned} \tag{6.40}$$

Thus, this last sum just adds a constant term to (5.38), and so does not affect the conclusion in (5.39): $\hat{f}^n$ has bounded distortion on η also in this case.

Finally, suppose that $\hat{f}^n(\eta)$ is not contained in E_0. Then we let $\nu = \nu_s$ be the last iterate for which $\hat{f}^\nu(\eta) \subset E_0$, and we do not define p_s. The previous case shows that $\hat{f}^\nu$ has bounded distortion on η, cf. (5.39).

$$\sum_{j=0}^{\nu-1} \big| \log|f'(f^j(x))| - \log|f'(f^j(y))| \big| \leq K_1\,.$$

In general, we can not expect $\hat{f}$ to have bounded distortion on $\hat{f}^\nu(\eta)$: for instance, this last interval may contain the critical point c. But it is easy to see that, given $\varepsilon_3 > 0$ there is $\varepsilon_4 > 0$ such that for every $\zeta \subset \hat{f}^\nu(\eta)$

$$\frac{m(\zeta)}{m(\hat{f}^\nu(\eta))} \leq \varepsilon_3 \quad \Rightarrow \quad \frac{m(\hat{f}(\zeta))}{m(\hat{f}^{\nu+1}(\eta))} \leq \varepsilon_4 \tag{6.41}$$

Finally, $\hat{f}^{n-\nu-1}$ has bounded distortion on $\hat{f}^{\nu+1}(\eta)$: as in (5.40), we find

$$\sum_{j=\nu+1}^{n-1} \big| \log|f'(f^j(x))| - \log|f'(f^j(y))| \big| \leq K_2\,,$$

for some $K_2 > 0$ independent of ν, n, and η. Now the conclusion of the lemma can be deduced as follows. Given $\varepsilon_2 > 0$, take $\varepsilon_4 = \big(\varepsilon_2/m(I)\big)\exp(-K_2)$. Next, take $\varepsilon_3 > 0$ as in (5.41). Finally, let $\varepsilon_1 = \varepsilon_3 \exp(-K_1)$. Then,

$$\frac{m(\xi)}{m(\eta)} \le \varepsilon_1 \;\Rightarrow\; \frac{m(\hat f^{\nu}(\xi))}{m(\hat f^{\nu}(\eta))} \le \varepsilon_3 \;\Rightarrow\; \frac{m(\hat f^{\nu+1}(\xi))}{m(\hat f^{\nu+1}(\eta))} \le \varepsilon_4 \;\Rightarrow\; \frac{m(\hat f^{n}(\xi))}{m(\hat f^{n}(\eta))} \le \frac{\varepsilon_2}{m(I)}$$

in particular, $m(\hat f^n(\xi)) \le \varepsilon_2$. □

Let $\mathcal{B}$ be the Borel σ-algebra of I and $\widehat{\mathcal{B}}$ be the Borel σ-algebra of $\hat I$. By definition, the invariant measure μ_0 is *exact* for f if

$$B \in \mathcal{B}_\infty = \bigcap_{n=0}^{\infty} f^{-n}(\mathcal{B}) \quad \Rightarrow \quad \mu_0(B) = 0 \;\text{ or }\; \mu_0(I \setminus B) = 0.$$

Analogously, we say that $\hat\mu_0$ is exact for $\hat f$ if

$$\hat B \in \widehat{\mathcal{B}}_\infty = \bigcap_{n=0}^{\infty} f^{-n}(\widehat{\mathcal{B}}) \quad \Rightarrow \quad \hat\mu_0(\hat B) = 0 \;\text{ or }\; \hat\mu_0(\hat I \setminus \hat B) = 0.$$

We continue to denote $\pi : \hat I \to I$ the canonical projection.

Lemma 6.4.3. *1. If $A \subset I$ belongs in $\mathcal{B}_\infty$ then $\pi^{-1}(A) \subset \hat I$ belongs in $\widehat{\mathcal{B}}_\infty$.*
2. For any $\hat A \subset \hat I$ in $\widehat{\mathcal{B}}_\infty$ there are $A_- \subset A_+ \subset I$ so that $\pi^{-1}(A_-) \subset \hat A \subset \pi^{-1}(A_+)$ and $A_+ \setminus A_-$ is a countable set.

Proof. The first part is easy: if $A = f^{-n}(A_n)$ for some Borel subset $A_n \subset I$, then

$$\begin{aligned} x \in \pi^{-1}(A) &\Leftrightarrow \pi(x) \in A \Leftrightarrow \pi(\hat f^n(x)) = f^n(\pi(x)) \in A_n \\ &\Leftrightarrow \hat f^n(x) \in \pi^{-1}(A_n) \Leftrightarrow x \in \hat f^{-n}(\pi^{-1}(A_n)), \end{aligned}$$

that is, $\pi^{-1}(A) = \hat f^{-n}(\pi^{-1}(A_n))$.

To prove part (2), let $A_+ = \pi(\hat A)$ and

$$A_- = A_+ \setminus \bigcup_{n=0}^{\infty} f^{-n}(\{c_j : j \ge 0\}).$$

It is clear that $\hat A \subset \pi^{-1}(A_+)$, so let us prove that $\pi^{-1}(A_-) \subset \hat A$. Given any $z \in A_-$ there exists some $\xi \in \hat A$ such that $\pi(\xi) = z$. Then we only have to show that any other $\eta \in \hat I$ such $\pi(\eta) = z$ also belongs in $\hat A$. Now, the elements of $\widehat{\mathcal{B}}_\infty$ are characterized by the property

$$\big[\zeta_1 \in \hat A \text{ and } \hat f^n(\zeta_1) = \hat f^n(\zeta_2) \text{ for some } n \ge 1\big] \quad \Rightarrow \quad \zeta_2 \in \hat A.$$

Therefore, we are left to show that for any ξ and η as above there is $n \ge 1$ such that $\hat f^n(\xi) = \hat f^n(\eta)$. To this end, since $\pi(\hat f^n(\xi)) = \pi(\hat f^n(\eta))$ for every $n \ge 1$, it suffices to show that there exists $n \ge 1$ such that $\hat f^n(\xi)$ and $\hat f^n(\eta)$ belong both in E_0.

To prove this we introduce the following notion. Given $x \in (-\delta, \delta)$, we define the *falling time* $p(x)$ of x to be the smallest integer $j \geq 1$ such that $\hat{f}^{j+1}(x, 0) \in E_0$. The same kind of argument as in (5.15) gives, recall also (U1),

$$e^{-\beta p(x)} \geq |f^{p(x)} - c_{p(x)}| \geq \frac{1}{C}|(f^{p(x)-1})'(c_1)|\,|c_1 - f(x)| \geq \frac{1}{C\lambda_c}\lambda_c^{p(x)}x^2. \tag{6.42}$$

Fix $\gamma = 1 - e^{\alpha-\beta} > 0$. Up to taking δ small, we may suppose that $p(x) \geq H(\delta)$ is large enough so that the previous relation implies

$$\lambda_c^{p(x)}x^2 \leq \gamma^2 \qquad \text{(in particular } x \neq 0 \Rightarrow p(x) < \infty). \tag{6.43}$$

Let us write $\xi = (z, k)$ and $\eta = (z, l)$. The definition of A_- ensures that the f-orbit of $z \in A_-$ is disjoint from the critical orbit, and so $p(f^n(z))$ is finite for every $n \geq 1$. Suppose there is no $n \geq 1$ such that both $\hat{f}^n(\xi)$ and $\hat{f}^n(\xi)$ are in E_0. Then each of their orbits must start climbing the tower again before the other one falls down to E_0. That is, there must be an infinite sequence of times $0 < \nu_1 < \nu_2 < \cdots$ such that $f^{\nu_i}(z) \in (-\delta, \delta)$ (one of the orbits moves from E_0 to E_1) and $\nu_{i+1} \leq \nu_i + p(f^{\nu_i}(z))$ (while the other is still climbing up) for all $i \geq 1$. To check that this leads to a contradiction, we write $p_i = p(f^{\nu_i}(z))$ and note that

$$\begin{aligned}\nu_{i+1} - \nu_i \leq p_i \quad &\Rightarrow \quad |f^{\nu_{i+1}}(z) - c_{(\nu_{i+1}-\nu_i)}| \leq e^{-\beta_1(\nu_{i+1}-\nu_i)} \\ &\Rightarrow \quad |f^{\nu_{i+1}}(z)| \geq \gamma e^{-\alpha(\nu_{i+1}-\nu_i)} \geq \gamma e^{-\alpha p_i}.\end{aligned}$$

(in the last implication we use (U2)) Combining this with (5.43) and $e^{2\alpha} < \sqrt{\lambda_c}$,

$$\gamma^2 \geq \lambda_c^{p_{i+1}}|f^{\nu_{i+1}}(z)|^2 \geq \gamma^2\lambda_c^{p_{i+1}}e^{-2\alpha p_i} \geq \gamma^2\lambda_c^{p_{i+1}-(p_i/2)},$$

and so $p_{i+1} \leq p_i/2$ for every $i \geq 1$. Since the p_i are positive integers, the sequence p_i can not be infinite. This gives us the contradiction we were looking for. □

Proposition 6.4.1. *The measure $\hat{\mu_0}$ is exact for $\hat{f}$ and μ_0 is exact for f.*

Proof. First we prove that $\hat{\mu}_0$ is exact. Let $\hat{A} \in \widehat{\mathcal{B}}_\infty$, that is, for every $j \geq 1$ there exists a Borel set $\hat{A}_j$ such that $\hat{A} = \hat{f}^{-j}(\hat{A}_j)$. We want to show that if $\hat{\mu_0}(\hat{A}) > 0$ then $\hat{\mu_0}(\hat{I} \setminus \hat{A}) = 0$. Fix $\varepsilon > 0$ once and for all, small enough so that $\mu_0(\hat{A}) > 3\varepsilon$. Let θ be an arbitrary constant in $(0, 1)$. By Lebesgue differentiation theorem and the fact that $\hat{\mu}_0$ is absolutely continuous with respect to Lebesgue measure m on $\hat{I}$, there exist $\hat{B}_\theta \subset \hat{A}$ and $r > 0$ such that $\mu_0(\hat{B}_\theta) \geq 2\varepsilon$ and

$$\frac{m(J \cap \hat{A})}{m(J)} \geq 1 - \theta$$

for every closed interval J with length less than r containing some point $\xi \in \hat{B}_\theta$. Let $N = N(\varepsilon)$ be as in Lemma 5.10, and fix $n \geq 1$ large enough so that all the elements of $\mathcal{P}^{(n+N)}$ have length less than r. Since $\mu_0(\hat{B}_\theta) \geq 2\varepsilon$, some element η_θ of $\mathcal{Q}(n, N)$ must intersect $\hat{B}_\theta$, recall Lemma 5.10(2). Then

$$\frac{m(\eta_\theta \setminus \hat{A})}{m(\eta_\theta)} \leq \theta.$$

Let $\varepsilon_2 > 0$ be any small number, and $\varepsilon_1 > 0$ be as given by Lemma 5.11. We choose $\theta \leq \varepsilon_1$, so that the previous relation implies

$$m\big(\hat{f}^n(\eta_\theta) \setminus \hat{A}_n\big) = m\big(\hat{f}^n(\eta_\theta \setminus \hat{A})\big) \leq \varepsilon_2$$

(given $\xi \in \eta_\theta$ then $\hat{f}^n(\xi)$ is in $\hat{A}_n$ if and only if ξ is in $\hat{A} = \hat{f}^{-n}(\hat{A}_n)$). By the construction of $\mathcal{Q}(n, N)$ in Lemma 5.10(1), $\zeta_\theta = \hat{f}(\eta_\theta)$ is an element of the partition $\mathcal{P}^{(n)}$ contained in some level E_l of the tower with $l \leq N$. Observe that there are only finitely many such intervals ζ_θ. Hence, there is $q \geq 1$ depending only on N (and so completely determined by $\varepsilon > 0$) such that

$$\pi(\hat{f}^q(\zeta_\theta)) = f^q(\pi(\zeta_\theta)) = I_* \tag{6.44}$$

Let ζ_i, $1 \leq i \leq \kappa$, be the (nonempty) intersections of ζ_θ with elements of the partition $\mathcal{P}^{(q)}$. Since $\zeta_\theta \subset \cup_{l \leq N} E_l$, the number κ of such intersections is finite and depends only on N (since q is also determined by N). It is easy to see that given any $\varepsilon_3 > 0$ there exists $\varepsilon_2 > 0$ such that

$$m(\hat{B}) \leq \varepsilon_2 \quad \Rightarrow \quad m(\hat{f}^q(\hat{B})) \leq \varepsilon_3$$

for every subset $\hat{B}$ of the $\hat{I}$: just take $\varepsilon_2 = \varepsilon_3/4^q$ and use $|(f^q)'| \leq 4^q$. Then, from

$$m(\zeta_i \setminus \hat{A}_n) \leq m(\zeta_\theta \setminus \hat{A}_n) \leq \varepsilon_2$$

we find

$$m\big(\hat{f}^q(\zeta_i) \setminus \hat{A}_{n+q}\big) = m\big(\hat{f}^q(\zeta_i \setminus \hat{A}_n)\big) \leq \varepsilon_3 \,.$$

As a consequence,

$$m\Big(\bigcup_{i=1}^{\kappa} \pi\big(\hat{f}^q(\zeta_i) \setminus \hat{A}_{n+q}\big)\Big) \leq \kappa\varepsilon_3 \leq \varepsilon_4$$

where $\varepsilon_4 > 0$ can be made arbitrarily small by reducing ε_3 (which corresponds to reducing θ) without changing κ (that is, keeping ε and N fixed). Recall also that μ_0 is absolutely continuous with respect to Lebesgue measure m on I. It follows that, given any $\varepsilon_5 > 0$, one has

$$\mu_0\Big(\bigcup_{i=1}^{\kappa} \pi\big(\hat{f}^q(\zeta_i) \setminus \hat{A}_{n+q}\big)\Big) \leq \varepsilon_5 \tag{6.45}$$

if θ (and so also ε_4) are taken small enough.

At this point we are close to our goal, which is to prove that $\hat{I} \setminus \hat{A}$ has zero measure. Observe that

$$\hat{\mu}_0(\hat{I} \setminus \hat{A}) = \hat{\mu}_0(\hat{f}^{n+q}(\hat{I}) \setminus \hat{A}_{n+q}),$$

simply because $\hat{\mu_0}$ is $\hat{f}$-invariant and $\hat{A} = \hat{f}^{-(n+q)}(\hat{A}_{n+q})$. We claim that

$$\pi\big(\hat{f}^{n+q}(\hat{I}) \setminus \hat{A}_{n+q}\big) \subset \left(\bigcup_{i=1}^{\kappa} \pi\big(\hat{f}^{q}(\zeta_i) \setminus \hat{A}_{n+q}\big)\right) \cup \left(\bigcup_{l \geq 0} f^{-l}(\{c_j : j \geq 0\}\right). \tag{6.46}$$

This last set is only countable, and so the combination of (5.45) and (5.46) implies that

$$\hat{\mu}_0(\hat{I} \setminus \hat{A}) = \hat{\mu}_0(\hat{f}^{n+q}(\hat{I}) \setminus \hat{A}_{n+q}) \leq \varepsilon_5.$$

Since ε_5 is arbitrary, this gives that $\hat{\mu}_0(\hat{I} \setminus \hat{A}) = 0$ as we wanted to show. Thus, all that remains to be done to conclude that $\hat{\mu}_0$ is exact is to prove the claim (5.46).

For this, let ξ_1 be any point in $\hat{f}^{n+q}(\hat{I}) \setminus \hat{A}_{n+q}$, and take $\tau_1 \in \hat{I}$ such that $\hat{f}^{n+q}(\tau_1) = \xi_1$. By (5.44) there exists $1 \leq i \leq \kappa$ and $\xi_2 \in \hat{f}^{n+q}(\zeta_i)$ such that $\pi(\xi_1) = \pi(\xi_2)$. Moreover, $\zeta_i \subset \hat{f}^{n+q}(\hat{I})$, by construction, and so we may take $\tau_2 \in \hat{I}$ such that $\hat{f}^{n+q}(\tau_2) = \xi_2$. We need to show that either

$$\pi(\xi_1) = \pi(\xi_2) \in \bigcup_{l \geq 0} f^{-l}(\{c_j : j \geq 0\}). \tag{6.47}$$

or else $\xi_2 \notin \hat{A}_{n+q}$. Suppose (5.47) does not hold. Then, as in Lemma 5.12, there exists $j \geq 0$ such that

$$\hat{f}^{n+q+j}(\tau_1) = \hat{f}^{j}(\xi_1) = \hat{f}^{j}(\xi_2) = \hat{f}^{n+q+j}(\tau_1)$$

As $\tau_1 \notin \hat{A}$, because $\xi_1 \notin \hat{A}_{n+q}$, it follows that $\tau_2 \notin \hat{A}$, and so

$$\xi_2 \in \hat{f}^{n+q}(\zeta_i) \setminus \hat{A}_{n+q}.$$

This establishes the claim (5.46). We have shown that $\hat{\mu}_0$ is an exact measure for $\hat{f}$.

The statement that μ_0 is exact is an immediate consequence: given any $A \in \mathcal{B}_\infty$ the preimage $\pi^{-1}(A)$ belongs in $\widehat{\mathcal{B}}_\infty$, by Lemma 5.12(1), and then $\mu_0(A) = \hat{\mu}_0(\pi^{-1}(A))$ is either 0 or 1. □

Proposition 6.4.2. *There exists $\tau < 1$ such that the spectrum of the operator $\mathcal{L}_0$ acting on $BV(\hat{I})$ may be written* $\operatorname{spec}(\mathcal{L}_0) = \{1\} \cup \Sigma_0$*, where 1 is a simple eigenvalue and Σ_0 is contained in the disk of radius τ. The corresponding invariant splitting is* $\mathrm{BV}(\hat{I}) = \mathbb{R}\varphi_0 \oplus X_0$*, where* $X_0 = \{\varphi \in \mathrm{BV}(\hat{I}) : \int \varphi \, dm_0 = 0\}$*. In particular, the spectral projection π_1 associated to the eigenvalue 1 is given by $\pi_1(\varphi) = \varphi_0 \int \varphi \, dm_0$.*

Proof. It follows from Proposition 5.6 that

$$\|\mathcal{L}_0^n\varphi\|_{\rm BV} \le C\bar\sigma^{-n}(\operatorname{var}\varphi + \sup|\varphi|) + C\int|\varphi|\,dm_0 \le C\|\varphi\|_{\rm BV}$$

for every $\varphi \in \mathrm{BV}(\hat I)$, where $C > 0$ and $\bar\sigma \in (1,\sigma)$ do not depend on φ nor on n. Then $\|\mathcal{L}_0^n\|_{\rm BV} \le C$ for every $n \ge 1$, and so the spectral radius of $\mathcal{L}_0$ is at most 1. As we already constructed a fixed point φ_0 for $\mathcal{L}_0$, the spectral radius is exactly 1. Now the proof has two main steps. First we show that the essential spectral radius of $\mathcal{L}_0$ is bounded by $1/\bar\sigma < 1$. This implies that the spectrum is the union of a finite set of eigenvalues of finite multiplicity contained in the unit circle, with a compact subset of a disc of radius $\tau_0 < 1$: τ_0 is either the essential spectral radius of $\mathcal{L}_0$ or the norm of the second largest eigenvalue. Then we deduce from Proposition 5.13 that 1 is the only eigenvalue in the unit circle, and that it has multiplicity 1.

The basic idea to bound the spectral radius of $\mathcal{L}_0$ is to show that the iterates $\mathcal{L}_0^n$ are well approximated by certain operators with finite-dimensional range. We begin by noting that, as a consequence of Remark 5.2 and Lemma 5.4, there exists $C > 0$ such that

$$\sup g^{(n)} \le C\sigma^{-n} \qquad \text{and} \qquad \operatorname{var} g^{(n)} \le C\sigma^{-n} \qquad \text{for all } n \ge 1.$$

Given any $n \ge 1$, fix $N \ge n$ such that $m_0(\cup_{k>N}E_k) < \bar\sigma^{-n}$, and then define

$$\alpha_n : BV(\hat I) \to BV(\hat I) \quad \text{and} \quad \alpha_{n,N} : BV(\hat I) \to BV(\hat I)$$

by choosing an arbitrary point x_η in each monotonicity interval $\eta \in \mathcal{P}^{(n)}$, and then setting

$$\alpha_n(\varphi) = \sum_{\eta\mathcal{P}^{(n)}} \varphi(x_\eta)\,\chi_\eta \qquad \text{and} \qquad \alpha_{n,N}(\varphi) = \alpha_n\big(\varphi\cdot\chi_{(\cup_{k\le N}E_k)}\big).$$

Observe that the range of $\alpha_{n,N}$ has finite dimension:

$$\dim \alpha_{n,N}\big(\mathrm{BV}(\hat I)\big) \le \#\{\eta \in \mathcal{P}^{(n)} : \eta \subset E_k \text{ for some } k \le N\} < \infty.$$

We claim that there is $C_0 > 0$ such that

$$\|\mathcal{L}_0^n - \mathcal{L}_0^n\alpha_{n,N}\|_{\rm BV} \le C_0\bar\sigma^{-n} \tag{6.48}$$

for every $n \ge 1$. Since each $\mathcal{L}_0^n\alpha_{n,N}$ has finite-dimensional range, it follows that the essential spectral radius of $\mathcal{L}_0$ is not bigger than $1/\bar\sigma$, as we claimed.

To prove claim (5.48) we use the relation

$$\|(\mathcal{L}_0^n - \mathcal{L}_0^n\alpha_{n,N})\varphi\|_{\rm BV} \le \|\mathcal{L}_0^n(\varphi-\varphi_N)\|_{\rm BV} + \|(\mathcal{L}_0^n - \mathcal{L}_0^n\alpha_n)\varphi_N\|_{\rm BV}\,, \tag{6.49}$$

where $\varphi_N = \varphi\cdot\chi_{(\cup_{k\le N}E_k)}$. To bound $\|\mathcal{L}_0^n(\varphi-\varphi_N)\|_{\rm BV}$ we apply Proposition 6.3.1 to the function $\varphi - \varphi_N = \varphi\cdot\chi_{(\cup_{k>N}E_k)}$. Since $\operatorname{var}(\varphi-\varphi_N) \le \operatorname{var}\varphi$ and $\sup|\varphi-\varphi_N| \le \sup|\varphi|$, we find

$$\operatorname{var} \mathcal{L}_0^n(\varphi - \varphi_N) \le C\bar{\sigma}^{-n}(\operatorname{var}\varphi + \sup|\varphi|) + C\int |\varphi - \varphi_N|\, dm_0$$

$$\sup |\mathcal{L}_0^n(\varphi - \varphi_N)| \le C\bar{\sigma}^{-n}(\operatorname{var}\varphi + \sup|\varphi|) + C\int |\varphi - \varphi_N|\, dm_0.$$

Moreover, in view of our choice of N,

$$\int |\varphi - \varphi_N|\, dm_0 \le m_0(\cup_{k>N} E_k)\, \sup|\varphi| \le \bar{\sigma}^{-n} \sup|\varphi|\,.$$

It follows that

$$\|\mathcal{L}_0^n(\varphi - \varphi_N)\|_{\mathrm{BV}} \le 4C\bar{\sigma}^{-n}(\operatorname{var}\varphi + \sup|\varphi|) \le 4C\bar{\sigma}^{-n}\|\varphi\|_{\mathrm{BV}}\,. \tag{6.50}$$

Next, we estimate $\|(\mathcal{L}_0^n - \mathcal{L}_0^n\alpha_n)\varphi_N\|_{\mathrm{BV}}$. Given a monotonicity interval η of $\hat{f}^n$ and given $y \in \hat{f}^n(\eta)$, we denote $y_\eta = (\hat{f}^n|\eta)^{-1}(y)$. Then, for every $\psi \in \mathrm{BV}(\hat{I})$,

$$\begin{aligned}\sup|(\mathcal{L}_0^n - \mathcal{L}_0^n\alpha_n)\psi| &\le \sup_y \sum_{y\in\hat{f}^n(\eta)} g^{(n)}(y_\eta)|\psi(y_\eta) - \psi(x_\eta)|\\ &\le \sum_\eta C\sigma^{-n}\operatorname*{var}_\eta\psi \quad \le C\sigma^{-n}\operatorname{var}\psi.\end{aligned}$$

Let $\psi_\eta = \psi - \psi(x_\eta)$. Note that $\sup_\eta|\psi_\eta| \le \operatorname{var}_\eta\psi = \operatorname{var}_\eta\psi_\eta$. Then, recall (5.16),

$$\begin{aligned}\operatorname{var}(\mathcal{L}_0^n - \mathcal{L}_0^n\alpha_n)\psi &= \operatorname{var}\sum_\eta \left(g^{(n)}\,\psi_\eta\,\chi_\eta\right)\circ(\hat{f}^n|\eta)^{-1}\\ &\le \sum_\eta \big(\operatorname*{var}_\eta g^{(n)} \sup_\eta|\psi_\eta| + \sup_\eta g^{(n)} \operatorname*{var}_\eta \psi_\eta + 2\sup_\eta g^{(n)}\sup_\eta|\psi_\eta|\big)\\ &\le \sum_\eta \big(4C\sigma^{-n}\operatorname*{var}_\eta\psi\big) \quad \le 4C\sigma^{-n}\operatorname{var}\psi\,,\end{aligned}$$

Finally,

$$\int |(\mathcal{L}_0^n - \mathcal{L}_0^n\alpha_n)\psi|\, dm_0 \le m_0(\hat{I})\,\sup|(\mathcal{L}_0^n - \mathcal{L}_0^n\alpha_n)\psi| \le m_0(\hat{I})\,C\,\sigma^{-n}\operatorname{var}\psi.$$

Summarizing,

$$\|(\mathcal{L}_0^n - \mathcal{L}_0^n\alpha_n)\psi\|_{\mathrm{BV}} \le K\sigma^{-n}\operatorname{var}\psi \le K\bar{\sigma}^{-n}\|\psi\|_{\mathrm{BV}}\,,$$

where $K = C(5 + m_0(\hat{I}))$. We use this relation for $\psi = \varphi_N$. Observing that $\|\varphi_N\|_{\mathrm{BV}} \le \|\varphi\|_{\mathrm{BV}}$, we get

$$\|(\mathcal{L}_0^n - \mathcal{L}_0^n\alpha_n)\varphi_N\|_{\mathrm{BV}} \le K\bar{\sigma}^{-n}\|\varphi_N\|_{\mathrm{BV}} \le K\bar{\sigma}^{-n}\|\varphi\|_{\mathrm{BV}}\,. \tag{6.51}$$

Combining in (5.49), (5.50), (5.51), we obtain (5.48) with $C_0 = 4C + K$:

$$\|(\mathcal{L}_0^n - \mathcal{L}_0^n \alpha_{n,N})\varphi\|_{\mathrm{BV}} \le (4C + K)\bar{\sigma}^{-n}\|\varphi\|_{\mathrm{BV}}.$$

Now we proceed to the second step of the proof. Let $\lambda_1 \in S^1$ be an eigenvalue of $\mathcal{L}_0$ and $\varphi_1 \in BV(\hat{I})$ be a corresponding eigenfunction: $\mathcal{L}_0\varphi_1 = \lambda_1\varphi_1$. Then $\mathcal{L}_0^n\varphi_1 = \lambda_1^n\varphi_1$ and this implies that $\varphi_1 = 0$ at every point in the complement of $\hat{f}^n(\hat{I})$, for each $n \ge 1$. Thus, by Remark 6.1.1, φ_1 is identically zero in the complement of

$$W_\delta = W_* \setminus \big([f^2(c), f^2(\delta)) \cup (f(\delta), f(c)]\big) \times \{0\}.$$

On the other hand, we showed in Lemma 5.9 that the fixed point φ_0 of $\mathcal{L}_0$ is strictly positive on W_δ. Then we may write $\varphi_1 = \phi\varphi_0$ for some function ϕ. Observe that ϕ belongs in $L^1(\hat{\mu}_0)$:

$$\int |\phi|\, d\hat{\mu}_0 = \int |\varphi_1|\, dm_0 \le \|\varphi_1\|_{\mathrm{BV}} < \infty.$$

On the other hand, Proposition 5.13 implies that the measure $\hat{\mu}_0$ is mixing for the map $\hat{f}$. Then, in particular,

$$\int (\psi \circ \hat{f}^n)\phi\, d\hat{\mu}_0 \to \int \psi\, d\hat{\mu}_0 \int \phi\, d\hat{\mu}_0,$$

for every bounded function ψ. Now, the left hand side may be written

$$\int (\psi \circ \hat{f}^n)\phi\, d\hat{\mu}_0 = \int (\psi \circ \hat{f}^n)\varphi_1\, dm_0 = \int \psi(\mathcal{L}_0^n\varphi_1)\, dm_0 = \int \psi(\lambda_1^n\varphi_1)\, dm_0,$$

and the right hand side

$$\int \psi\, d\hat{\mu}_0 \int \phi\, d\hat{\mu}_0 = \int \psi\big(\varphi_0 \int \phi\, d\hat{\mu}_0\big) dm_0 = \int \psi\big(\varphi_0 \int \varphi_1\, dm_0\big) dm_0.$$

Thence, $\lambda_1^n\varphi_1$ converges to $\varphi_0 \int \varphi_1\, dm_0$ weakly in $L^1(m_0)$. Clearly, this implies that $\lambda_1 = 1$ and $\varphi_1 = \varphi_0 \int \hat{\varphi}_1\, dm_0$. This proves that 1 is the only eigenvalue in the unit circle, and that its eigenspace kernel$(\mathcal{L}_0 - \mathrm{id})$ has dimension 1.

In fact, one can say more: $\lambda_1 = 1$ has algebraic multiplicity 1, meaning that

$$\dim \operatorname{kernel}\big((\mathcal{L}_0 - \mathrm{id})^n\big) = 1 \quad \text{for every } n \ge 1.$$

Otherwise, there would exist a nonzero function $\psi_0 \in \mathrm{BV}(\hat{I})$ such that $\mathcal{L}_0\psi_0 = \psi_0 + \varphi_0$. Then, by recurrence,

$$\mathcal{L}_0^n\psi_0 = n\psi_0 + \varphi_0 \quad \text{for every } n \ge 1,$$

which would contradict the conclusion obtained previously that the norms $\|\mathcal{L}_0^n\|$ are uniformly bounded.

Thus far we have shown that $\operatorname{spec}(\mathcal{L}_0) = \{1\} \cup \Sigma_0$, where Σ_0 is contained in a disk of radius τ, for some $\tau < 1$, and 1 is a simple eigenvalue. Finally, the splitting

$$\mathrm{BV}(\hat{I}) = \mathbb{R}\varphi_0 \oplus X_0, \qquad X_0 = \{\varphi \in \mathrm{BV} : \int \varphi \, dm_0 = 0\}$$

is invariant under $\mathcal{L}_0$, with $\operatorname{spec}(\mathcal{L}_0|\mathbb{R}\varphi_0) = \{1\}$ and $1 \notin \operatorname{spec}(\mathcal{L}_0|X_0)$. Hence, it must be the spectral splitting associated to the decomposition $\operatorname{spec}(\mathcal{L}_0) = \{1\} \cup \Sigma_0$. Clearly, the projection π_1 onto the first factor is given by $\pi_1(\varphi) = \varphi_0 \int \varphi \, dm_0$. □

Proposition 6.4.3. *Let $\bar{\tau} \in (\tau, 1)$ where τ is as in Proposition 5.14. There exists $\bar{C} > 0$ such that*

1. *given any $\hat{\varphi} \in \mathrm{BV}(\hat{I})$, any $\hat{\psi} \in L^1(m_0)$, and any $n \geq 1$,*

$$\left| \int (\hat{\psi} \circ \hat{f}^n) \hat{\varphi} \, d\hat{\mu}_0 - \int \hat{\psi} \, d\mu_0 \int \hat{\varphi} \, d\hat{\mu}_0 \right| \leq \bar{C} \, \bar{\tau}^n \, \|\hat{\varphi}\|_{\mathrm{BV}} \, \|\hat{\psi}\|_1$$

for every $n \geq 1$;

2. *given any $\varphi \in \mathrm{BV}(I)$, any bounded function $\psi : I \to \mathbb{R}$, and any $n \geq 1$,*

$$\left| \int (\psi \circ f^n) \varphi \, d\mu_0 - \int \psi \, d\mu_0 \int \varphi \, d\mu_0 \right| \leq \bar{C} \, \bar{\tau}^n \, \|\varphi\|_{\mathrm{BV}} \, \sup |\psi|$$

Proof. To prove (1) we write

$$\begin{aligned}\int (\hat{\psi} \circ \hat{f}^n) \hat{\varphi} \, d\hat{\mu}_0 - \int \hat{\psi} \, d\mu_0 \int \hat{\varphi} \, d\hat{\mu}_0 &= \int \hat{\psi} \, \mathcal{L}_0^n(\hat{\varphi} \varphi_0) \, dm_0 - \int \hat{\psi} \varphi_0 \big(\int \hat{\varphi} \varphi_0 \, dm_0 \big) \, dm_0 \\ &= \int \hat{\psi} \Big(\mathcal{L}_0^n(\hat{\varphi} \varphi_0) - \varphi_0 \big(\int \hat{\varphi} \varphi_0 \, dm_0 \big) \Big) dm_0\end{aligned}$$

Observe that

$$\begin{aligned}\operatorname{var} \hat{\varphi}\varphi_0 &= \sum_{k \geq 0} \operatorname{var}(\hat{\varphi}\varphi_0 | E_k) \\ &\leq \sum_{k \geq 0} \operatorname{var}(\hat{\varphi}|E_k) \sup(\varphi_0|E_k) + \sup |(\hat{\varphi}|E_k)| \operatorname{var}(\varphi_0|E_k) \\ &\leq \big(\sup_{k \geq 0} \operatorname{var}(\hat{\varphi}|E_k) \big) \sum_{k \geq 0} \sup(\varphi_0|E_k) + \big(\sup_{k \geq 0} \sup |(\hat{\varphi}|E_k)| \big) \sum_{k \geq 0} \operatorname{var}(\varphi_0|E_k) \\ &\leq \operatorname{var} \hat{\varphi} \sum_{k \geq 0} \sup |(\varphi_0|E_k)| + \sup |\hat{\varphi}| \sum_{k \geq 0} \operatorname{var}(\varphi_0|E_k)\end{aligned} \tag{6.52}$$

recall Remark 6.3.1. Moreover,

$$\sup|\hat\varphi\varphi_0| = \sup_{k\geq 0}\sup|(\hat\varphi\varphi_0|E_k)| \leq \sup_{k\geq 0}\big(\sup|(\hat\varphi|E_k)|\,\sup(\varphi_0|E_k)\big) \leq \sup|\hat\varphi|\,\sup\varphi_0$$

and

$$\int|\hat\varphi\varphi_0|\,dm_0 \leq \sup|\hat\varphi|\int\varphi_0\,dm_0\,.$$

This proves that,

$$\|\hat\varphi\varphi_0\|_{\mathrm{BV}} \leq K_1\|\hat\varphi\|_{\mathrm{BV}}\,,\qquad\text{and so}\qquad \hat\varphi\varphi_0\in\mathrm{BV}(\hat I), \tag{6.53}$$

where $K_1 = \sum_{l\geq 0}\sup(\varphi_0|E_k) + \operatorname{var}\varphi_0 + \int\varphi_0\,dm_0$, recall Remark 6.3.1.

Now, φ_0 being a fixed point of $\mathcal{L}_0$,

$$\mathcal{L}_0^n(\hat\varphi\,\varphi_0) - \varphi_0\Big(\int\hat\varphi\,\varphi_0\,dm_0\Big) = \mathcal{L}_0^n\Big(\hat\varphi\,\varphi_0 - \varphi_0\int\hat\varphi\,\varphi_0\,dm_0\Big) = \mathcal{L}_0^n\big(\pi_0(\hat\varphi\,\varphi_0)\big),$$

where $\pi_0(\phi) = \phi - \varphi_0\int\phi\,d\hat\mu_0$ is the projection onto the factor X_0 of the spectral splitting $\mathbb{R}\varphi_0\oplus X_0$, recall Proposition 5.14. Since $\operatorname{spec}(\mathcal{L}_0|X_0) = \Sigma_0$, which is contained in the disk of radius τ, we get that

$$\sup\big|\mathcal{L}_0^n\big(\pi_0(\hat\varphi\,\varphi_0)\big)\big| \leq \|\mathcal{L}_0^n\big(\pi_0(\hat\varphi\,\varphi_0)\big)\| \leq K_0\bar\tau^n\|\hat\varphi\,\varphi_0\|_{\mathrm{BV}} \leq (K_0K_1\,\|\varphi_0\|_{\mathrm{BV}})\,\bar\tau^n\,\|\hat\varphi\|_{\mathrm{BV}}$$

for some $K_0>0$ and every $n\geq 1$. Replacing above,

$$\begin{aligned}\Big|\int(\hat\psi\circ\hat f^n)\hat\varphi\,d\hat\mu_0 - \int\hat\psi\,d\mu_0\int\hat\varphi\,d\hat\mu_0\Big| &\leq \int|\hat\psi|\,dm_0\cdot\sup\big|\mathcal{L}_0^n\big(\pi_0(\varphi\,\varphi_0)\big)\big| \\ &\leq (K_0K_1\,\|\varphi_0\|_{\mathrm{BV}})\,\bar\tau^n\,\|\hat\varphi\|_{\mathrm{BV}}\,\|\psi\|_1\end{aligned}$$

and so it suffices to take $\bar C\geq K_0K_1\,\|\varphi_0\|_{\mathrm{BV}}$.

Now part (2) can be deduced easily. Given $\varphi,\psi\in\mathrm{BV}(I)$ define $\hat\varphi(x,k) = \varphi(x)$ and $\hat\psi(x,k) = \psi(x)$. Then

$$\int|\hat\psi|\,dm_0 \leq m_0(\hat I)\sup|\hat\psi| \leq m_0(\hat I)\sup|\psi|, \tag{6.54}$$

in particular $\hat\psi\in L^1(m_0)$. Moreover, the function $\hat\varphi$ is bounded and satisfies

$$\sup_{k\geq 0}\operatorname{var}(\hat\varphi|E_k) = \sup_{k\geq 0}\operatorname{var}(\varphi|B_k) \leq \operatorname{var}\varphi < \infty.$$

Then, as in (5.52), (5.53), $\|\hat\varphi\varphi_0\|_{\mathrm{BV}}\leq K_1\|\varphi\|_{\mathrm{BV}}$, which ensures that $\hat\varphi\varphi_0\in\mathrm{BV}(\hat I)$. So, in just the same way as in the previous situation,

$$\begin{aligned}\Big|\int(\psi\circ f^n)\varphi\,d\mu_0 - \int\psi\,d\mu_0\int\varphi\,d\mu_0\Big| &= \Big|\int(\hat\psi\circ\hat f^n)\hat\varphi\,d\hat\mu_0 - \int\hat\psi\,d\mu_0\int\hat\varphi\,d\hat\mu_0\Big| \\ &\leq (K_0K_1\,\|\varphi_0\|_{\mathrm{BV}})\,\bar\tau^n\,\|\varphi\|_{\mathrm{BV}}\,m_0(\hat I)\sup|\psi|\,.\end{aligned} \tag{6.55}$$

We just take $\bar C\geq K_0K_1\,m_0(\hat I)\,\|\varphi_0\|_{\mathrm{BV}}$. □

Corollary 6.4.1. *Let $\varphi \in \mathrm{BV}(I)$ and*

$$\sigma^2 = \int \phi^2 \, d\mu_0 + 2\sum_{j=1}^{\infty} \int \phi(\phi \circ f^j) d\mu_0, \quad \textit{where} \quad \phi = \varphi - \int \varphi \, d\mu_0.$$

Then $\sigma < \infty$ and $\sigma = 0$ if and only if $\phi = \mu_0 \circ f - u$ for some $u \in L^2(\mu_0)$. Moreover, if $\sigma > 0$ then for every interval $A \subset \mathbb{R}$

$$\mu_0 \left\{ x \in I : \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \phi(f^j(x)) \in A \right\} \longrightarrow \int_A \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{t^2}{2\sigma^2}} \, dt \qquad \textit{as } n \to +\infty.$$

Proof. This follows the steps of Corollary —refc.2.10 and Proposition **??**. It is no restriction to suppose $\int \varphi \, d\mu_0 = \int \varphi \, \varphi_0 \, dm = 0$, and we do so. Let $\mathcal{F}_n = f^{-n}(\mathcal{B})$, where $\mathcal{B}$ is the Borel σ-algebra of I. Then

$$\begin{aligned} \|E(\varphi \mid \mathcal{F}_n)\|_2 &= \sup\{\int \psi\varphi \, d\mu_0 : \psi \in L^2(\mathcal{F}_n) \text{ and } \|\psi\|_2 = 1\} \\ &= \sup\{\int (\psi \circ f^n)\varphi \, d\mu_0 : \psi \in L^2(\mu_0) \text{ and } \|\psi\|_2 = 1\}. \end{aligned}$$

Define $\hat{\varphi}(x,k) = \varphi(x)$ and $\hat{\psi}(x,k) = \psi(x)$, for each $\psi \in L^2(\mu_0)$. We claim that the function $\hat{\psi} \in L^1(m_0)$. Indeed, by Remark 5.1 and Cauchy-Schwarz inequality,

$$\begin{aligned} \int |\hat{\psi}| \, dm_0 &= \int |\hat{\psi}| \varphi_0^{-1} \, d\hat{\mu}_0 \le \left(\int \varphi_0^{-2} \, d\hat{\mu}_0 \right)^{1/2} \left(\int |\hat{\psi}|^2 \, d\hat{\mu}_0 \right)^{1/2} \\ &= \left(\int (\varphi_0 | W_*)^{-1} \, dm_0 \right)^{1/2} \left(\int |\psi|^2 \, d\mu_0 \right)^{1/2}. \end{aligned}$$

Using (5.30) and (5.32), there exists some constant $K_3 > 0$ so that

$$\begin{aligned} \int (\varphi_0 | W_*)^{-1} \, dm_0 &\le \sum_{k=0}^{\infty} m_0(E_k) \sup(\varphi_0 | W_* \cap E_k)^{-1} \le \sum_{k=0}^{\infty} K_3 \lambda^{2k} \lambda_c^{-k/2} e^{-\beta k/2} \\ &\le \sum_{k=0}^{\infty} K_3 e^{-(\alpha+\beta)k} < \infty \end{aligned}$$

since we have chosen $\lambda < \rho$ and $\lambda \rho e^{\alpha} < \sqrt{\lambda_c}$. Thus, denoting K_4 the square root of the last term in the previous expression,

$$\int |\hat{\psi}| \, dm_0 \le K_4 \|\psi\|_2 < \infty,$$

proving our claim. Then, compare (5.55),

$$\|E(\varphi \mid \mathcal{F}_n)\|_2 \leq \sup\{\int (\hat{\psi} \circ \hat{f}^n)\hat{\varphi}\,\varphi_0\, dm_0 : \hat{\psi} \in L^2(m_0) \text{ and } \|\hat{\psi}\|_1 \leq K_2\}$$
$$\leq (K_0 K_1 \|\varphi_0\|_{\mathrm{BV}})\bar{\tau}^n \|\varphi\|_{\mathrm{BV}} K_2 \quad \leq K_5 \bar{\tau}^n \|\varphi\|_{\mathrm{BV}} \,,$$

where $K_5 = K_0 K_1 K_2 \|\varphi_0\|_{\mathrm{BV}}$. In particular $\sum_{n=0}^{\infty} \|E(\varphi \mid \mathcal{F}_n)\|_2^2$ is finite, and so the corollary follows directly from Theorem 2.11. □

Abundance (positive Lebesgue measure set of parameter values) of quadratic maps with absolutely continuous invariant measure is a result of Jakobson [59]. Several other proofs and extensions appeared since then, e.g. [**?**], [12], [13], [86].

Exponential decay of correlations was proved independently by [66] and [**?**]. They used (other) tower extensions and cocycles, in much the same way we do here.

Stochastic stability was proved by [61] when the critical point is non-recurrent, corresponding to an uncountable zero Lebesgue measure set of parameters, and by [14] for a set of parameters with positive Lebesgue measure slightly different from the one we consider here. [61] considered a more general random perturbation scheme. Both papers dealt with stochastic stability in the weak sense: weak* convergence of the stationary measure to the absolutely continuous invariant measure. Strong stochastic stability (L^1 convergence) had been obtained in a unpublished paper of [31].

Our presentation is based on [10], who proved the full statement of Theorem 5.1, together with a result of exponential decay of correlations for the random perturbations scheme. The approach is inspired by the treatment of uniformly expanding maps in [11].

7. Non-Uniformly Hyperbolic Systems

7.1 Lorenz-like Attractors of Flows

In this section we prove that Lorenz-like attractors Λ support a unique physical measure μ_X, whose basin contains a full Lebesgue measure subset of the topological basin of attraction $B(\Lambda)$. We obtain this physical measure from the unique ergodic absolutely continuous invariant measure μ_f of the quotient map f (we keep the notations of the previous section), through two general constructions that we present in detail in Subsection 7.1.1.

We want to explain how physical measures of Lorenz-like attractors of flows may be constructed from the information on Lorenz-like maps we gathered in Chapter 1. See also [101] and [32].

We begin by recalling some basic facts concerning the geometric Lorenz models of [2], [51]. By construction, these systems have a two-dimensional submanifold Σ as a partial cross-section to the flow. More precisely, there is a curve $\Gamma \subset \Sigma$ and a well-defined Poincaré first-return map

$$F : \Sigma \setminus \Gamma \to \Sigma.$$

The curve Γ corresponds to the intersection of Σ with the stable manifold of the singularity O contained in the attractor, and future trajectories of points in Γ do not come back to Σ. The Poincaré map F is hyperbolic, in the following sense:

1. F admits an invariant contracting smooth foliation $\mathcal{F}^s$ containing Γ as a leaf; that is, every leaf $\mathcal{F}^s_z$ is mapped by F completely inside some leaf $\mathcal{F}^s_{F(z)}$, and $F|\mathcal{F}^s_z$ is a uniform contraction;
2. the space of leaves $\Sigma/\mathcal{F}^s$ of this foliation $\mathcal{F}^s$ is diffeomorphic to an interval, and the map f induced by F on $\Sigma/\mathcal{F}^s$ is a Lorenz-like map: it is uniformly expanding, with derivative tending to infinity as one approaches Γ.

The first step is to construct a physical measure μ_F for the Poincaré return map F.
***

Moreover, μ is ergodic and so it is an SRB measure for f. Recall also that the density $d\mu/dm$ is a function with bounded variation, in particular it is bounded.

From μ we may now construct an SRB measure μ_F for the , through the following general procedure [22]. Since μ is defined on the interval, identified to the space of leaves of the contracting foliation $\mathcal{F}^s$, we may also think of it as a measure on the σ-algebra of Borel subsets of Σ which are union of entire leaves of $\mathcal{F}^s$. Using the fact that F is uniformly contracting on leaves of $\mathcal{F}^s$, one concludes that the sequence

$$F_*^n \mu, \qquad n \geq 1,$$

of push-forwards of μ under F is weak*-Cauchy: given any continuous $\varphi : \Sigma \to \mathbb{R}$

$$\int \varphi \, d(F_*^n \mu) = \int (\varphi \circ F^n) \, d\mu, \qquad n \geq 1,$$

is a Cauchy sequence in $\mathbb{R}$. Define, μ_F to be the weak*-limit of this sequence, that is,

$$\int \varphi \, d\mu_F = \lim \int \varphi \, d(F_*^n \mu)$$

for each continuous φ. Then μ_F is invariant under F, and it is an SRB measure for the Poincaré return map: this last statement follows from the fact that μ is an SRB measure for f, together with the remark that asymptotic time-averages of continuous functions $\varphi : \Sigma \to \mathbb{R}$ are constant on the leaves of $\mathcal{F}^s$.

Next, another general procedure yields an SRB measure μ_X for the flow. Denote $\tau : \Sigma \setminus \Gamma \to (0, +\infty)$ the return time to Σ, defined by,

$$F(z) = X_{\tau(z)}.$$

Then τ is bounded away from zero. Moroever,

$$F(z) \approx |\log d(z, \Gamma)|$$

for z close to Γ. Combining this with the definition of μ_F and the remark made above that $d\mu/dm$ is a bounded function, one may conclude that

$$\tau_0 = \int \tau \, d\mu_F < \infty. \tag{7.1}$$

Denote by $\sim$ the equivalence relation on $\Sigma \times \mathbb{R}$ generated by $(z, \tau(z)) \sim (F(z), 0)$. Let $N = (\Sigma \times \mathbb{R})/\sim$ and $\nu = \pi_*(\mu_F \times dt)$, where $\pi : \Sigma \times \mathbb{R} \to N$ is the quotient map and dt is Lebesgue measure in $\mathbb{R}$. Observe that (6.6) means that ν is a finite measure. Define $\phi : N \to M$ by $\phi(z,t) = X_t(z)$, and let $\mu_X = \phi_* \mu_F$. The measure μ_X is invariant under the flow, and one can check that it is an SRB for the flow X:

$$\frac{1}{T} \int_0^T \varphi(X_t(z)) \, dt \to \int \varphi \, d\mu_X \quad \text{as } T \to +\infty$$

for every continuous function $\varphi : M \to \mathbb{R}$, and Lebesgue almost every point $z \in \phi(N)$.

7.1.1 Suspending Invariant Measures

Let Σ be a compact metric space, $\Gamma \subset \Sigma$, and $F : (\Sigma \setminus \Gamma) \to \Sigma$ be a measurable map. We assume that there exists a partition $\mathcal{F}$ of Σ into measurable subsets, having Γ as an element, which is

- *invariant:* the image of any $\xi \in \mathcal{F}$ distinct from Γ is contained in some element η of $\mathcal{F}$;
- *contracting:* the diameter of $F^n(\xi)$ goes to zero when $n \to \infty$, uniformly over all the $\xi \in \mathcal{F}$ for which $F^n(\xi)$ is defined.

We denote $p : \Sigma \to \mathcal{F}$ the canonical projection, i.e. p assigns to each point $x \in \Sigma$ the atom $\xi \in \mathcal{F}$ that contains it. By definition, $A \subset \mathcal{F}$ is measurable if and only if $p^{-1}(A)$ is a measurable subset of Σ.

The invariance condition means that there is a uniquely defined map

$$f : (\mathcal{F} \setminus \{\Gamma\}) \to \mathcal{F} \quad \text{such that} \quad f \circ p = p \circ F.$$

Clearly, f is measurable with respect to the measurable structure we introduced in $\mathcal{F}$. Let μ_f be any probability measure on $\mathcal{F}$ invariant under the transformation f. For any bounded function $\psi : \Sigma \to \mathbb{R}$, let $\psi_- : \mathcal{F} \to \mathbb{R}$ and $\psi_+ : \mathcal{F} \to \mathbb{R}$ be defined by

$$\psi_-(\xi) = \inf_{x\in\xi} \psi(x) \qquad \psi_+(\xi) = \sup_{x\in\xi} \psi(x).$$

Lemma 7.1.1. *Given any continuous function $\psi : \Sigma \to \mathbb{R}$, both limits*

$$\lim_n \int (\psi \circ F^n)_- \, d\mu_F \quad \textit{and} \quad \lim_n \int (\psi \circ F^n)_+ \, d\mu_F$$

exist, and they coincide.

Proof. Let ψ be fixed as in the statement. Given $\varepsilon > 0$, let $\delta > 0$ be such that $|\psi(x_1) - \psi(x_2)| \le \varepsilon$ for all x_1, x_2 with $d(x_1, x_2) \le \delta$. Since the partition $\mathcal{F}$ is assumed to be contractive, there exists $n_0 \ge 0$ such that $\operatorname{diam}(F^n(\xi)) \le \delta$ for every $\xi \in \mathcal{F}$ and any $n \ge n_0$. Let $n + p \ge n \ge n_0$. By definition,

$$(\psi \circ F^{n+p})_-(\xi) - (\psi \circ F^n)_-(f^p(\xi)) = \inf\big(\psi \mid F^{n+p}(\xi)\big) - \inf\big(\psi \mid F^n(f^p(\xi))\big).$$

Observe that $F^{n+p}(\xi) \subset F^n(f^p(\xi))$. So the difference on the right hand side is bounded by

$$\sup\big(\psi \mid F^n(f^p(\xi))\big) - \inf\big(\psi \mid F^n(f^p(\xi))\big) \le \varepsilon.$$

Therefore,

$$\Big| \int (\psi \circ F^{n+p})_- \, d\mu_f - \int (\psi \circ F^n)_- \circ f^p \, d\mu_f \Big| \le \varepsilon.$$

Moreover, one may replace the second integral by $\int(\psi \circ F^n)_- \, d\mu_f$, because μ_f is f-invariant.

At this point we have shown $\int(\psi \circ F^n)_- \, d\mu_F$, $n \geq 1$, is a Cauchy sequence in $\mathbb{R}$. In particular, it converges. The same argument proves that $\int(\psi \circ F^n)_+ \, d\mu_F$ is also convergent. Moreover, keeping the previous notations,

$$(\psi \circ F^n)_+(\xi) - (\psi \circ F^n)_-(\xi) = \sup\big(\psi \mid F^n(\xi)\big) - \inf\big(\psi \mid F^n(\xi)\big) \leq \varepsilon$$

for every $n \geq n_0$. So the two sequences must have the same limit. The lemma is proved.

Corollary 7.1.1. *There exists a unique probability measure μ_F on Σ such that*

$$\int \psi \, d\mu_F = \lim \int (\psi \circ F^n)_- \, d\mu_f = \lim \int (\psi \circ F^n)_+ \, d\mu_f.$$

for every continuous function $\psi : \Sigma \to \mathbb{R}$. Besides, μ_F is invariant under F.

Proof. Let $\hat{\mu}(\psi)$ denote the value of the two limits. Using the expression for $\hat{\mu}(\psi)$ in terms of $(\psi \circ F^n)_-$ we immediately get that

$$\hat{\mu}(\psi_1 + \psi_2) \geq \hat{\mu}(\psi_1) + \hat{\mu}(\psi_2).$$

Analogously, the expression of $\hat{\mu}(\psi)$ in terms of $(\psi \circ F^n)_+$ gives the opposite inequality. So, the function $\hat{\mu}(\cdot)$ is additive. Moreover, $\hat{\mu}(c\psi) = c\hat{\mu}(\psi)$ for every $c \in \mathbb{R}$ and every continuous function ψ. Therefore, $\hat{\mu}(\cdot)$ is a linear real operator in the space of continuous functions $\psi : \Sigma \to \mathbb{R}$.

Clearly, $\hat{\mu}(1) = 1$ and the operator $\hat{\mu}$ is non-negative: $\hat{\mu}(\psi) \geq 0$ if $\psi \geq 0$. By the Riesz-Markov theorem, there exists a unique measure μ_F on Σ such that $\hat{\mu}(\psi) = \int \psi \, d\mu_F$ for every continuous ψ. To conclude that μ_F is invariant under F it suffices to note that

$$\hat{\mu}(\psi \circ F) = \lim_n \int (\psi \circ F^{n+1})_- \, d\mu_f = \hat{\mu}(\psi)$$

for every ψ. This finishes the proof of the corollary.

Remark 7.1.1. Note that $\int \psi \, d\mu_F = \lim_n \int (\psi \circ F^n)_\# \, d\mu_F$ for any choice of a sequence $(\psi \circ F^n)_\# : \mathcal{F} \to \mathbb{R}$ with

$$\inf(\psi \mid F^n(\xi)) \leq (\psi \circ F^n)_\#(\xi) \leq \sup(\psi \mid F^n(\xi)).$$

Lemma 7.1.2. *Let $\psi : \Sigma \to \mathbb{R}$ be a continuous function, and $\xi \in \mathcal{F}$ be such that*

$$\lim_n \frac{1}{n} \sum_{j=0}^{n-1} (\psi \circ F^k)_-(f^j(\xi)) = \int (\psi \circ F^k)_- \, d\mu_f$$

for every $k \geq 1$. Then $\lim_n \frac{1}{n} \sum_{j=0}^{n-1} \psi(F^j(x)) = \int \psi \, d\mu_F$ for every $x \in \xi$.

Proof. ***

Corollary 7.1.2. *If μ_f is ergodic for f then μ_F is ergodic for F.*

Let Σ be a measurable space, Γ be some measurable subset of Σ, and $F : (\Sigma \setminus \Gamma) \to \Sigma$ be a measurable map. Let $\tau : \Sigma \to (0, +\infty]$ be a measurable function such that $\inf \tau > 0$ and $\tau \equiv +\infty$ on Γ.

We denote by $\sim$ the equivalence relation on $\Sigma \times [0, +\infty)$ generated by $(x, \tau(x)) \sim (F(x), 0)$. That is, $(x, s) \sim (\tilde{x}, \tilde{s})$ if and only if there exist

$$(x, s) = (x_0, s_0),\ (x_1, s_1),\ \ldots,\ (x_N, y_N) = (\tilde{x}, \tilde{s})$$

in $\Sigma \to (0, +\infty]$ such that, for every $1 \le i \le N$, either

$$\begin{aligned} \text{either} \quad & x_i = F(x_{i-1}) \quad \text{and} \quad s_i = s_{i-1} - \tau(x_{i-1}) \\ \text{or} \quad & x_{i-1} = F(x_i) \quad \text{and} \quad s_{i-1} = s_i - \tau(x_i). \end{aligned}$$

We denote $V = \Sigma \times [0, +\infty)/ \sim$ the corresponding quotient space, and let $\pi : \Sigma \to V$ be the canonical projection.

Definition 7.1.1. *The* suspension of F with return-time τ *is the semi-flow* $(X^t)_{t \ge 0}$ *defined on* V *by*

$$X^t(\pi(x, s)) = \pi(x, s + t) \quad \textit{for every } (x, s) \in \Sigma \times [0, +\infty) \textit{ and } t > 0.$$

It is easy to see that this is indeed well defined.

Remark 7.1.2. If F is injective then we can also define

$$X^{-t}\big(\pi(x, s)\big) = \pi\big(F^{-n}(x), s + \tau(F^{-n}(x)) + \cdots + \tau(F^{-1}(x)) - t\big)$$

for every $x \in F^n(\Sigma)$ and $0 < t \le s + \tau(F^{-n}(x)) + \cdots + \tau(F^{-1}(x))$. The expression on the righ does not depend on the choice of $n \ge 1$. In particular, the restriction of the semi-flow $(X_t)_{t \ge 0}$ to the maximal invariant set

$$\Lambda = \left\{ (x, t) : x \in \bigcap_{n \ge 0} F^n(\Sigma) \quad \text{and} \quad t \ge 0 \right\}$$

extends, in this way, to a flow $(X^t)_{t \in \mathbb{R}}$ on Λ.

Let μ_F be any probability measure on Σ that is invariant under F. Then the product $\mu_F \times dt$ of μ_F by Lebesgue measure on $[0, +\infty)$ is an infinite measure, invariant under the trivial flow $(x, s) \mapsto (x, s + t)$ in $\Sigma \times [0, +\infty)$. In what follows we assume that

$$\int \tau \, d\mu_F < \infty. \tag{7.2}$$

In particular, $\mu_F(\Gamma) = 0$. Then we introduce the probability measure μ_X on V defined by

$$\int \varphi \, d\mu_X = \frac{1}{\int \tau \, d\mu_F} \int d\mu_F(x) \int_0^{\tau(x)} \varphi(\pi(x,t)) \, dt$$

for each $\varphi : V \to \mathbb{R}$.

Lemma 7.1.3. *The measure μ_X is invariant under the semi-flow $(X^t)_{t \geq 0}$.*

Proof. It is enough to show that $\mu_X\big((X^t)^{-1}(B)\big) = \mu_X(B)$ for every measurable set $B \subset V$ and any $0 < t < \inf \tau$. Moreover, we may suppose that B is of the form $B = \pi(A \times J)$ for some $A \subset \Sigma$ and J a bounded interval in $[0, \inf(\tau \mid A))$. This is because these sets form a basis for the σ-algebra of measurable subsets of V.

Let B be of this form and (x, s) be any point in Σ with $0 \leq s < \tau(x)$. Then, $X^t(x,s) \in B$ if and only if $\pi(x, s+t) = \pi(\tilde{x}, \tilde{s})$ for some $(\tilde{x}, \tilde{s}) \in A \times J$. In other words, $(x,s) \in (X^t)^{-1}(B)$ if and only if there exists some $n \geq 0$ such that

$$\tilde{x} = F^n(x) \quad \text{and} \quad \tilde{s} = s + t - \tau(x) - \cdots - \tau(F^{n-1}(x)).$$

Since $s < \tau(x)$, $t < \inf \tau$, and $\tilde{s} \geq 0$, it is impossible to have $n \geq 2$. So,

- either $\tilde{x} = x$ and $\tilde{s} = s + t$ (corresponding to $n = 0$),
- or $\tilde{x} = F(x)$ and $\tilde{s} = s + t - \tau(x)$ (corresponding to $n = 1$)

The two possibilities are mutually exclusive: for the first one (x, s) must be such that $s + t < \tau(x)$, whereas in the second case $s + t \geq \tau(x)$. This shows that we can write $(X^t)^{-}(B)$ as a disjoint union $(X^t)^{-}(B) = B_1 \cup B_2$, with

$$B_1 = \pi\big\{(x,s) : x \in A \text{ and } s \in (J - t) \cap [0, \tau(x))\big\}$$
$$B_2 = \pi\big\{(x,s) : F(x) \in A \text{ and } s \in (J + \tau(x) - t) \cap [0, \tau(x))\big\}.$$

Since $t > 0$ and $\sup J < \tau(x)$, we have $(J - t) \cap [0, \tau(x)) = (J - t) \cap [0, +\infty)$ for every $x \in A$. So, by definition, $\mu_X(B_1)$ equals

$$\int_A d\mu_F(x) \text{ length } \big[(J-t) \cap [0, \tau(x))\big] = \mu_F(A) \text{ length } \big[(J-t) \cap [0, +\infty)\big].$$

Similarly, $\inf J \geq 0$ and $t < \tau(x)$ imply that

$$(J + \tau(x) - t) \cap [0, \tau(x)) = \tau(x) + \big[(J - t) \cap (-\infty, 0)\big].$$

So, $\mu_X(B_2)$ is given by

$$\int_{F^{-1}(A)} d\mu_F(x) \text{ length } \big[(J-t) \cap (-\infty, 0)\big]$$
$$= \mu_F(F^{-1}(A)) \text{ length } \big((J-t) \cap (-\infty, 0)\big).$$

Since μ_F is invariant under F, we may replace $\mu_F(F^{-1}(A))$ by $\mu_F(A)$ in the last expression. It follows that

$$\mu_X\big((X^t)^-(B)\big) = \mu_X(B_1) + \mu_X(B_2) = \mu_F(A)\ \text{length}(J-t).$$

Clearly, the last term may be written as $\mu_F(A)\,\text{length}(J)$ which, by definition, is the same as $\mu_X(B)$. This proves that μ_X is invariant under the semi-flow, thus the lemma is proved.

Given $\varphi : V \to \mathbb{R}$ a bounded measurable function, let $\hat{\varphi} : \Sigma \to \mathbb{R}$ be defined by

$$\hat{\varphi}(x) = \int_0^{\tau(x)} \varphi(\pi(x,r))\,dt\,. \tag{7.3}$$

Observe that $\hat{\varphi}$ is integrable with respect to μ_F, and

$$\int \hat{\varphi}\,d\mu_F = \int \varphi\,d\mu_X,.$$

by the definition of μ_X.

Lemma 7.1.4. *Let $\varphi : V \to \mathbb{R}$ be a bounded function, and $\hat{\varphi}$ be as above. Suppose $x \in \Sigma$ is such that $\tau(F^j(x))$ and $\hat{\varphi}(F^j(x))$ are finite for every $j \geq 0$,*

(a) $\displaystyle\lim_n \frac{1}{n}\sum_{j=0}^{n-1} \tau(F^j(x)) = \int \tau\,d\mu_F$, *and*

(b) $\displaystyle\lim_n \frac{1}{n}\sum_{j=0}^{n-1} \hat{\varphi}(F^j(x)) = \int \hat{\varphi}\,d\mu_F$.

Then $\displaystyle\lim_{T\to+\infty} \frac{1}{T}\int_0^T \varphi(\pi(x,s+t))\,dt = \int \tau\,d\mu_X$ *for every* $\pi(x,s) \in V$.

Proof. Let x be fixed, satisfying (a), (b). Given any $T > 0$, define $n = n(T)$ by

$$T_{n-1} \leq T < T_n \quad \text{where} \quad T_j = \tau(x) + \cdots \tau(F^j(x)) \text{ for } j \geq 0$$

Then, using $(y, \tau(y)) \sim (F(y), 0)$,

$$\begin{aligned}\frac{1}{T}\int_0^T \varphi(\pi(x,s+t))\,dt = \frac{1}{T}\Bigg[&\sum_{j=0}^{n-1}\int_0^{\tau(F^j(x))} \varphi(\pi(F^j(x),t))\,dt \\ &+ \int_0^{T-T_{n-1}} \varphi(\pi(F^n(x),t))\,dt - \int_0^s \varphi(\pi(x,t))\,dt\Bigg],\end{aligned} \tag{7.4}$$

Using the definition of $\hat{\varphi}$, we may rewrite the first term on the right hand side as

$$\frac{n}{T}\frac{1}{n}\sum_{j=0}^{n-1}\hat{\varphi}(F^j(x)). \tag{7.5}$$

Let $\varepsilon > 0$. Assumption (a) and the definition of n imply that,

$$n\Big(\int \tau\, d\mu_F - \varepsilon\Big) \le T_{n-1} \le T \le T_n \le (n+1)\Big(\int \tau\, d\mu_F + \varepsilon\Big),$$

for every large enough n. Observe also that n goes to infinity as $T \to +\infty$, since $\tau(F^j(x)) < \infty$ for every j. So, for every large T,

$$\int \tau\, d\mu_F - \varepsilon \le \frac{T}{n} \le \frac{n+1}{n}\int \tau\, d\mu_F + \varepsilon \le \int \tau\, d\mu_F + 2\varepsilon.$$

This proves that T/n converges to $\int \tau\, d\mu_F$ when $T \to +\infty$. Consequently, assumption (b) implies that (7.5) converges to

$$\frac{1}{\int \tau\, d\mu_F}\int \hat{\varphi}\, d\mu_F = \int \varphi\, d\mu_X\,.$$

Now we prove that the remaining terms in (7.4) converge to zero when T goes to infinity. Since φ is bounded,

$$\Big|\frac{1}{T}\int_0^{T-T_{n-1}} \varphi(\pi(F^n(x),t))\,dt\Big| \le \frac{T-T_{n-1}}{T}\sup|\varphi|. \tag{7.6}$$

Let $\varepsilon > 0$. Using the definition of n once more,

$$T - T_{n-1} \le T_n - T_{n-1} \le (n+1)\Big(\int \tau\, d\mu_F + \varepsilon\Big) - n\Big(\int \tau\, d\mu_F - \varepsilon\Big)$$

whenever n is large enough. Then

$$\frac{T-T_{n-1}}{T} \le \frac{\int \tau\, d\mu_F + (2n+1)\varepsilon}{n\big(\int \tau\, d\mu_F - \varepsilon\big)} \le \frac{4\varepsilon}{\int \tau\, d\mu_F - \varepsilon}$$

for all large enough T. This proves that $(T-T_{n-1})/T$ converges to zero, and then so does (7.6). Finally, it is clear that

$$\frac{1}{T}\int_0^s \varphi(\pi(x,t))\,dt \to 0 \quad \text{when} \quad T \to +\infty.$$

This completes the proof of the lemma.

Corollary 7.1.3. *If μ_F is ergodic then μ_X is ergodic.*

Proof. Let $\varphi : V \to \mathbb{R}$ be any bounded measurable function, and $\hat{\varphi}$ be as in 7.3. As noted before, $\hat{\varphi}$ is μ_F-integrable. It follows that $\hat{\varphi}(F^j(x)) < \infty$ for every $j \ge 0$, at μ_F-almost every point $x \in \Sigma$. Moreover, by the ergodic theorem, condition (b) in Lemma 7.1.4 holds μ_F-almost everywhere. For the same reasons, $\tau(F^j(x))$ is finite for all $j \geq 0$, and condition (a) in the lemma is satisfied, for μ_F-almost all $x \in \Sigma$.

This shows that Lemma 7.1.4 applies to every point x in a subset $A \subset \Sigma$ with $\mu_F(A) = 1$. It follows that

$$\lim_{T \to +\infty} \frac{1}{T} \int_0^T \varphi(X^t(z))\, dt = \int \varphi\, d\mu_X$$

for every point z in $B = \pi(A \times [0, +\infty))$. Since the latter has $\mu_X(B) = 1$, we have shown that the Birhkhoff average of φ is constant μ_X-almost everywhere. Then the same is true for any integrable function, as bounded functions are dense in $L^1(\mu_X)$. Thus μ_X is ergodic, and the corollary is proved.

7.1.2 Robust attractors of flows

Let $(X^t)_{t \in \mathbb{R}}$ be a C^1 flow on some compact manifold M. A set $\Lambda \subset M$ is *invariant* under the flow if $X^t(\Lambda) = \Lambda$ for every $t \in \mathbb{R}$.

Definition 7.1.2. *An* attractor *of the flow X^t is a compact invariant set $\Lambda \subset M$ such that*

1. *$X^t \mid \Lambda$ is* transitive*: there exists $x \in \Lambda$ whose forward orbit $\{X^t(x) : t > 0\}$ is dense in Λ;*
2. *the* basin $B(\Lambda)$ *of Λ*

$$B(\Lambda) = \{x \in M : d(X^t(x), \Lambda) \to 0 \text{ as } t \to +\infty\}$$

has positive Lebesgue measure.

An important class of examples are the *hyperbolic attractors*.
***

Attractors of flows present important new features with respect to the discrete time case, specially when they involve singularities (equilibrium points) interacting with regular orbits.

Such singular attractors were first introduced by [2], [51], as models for the observations made by Lorenz in [72]. A key fact about these geometric Lorenz models is that they are robust: any flow close to the initial one has an attractor with similar properties.

There is now a vast literature on the geometric, dynamical, and ergodic properties of these attractors. See e.g. [120], [27], [32], [93], [?], and references therein. However, the actual occurrence of a strange attractor in Lorenz original equations remained a conjecture for more than three decades, until it was settled in the affirmative by [122].

On the other hand, similar types of robust attractors have been shown to appear in various types of bifurcations of flows [107], [?], [3], [80], [?], [?].

Recently, [79], [?] have been developing a theory of singular attractors of flows in three dimensions, characterizing robustness in terms of a hyperbolicity property. For a precise statement of their results we need a few definitions.

Let $(X_t)_{t\in\mathbb{R}}$ be a C^k flow on a manifold M, $k \geq 1$, and $\Lambda \subset M$ be compact and invariant under the flow. One calls Λ a *singular* (or *Lorenz-like*) *attractor* if

(a) the flow is transitive on Λ;
(b) there exists an open neighbourhood U of Λ such that $\Lambda = \cap_{t>0} X_t(U)$;
(c) Λ contains some singularity of the flow (and is not reduced to it).

Singular transitive sets are defined analogously, just taking the intersection over all $t \in \mathbb{R}$ in (b). A singular attractor is called C^1 *robust* if

$$\Lambda_Y = \bigcup_{t>0} Y_t(U)$$

is again a singular attractor, for any C^1 near flow $(Y_t)_{t\in\mathbb{R}}$. A similar notion applies to singular transitive sets, with the intersection taken over $t \in \mathbb{R}$.

We say that $\Lambda \subset M$ is a *singular hyperbolic set* for the flow if it is invariant and there exists a continuous splitting

$$T_\Lambda M = E^{ss} \oplus E^{cu}$$

of the tangent bundle over Λ, which is invariant under the flow and satisfies

(i) E^{ss} has dimension 1 and is uniformly contracting;
(ii) E^{cu} is volume expanding and is dominated by E^{ss}.

Denote $-X$ the flow obtained from $X = (X_t)_{t\in\mathbb{R}}$ by reversing the direction of time. The following result was proved by [79], [?].

Theorem 7.1.1. *Let Λ be a C^1 robust singular transitive set of a C^1 flow on a three dimensional manifold M. Then Λ is a singular hyperbolic set, and a singular attractor, for either X or $-X$.*

They also characterize the type of singularities that may be contained in a C^1 robust singular transitive set: the eigenvalues are necessarily real and satisfy

$$\text{either} \quad \lambda_2 < \lambda_3 < 0 < -\lambda_3 < \lambda_1, \quad \text{or} \quad \lambda_2 > \lambda_3 > 0 > -\lambda_3 > \lambda_1 . \qquad (7.7)$$

Moreover, all the singularities have the same number of contracting eigenvalues.

Other types of singular attractors of flows have also been constructed in recent years, displaying more subtle forms of persistence under perturbations. [111] considered a modification of the classical geometric Lorenz models where the eigenvalues at the singularity satisfy

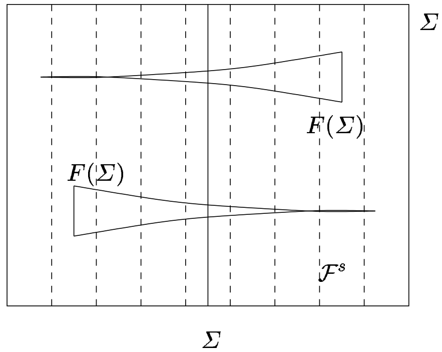

Fig. 7.1. Poincaré map of a Lorenz-like flow

$$\lambda_2 < \lambda_3 < 0 < \lambda_1 < -\lambda_3.$$

In other words, the expanding eigenvalue is dominated by both contracting eigenvalues, compare (7.7). The singular attractors he obtains in this way have rather different properties, if compared with the cases we have been discussing, in particular they are persistent only in a measure-theoretical sense: a singular attractor exists for a positive Lebesgue measure set of parameters, in generic parametrized families of flows through the initial one.

The results we describe also apply to the multidimensional Lorenz-like attractors in [20].

7.2 Stochastic Stability of Flows

7.3 Non-Uniformly Expanding Maps

We have seen in Theorem 1.2.1 that uniformly expanding maps on compact manifolds always admit invariant measures absolutely continuous with respect to Lebesgue measure. In this section we prove, following [5], that this conclusion extends to a non-uniformly hyperbolic setting.

Definition 7.3.1. *Let $f : M \to M$ be a C^1 local diffeomorphism on a compact manifold M, and H be a subset of M. We say that f is* non-uniformly expanding on *H if there exists $c > 0$ and a Riemannian norm $\|\cdot\|$ on M such that for every $x \in H$*

$$\limsup_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} -\log \left\|Df(f^j(x))^{-1}\right\| > c. \tag{7.8}$$

Here $Df(y)^{-1} : T_{f(y)}M \to T_yM$ denotes the inverse of the linear map $Df(y)$. More generally, we use $Df^k(y)^{-1} : T_{f^k(y)}M \to T_yM$ to represent the inverse of $Df^k(y)$, for $k \geq 1$.

Example 7.3.1. Let $f : M \to M$ be C^1 and locally volume-expanding: $\det Df > 1$. Assume there is an open set $V \subset M$ such f is expanding in the complement of V,

$$\|Df(y)^{-1}\| < 1 \quad \text{for every} \quad y \in M \setminus V,$$

and the restriction of f to V is injective and not too coontracting:

$$\|Df(y)^{-1}\| < 1 + \delta \quad \text{for every} \quad y \in V$$

and some sufficiently small $\delta > 0$. Then f is non-uniformly expanding. This is not a difficult exercise, the reader may also check [5, Appendix] for a proof.

Theorem 7.3.1. *Let $f : M \to M$ be $C^{1+\nu_0}$, for some $0 < \nu_0 \leq 1$, and non-uniformly expanding on a positive Lebesgue measure set $H \subset M$. Then f admits a finite number of invariant ergodic absolutely continuous probability measures $\mu_1, \cdots, \mu_\ell$, such that the union of their basins contains H up to a zero Lebesgue measure subset.*

The proof of Theorem 7.3.1 occupies the remaining of this section.

Iterating Lebesgue Measure. We denote by m the Lebesgue measure on M, and let $c > 0$ and $\|\cdot\|$ be as in Definition 7.3.1. The assumption (7.8) means that for any $N \geq 1$ and $x \in H$ there exists $n \geq N$ such that

$$\frac{1}{n} \sum_{j=0}^{n-1} -\log \left\| Df(f^j(x))^{-1} \right\| > c. \tag{7.9}$$

Let $H_N(n)$ be the set of points $x \in H$ for which n is the smallest integer larger or equal than N for which (7.9) holds. We consider the sequence of measures μ_N defined by

$$\mu_N = \frac{1}{m(H)} \sum_{n=N}^{\infty} \frac{1}{n} \sum_{j=0}^{n-1} f_*^j(m \mid H_N(n)). \tag{7.10}$$

Each μ_N is a probability, since $m(H) = \sum_{n \geq N} m(H_N(n))$.

Lemma 7.3.1. *Any weak* accumulation point of the sequence $(\mu_N)_N$ is an invariant measure for f.*

Proof. For every $N \geq 1$, we have

$$f_*\mu_N = \frac{1}{m(H)} \sum_{n=N}^{\infty} \frac{1}{n} \sum_{j=1}^{n} f_*^j(m \mid H_N(n))$$
$$= \mu_N + \frac{1}{m(H)} \sum_{n=N}^{\infty} \frac{1}{n} f_*^n(m \mid H_N(n)) - \frac{1}{n}(m \mid H_N(n)).$$

Hence, for any continuous function $\varphi : M \to \mathbb{R}$,

$$\left| \int \varphi \, d(f_*\mu_N) - \int \varphi \, d\mu_N \right|$$
$$\leq \frac{1}{m(H)} \sum_{n=N}^{\infty} \frac{1}{n} \left| \int \varphi \, df_*^n(m \mid H_N(n)) - \int \varphi \, d(m \mid H_N(n)) \right|$$
$$\leq \frac{1}{Nm(H)} \sum_{n=N}^{\infty} \left| \int_{H_N(n)} (\varphi \circ f^n) \, dm - \int_{H_N(n)} \varphi \, dm \right|.$$

For each N fixed, the $H_N(n)$, $n \geq N$, form a partition of H. So, the right hand side does not exceed

$$\frac{1}{Nm(H)} \left(\int_H |\varphi \circ f^n| \, dm + \int_H |\varphi| \, dm \right) \leq \frac{2}{Nm(H)} \sup |\varphi| .$$

Therefore,

$$\left| \int \varphi \, d(f_*\mu_N) - \int \varphi \, d\mu_N \right| \leq \frac{2}{Nm(H)} \sup |\varphi| . \tag{7.11}$$

Let μ be any accumulation point of μ_N. Taking limits in (7.11) we get that

$$\int \varphi \, d(f_*\mu) - \int \varphi \, d\mu = 0$$

for every continuous function φ. In other words, μ is f-invariant. □

Definition 7.3.2. *We say that* $n \geq 1$ *is a* c-hyperbolic time *for* $x \in M$ *if*

$$\sum_{j=n-k}^{n-1} -\log \|Df(f^j(x))^{-1}\| \geq \frac{c}{2}$$

for all $1 \leq k \leq n$.

In particular, if n is a c-hyperbolic time for x then $Df^k(f^{n-k}(x))$ is an expansion for every $1 \leq k \leq n$:

$$\|Df(f^{n-k}(x))^{-k}\| \leq \prod_{j=n-k}^{n-1} \|Df(f^j(x))^{-1}\| \leq e^{-ck/2}.$$

Moreover, this remains true in a neighbourhood of $f^n(x)$ whose size depends only on c and the map f:

Lemma 7.3.2. *Given $c > 0$ there exists $\delta > 0$ such that if $n \geq 1$ a c-hyperbolic time for x then*

$$\operatorname{dist}(f^{n-k}(y), f^{n-k}(x)) \leq e^{-ck/2} \operatorname{dist}(f^n(y), f^n(x)).$$

for any point $y \in D$ with $\operatorname{dist}(f^n(x), f^n(y)) \leq \delta$.

Proof. Choose $\delta > 0$ small enough so that,

$$\|Df(f(z))^{-1}v\| \leq e^{c/2}\|Df(f(w))^{-1}\| \, \|v\| \tag{7.12}$$

for any $z, w \in M$ with $\operatorname{dist}(z, w) \leq \delta$, and any $v \in T_{f(z)}M$. Here $\operatorname{dist}(\cdot, \cdot)$ is the distance function corresponding to the Riemannian norm $\|\cdot\|$. Let η_0 be a curve of minimal length connecting $f^n(x)$ to $f^n(y)$. For $1 \leq k \leq n$ write $\eta_k = f^{n-k}(\eta_0)$. We prove the lemma by induction. Let $1 \leq k \leq n$ and assume that

$$\operatorname{length}(\eta_j) \leq \delta \quad \text{for } 0 \leq j \leq k-1.$$

Denote by $\dot\eta_0(z)$ the tangent vector to the curve η_0 at the point z. Then, in view of the choice of δ in (7.12) and the definition of c-hyperbolic times,

$$\|Df^{-k}(z)\dot\eta_0(z)\| \leq e^{ck/2} \, \|\dot\eta_0(z)\| \prod_{j=n-k+1}^{n} \|Df^{-1}|E^{cu}_{f^j(x)}\| \leq e^{ck/2}\|\dot\eta_0(z)\|.$$

As a consequence,

$$\operatorname{length}(\eta_k) \leq e^{-ck/2} \operatorname{length}(\eta_0) = e^{-ck/2} \operatorname{dist}_{f^{n-k}(D)}(f^{n-k}(y), f^{n-k}(x)) \leq \delta\,.$$

This completes our induction, thus proving the lemma. □

Let $H_N(n, j)$, $j < n \leq N$, be the set of points $x \in H_N(n)$ for which j is a c-hyperbolic time. We decompose each measure μ_N as $\mu_N = \nu_N + \eta_N$, where

$$\nu_N = \frac{1}{m(H)} \sum_{n=N}^{\infty} \frac{1}{n} \sum_{j=0}^{n-1} f_*^j(m \mid H_N(n, j))$$

and

$$\eta_N = \frac{1}{m(H)} \sum_{n=N}^{\infty} \frac{1}{n} \sum_{j=0}^{n-1} f_*^j(m \mid H_N(n) \setminus H_N(n, j)).$$

The main step in the proof of Theorem 7.3.1 is

Proposition 7.3.1. *There exists constants $C_1, C_2 > 0$ such that, for every $N \geq 1$,*

(a) the density $d\nu_N/dm$ of ν_N is bounded above by C_1;
(b) the total mass $\nu_N(M)$ of ν_N is bounded below by C_2.

Let $(N_k)_k$ be such that the three sequences $(\mu_{N_k})_k$, $(\nu_{N_k})_k$, $(\eta_{N_k})_k$ converge, in the weak* topology, to measures μ, ν, η, respectively. Then $\mu = \nu + \eta$ and, by Lemma 7.3.1, μ is an invariant probability measure for f. On the other hand, by Proposition 7.3.1, ν has positive total mass $\nu(M)$, and it is absolutely continuous with respect to Lebesgue measure, with bounded density. Let $\eta = \eta^a + \eta^s$ denote the Lebesgue decomposition of η: η^a is absolutely continuous and η^s is totally singular, with respect to Lebesgue measure. Then the Lebesgue decomposition $\mu = \mu^a + \mu^s$ of μ is given by

$$\mu^a = \nu + \eta^a \quad \text{and} \quad \mu^s = \eta^s .$$

The map f is such that the images and pre-images of zero Lebesgue measure sets have zero Lebesgue measure. This, together with the fact that μ is f-invariant, implies that μ_a and μ_s are also f-invariant measures. Moreover, we are assured that $\mu^a(M) \geq \nu(M) > 0$. Thus, normalizing μ_a we get an absolutely continuous invariant probability measure for f.

So far we have explained that f has some invariant absolutely continuous invariant measure. For Theorem 7.3.1 we also have to show that such measures can be taken ergodic, and that the basins of finitely many of them cover almost all of the set H. This follows Subsection 1.2.3 fairly closely, we give an outline at the end of the present section. But, before that, let us prove Proposition 7.3.1.

Hyperbolic times and distortion bounds. Suppose n is a c-hyperbolic time for x, and $\delta > 0$ has been chosen as in Lemma 7.3.2.

Proposition 7.3.2. *There exists $C_0 > 1$ such that*

$$\frac{1}{C_0} \leq \frac{|\det Df^n(x)|}{|\det Df^n(y)|} \leq C_0$$

for every $y \in M$ such that $\operatorname{dist}(f^n(y), f^n(x)) \leq \delta$.

Proof. We have

$$\log \frac{|\det Df^n(x)|}{|\det Df^n(y)|} = \sum_{i=0}^{n-1} \big(\log|\det Df(f^j(x))| - \log|\det Df(f^j(y))| \big)$$

Since f is a C^2 local diffeomorphism, $\log|\det Df|$ is a C_1-Lipschitz for some $C_1 > 0$. By Lemma 7.3.2, the sum of all $\operatorname{dist}(f^j(x), f^j(y))$ over $0 \leq j \leq n$ is bounded by $\delta/(1 - e^{-c/2})$. So, it suffices to take $C_2 = \exp(C_1\delta/(1 - e^{-c/2}))$. □

As a consequence we get the first claim in Proposition 7.3.1:

Corollary 7.3.1. *There exists $C_1 > 1$ such that $\nu_N(E) \leq C_2 m(E)$ for every $N \geq 1$ and any measurable set $E \subset M$.*

Proof. Let □

Abundance of hyperbolic times. Using the following lemma, due to Pliss [95], we show that for any point $x \in H_N(n)$, $n \geq N$, a definite fraction of the iterates preceding n are hyperbolic times for x. This ensures that the total mass of the measures ν_N is bounded away from zero.

Lemma 7.3.3. *Given $A \geq c_2 > c_1 > 0$, let $\theta_0 = (c_2 - c_1)/(A - c_1)$. Then, given any real numbers $a_1, \dots, a_n$ such that $a_j \leq A$ for every $1 \leq j \leq n$ and*

$$\sum_{j=1}^{n} a_j \geq c_2 n,$$

there are $l > \theta_0 n$ and $1 < n_1 < \cdots < n_l \leq n$ so that

$$\sum_{j=k+1}^{n_i} a_j \geq c_1(n_i - k)$$

for every $0 \leq k < n_i$ and $i = 1, \dots, l$.

Proof. Define $S(k) = \sum_{j=1}^{k}(a_j - c_1)$, for each $1 \leq k \leq n$, and also $S(0) = 0$. Then define $1 < n_1 < \cdots < n_l \leq n$ to be the maximal sequence such that $S(n_i) \geq S(k)$ for every $0 \leq k < n_i$ and $i = 1, \dots, l$. Note that l can not be zero, since $S(n) > S(0)$. Moreover, the definition means that

$$\sum_{j=k+1}^{n_i} a_j \geq c_1(n_i - k) \quad \text{for} \quad 0 \leq k < n_i \quad \text{and} \quad i = 1, \dots, l.$$

So, we only have to check that $l > \theta_0 n$. Observe that, by definition,

$$S(n_i - 1) < S(n_{i-1}) \quad \text{and so} \quad S(n_i) < S(n_{i-1}) + (A - c_1)$$

for every $1 < i \leq l$. Moreover, $S(n_1) \leq (A - c_1)$ and $S(n_l) \geq S(n) \geq n(c_2 - c_1)$. This gives,

$$n(c_2 - c_1) \leq S(n_l) = \sum_{i=2}^{l} \big(S(n_i) - S(n_{i-1})\big) + S(n_1) < l(A - c_1),$$

which completes the proof. □

Corollary 7.3.2. *There exists $C_2 > 0$ such that, given any $n \geq N \geq 1$ and $x \in H_N(n)$, there exist at least $C_2 n$ values of j with $0 \leq j < n$ which are hyperbolic times for x.*

Proof. Let A be an upper bound for $-\log \|Df(y)^{-1}\|$ over all $y \in M$, and $a_j = -\log \|Df(f^{j-1})^{-1}\|$ for each $0 \leq j < n$. □

Part (b) of Proposition 7.3.1 is a simple consequence:

Corollary 7.3.3. *There exists* $C_2 > 0$ *such that* $\nu_N(M) \geq C_2$ *for every* $N \geq 1$.

Proof. For each $n \geq 1$, let ξ_n be the normalized counting measure in $\{0, 1, \ldots, n\}$. That is, $\xi_n(J) = \#J/n$ for any subset J. The previous corollary states that, give any $n \geq N \geq 1$ and $x \in H_N(n)$, the ξ_n-measure of the set of c-hyperbolic times of x is at least C_2.

□

Ergodicity and finiteness. We begin by fixing some partition $\mathcal{P}_0 = \{U_1, \ldots, U_s\}$ of M into regions with non-empty interior and diameter less than ρ_0. Then, for each $n \geq 1$, we let $\mathcal{P}_n$ be the partition of M consisting of the images of each of the U_i, $1 \leq i \leq s$, under corresponding inverse branches of f^n. The diameter of $\mathcal{P}_n$, defined as the supremum of the diameters of its elements, is less than $\rho_0 \sigma^{-n}$.

Lemma 7.3.4. *Let* $\mathcal{P}_n$, $n \geq 1$, *be a sequence of partitions in a compact metric space with diameters converging to zero as* $n \to \infty$. *Let* ν *be a probability measure in that space, and* B *be any measurable subset such that* $\nu(B) > 0$. *Then there are* $V_n \in \mathcal{P}_n$, *for* $n \geq 1$, *so that*

$$\nu(V_n) > 0 \quad \textit{and} \quad \frac{\nu(B \cap V_n)}{\nu(V_n)} \to 1 \quad \textit{as } n \to \infty.$$

Proof. Given any $0 < \varepsilon < \nu(B)$, let K_ε be some compact subset of B with $\nu(B \setminus K_\varepsilon) < \varepsilon$. As the diameter of the partitions converges to zero, the measure of the union $A_{\varepsilon,n}$ of all the elements of $\mathcal{P}_n$ that intersect K_ε satisfies $\nu(A_{\varepsilon,n} \setminus K_\varepsilon) < \varepsilon$ for every large enough n. If we had

$$\nu(K_\varepsilon \cap V_n) \leq \frac{\nu(B) - \varepsilon}{\nu(B) + \varepsilon}\, \nu(V_n)$$

for every $V_n \in \mathcal{P}_n$ that intersects K_ε, it would follow that

$$\nu(K_\varepsilon) \leq \frac{\nu(B) - \varepsilon}{\nu(B) + \varepsilon}\, \nu(A_{\varepsilon,n}) \leq \frac{\nu(B) - \varepsilon}{\nu(B) + \varepsilon}\, (\nu(K_\varepsilon) + \varepsilon) \leq \nu(B) - \varepsilon,$$

a contradiction. So, there must be some $V_n \in \mathcal{P}_n$ with

$$\nu(B \cap V_n) \geq \nu(K_\varepsilon \cap V_n) > \frac{\nu(B) - \varepsilon}{\nu(B) + \varepsilon}\, \nu(V_n)$$

and this also implies $\nu(V_n) > 0$. The statement follows, taking $\varepsilon \to 0$. □

We say that $A \subset M$ is an *invariant set* of $f : M \to M$ if $f^{-1}(A) = A$.

Lemma 7.3.5. *Let* $A \subset M$ *be an invariant set of a* $C^{1+\nu_0}$ *expanding map* f *such that* $m(A) > 0$. *Then* A *has full Lebesgue measure in some* $U_i \in \mathcal{P}_0$, *that is, there exists* $1 \leq i \leq s$ *so that* $m(U_i \setminus A) = 0$.

Proof. By Lemma 1.2.3, there exist $V_n \in \mathcal{P}_n$ such that $m(V_n \setminus A)/m(V_n)$ converges to zero as $n \to \infty$. Let $U_{i(n)} = f^n(V_n)$. Applying Lemma 1.2.1 to the inverse branch of f^n mapping $U_{i(n)}$ to V_n, we conclude that

$$\frac{m(U_{i(n)} \setminus A)}{m(U_{i(n)})} = \frac{m(f^n(V_n) \setminus A)}{m(f^n(V_n))} \le \exp\left(C_1(2\rho_0)^{\nu_0}\right) \frac{m(V_n \setminus A)}{m(V_n)}$$

also converges to zero. Since $\mathcal{P}_0$ is finite, there must exist $1 \le i \le s$ such that $i(n) = i$ for infinitely many values of n. Then $m(U_i \setminus A) = 0$. □

Corollary 7.3.4. *Any $C^{1+\nu_0}$ expanding map $f : M \to M$ has some ergodic absolutely continuous invariant measure.*

Proof. As a consequence of the lemma, there exist at most $\#\mathcal{P}_0$ two-by-two disjoint invariant sets with positive Lebesgue measure. It follows that M can be partitioned into finitely many minimal positive Lebesgue measure invariant sets $A_1, \ldots, A_s$, $s \le \#\mathcal{P}_0$: minimality means there are no invariant subsets $B_i \subset A_i$ with $0 < m(B_i) < m(A_i)$. Given any f-invariant absolutely continuous measure μ, there is some i such that $\mu(A_i) > 0$. Then the normalized restriction μ_i of μ to A_i,

$$\mu_i(B) = \frac{\mu(B \cap A_i)}{\mu(A_i)}$$

is invariant, absolutely continuous, and ergodic (because A_i is minimal). □

7.4 Partial Hyperbolicity and Robustness

Let us introduce the notion of robust attractor meaning, roughly, that it can not be destroyed by small perturbations of the system. On the other hand, the dynamics on the attractor may be changed by the perturbation.

Definition 7.4.1. *We say that a compact invariant set Λ is a C^r* robust attractor *for f if there exists a neighbourhood U of Λ such that*

$$\operatorname{clos}(f(U)) \subset U \quad \textit{and} \quad \Lambda = \bigcap_{n=0}^{\infty} f^n(U),$$

and $\Lambda_g = \cap_{n=0}^{\infty} g^n(U)$ is a transitive set for any diffeomorphism g in a C^r neighbourhood of f.

When $r = 1$, which is the case we are most interested in, we simply say that Λ is a robust attractor. Observe also that, as a consequence of the definition, the neighbourhood U is contained in the stable set of Λ_g, for any g near f.

Hyperbolic attractors are robust attractors, as we have seen in the previous section. The converse is true in two dimensions, according to [74]: robust attractors of surface diffeomorphisms are always hyperbolic. On the other hand, non-hyperbolic robust attractors were exhibited by [115], in dimension at least 4, and by [74], in dimension 3 or larger. Several other constructions were proposed in recent years, see for instance [19] and [21], and references therein.

Remarkably, robustness does imply some weak form of hyperbolicity: [?] proved recently that robust attractors in any dimension always admit an invariant dominated splitting; and [37] had shown that robust attractors in dimension 3 are always partially hyperbolic.

It is easy to see that a (transitive) partially hyperbolic attractor may support infinitely many ergodic probability measures absolutely continuous along the strong-unstable foliation. Moreover, in general they are *not* SRB measures for the map.

Example 7.4.1. Let $f_1 : M_1 \to M_1$ and $f_2 : M_2 \to M_2$ be Anosov diffeomorphisms, with splittings $TM_i = E_i^u \oplus E_i^s$ for $i = 1, 2$. Then $f_1 \times f_2$ is an Anosov diffeomorphism on $M_1 \times M_2$. Taking f_1 and f_2 to be transitive, i.e. so that there is some point whose forward orbit is dense in the corresponding manifold, then $f_1 \times f_2$ is also transitive. See e.g. Remark A.1.1. Assume that the contraction of f_1 along E_1^s is stronger than the contraction of f_2 along E_2^s, and the expansion of f_1 along E_1^u is also stronger than the expansion of f_2 along E_2^u. More precisely, there exists $\lambda < 1$ such that

$$\|Df_1 \mid E_1^s\| \, \|(Df_2 \mid E_2^s)^{-1}\| \le \lambda \quad \text{and} \quad \|Df_2 \mid E_2^u\| \, \|(Df_1 \mid E_1^u)^{-1}\| \le \lambda.$$

Then we may also think of $M_1 \times M_2$ as a strongly partially hyperbolic attractor for $f_1 \times f_2$, with splitting

$$T(M_1 \times M_2) = \big(E_1^u \times \{0\}\big) \oplus TM_2 \oplus \big(E_1^s \times \{0\}\big).$$

The foliation $\mathcal{F}^u$ tangent to the strong-unstable bundle $E_1^u \times \{0\}$ is given by $\mathcal{F}^u(x_1, x_2) = \mathcal{F}_1^u(x_1) \times \{x_2\}$, where $\mathcal{F}_1^u$ is the unstable foliation of f_1. Let μ_i be the SRB measure of f_i for $i = 1, 2$, and ν be an f_2-invariant measure supported on a periodic orbit of f_2. That is, ν is the average of the Dirac measures supported on the points of the periodic orbit. Then $\mu_1 \times \nu$ is an invariant ergodic measure for $f_1 \times f_2$, absolutely continuous along $\mathcal{F}^u$. But the Anosov diffeomorphism $f_1 \times f_2$ has a unique SRB measure $\mu_1 \times \mu_2$.

There are, however, relevant situations in which SRB measures can be constructed via Theorem 2.3.1. The following sufficient condition was proposed by [21], extending [?]. It holds for a C^1 open set of diffeomorphisms with partially hyperbolic, possibly non-hyperbolic, attractors.

Let Λ be a partially hyperbolic attractor of type $E^u \oplus E^{cs}$ for a diffeomorphism $f : M \to M$. We say that E^{cs} is *mostly contracting* if $Df^n \mid E^{cs}$ is asymptotically contracting, as $n \to \infty$, over a large set of points: given any domain U inside a strong-unstable leaf, there exists a positive Lebesgue measure subset U_0 of U, such that

$$\limsup_{n\to+\infty} \frac{1}{n} \log \|Df^n \mid E_x^{cs}\| < 0 \tag{7.13}$$

for every $x \in U_0$.

Theorem 7.4.1. *Let Λ be a partially hyperbolic attractor of type $E^u \oplus E^{cs}$ for a C^2 diffeomorphism, such that E^{cs} is mostly contracting. Then any ergodic measure absolutely continuous along the strong-unstable foliation $\mathcal{F}^u$ is an SRB measure. There are finitely many such measures and their basins contain a full Lebesgue measure subset of the basin of the attractor Λ. If the orbit of any strong-unstable leaf is dense in the attractor then the SRB measure is unique.*

Even if Λ is a transitive attractor, there may be more than one SRB measure supported on it. Indeed, [60] constructed diffeomorphisms of $T^2 \times [0,1]$ having two SRB measures (one supported on each boundary component), whose basins are both dense, their union covering a full Lebesgue measure subset. For these maps, $\Lambda = T^2 \times [0,1]$ is transitive and strongly partially hyperbolic, with $E^{cs} = E^c \oplus E^s$ mostly contracting. Similar examples can be constructed in T^3, gluing two maps on $T^2 \times [0,1]$ as above along the boundary. Most interesting, in $T^2 \times [0,1]$, the construction of [60] is robust: any nearby diffeomorphism is transitive and has two SRB measures. This relies on the fact that diffeomorphisms of $T^2 \times [0,1]$ have to preserve the boundary of the manifold.

For manifolds without boundary, it is a very interesting open problem whether there are robust examples of coexistence of several SRB measures on a same transitive attractor. For instance, let $\Lambda = \cap_{n\geq 0} f^n(U)$ be a partially hyperbolic attractor of a C^2 diffeomorphism f. Here U is an open neighbourhood of Λ such that the closure of $f(U)$ is contained in U. Assume that for every g in a neighbourhood of f, the maximal invariant set $\Lambda_g = \cap_{n\geq 0} g^n(U)$ is transitive and partially hyperbolic of type $E^u \oplus E^{cs}$, with E^{cs} mostly contracting. Is it true that *for the generic diffeomorphism g close to f the attractor Λ_g supports a unique SRB measure ?*

7.5 Hénon-Like Attractors

A. Appendices

A.1 Global Dynamics of Hyperbolic Diffeomorphisms

In this section we review a number of results on the global dynamics of hyperbolic (Axiom A) diffeomorphisms. More information and references can be found e.g. in [119, 84, 92].

Definition A.1.1. *A point p is* non-wandering *for f if for any neighbourhood V of p there exists $n \geq 1$ such that $f^n(V) \cap V \neq \emptyset$. The set of non-wandering points, or* non-wandering set*, is denoted $\Omega(f)$.*

In other words, $p \in M$ is non-wandering if and only if there are points arbitrarily close to it that return arbitrarily close to p in future times (p itself may never return).

It follows directly from the definition that $\Omega(f)$ is a closed set. It is also easy to check that any point that is in the accumulation set

$$L(f, z) = \{w \in M : \text{ there exists } n_j \to \pm\infty \text{ so that } f^{n_j}(z) \to w\}$$

of some $z \in M$ is a non-wandering point. Therefore, $\Omega(f)$ always contains the *limit set* $L(f)$ of f, which is the closure of the union of all the accumulation sets for all the points $z \in M$. Note also that if z is a periodic point then $L(f, z)$ is just the orbit of z. Therefore, the closure of the set $\mathrm{Per}(f)$ of periodic points of f is always contained in $L(f)$. Summarizing,

$$\operatorname{clos}\big(\mathrm{Per}(f)\big) \subset L(f) \subset \Omega(f)$$

for any diffeomorphism (or even homeomorphism) f. The inclusions may be strict, in general.

Definition A.1.2. *A diffeomorphism $f : M \to M$ is* hyperbolic *(or* Axiom A*) if its non-wandering set $\Omega(f)$ is hyperbolic for f and coincides with the closure* $\operatorname{clos}\big(\mathrm{Per}(f)\big)$ *of the set of periodic points of f.*

A fundamental property of hyperbolic systems is that the dynamics on the non-wandering set can be decomposed into finitely many hyperbolic *basic pieces*, cf. the next theorem [119]. Recall that transitivity means that the forward orbit of some point is dense.

Theorem A.1.1. *If f is hyperbolic then its non-wandering set can be written as a disjoint union*

$$\Omega(f) = \Lambda_1 \cup \cdots \cup \Lambda_N ,$$

of isolated hyperbolic sets $\Lambda_1, \ldots, \Lambda_N$, such that the restriction of f to Λ_i is transitive for every $1 \le i \le N$.

More generally, there is a similar decomposition for the closure of the set of periodic points whenever it is a hyperbolic set for f. It should be noted that, cf. [83], if the limit set $L(f)$ is hyperbolic then $\operatorname{clos}\big(\operatorname{Per}(f)\big) = L(f)$. This is not always true for the non-wandering set.

Remark A.1.1. An Anosov diffeomorphism $f : M \to M$ is transitive if and only if its periodic points are dense in M. Indeed, transitivity implies $L(f)$ is the whole manifold M, and then $\operatorname{Per}(f)$ is dense in M by the result of [83] mentioned above. In the converse direction, if $\operatorname{Per}(f)$ is dense in M then f is Axiom A. So, by Theorem A.1.1, M can be split into a finite number of compact transitive sets. By connectedness, there must be exactly one such set. This means that f is transitive.

Actually, any Anosov diffeomorphism is hyperbolic [7]. Let us also mention that all known examples of Anosov diffeomorphisms are transitive. However, [46] exhibited non-transitive Anosov flows (recall the definition in Example 2.2.2).

If $\Lambda_1 \cup \cdots \cup \Lambda_N$ is the decomposition of $\Omega(f)$, or even of $L(f)$, then every point in M is in the stable set, respectively unstable set, of some basic piece:

$$\bigcup_{i=1}^{N} W^s(\Lambda_i) = M = \bigcup_{i=1}^{N} W^u(\Lambda_i).$$

If f is C^2 then the stable set of Λ_i has positive Lebesgue measure if and only if Λ_i is a attractor, see [22]. Recall that the stable set (or basin) of a hyperbolic attractor contains a neighbourhood of it. Thus, *for hyperbolic C^2 diffeomorphisms Lebesgue almost every point in M is in the basin of some attractor*. On the contrary, C^1 diffeomorphisms may exhibit transitive hyperbolic sets that are not attractors and have positive Lebesgue measure [23].

Definition A.1.3. *Suppose f is a hyperbolic diffeomorphism. A* cycle *in $\Omega(f)$ is a sequence of basic pieces $\Lambda_{i_0}, \ldots, \Lambda_{i_{k-1}}, \Lambda_{i_k} = \Lambda_{i_0}$ of the non-wandering set such that*

$$\big(W^u(\Lambda_{i_{j-1}}) \setminus \Lambda_{i_{j-1}}\big) \cap \big(W^s(\Lambda_{i_j}) \setminus \Lambda_{i_j}\big) \neq \emptyset.$$

for $1 \le j \le k$. We say that f has no cycles *if there are no cycles in $\Omega(f)$.*

If f has no cycles then there exists a *filtration* [119] for it, that is, a sequence $\emptyset = M_0 \subset M_1 \subset \cdots \subset M_N = M$ of compact submanifolds with boundary such that f maps each M_i into its interior, and the set of points whose orbits never leave $M_i \setminus M_{i-1}$ coincides with Λ_i for every $1 \le i \le N$:

$$\Lambda_i = \bigcap_{n \in \mathbb{Z}} f^n(M_i \setminus M_{i-1}). \tag{A.1}$$

If the limit set of f is hyperbolic, one defines cycles in it in the same way as in Definition A.1.3. As before, if there are no cycles in $L(f)$ then there is a filtration $M_1, \ldots, M_N$ such that the basic pieces of the limit set coincide with the maximal invariant sets in each $M_i \setminus M_{i-1}$. Now, existence of such a filtration forces the non-wandering set to coincide with $L(f)$. Therefore, if the limit set is hyperbolic and there are no cycles in it, then $\Omega(f) = L(f)$, and so f is a hyperbolic diffeomorphism with no cycles [83]. Clearly, the converse is also true.

Another important conclusion is that hyperbolic diffeomorphisms with no cycles are C^r *Ω-stable* for any $r \ge 1$: any g in a small C^r neighbourhood of f is topologically conjugate to f, restricted to their non-wandering sets. That is, there exists a homeomorphism $h : \Omega(f) \to \Omega(g)$ such that

$$(f \mid \Omega(f)) \circ h = h \circ (g \mid \Omega(g)).$$

Cf. [119], this follows from the existence of a filtration as in (A.1), combined with the local stability Theorem 2.1.1. On the other hand, a C^r Ω-stable hyperbolic diffeomorphism can not have cycles [87].

Inspired on these facts, as well as on the stability results for Anosov systems [7] and for Morse-Smale systems [90], Palis and Smale conjectured that hyperbolicity together with the no cycle condition completely characterize the Ω-stable systems. In view of the results mentioned before, this *Ω-stability conjecture* reduced to proving that C^r Ω-stable systems are hyperbolic, for every $r \ge 1$.

They also proposed a similar characterization for structural stability. After [6], one says that a diffeomorphism f is C^r *structurally stable*, $r \ge 1$, if any g in a small C^r neighbourhood of f is topologically conjugate to f: there exists a homeomorphism $h : M \to M$ that

$$f \circ h = h \circ g.$$

The *stability conjecture* in [90] claims that a system is C^r structurally stable in and only if it is hyperbolic and satisfies the *strong transversality condition*: the stable manifold and the unstable manifold of any two points in $\Omega(f)$ are transverse.

The fact that hyperbolic systems satisfying the transversality condition are structurally stable was established in the seventies by [102], [34], [104], [105], [106]. Moreover, [103] implied that the strong-transversality condition is indeed necessary for structural stability. In this way, the stability conjecture was also reduced to proving that hyperbolicity is necessary for structural stability.

The proof came only after a decade. By the mid-eighties Mañé [77] proved the remarkable fact that C^1 structurally stable diffeomorphisms must be hyperbolic, thus settling the C^1 stability conjecture. Based on his methods, [88] extended this conclusion to C^1 Ω-stable case. Other important contributions to these problems had been given, specially, by [95], [?], [74], [75], [113]. The extension of both conjectures for C^1 flows was achieved very recently by [52].

Some of the key tools in the proofs, such as the closing lemma of [97] and the connecting lemma of [52] are available only in the C^1 topology. Their C^r versions, as well as the C^r versions of the results in the previous paragraph, remain outstanding open problems for any $r \geq 2$.

A.2 Rokhlin's Disintegration Theorem

Let Z be a compact metric space, μ be a Borel probability measure on Z, and $\mathcal{P}$ be a partition of Z into measurable subsets. Let $\pi : Z \to \mathcal{P}$ be the map associating to each $z \in Z$ the atom $P \in \mathcal{P}$ that contains it. By definition, $\mathcal{Q}$ is a measurable subset of $\mathcal{P}$ if and only if $\pi^{-1}(\mathcal{Q})$ is a measurable subset of Z. Let $\hat{\mu}$ be the push-forward of μ under π, in other words, $\hat{\mu}$ is the probability measure on $\mathcal{P}$ defined by $\hat{\mu}(\mathcal{Q}) = \mu(\pi^{-1}(\mathcal{Q}))$ for every measurable set $\mathcal{Q} \subset \mathcal{P}$.

Definition A.2.1. *A system of conditional measures of μ with respect to $\mathcal{P}$ is a family $(\mu_P)_{P \in \mathcal{P}}$ of probability measures on Z such that*

(1) $\mu_P(P) = 1$ for $\hat{\mu}$-almost every $P \in \mathcal{P}$;
(2) given any continuous $\varphi : Z \to \mathbb{R}$, the function $\mathcal{P} \ni P \mapsto \int \varphi \, d\mu_P$ is measurable and $\int \varphi \, d\mu = \int \left(\int \varphi \, d\mu_P \right) d\hat{\mu}(P)$.

Lemma A.2.1. *If $(\mu_P)_{P \in \mathcal{P}}$ satisfies condition (2) above then:*

(a) For any bounded measurable $\psi : Z \to \mathbb{R}$, the function $\mathcal{P} \ni P \mapsto \int \psi \, d\mu_P$ is measurable and satisfies

$$\int \psi \, d\mu = \int \left(\int \psi \, d\mu_P \right) d\hat{\mu}(P).$$

(b) In particular, a measurable set $E \subset Z$ has zero measure for μ if and only if it has zero μ_P-measure for $\hat{\mu}$-almost every $P \in \mathcal{P}$.

Proof. Let $\mathcal{A}$ represent the class of functions that satisfy the conclusion of (a). By definition, $\mathcal{A}$ contains all continuous functions. The characteristic function of any compact subset is the pointwise limit of some uniformly bounded sequence of continuous functions. So, by the dominated convergence theorem,

$\mathcal{A}$ contains all the characteristic functions of compact sets. Given any measurable subset E of Z, there exist a compact subsets K of E with $\mu(E \setminus K)$ arbitrarily small. It follows that the characteristic function of E is also in the class $\mathcal{A}$. Then, by linearity, $\mathcal{A}$ contains all simple functions, that is, finite linear combinations of characteristic functions of measurable sets. Finally, any bounded measurable function is the pointwise limit of some uniformly bounded sequence of simple functions. Hence, using the dominated convergence theorem once more, $\mathcal{A}$ contains all bounded measurable functions. This proves the first statement. Taking as ψ the characteristic function of E, part (b) follows. □

Conditional measures, when they exist, are unique almost everywhere:

Proposition A.2.1. *If $(\mu_P)_{P\in\mathcal{P}}$ and $(\nu_P)_{P\in\mathcal{P}}$ are two systems of conditional measures of μ with respect to $\mathcal{P}$, then $\mu_P = \nu_P$ for $\hat{\mu}$-almost every $P \in \mathcal{P}$.*

Proof. Suppose there exists a measurable set $\mathcal{Q}_0 \subset \mathcal{P}$ with $\hat{\mu}(\mathcal{Q}_0) > 0$ such that $\mu_P \neq \nu_P$ for every $P \in \mathcal{Q}_0$. In other words, for each $P \in \mathcal{Q}_0$ there exists some continuous function $\varphi : Z \to \mathbb{R}$ such that $\int \varphi\, d\mu_P \neq \int \varphi\, d\nu_P$. Using the fact that the space $C^0(Z,\mathbb{R})$ of continuous real functions on Z admits countable dense subsets, see e.g. [124, Theorem 0.19], we conclude that there exists a continuous function φ and a subset $\mathcal{Q}$ of $\mathcal{Q}_0$ such that $\hat{\mu}(\mathcal{Q}) > 0$ and (reversing the roles of μ_P and ν_P if necessary) $\int \varphi\, d\mu_P > \int \varphi\, d\nu_P$ for every $P \in \mathcal{Q}$. Then

$$\int_{\mathcal{Q}} \left(\int \varphi\, d\mu_P \right) d\hat{\mu}(P) > \int_{\mathcal{Q}} \left(\int \varphi\, d\nu_P \right) d\hat{\mu}(P). \tag{A.2}$$

On the other hand, by Lemma A.2.1,

$$\int (\varphi \mathcal{X}_{\pi^{-1}(\mathcal{Q})})\, d\mu = \int \left(\int (\varphi \mathcal{X}_{\pi^{-1}(\mathcal{Q})})\, d\mu_P \right) d\hat{\mu}(P).$$

By assumption, $\mu_P(P) = 1$ for $\hat{\mu}$-almost every $P \in \mathcal{P}$. For any such P,

$$\int (\varphi \mathcal{X}_{\pi^{-1}(\mathcal{Q})})\, d\mu_P = \mathcal{X}_{\mathcal{Q}}(P) \int \varphi\, d\mu_P.$$

Therefore,

$$\int (\varphi \mathcal{X}_{\pi^{-1}(\mathcal{Q})})\, d\mu = \int \left(\mathcal{X}_{\mathcal{Q}}(P) \int \varphi\, d\mu_P \right) d\hat{\mu}(P) = \int_{\mathcal{Q}} \left(\int \varphi\, d\mu_P \right) d\hat{\mu}(P).$$

Analogously, we find

$$\int (\varphi \mathcal{X}_{\pi^{-1}(\mathcal{Q})})\, d\mu = \int_{\mathcal{Q}} \left(\int \varphi\, d\nu_P \right) d\hat{\mu}(P).$$

These two last equalities contradict (A.2). Therefore, $\mu_P = \nu_P$ for $\hat{\mu}$-almost every P, as claimed. □

Definition A.2.2. *$\mathcal{P}$ is a* measurable partition *if there exist measurable subsets $E_1, E_2, \ldots, E_n, \ldots$ of Z such that*

$$\mathcal{P} = \{E_1, Z \setminus E_1\} \vee \{E_2, Z \setminus E_2\} \vee \cdots \vee \{E_n, Z \setminus E_n\} \vee \cdots \qquad \text{mod } 0.$$

In other words, there exists some full μ-measure subset $F_0 \subset Z$ such that, given any atom P of $\mathcal{P}$ we may write

$$P \cap F_0 = E_1^* \cap E_2^* \cap \cdots \cap E_n^* \cap \cdots \cap F_0 \tag{A.3}$$

where E_j^* is either E_j or its complement $Z \setminus E_j$, for every $j \geq 1$.

Example A.2.1. Every finite or countable partition is a measurable partition. In fact, $\mathcal{P}$ is measurable if and only if there exists a non-decreasing sequence of countable (possibly finite) partitions $\mathcal{P}_1 \preceq \mathcal{P}_2 \preceq \cdots \preceq \mathcal{P}_n \preceq \cdots$ such that $\mathcal{P} = \vee_{n=1}^{\infty} \mathcal{P}_n$ restricted to some full measure subset.

Example A.2.2. Let $Z = X \times Y$, where X and Y are compact metric spaces, and $\mathcal{P}$ be the partition of Z into horizontal lines $X \times \{y\}$, $y \in Y$. Then $\mathcal{P}$ is a measurable partition of Z: just take $E_j = X \times B_j$ where $(B_j)_j$ is any countable basis for the topology of Y.

The following result is due to Rokhlin [109, 110]:

Theorem A.2.1. *If $\mathcal{P}$ is a measurable partition, then there exists some system of conditional measures of μ relative to $\mathcal{P}$.*

Proof. The conclusion is not affected if we replace the space Z by any full measure subset. So, it is no restriction to suppose that the set F_0 in (A.3) coincides with Z, and we do so in all that follows. Let ψ be any bounded measurable real function on Z. For each $n \geq 1$ let

$$\mathcal{P}_n = \{E_1, Z \setminus E_1\} \vee \{E_2, Z \setminus E_2\} \vee \cdots \vee \{E_n, Z \setminus E_n\}$$

that is, $\mathcal{P}_n$ is the partition of Z whose atoms are the sets $E_1^* \cap \cdots \cap E_n^*$, with $E_j^* = E_j$ or $E_j^* = Z \setminus E_j$, for each $1 \leq j \leq n$. It is also no restriction to suppose that all the atoms of the partitions $\mathcal{P}_n$ have positive μ-measure: if $P_n \in \mathcal{P}_n$ has zero measure we just remove it from Z; this can occur at most countably often, because the set of all partitions' atoms is countable, so at the end we keep a full measure subset of Z. Now, we may define $\tilde{\psi}_n : Z \to \mathbb{R}$ by

$$\tilde{\psi}_n(z) = \frac{1}{\mu(P_n(z))} \int_{P_n(z)} \psi \, d\mu, \tag{A.4}$$

where $P_n(z)$ represents the atom of $\mathcal{P}_n$ that contains z.

Lemma A.2.2. *Given any bounded measurable function $\psi : Z \to \mathbb{R}$, there exists a full μ-measure subset $F = F(\psi)$ of Z such that $\tilde{\psi}_n(z)$, $n \geq 1$, converges to some real number $\tilde{\psi}(z)$, for every $z \in F$.*

Proof. We may always write $\psi = \psi^+ - \psi^-$, where $\psi^\pm$ are measurable, bounded, and non-negative: for instance, $\psi^\pm = (|\psi| \pm \psi)/2$. Then $\tilde\psi_n = \tilde\psi_n^+ - \tilde\psi_n^-$ for $n \geq 1$, and so the conclusion holds for ψ if it holds for ψ^+ and ψ^-. This shows that it is no restriction to assume that ψ is non-negative. We do so from now on.

For any $\alpha < \beta$, let $S(\alpha, \beta)$ be the set of points $z \in Z$ such that

$$\liminf \tilde\psi_n(z) < \alpha < \beta < \limsup \tilde\psi_n(z).$$

Clearly, given $z \in Z$, the sequence $\tilde\psi_n(z)$ diverges if and only if z is in $S(\alpha, \beta)$ for some pair of rational numbers α and β. So, the lemma will follow if we show that $S = S(\alpha, \beta)$ has zero μ-measure for all α and β.

For each $z \in S$ fix some sequence of integers $1 \leq a(1) < b(1) < \cdots < a(i) < b(i) < \cdots$ such that

$$\tilde\psi_{a(i)}(z) < \alpha \quad \text{and} \quad \tilde\psi_{b(i)}(z) > \beta \quad \text{for every } i \geq 1.$$

Define A_i to be the union of the partition sets $P_{a(i)}(z)$, and B_i to be the union of the partition sets $P_{b(i)}(z)$ obtained in this way, for all the points $z \in S$. By construction,

$$S \subset A_{i+1} \subset B_i \subset A_i \quad \text{for every } i \geq 1.$$

In particular, S is contained in the set

$$\tilde S = \bigcap_{i=1}^{\infty} B_i = \bigcap_{i=1}^{\infty} A_i\,.$$

Given any two of the sets $P_{a(i)}(z)$ that form A_i, either they are disjoint or one is contained in the other. This is because $\mathcal{P}_n$, $n \geq 1$, is a non-decreasing sequence of partitions. Consequently, A_i may be written as a two-by-two disjoint union of such sets $P_{a(i)}(z)$. Hence,

$$\int_{A_i} \psi\, d\mu = \sum_{P_{a(i)}(z)} \int_{P_{a(i)}} \psi\, d\mu < \sum_{P_{a(i)}(z)} \alpha\mu(P_{a(i)}) = \alpha\mu(A_i),$$

for any $i \geq 1$ (the sums are over that disjoint union). Analogously,

$$\int_{B_i} \psi\, d\mu = \sum_{P_{b(i)}(z)} \int_{P_{b(i)}} \psi\, d\mu > \sum_{P_{b(i)}(z)} \beta\mu(P_{b(i)}) = \beta\mu(B_i).$$

Since $A_i \supset B_i$ and we are assuming $\psi \geq 0$, it follows that

$$\alpha\mu(A_i) > \int_{A_i} \psi\, d\mu \geq \int_{B_i} \psi\, d\mu > \beta\mu(B_i),$$

for every $i \geq 1$. Taking the limit as $i \to \infty$, we find

$$\alpha\mu(\tilde S) \geq \beta\mu(\tilde S).$$

This implies that $\mu(\tilde S) = 0$, and so $S \subset \tilde S$ also has zero μ-measure. □

Given any bounded measurable function $\psi : Z \to \mathbb{R}$, we shall represent as $e_n(\psi)$, $e(\psi)$, respectively, the functions $\tilde{\psi}_n$, $\tilde{\psi}$ defined by (A.4) and Lemma A.2.2.

Let $\{\varphi_k : k \in \mathbb{N}\}$ be some countable dense subset of $C^0(Z, \mathbb{R})$, and let

$$F_* = \bigcap_{k=1}^{\infty} F(\varphi_k),$$

where $F(\varphi_k)$ is as given by Lemma A.2.2. Given any $z \in F_*$, we have

$$\lim_n e_n(\varphi)(z) = e(\varphi)(z)$$

for every continuous function $\varphi : Z \to \mathbb{R}$. Indeed, the class of functions for which this happens is, clearly, closed under uniform convergence. Since it contains a dense subset $\{\varphi_k : k \in \mathbb{N}\}$, it must contain all continuous functions.

Let $\varphi : Z \to \mathbb{R}$ be continuous. By construction, for each $n \geq 1$ the function $e_n(\varphi)$ is constant on every $P_n \in \mathcal{P}_n$, and so it is also constant on every atom P of $\mathcal{P}$. Therefore, $e(\varphi)$ is constant on $P \cap F_*$ for every $P \in \mathcal{P}$. Let $e_n(\varphi)(P_n)$ represent the value of $e_n(\varphi)$ on each $P_n \in \mathcal{P}_n$. Similarly, $e(\varphi)(P)$ represents the value of $e(\varphi)$ on $P \cap F_*$ whenever the latter set is nonempty. Then

$$\int \varphi \, d\mu = \sum_{P_n \in \mathcal{P}_n} \int_{P_n} \varphi \, d\mu = \sum_{P_n \in \mathcal{P}_n} \mu(P_n) \, e_n(\varphi)(P_n) = \int e_n(\varphi) \, d\mu.$$

Observe also that $|e_n(\varphi)| \leq \sup |\varphi| < \infty$ for every $n \geq 1$. Therefore, we may use the dominated convergence theorem to conclude that

$$\int \varphi \, d\mu = \int e(\varphi) \, d\mu. \tag{A.5}$$

Now we are in a position to exhibit a system of conditional measures of μ with respect to $\mathcal{P}$. Let P be any atom of $\mathcal{P}$ such that $P \cap F_*$ is nonempty. It is easy to see that

$$\varphi \mapsto e(\varphi)(P) \in \mathbb{R}$$

is a non-negative linear functional on $C^0(Z, \mathbb{R})$ with $e(1)(P) = 1$. By the Riesz-Markov theorem, there exists a unique probability measure μ_P on Z such that

$$\int \varphi \, d\mu_P = e(\varphi)(P) \tag{A.6}$$

for every $\varphi \in C^0(Z, \mathbb{R})$. For completeness, we should define μ_P also when P does not intersect F_*. In this case we let μ_P be any probability measure on Z: since the set of all these atoms P has zero $\hat{\mu}$-measure in $\mathcal{P}$ (in other words, their union has zero μ-measure in Z), the choice is not relevant. In view of these definitions, (A.5) may be rewritten as

$$\int \varphi \, d\mu = \int \left(\int \varphi \, d\mu_P \right) \, d\hat{\mu}(P),$$

the fact that $\mathcal{P} \ni P \mapsto \int \varphi \, d\mu_P$ is a measurable function being a direct consequence of (A.6). Therefore, to conclude that $(\mu_P)_{P \in \mathcal{P}}$ do form a system of conditional measures of μ with respect to $\mathcal{P}$ we only have to prove

Lemma A.2.3. *$\mu_P(P) = 1$ for $\hat{\mu}$-almost every $P \in \mathcal{P}$.*

This will follow from a version of (A.6) for the characteristic functions of the partitions' atoms :

Lemma A.2.4. *Given $l \geq 1$ and $P_l \in \mathcal{P}_l$ there exists a full $\hat{\mu}$-measure subset $\mathcal{F}(P_l)$ of $\mathcal{P}$ such that*

$$\mu_P(P_l) = e(\mathcal{X}_{P_l})(P) = \begin{cases} 1 \text{ if } P \subset P_l \\ 0 \text{ otherwise} \end{cases}$$

for any $P \in \mathcal{F}(P_l)$.

Proof. The last equality in the statement is a direct consequence of the definition of $e(\cdot) = \lim_n e_n(\cdot)$. Now, let us fix a sequence of continuous functions $\varphi_k : Z \to [0, 1]$ converging μ-almost everywhere to the characteristic function $\mathcal{X}_{P_l}$. By part (b) of Lemma A.2.1, there exists a full $\hat{\mu}$-measure set $\mathcal{F}_1 \subset \mathcal{P}$ such that φ_k converges to $\mathcal{X}_{P_l}$ at μ_P-almost every point, for every $P \in \mathcal{F}_1$. Replacing $\mathcal{F}_1$ by a full measure subset if necessary, we may suppose that every $P \in \mathcal{F}_1$ intersects F_*. Then

$$\mu_P(P_l) = \int \mathcal{X}_{P_l} \, d\mu_P = \lim_k \int \varphi_k \, d\mu_P = \lim_k e(\varphi_k)(P) \tag{A.7}$$

for every $P \in \mathcal{F}_1$. We claim that, as $k \to \infty$, the sequence $e(\varphi_k)$ converges in $\hat{\mu}$-measure to $e(\mathcal{X}_{P_l})$. Combining this with the pointwise convergence in (A.7), we get that $\mu_P(P_l) = e(\mathcal{X}_{P_l})(P)$ for every P in some full $\hat{\mu}$-measure set $\mathcal{F}_2 \subset \mathcal{F}_1$. This gives the lemma, with $\mathcal{F}(P_l) = \mathcal{F}_2$.

We are left to prove the claim above. We begin by noting that the same calculation as in (A.5) shows that

$$\int_{P_l} \varphi \, d\mu = \int_{P_l} e(\varphi) \, d\mu \tag{A.8}$$

for any continuous function φ. So,

$$\int_{P_l} (1 - \varphi_k) \, d\mu = \int_{P_l} (1 - e(\varphi_k)) \, d\mu.$$

In view of our choice of the φ_k, the left hand side of the last equation converges to zero as $k \to \infty$. Moreover, $e(\varphi_k) \leq 1$ for all $k \geq 1$. It follows that the restriction of $e(\varphi_k)$ to P_l converges to 1 in μ-measure. Dually, (A.5) and (A.8) imply

$$\int_{Z\setminus P_l} \varphi_k \, d\mu = \int_{Z\setminus P_l} e(\varphi_k) \, d\mu.$$

Since the left hand side converges to zero as $k \to \infty$, and $e(\varphi_k)$ is non-negative, the restriction of $e(\varphi_k)$ to the complement of P_l converges to 0 in μ-measure. Altogether, this shows that the sequence $e(\varphi_k)$ converges in μ-measure to the characteristic function of P_l. Thinking of each $e(\varphi_k)$ as a function on $\mathcal{P}$, as we were before, this means that the sequence converges in $\hat{\mu}$-measure to $P \mapsto e(\mathcal{X}_{P_l})(P)$. This is precisely what we claimed. □

Now we can prove Lemma A.2.3:

Proof. Define $\mathcal{F}_* = \cap_{l,P_l} \mathcal{F}(P_l)$, where the intersection is over the set of all the atoms $P_l \in \mathcal{P}_l$, and every $l \geq 1$. Since this is a countable set, $\mathcal{F}_*$ has full $\hat{\mu}$-measure. We claim that the conclusion of the lemma holds for every $P \in \mathcal{F}_*$. Indeed, let P_l be the element of $\mathcal{P}_l$ that contains P. By Lemma A.2.4, we have $\mu_P(P_l) = e(\mathcal{X}_{P_l})(P) = 1$ for every $l \geq 1$. Since the P_l are a decreasing sequence whose intersection is P, we get $\mu_P(P) = \lim_l \mu_P(P_l) = 1$. □

The proof of Theorem A.2.1 is complete. □

Example A.2.3. Let Z be the 2-dimensional torus, α be some irrational number, and $\mathcal{P}$ be the partition of Z into the straight lines of slope α. Then $\mathcal{P}$ is *not* a measurable partition. Indeed, the Haar (Lebesgue) measure μ on Z admits no system of conditional measures with respect to $\mathcal{P}$. To see this, suppose $(\mu_P)_P$ was such a system of conditional measures. Let $g : Z \to Z$, $g(\theta_1, \theta_2) = (\theta_1 + 1, \theta_2 + \alpha)$, be the translation along the direction $(1, \alpha)$. Clearly, g maps each $P \in \mathcal{P}$ into itself, and also preserves the Haar measure μ. It follows that the push-forwards $(g_*\mu_P)_P$ form a system of conditional measures of $g_*\mu = \mu$ with respect to $\mathcal{P}$. So, by Proposition A.2.1, we must have $g_*\mu_P = \mu_P$ for $\hat{\mu}$-almost every $P \in \mathcal{P}$. However, this is a contradiction: each μ_P is a probability on P, and translations on the real line have no invariant finite measures.

Another application of Theorem A.2.1 is the theorem of *ergodic decomposition of invariant measures:*

Example A.2.4. Let $f : Z \to Z$ be a continuous transformation on a compact metric space Z, and μ be any f-invariant probability measure. Let B_f be the subset of points $z \in Z$ such that time averages are well-defined on the orbit of z: given any continuous function $\varphi : Z \to \mathbb{R}$, the limit

$$\tilde{\varphi}(z) = \lim_n \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(z))$$

exists. By Birkhoff's ergodic theorem, B_f has full measure with respect to μ. Let $\mathcal{P}$ be the partition of Z defined by

(i) two points z_1, z_2 in B_f are in the same atom of $\mathcal{P}$ if and only if they have the same time averages: $\tilde{\varphi}(z_1) = \tilde{\varphi}(z_2)$ for every continuous function φ;
(ii) $Z \setminus B_f$ is an atom of $\mathcal{P}$.

Let $\{\varphi_k : k \geq 1\}$ be any countable dense subset of $C^0(Z, \mathbb{R})$, and $\{a_l : l \geq 1\}$ be any countable dense subset of $\mathbb{R}$. Define $E_{k,l}$ as the set of points $x \in B_f$ for which $\tilde{\varphi}_k(x) < a_l$. Then,

$$\mathcal{P} = \bigvee_{k,l} \{E_{k,l}, Z \setminus E_{k,l}\}$$

restricted to the full measure subset $F_0 = B_f$. This proves that $\mathcal{P}$ is a measurable partition. Let $(\mu_P)_P$ be a system of conditional measures of μ with respect to $\mathcal{P}$. Since f maps each element of $\mathcal{P}$ to itself, $(f_*\mu_P)_P$ is a system of conditional measures of $f_*\mu = \mu$ with respect to $\mathcal{P}$. So, by Proposition A.2.1, μ_P is f-invariant for $\hat{\mu}$-almost every P. Finally, the property $\mu_P(P) = 1$ means that time-averages of continuous functions are constant μ_P-almost everywhere, and so μ_P is ergodic, for $\hat{\mu}$-almost every $P \in \mathcal{P}$.

A.3 Hölder Continuity of Invariant Foliations

Here we prove Proposition 2.2.4: the strong-unstable foliation of a partially hyperbolic attractor of type $E^u \oplus E^{cs}$ is transversely Hölder continuous, if the diffeomorphism is twice differentiable. If Λ is hyperbolic then both the stable foliation and the unstable foliation are transversely Hölder continuous.

Up to replacing f by some f^N, with N large, we may suppose that (2.7) and the condition in Definition 2.2.2 hold with $C = 1$:

$$\|Df^{-1} \mid E^u_\xi\| \leq \lambda \quad \text{and} \quad \|Df^{-1} \mid E^u_{f(\xi)}\| \, \|Df \mid E^{cs}_\xi\| \leq \lambda, \tag{A.9}$$

for all $\xi \in \Lambda$. Fix $\varepsilon > 0$ such that $e^{4\varepsilon} < \lambda$. Then take $\delta > 0$ and $a > 0$ small enough so that

$$\|Df^{-1} \mid E^u_\xi\| \leq e^{\varepsilon} \|Df^{-1} \mid E^u_\eta\| \quad \text{and} \quad \|Df \mid V\| \leq e^{\varepsilon} \|Df \mid E^{cs}_\eta\|, \tag{A.10}$$

for all $\xi, \eta \in \Lambda$ with $d(\xi, \eta) \leq \delta$ and any subspace V of $T_\xi M$ contained in the cone $C_a(E^{cs}, \xi)$. Extend E^u and E^{cs} continuously to a neighbourhood U of Λ, small enough so that the extended cone field $C_a(E^{cs})$ remains forward invariant. Reducing δ if necessary, we may suppose that U contains the 2δ-neighbourhood of Λ, and conditions (A.9), (A.10) remain true for every $\xi \in U$ and $\eta \in \Lambda$ with $d(\xi, \eta) \leq 2\delta$. Moreover, any $\xi, \eta \in \Lambda$ with $d(\xi, \eta) \leq 2\delta$ are contained in the domain of some foliated chart of $\mathcal{F}^u$.

We want to prove that any local holonomy map π between cross-sections Σ_1 and Σ_2 satisfies a Hölder condition. For this it is no restriction to suppose that Σ_1 and Σ_2 are nearby, and their tangents are close to E^{cs} at each point:

(a) $d(x, \pi(x)) \leq \delta$ for any $x \in \Sigma_1$;

(b) $T_x \Sigma_i \subset C_a(E^{cs}, x)$ for every $x \in \Sigma_i$, $i = 1, 2$.

Indeed, (a) and (b) can always be enforced by considering backward iterates $f^{-n}(\Sigma_1)$ and $f^{-n}(\Sigma_2)$, with n large, in the place of Σ_1 and Σ_2. Observe that the holonomy π_n from $f^{-n}(\Sigma_1)$ to $f^{-n}(\Sigma_2)$ is given by $\pi_n = f^{-n} \circ \pi \circ f^n$, since the foliation $\mathcal{F}^u$ is invariant under f. Hence, π is Hölder continuous if and only if π_n is, so that replacing the cross-sections does not affect the validity of the conclusion. Thus, in what follows we assume (a) and (b).

By the compactness of Λ and the continuity of foliated charts, we may fix $\delta_1 > 0$ small enough so that

$$d(\xi, \eta) < \delta_1 \quad \Rightarrow \quad d(\pi(\xi), \pi(\eta)) < \delta,$$

for any $\xi, \eta \in \Sigma_1$ and any pair of cross-sections Σ_1 and Σ_2 as before. We are going to show that there are constants $K > 0$ and $0 < \gamma \leq 1$, depending only on the previous choices, such that

$$d_\Sigma(\pi(x_1), \pi(y_1)) \leq K d_\Sigma(x_1, y_1)^\gamma \tag{A.11}$$

for every $x_1, y_1 \in \Sigma_1$ and any pair of cross-sections Σ_1 and Σ_2 as above. Here $d_\Sigma(\cdot\,,\cdot)$ represents the distance measured along a corresponding cross-section to the foliation (it is always clear from the context which one is meant).

Clearly, we may restrict ourselves to the case when $d_\Sigma(x_1, y_1) \leq \delta$. We denote $x_2 = \pi(x_1)$ and $y_2 = \pi(y_1)$. As before, we use $d_u(\cdot\,,\cdot)$ to represent the distance measured along leaves of $\mathcal{F}^u$. Conditions (A.9) and (a) imply

$$d_u(f^{-j}(y_1), f^{-j}(y_2)) \leq \lambda^j d_u(y_1, y_2) \leq \lambda^j \delta \leq \delta \tag{A.12}$$

for every $j \geq 0$, and analogously for $d_u(f^{-j}(x_1), f^{-j}(x_2))$. We claim that there exists $n \leq 4 \log_\lambda d_\Sigma(x_2, y_2)$ such that

(i) $d_\Sigma(f^{-j}(x_2), f^{-j}(y_2)) \leq \delta$ for all $0 \leq j < n$, and

(ii) $\delta_1\, d_\Sigma(f^{-n}(x_2), f^{-n}(y_2)) \leq d_\Sigma(f^{-n}(x_1), f^{-n}(y_1))$.

Before proving this claim, let us show how the proposition follows from it.

From (i) and (A.10) we get

$$d_\Sigma(x_2, y_2) \leq \prod_{j=1}^{n} \left(\|Df \mid E^{cs}_{f^{-j}(x_2)}\| e^\varepsilon\right) d_\Sigma(f^{-n}(x_2), f^{-n}(y_2)).$$

We have seen in (A.12) that $d_u(f^{-j}(x_1), f^{-j}(x_2)) \leq \delta$ for all $0 \leq j < n$. Moreover, (i) and (A.12) imply $d(f^{-j}(y_1), f^{-j}(x_2)) \leq 2\delta$. Then, using (A.10) and the mean value theorem for f^{-1},

$$d_\Sigma(x_1, y_1) \geq \prod_{j=0}^{n-1} \left(\|Df^{-1} \mid E^{cs}_{f^{-j}(x_2)}\| e^\varepsilon\right)^{-1} d_\Sigma(f^{-n}(x_1), f^{-n}(y_1)).$$

Let $K_1 = e^{2\varepsilon} \sup\{\|Df \mid E^{cs}_\eta\| \, \|Df^{-1} \mid E^{cs}_{f(\eta)}\| : \eta \in \Lambda\}$. The previous inequalities, combined with (ii), give

$$\frac{d_\Sigma(x_2, y_2)}{d_\Sigma(x_1, y_1)} \leq \frac{d_\Sigma(f^{-n}(x_2), f^{-n}(y_2))}{d_\Sigma(f^{-n}(x_1), f^{-n}(y_1))} K_1^n \leq \frac{1}{\delta_1} K_1^n.$$

Since $n \leq 4 \log_\lambda d_\Sigma(x_2, y_2)$, this implies

$$\frac{d_\Sigma(x_2, y_2)}{d_\Sigma(x_1, y_1)} \leq \frac{1}{\delta_1} d_\Sigma(x_2, y_2)^\theta,$$

where $\theta = 4 \log K_1 / \log \lambda < 0$. Then (A.11) follows, with $\gamma = 1/(1-\theta) < 1$ and $K = \delta_1^{-\gamma}$.

All that is left to do is to prove the claims (i) and (ii). Suppose first that $d_\Sigma(f^{-j}(x_2), f^{-j}(y_2))$ is less that δ for all $j \leq 4 \log_\lambda d_\Sigma(x_2, y_2)$. Fix any n between $2 \log_\lambda d_\Sigma(x_2, y_2)$ and $4 \log_\lambda d_\Sigma(x_2, y_2)$. From

$$d_u(f^{-n}(x_1), f^{-n}(x_2)) \leq \prod_{j=0}^{n-1} \big(\|Df^{-1} \mid E^u_{f^{-j}(x_2)}\| e^\varepsilon\big) d_u(x_1, x_2)$$

$$d_\Sigma(f^{-n}(x_2), f^{-n}(y_2)) \geq \prod_{j=1}^{n} \big(\|Df \mid E^{cs}_{f^{-j}(x_2)}\| e^\varepsilon\big)^{-1} d_\Sigma(x_2, y_2)$$

and (A.9), we obtain

$$\frac{d_u(f^{-n}(x_1), f^{-n}(x_2))}{d_\Sigma(f^{-n}(x_2), f^{-n}(y_2))} \leq (\lambda e^{2\varepsilon})^n \frac{d_u(x_1, x_2)}{d_\Sigma(x_2, y_2)} \leq \lambda^{n/2} \frac{\delta}{d_\Sigma(x_2, y_2)}.$$

In view of our choice of n, the last term is bounded by δ. Analogously,

$$d_u(f^{-n}(y_1), f^{-n}(y_2)) \leq \delta d_\Sigma(f^{-n}(x_2), f^{-n}(y_2)).$$

Since δ and δ_1 were taken small, the triangular inequality gives

$$d_u(f^{-n}(x_1), f^{-n}(y_1)) \geq \frac{1}{2} d_\Sigma(f^{-n}(x_2), f^{-n}(y_2)) \geq \delta_1 d_\Sigma(f^{-n}(x_2), f^{-n}(y_2)).$$

This proves the claim in this case. Now suppose that, on the contrary, there exists $n \leq 4 \log_\lambda d_\Sigma(x_2, y_2)$ such that $d_\Sigma(f^{-n}(x_2), f^{-n}(y_2))$ is larger than δ. Take such an n minimum. Then, by the choice of δ_1,

$$d_\Sigma(f^{-n}(x_1), f^{-n}(y_1)) \geq \delta_1 \geq \delta_1 d_\Sigma(f^{-n}(x_2), f^{-n}(y_2)),$$

because $f^{-n}(x_2)$ and $f^{-n}(y_2)$ are the images of $f^{-n}(x_1)$ and $f^{-n}(y_1)$, respectively, under a local holonomy map from $f^{-n}(\Sigma_1)$ to $f^{-n}(\Sigma_2)$. The proof of the claim is complete.

This finishes the proof of Proposition 2.2.4

Remark A.3.1. The Hölder constants provided by the proof depend only on the distance between the cross-sections Σ_1 and Σ_2, and the angle they make with E^{cs} (through the iterate $n \geq 1$ required to obtain properties (a) and (b) in the proof).

Remark A.3.2. If E^{cs} has dimension 1 or, more generally, if Df is conformal in the direction E^{cs}

$$\|Df^{-1} \mid E^{cs}_{f(\xi)}\| = \|Df \mid E^{cs}_{\xi}\|^{-1}$$

then $K_1 = e^{2\varepsilon}$, and then $\theta = 8\varepsilon/\log\lambda$. So, in this case the Hölder constant γ may be taken arbitrarily close to 1. This has a number of interesting consequences, for instance, the attractor Λ has well-defined transverse Hausdorff dimension; see [91].

A.4 Absolute Continuity of Invariant Foliations

Here, we prove that invariant strong-stable (or strong-unstable) foliations of partially hyperbolic C^2 diffeomorfisms are absolutely continuous.

Let $f : M \to M$ be the diffeomorphism and $TM = E^{cu} \oplus E^s$ be the Df-invariant splitting. Let $\mathcal{F}^s$ be the strong-stable foliation, tangent to E^s at every point. We are going to prove

Theorem A.4.1. *For any holonomy map $\pi : \Sigma_1 \to \Sigma_2$ of $\mathcal{F}^s$ there exists a constant $K > 0$ such that*

$$\frac{1}{K} < \frac{m_{\Sigma_1}(D)}{m_{\Sigma_2}(\pi(D))} < K$$

for any disk $D \subset \Sigma_1$.

Theorem 2.2.3 is a direct consequence. Indeed, although the conclusion of Theorem A.4.1 seems weaker, because it refers to disks instead of general measurable sets, it is quite easy to deduce the full statement. Given any measurable set B let $\mathcal{D}$ be any family of disks covering B. By Theorem A.4.1,

$$m_{\Sigma_2}(\pi(B)) \leq \sum_{D\in\mathcal{D}} m_{\Sigma_2}(\pi(D)) \leq K \sum_{D\in\mathcal{D}} m_{\Sigma_1}(D).$$

Since, $\mathcal{D}$ may be taken such that $\sum_{D\in\mathcal{D}} m_{\Sigma_1}(D)$ is arbitrarily close to $m_{\Sigma_1}(B)$, it follows that $m_{\Sigma_2}(\pi(B)) \leq K m_{\Sigma_1}(B)$. Therefore, we may take $C_2 = K$ in Theorem 2.2.3. Similarly, using the inverse holonomy map $\pi^{-1} : \Sigma_2 \to \Sigma_1$, we may take $C_1 = 1/K$. This shows that Theorem 2.2.3 does follow from Theorem A.4.1.

Outline of the proof. Before starting the proof of Theorem A.4.1, let us explain what are the main steps. Instead of trying to compare the volumes of D and $\pi(D)$ directly, one looks at iterates $f^n(D) \subset f^n(\Sigma_1)$ and $f^n(\pi(D)) \subset f^n(\Sigma_2)$ for some large $n \geq 1$. The point is that $f^n(\Sigma_1)$ and $f^n(\Sigma_2)$ are very close to each other, because Σ_1 and Σ_2 are transverse to the strong-stable foliation $\mathcal{F}^s$, and the leaves of $\mathcal{F}^s$ are exponentially contracted by f^n. This makes it possible to compare the volumes of appropriate subsets of $f^n(\Sigma_1)$ and $f^n(\Sigma_2)$) using the holonomy map $\pi_n = \pi(f^n(\Sigma_1), f^n(\Sigma_2))$. More precisely, we consider balls $\mathcal{B}(n,x) \subset f^n(\Sigma_1)$ around each $f^n(x) \in f^n(\Sigma_1)$, with radius $r(n,x)$ chosen in a judicious way. One important condition is that $r(n,x)$ *should be much larger than the distance between* $f^n(x)$ *and* $\pi_n(f^n(x))$*:* this ensures that the volume of $\mathcal{B}(n,x)$ is approximately equal to the volume of $\pi_n(\mathcal{B}(n,x))$:

$$\frac{m_{f^n(\Sigma_1)}\big(\mathcal{B}(n,x)\big)}{m_{f^n(\Sigma_2)}\big(\pi_n(B(n,x))\big)}$$

is uniformly close to 1. See Figure A.1.

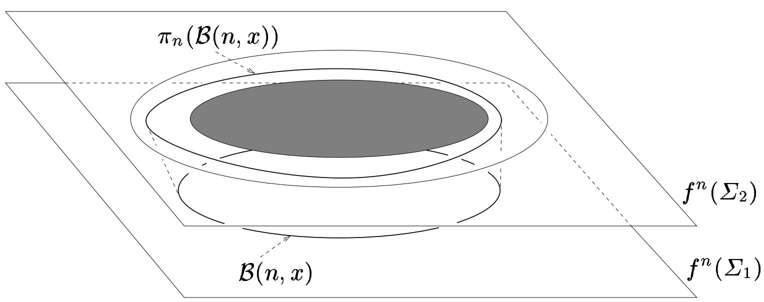

Fig. A.1. Volumes of large balls are almost preserved by the holonomy of nearby cross-sections

Then one considers preimages of the $\mathcal{B}(n,x)$ under f^n. The volume of $f^{-n}(\mathcal{B}(n,x))$ is given by the volume of $\mathcal{B}(n,x)$ divided by the Jacobian $J_{\Sigma_1} f^n(\xi)$ of f^n along Σ_1, at some point ξ in $f^{-n}(\mathcal{B}(n,x))$. Another main condition is that $r(n,x)$ *should be small enough* so that the Jacobians at different points of $f^{-n}(\mathcal{B}(n,x))$ be comparable, up to some factor close to 1. Then, one may take $\xi = x$. Similarly, the volume of

$$f^{-n}(\pi_n(\mathcal{B}(n,x))) = \pi(f^{-n}(\mathcal{B}(n,x)))$$

equals the volume of $\pi_n(\mathcal{B}(n,x))$ divided by the Jacobian $J_{\Sigma_2} f^n(\eta)$ at some point η, and one may take $\eta = \pi(x)$. Now, $J_{\Sigma_1} f^n(\xi)$ and $J_{\Sigma_2} f^n(\eta)$ are comparable up to some factor bounded from zero and infinity. The main reason for this is that $\xi = x$ and $\eta = \pi(x)$ are in the same strong-stable leave, and so their forward iterates remain for ever close; one also needs a Hölder type estimate for the tangent spaces to iterates of Σ_1 and Σ_2. It follows that the volumes of $f^{-n}(\mathcal{B}(n,x))$ and $\pi(f^{-n}(\mathcal{B}(n,x)))$ are comparable:

$$\frac{m_{\Sigma_1}\big(f^{-n}(\mathcal{B}(n,x))\big)}{m_{\Sigma_2}\big(\pi(f^{-n}(B(n,x)))\big)} \approx \frac{m_{f^n(\Sigma_1)}\big(\mathcal{B}(n,x)\big)}{m_{f^n(\Sigma_2)}\big(\pi(B(n,x))\big)} \frac{J_{\Sigma_2} f^n(\eta)}{J_{\Sigma_1} f^n(\xi)}$$

is uniformly bounded from zero and infinity.

Finally, to prove that the same is true for the volumes D and $\pi(D)$, it suffices to find an efficient covering of D by sets of the $f^{-n}(\mathcal{B}(n,x))$: there is a uniform upper bound for the number of elements of the covering that contain a given point. That is provided by Besicovich's covering lemma.

Preliminaries. Now, let us get into the details of the proof. According to Definitions 2.2.1 and 2.2.2, there exist $m \geq 1$ and $\theta < 1$ such that

(a) $\|Df^m \mid E^s_x\| < \theta^3$ and
(b) $\|Df \mid E^s_x\| \leq \theta^3 \, \|Df^{-m} \mid E^{cu}_{f^m(x)}\|^{-1}$

for every $x \in M$. Up to replacing f by f^m, we may suppose that $m = 1$, and we do so in all that follows. Let us define

$$\tilde{a}(x) = \|Df \mid E^s_x\| \quad \text{and} \quad \tilde{b}(x) = \|Df^{-1} \mid E^{cu}_{f(x)}\|^{-1}.$$

Clearly, $\tilde{a}$ and $\tilde{b}$ are continuous functions, and $\tilde{a}(x) < \theta^3$ and $\tilde{a}(x) < \theta^3 \, \tilde{b}(x)$ for every $x \in M$, by conditions (a) and (b).

Let $\delta > 0$ be some fixed small constant, and

$$a(x) = \sup\{\tilde{a}(y) : d(x,y) < \delta\} \quad \text{and} \quad b(x) = \sup\{\tilde{b}(y) : d(x,y) < \delta\}.$$

Assuming δ is sufficiently small, we have $a(x) < \theta^2$ and $a(x) < \theta^2 b(x)$, for every $x \in M$. Now, define

$$\mu(n,x) = \prod_{i=0}^{n-1} a(f^i(x)) \quad \text{and} \quad \sigma(n,x) = \prod_{i=0}^{n-1} b(f^i(x)).$$

Thus, $\mu(x,n)$ is an upper bound for the contraction along the direction of $\mathcal{F}^s$, over any orbit that stays within δ from the orbit of x during the first n iterates. Similarly, $\sigma(x,n)$ is a lower bound for the least expansion (or strongest contraction) along E^{cu} over all such orbits. The previous estimates imply that

$$\mu(n,x) < \theta^{2n} \quad \text{and} \quad \mu(n,x) < \theta^{2n} \, \sigma(n,x) \tag{A.13}$$

for every x and $n \geq 1$.

Let $c(\cdot)$ be a continuous function such that $\theta^{-1} a(y) \leq c(y) \leq \theta b(y)$ and $c(y) \leq \theta$ for every $y \in M$. For instance, $c(y) = \theta^{-1} a(y)$. Then define

$$r(n,x) = \prod_{i=0}^{n-1} c(f^i(x)). \tag{A.14}$$

The following properties are direct consequences of the definition:

(R1) $r(n,x) \le \theta^n$, (R2) $\mu(n,x) \le \theta^n r(n,x)$, (R3) $r(n,x) \le \theta^n \sigma(n,x)$.

For each $x \in \Sigma_1$ and $n \ge 1$, we denote by $\mathcal{B}(n,x)$ the ball of radius $r(n,x)$ around $f^n(x)$ inside $f^n(\Sigma_1)$. Conditions (R1) and (R3) are saying that these radii are uniformly small if n is large. Yet, according (R2), they are much larger than the distance between $f^n(x)$ and $\pi_n(x)$.

Step 1. The first main step in the proof of Theorem A.4.1 is to show that the volume of the ball $\mathcal{B}(n,x)$ of radius $r(n,x)$ around $f^n(x)$ inside Σ_1 is approximately equal to the volume of its image $\pi_n(\mathcal{B}(n,x))$ under the holonomy map π_n from $f^n(\Sigma_1)$ to $f^n(\Sigma_2)$.

Proposition A.4.1. *There exists a sequence $(\varepsilon_n)_n$ converging to zero such that*

$$\left| \frac{m_{f^n(\Sigma_1)}(\mathcal{B}(n,x))}{m_{f^n(\Sigma_2)}(\pi_n(\mathcal{B}(n,x)))} - 1 \right| \le \varepsilon_n$$

for every $n \ge 1$ and $x \in \Sigma_1$.

For the proof of Proposition A.4.1 we need a few auxiliary results. Let $x \in \Sigma_1$ and $n \ge 1$ be fixed throughout. At a few places we assume that δ is small, and n is sufficiently large (all the conditions are independent of the point x). The constants $K_1, \dots, K_j, \dots$ that appear in the sequel depend only on Σ_1, Σ_2, and the map f.

Preparatory lemmas. Let $1 \le k \le d$, where d is the dimension of the manifold M. For each $x \in M$ and k-dimensional subspaces V^1 and V^2 of T_xM, we define

$$\operatorname{angle}(V^1, V^2) = \max_{u_1 \in V^1} \min_{u^2 \in V^2} \angle(u_1, u_2) \tag{A.15}$$

We are going to use the following elementary fact:

$$\angle(u_1, u_2) \le \frac{\|u_1 - u_2\|}{\|u_2\|} \tag{A.16}$$

for every nonzero vectors u_1 and u_2 in any Hilbert space.

Our first lemma, which is a consequence of the domination property (b), says that the tangent spaces of $f^n(\Sigma_1)$ and $f^n(\Sigma_2)$ approach the center-unstable bundle E^{cu} exponentially fast as n increases.

Lemma A.4.1. *There exists $K_1 > 0$ such that*

$$\operatorname{angle}\big(T_{f^n(\xi)} f^n(\Sigma_s), E^{cu}(f^n(\xi))\big) \le K_1 \theta^{3n}$$

for any $n \ge 1$, $\xi \in \Sigma_s$, and $s = 1, 2$.

Proof. We consider $j = 1$, the other case is entirely analogous. Every nonzero vector $\tilde{v} \in T_{f^n(\xi)} f^n(\Sigma_1)$ may be written as $Df^n(\xi)v$ for some $v \in T_\xi \Sigma_1$. Let us write $v = v_1 + v_2$ where $v_1 \in E^s_\xi$ and $v_2 \in E^{cu}_\xi$. Since Σ_1 is transverse to the direction of E^s, there exists a constant $K_1 > 0$ that depends only on Σ_1 such that $\|v_1\| \le K_1 \|v_2\|$. From

$$\|Df^n(\xi)v_1\| \le \|Df^n \mid E^s_\xi\| \|v_1\| \quad \text{and} \quad \|v_2\| \le \|Df^{-n} \mid E^{cu}_{f^n(\xi)}\| \, \|Df^n(\xi)v_2\|$$

and the domination condition (b), we conclude that

$$\frac{\|Df^n(\xi)v_1\|}{\|Df^n(\xi)v_2\|} \le \theta^{3n} \frac{\|v_1\|}{\|v_2\|} \le K_1 \theta^{3n}. \tag{A.17}$$

Then, by (A.16), $\text{angle}(Df^n(\xi)v, Df^n(\xi)v_2) \le K_1\theta^{3n}$. Since $Df^n(\xi)v_2$ is in $E^{cu}_{f^n(\xi)}$, the definition (A.15) gives

$$\text{angle}(T_{f^n(\xi)} f^n(\Sigma_1), E^{cu}(f^n(\xi))) \le K_1 \theta^{3n},$$

as we claimed. □

Next, we use (R1) and (R3) to conclude that any point whose nth iterate is in $\mathcal{B}(n,x)$ remains close to the orbit of x during the first n iterates.

We denote by d_N the distance induced on a submanifold $N \subset M$ by the Riemannian metric of M. That is, $d_N(p,q)$ is the shortest (infimum) length of a piecewise smooth curve connecting p and q inside N. On the other hand, $d(p,q)$ is the distance between the points p and q in the ambient M.

Lemma A.4.2. *There exists $K_2 > 0$ such that $d(f^j(x), f^j(\xi)) \le K_2\theta^n$ for any $0 \le j \le n$ and any $\xi \in \Sigma_1$ such that $f^n(\xi) \in \mathcal{B}(n,x)$.*

Proof. Take $K_2 = 3$. The case $j = n$ is clear: by the definition of $\mathcal{B}(n,x)$,

$$d(f^n(x), f^n(\xi)) \le d_{f^n(\Sigma_1)}(f^n(x), f^n(\xi)) < r(n,x) \le \theta^n.$$

Now, given $0 \le j < n$, suppose the statement is known for all $j < i \le n$. More precisely, there exists some piecewise smooth curve γ_n connecting $f^n(x)$ to $f^n(\xi)$ inside $f^n(\Sigma_1)$, whose length is less than $r(n,x)$ and such that the length of $\gamma_i = f^{i-n}(\gamma_n)$ is less than $K_2\theta^n$ for every $j < i \le n$. We are going to prove that this remains true for $i = j$. Let $\dot\gamma_i$ denote the velocity vector of each γ_i. We decompose

$$\dot\gamma_i = \dot\gamma_i^s + \dot\gamma_i^{cu} \in E^s \oplus E^{cu}.$$

Just as in (A.17), we have $\|\dot\gamma_i^s\| / \|\dot\gamma_i^{cu}\| \le K_1\theta^{3n}$ for every $0 \le i \le n$. Assuming n is large enough, this is smaller than $1/2$. Then the cases $i = j$ and $i = n$ give

$$\|\dot\gamma_j\| \le \frac{3}{2} \|\dot\gamma_j^{cu}\| \quad \text{and} \quad \|\dot\gamma_n\| \ge \frac{1}{2} \|\dot\gamma_n^{cu}\|. \tag{A.18}$$

Now, the induction assumption implies that the length of γ_j is less than

$$\|Df^{-1}\| \operatorname{length}(\gamma_{j+1}) \leq \|Df^{-1}\| K_2 \theta^n.$$

Assuming n is large enough, this is smaller than δ. So, γ_i is contained in the δ-neighbourhood of $f^i(x)$ for all $i \leq j \leq n$. Thus, by the definition of $\sigma(\cdot\,,\cdot)$,

$$\|\dot\gamma_j^{cu}\| = \|Df^{j-n} \cdot \dot\gamma_n^{cu}\| \leq \frac{\|\gamma_n^{cu}\|}{\sigma(n-j, f^j(x))}.$$

Together with (A.18), this gives $\|\dot\gamma_j\| \leq 3\,\|\dot\gamma_n\|\,\sigma(n-j, f^j(x))^{-1}$. Then,

$$\operatorname{length}(\gamma_j) \leq \frac{3 \operatorname{length}(\gamma_n)}{\sigma(n-j, f^j(x))} \leq \frac{3\, r(n,x)}{\sigma(n-j, f^j(x))}.$$

By (R1) and (R3), $r(n,x) = r(j,x)\, r(n-j, f^j(x)) \leq \theta^n \sigma(n-j, f^j(x))$. So, the previous inequality gives $\operatorname{length}(\gamma_j) \leq 3\theta^n = K_2\theta^n$. □

For notational simplicity, given ξ and η in the same strong-stable leaf, we represent by $d_s(\xi, \eta)$ the distance between the two points inside that leaf.

Lemma A.4.3. *There exists $K_3 > 0$ such that $d_s(y, \pi_n(y)) \leq K_3 \mu_n(n,x)$ for every $y \in \mathcal{B}(n,x)$.*

Proof. Recall that $\mu(k,z)$ is an upper bound for the derivative of f^k along the stable direction, for orbits that remain within δ from the orbit of z. Given $y \in \mathcal{B}(n,x)$, let $\xi = f^{-n}(y)$. Since Σ_1 and Σ_2 are compact, there exists a uniform upper bound C_3 for the distance between ξ and $\pi(\xi)$ inside the leaf of $\mathcal{F}^s$ that contains the two points. As f contracts strong-stable leaves, by property (a), it follows that

$$d_s(f^j(\xi), f^j(\pi(\xi))) \leq C_3 \sup \|Df^j \mathcal{E}^s\| \leq C_3 \theta^{3j}$$

for all $j \geq 1$. In particular, fixing $p \geq 1$ so that $C_3\theta^{3p} < \delta/2$, we have

$$d(f^j(\xi), f^j(\pi(\xi))) \leq d_s(f^j(\xi), f^j(\pi(\xi))) < \delta/2$$

for all $j \geq p$. Lemma A.4.2 gives $d(f^j(\xi), f^j(x) \leq K_2\theta^n$ for all $0 \leq j \leq n$. Assume n is large enough so that $K_2\theta^n < \delta/2$. Then the last two inequalities imply that $f^p(\xi)$ and $f^p(\pi(\xi))$ remain within δ from $f^p(x)$ for, at least, $n-p$ iterates. Therefore,

$$d_s(y, \pi_n(y)) = d_s\big(f^n(\xi), f^n(\pi(\xi))\big) \leq \mu(n-p, f^p(x))\, d_s\big(f^p(\xi), f^p(\pi(\xi))\big).$$

Observe that $\mu(n-p, f^p(x)) = \mu(n,x)/\mu(p,x)$. Moreover, $\mu(p,x)$ admits a uniform lower bound $c_3 > 0$, because f is a diffeomorphism and p has been fixed. It follows that $d_s(y, \pi_n(y)) \leq c_3\mu(n,x)$, where $K_3 = \delta/c_3$. □

Local coordinates. Now we are going to show that $\mathcal{B}(n,x)$ and $\pi_n(\mathcal{B}(n,x))$ may be written as graphs, of C^1-nearby maps, over the center-unstable direction at $f^n(x)$. This will allow us to compare the measures induced by the Riemannian structure of M on $\mathcal{B}(n,x)$ and on $\pi_n(\mathcal{B}(n,x))$, and, thus prove Proposition A.4.1. For the precise statement, it is convenient to introduce local coordinates near $f^n(x)$.

Let $\exp_z : T_zM \to M$ be the exponential map of M at any z. In the tangent space T_zM we consider the inner product defined by the Riemannian metric of M. Let $B^{cu}(z,\rho)$ and $B^s(z,\rho)$ be the balls of radius ρ around the origin inside E^{cu}_z and E^s_z , respectively, and let $B(z,\rho) = B^{cu}(z,\rho) \times B^{cu}(z,\rho)$.

Assuming ρ is small enough, the exponential map is a diffeomorphism from

Given $z \in M$,

Assuming δ has been fixed sufficiently small, the inverse map $\phi_z = \exp_z^{-1}$ is wel-defined in the δ-neighbourhood $B(z,\delta)$ of every $z \in M$. In what follows we often identify points $y \in B(z,\delta)$ with the corresponding images under these local charts ϕ_z.

Let .

Lemma A.4.4. *There exist $\rho_0 > 0$ such that for every ρ*

For any $\epsilon \le \delta$ and any

Proof. □

Moreover, if n is large then $r(n,x) \le \theta^n$ is much smaller than δ, and so $\mathcal{B}(n,x)$ is contained in $B(f^n(x),\delta)$.

Lemma A.4.5. *There exist subsets D_1 and D_2 of $E^{cu}_{f^n(x)}$, and maps $g_1 : D_1 \to E^s_{f^n(x)}$ and $g_2 : D_2 \to E^s_{f^n(x)}$, such that $\mathcal{B}(n,x) = \operatorname{graph}(g_1)$ and $\pi_n(\mathcal{B}(n,x)) = \operatorname{graph}(g_2)$.*

Proof. □

Lemma A.4.6. *There exists $K_6 > 0$ such that D_1 and D_2 contain the ball of radius $r(n,x)(1 - K_6\theta^n)$, and they are contained in the ball of radius $r(n,x)(1 + K_6\theta^n)$ around the origin.*

Proof. □

Lemma A.4.7. *There exists $K_7 > 0$ such that $\|Dg_1\| \le K_7\theta^{\alpha n}$ and $\|Dg_1\| \le K_7\theta^{\alpha n}$;*

Proof. First, we observe that the tangent space to $\mathcal{B}(n,x)$ at every point $y \in \mathcal{B}(n,x)$ may be written as a graph $T_yf^n(\mathcal{B}(n,x)) = \operatorname{graph}(h_y)$ of some linear map $h_y : E^{cu}(f^n(x)) \to E^s(f^n(x))$ with $\|h_y\| \le C_4\theta^{\alpha n}$ for some uniform constants $C > 0$ and $\alpha \in (0,1]$. This is a simple consequence of Lemma **??**, together with Proposition 2.2.4. Indeed, according to the proposition, the subbundle E^{cu} is Hölder continuous. So, there exist constants $C > 0$ and $\alpha \in (0,1]$, depending only on f, such that, for every y in the δ-neighbourhood of $f^n(x)$, the subspace $E^{cu}(y)$ may be written as the graph of a linear map $\xi_y : E^{cu}(f^n(x)) \to E^s(f^n(x))$ with $\|\xi_y\| \le Cd(f^n(x), y)^\alpha$.

By definition, $d_{f^n(\Sigma_1)}(f^n(x), y) \le r(n, x) \le \theta^n$. So, $d(f^n(x), y) \le 2d_{f^n(\Sigma_1)}(f^n(x), y) \le 2\theta^n$. The factor 2 accounts for the fact that the local chart ϕ may be slighty expanding. Hence, in this case ξ_y is exponentially close to zero:

$$\|\xi_y\| \le C2^\alpha\theta^{n\alpha}.$$

On the other hand, by Lemma **??**,

$$d(T_y\mathcal{B}(n, x), E^{cu}(y)) \le K_1\theta^{2n}$$

As long as n is sufficiently large, this implies that $T_y\mathcal{B}(n, x)$ si also a graph over $E^{cu}(f^n(x))$

$$T_y\mathcal{B}(n, x) = \text{graph}(h_y), \quad h_y : E^{cu}_{f^n(y)} \to E^s_{f^n(y)}$$

and, cf. Remark **??**, $\|h_y - \xi_y\| \le Kd(T_y\mathcal{B}(n, x), E^{cu}(y)) \le KK_1\theta^{2n}$. It follows that $\|h_y\| \le K_2\theta^{n\alpha}$, as long as we choose $K_2 \ge KK_1 + C2^\alpha$. □

Now we can prove Proposition A.4.1:

Proof. Now, we start the study of the metrics $m_{f^n(\Sigma_1)}$ and $m_{f^n(\Sigma_2)}$. If we define $\gamma_1(u) = (u, g_1(u))$ $\gamma_2(u) = (u, g_2(u))$, this metrics are determineted by the first fundamental form $g^s_{ij} = \frac{\partial\gamma_s}{\partial u_i}\frac{\partial\gamma_s}{\partial u_j}$, where $s = 1, 2$ and $i, j = 1, ..., n-k$, i.e.,

$$g^s_{ij} = \delta_{ij} + \frac{\partial g_s}{\partial u_i}\frac{\partial g_s}{\partial u_j}(5)$$

, k=1,2.

Consider the sets

$A(n, x)$ = ball of radius $r(n, x) - K_4\mu(n, x)$ and center $f^n(x)$

$C(n, x)$ = ball of radius $r(n, x) + K_4\mu(n, x)$ and center $f^n(x)$

Considering that the $\|Dg_1\|$ and $\|Dg_2\|$ are uniformly bounded and the fact that $\lim_{n\to\infty} \frac{\mu(n,x)}{r(n,x)} = 0$, we have that there exists a sequence $\delta(n)$ converging to zero, such that for every $x \in \Sigma_1$

$$\|\frac{m_{f^n(\Sigma_1)}A(n, x)}{m_{f^n(\Sigma_1)}C(n, x)} - 1\| < \delta(n)$$

and

$$\|\frac{m_{f^n(\Sigma_2)}\pi_n(A(n, x))}{m_{f^n(\Sigma_2)}\pi_n(C(n, x))} - 1\| < \delta(n)$$

We denote by P the projection along the k-plane $E^s(f^n(\tilde{x}))$, i.e., $P(u, g_s(u)) = u$ for s=1,2. Using the property (*) is easy see that

$$P(A(n, x)) \subset P(\pi_n(B(n, x))) \subset P(C(n, x))(6)$$

We have, by the definition of $m_{f^n(\Sigma_1)}$ and $m_{f^n(\Sigma_2)}$ that

$$m_{f^n(\Sigma_s)}(D) = \iint_{P(D)} \sqrt{det(g^s_{ij})} du_1 du_2 .. du_{n-k} (7)$$

where $s = 1, 2$ and $D \subset f^n(\Sigma_1)$ or $D \subset f^n(\Sigma_2)$.

Then using 4,5,6 and 7 we have that

$$\begin{aligned} m_{f^n(\Sigma_1)}(A(n,x)) &= \iint_{P(A(n,x))} \sqrt{\det(g^1_{ij})} du_1 du_2 \dots du_{n-k} \\ &\leq \iint_{P(\pi_n(B(n,x)))} \sqrt{\det(g^2_{ij})} du_1 \dots du_{n-k} + \varepsilon m(P(\pi_n(B(n,x))) \end{aligned}$$

where m is a euclidean measure in $T_{f^n(x)}B(n,x)$ Analougously, we can obtain a over estimate for $m_{f^n(\Sigma_2)}\pi_n(B(n,x))$ using $m_{f^n(\Sigma_1)}C(n,x)$. From (5) we conclude the lemma.

□

Step 2. The next main result in the proof of Theorem A.4.1 is

Proposition A.4.2. *There exists K_5 such that*

$$\frac{1}{K_5} \leq \frac{m_{\Sigma_1} f^{-n}(B(n,x)}{m_{\Sigma_2}\pi(f^{-n}(B(n,x))} \leq K_5$$

This will follow from Proposition A.4.1 and the following distortion lemma for the Jacobian of f along Σ_1 and Σ_2:

Lemma A.4.8. *There exists $K_1 1 > 0$ such that*

$$|\log\det Df^{-n}(z_1)|T_{z_1}\mathcal{B}(n,x) - \log\det Df^{-n}(z_2)|T_{z_2}\mathcal{B}(n,x)| \leq K_{11}$$

for all z_1 , z_2 in $\mathcal{B}(n,x)$. This result remains true with $\pi_n(\mathcal{B}(n,x))$ in the place of $\mathcal{B}(n,x)$.

Proof. We have, by the last appendix, that the function $w_2 : x \to E^{cu}(x)$ is a (C,α) hölder continuous function. Then, if $\rho : x \to \log\det Df^{-n}(x)|E^{cu}$ is a hölder continuous funcion with constants (Q,α). In this way, we have that, if $z_1, z_2 \in f^{-n}(B(n,x))$ then

$$
\begin{aligned}
|\log \frac{\det Df^{-n}(z_1)|T_{z_1}B(n,x)}{\det Df^{-n}(z_2)|T_{z_2}B(n,x)}| &\le |\log \frac{\det Df^{-n}(z_1)|T_{z_1}B(n,x)}{\det Df^{-n}(z_1)|E^{cu}(z_1)}| \\
&+ |\log \frac{\det Df^{-n}(z_2)|T_{z_2}B(n,x)}{\det Df^{-n}(z_2)|E^{cu}(z_2)}| \\
&+ |\log \frac{\det Df^{-n}(z_1)|E^{cu}(z_1)}{\det Df^{-n}(z_2)|E^{cu}(z_2)}| \\
&\le Q\sum_{i=0}^{n-1} d(f^{-i}(z_1), f^{-i}(z_2))^{\alpha} \\
&+ R_2 \sum_{i=0}^{n-1} \text{angle}(T_{f^{-i}(z_1)}f^{-i}(B(n,x)), E^{cu}(f^{-i}(z_1)) \\
&+ R_2 \sum_{i=0}^{n-1} \text{angle}(T_{f^{-i}(z_2)}f^{-i}(B(n,x)), E^{cu}(f^{-i}(z_2))) \\
&\le (Q+2R_2)\sum_{i=0}^{n-1} \theta^{\alpha(n-i)} d(z_1,z_2) < K_{10}
\end{aligned}
$$

In a similar way, we prove that

$$|\log \frac{\det Df^{-n}(z_1)|T_{z_1}\pi_n(B(n,x))}{\det Df^{-n}(z_2)|T_{z_2}\pi_n(B(n,x))}| \le K_{11}$$

for any z_1 and z_2 in $\pi_n(B(n,x))$. □

Now we can prove Proposition A.4.2:

Proof. Observe that

$$m_{\Sigma_1} f^{-n}(\mathcal{B}(n,x)) = \int_{\mathcal{B}(n,x)} \left|\det Df^{-n}(z)|T_z\mathcal{B}(n,x)\right| \, dm_{f^n(\Sigma_1)}(z)\,. \quad \text{(A.19)}$$

and similarly for $m_{\Sigma_2}\pi(f^{-n}(B(n,x)))$.

Then, as we know compair $m_{f^n(\Sigma_1)}B(n,x)$ and $m_{f^n(\Sigma_2)}\pi_n(B(n,x))$, we just need compair the expresions inside the integrals.

It easy to see, using the fact that f is a C^2 function, that there exist constants R_1 and R_2 such that, if $z_1, z_2 \in M, d(z_1,z_2) \le 1$ and A_1, A_2 are subspaces of R^n with dimension n-k, then

$$(**)\| \log\det Df^{-1}(z_1)|A_1 - \log\det Df^{-1}(z_2)|A_2\| \le R_1 d(z_1,z_2) + R_2 d(A_1,A_2)$$

Denoting by $K_{12} = max\{K_{10}, K_{11}\}$, we know that if we replace the integrands in (9) and (10) by their values at certain point then the numbers $m_{\Sigma_1} f^{-n}(B(n,x))$ and $m_{\Sigma_2}\pi(f^{-n}(B(n,x)))$ change less a factor $e^{K_{12}}$. It remains now, estimates the expression

$\|\log\det Df^{-n}(z_1)|T_{z_1}B(n,x) - \log\det Df^{-n}(\pi_n(z_1))|T_{\pi_n(z_1)}\pi_n(B(n,x))\|$

. But, following by the same arguments

$$\|\log\det Df^{-n}(z_1)|T_{z_1}B(n,x) - \log\det Df^{-n}(z_1)|E^{cu}(z_1)\|+$$

$$\|\log\det Df^{-n}(\pi_n(z_1))|T_{\pi_n(z_1)}\pi_n(B(n,x)) - \log\det Df^{-n}(\pi_n(z_1))|E^{cu}(\pi_n(z_1))\|+$$

$$\|\log\det Df^{-n}(z_1)|E^{cu}(z_1) - \log\det Df^{-n}(\pi_n(z_1))|E^{cu}(\pi_n(z_1))\| \leq$$

$$Q\sum_{i=0}^{n-1} d(f^{-i}(z_1), f^{-i}(\pi_n(z_1)))^{\alpha} + R_2\sum_{i=0}^{n-1} d(T_{f^{-i}(z_1)}f^{-i}(B(n,x)), E^{cu}(f^{-i}(z_1))$$

$$R_2\sum_{i=0}^{n-1} d(T_{f^{-i}(\pi_n(z_1))}f^{-i}(\pi_n(B(n,x))), E^{cu}(f^{-i}(\pi_n(z_1)))) \leq$$

$$(Q+2R_2)\sum_{i=0}^{n-1}\theta^{\alpha(n-i)}d(z_1,z_2) < K_1 0$$

It follows from last inequalities that the assertion of the lemma is valid.

At last, we are in a position to complete the proof of the theorem:

Proof. Consider $\varepsilon > 0$ such that the set $D_\varepsilon = \{y \in D/d(y, D^c) > \varepsilon$ satisfies

$$\|\frac{m_{\Sigma_1}(D_\varepsilon)}{m_{\Sigma_1}(D)} - 1\| < \frac{1}{2}$$

$$\|\frac{m_{\Sigma_1}(\pi(D_\varepsilon))}{m_{\Sigma_1}(\pi(D))} - 1\| < \frac{1}{2}$$

Consider in $f^n(\Sigma_1)$ the set $\mathcal{B}_n = \{B(n,x)/x \in D_\varepsilon\}$. Following from the fact that $\frac{r(n,x)}{\sigma(n,x)} \leq \theta^n$ that if n is large, then $B(n,x) \subset f^n(D)$ for all $B(n,x) \in \mathcal{B}_n$, as $d(\partial f^n(D), \partial f^n(D_\varepsilon) > K\sigma(n,x)$.

Then, using the Besicovich's Covering Theorem(see **??**, for instance) we can cover D_ε by a countable set of balls $\mathcal{G}_n \subset \mathcal{B}_n$, such that each ball in $\mathcal{G}_n$ intersects, at most, l other balls in $\mathcal{G}_n$, where $l \in \mathbb{N}$ depends only the dimension of Σ_1. Then

$$\frac{1}{l}m_{\Sigma_1}D_\varepsilon \leq \sum_{B\in\mathcal{G}_n} m_{\Sigma_1}f^{-n}(B) \leq l m_{\Sigma_1}D$$

Analougously,

$$\frac{1}{l}p(D_\varepsilon) \leq \sum_{B\in\mathcal{G}_n} m_{\Sigma_1}\pi(f^{-n}(B)) \leq l m_{\Sigma_1}p(D)$$

Then, using the lemma **??**, we have the theorem.

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