Strange attractors in saddle-node cycles: prevalence and globality

L. J. Díaz, J. Rocha, and M. Viana

Abstract

We consider parametrized families of diffeomorphisms bifurcating through the creation of critical saddle-node cycles. We show that they always exhibit Hénon-like strange attractors for a set of parameter values with positive Lebesgue density at the bifurcation value. In open classes of such families the strange attractors are of global type: their basins contain an *a priori* defined neighbourhood of the cycle. Furthermore, the bifurcation parameter may also be a point of positive density of hyperbolic dynamics.

1 Introduction

In this paper we address the problem of determining which dynamical features are more frequently displayed by parametrized families of diffeomorphisms in the sequel of a bifurcation. *Frequency* is to be understood in the sense of Lebesgue measure in parameter space and the general question may be stated as follows, see e.g. [PT2, Chapter 7]. Let M be a smooth manifold of dimension $m \geq 2$ and $f\colon M \longrightarrow M$ be a diffeomorphism exhibiting a bifurcation of codimension $l \geq 1$. Let $k \geq l$ and $(f_\mu)_\mu$ be a generic k-parameter family of diffeomorphisms on M passing through f, say $f_0 = f$. Then one aims at describing the dynamical phenomena exhibited by f_μ for a set E of values of μ with positive Lebesgue measure (*persistent* phenomena) and at determining which of these are even *prevalent* near the bifurcation, meaning that $m(E \cap (-\varepsilon, \varepsilon)^k) \geq \text{const}\, \varepsilon^k$ for some const > 0 and every small $\varepsilon > 0$.

We deal with the unfolding of *critical saddle-node cycles* by smooth 1-parameter families $(f_\mu)_{\mu \in \mathrm{R}}$; *smooth* means that $\mathrm{R} \times M \ni (\mu, z) \mapsto f_\mu(z) \in M$ is a C^∞ map. Critical saddle-node cycles present a special interest from the point of view of the problem we have just stated. On the one hand, they exhibit a very complex bifurcation diagram, including nearly every known form of complicated and/or unstable dynamics. A converse is also true: the formation/destruction of such cycles is part of the unfolding of other main bifurcation mechanisms, e.g. homoclinic tangencies. On the other hand, for the unfolding of critical saddle-node cycles we are able to give here a fairly precise description of the dynamics related to the bifurcation, leading to stronger statements than have been obtained so far for such related mechanisms with a similar degree of complexity. We focus on analysing the occurrence of strange attractors *vs.* hyperbolicity of the dynamics, see below. In very brief terms, we show that Hénon-like strange attractors are always a prevalent phenomenon in the unfolding of such a cycle and the same holds for hyperbolicity in an open set of cases. Before we explain these ideas in more detail and state our results, let us give the precise definitions of the main notions involved.

By a *saddle-node* of a diffeomorphism $f\colon M \longrightarrow M$ we understand a periodic orbit $\mathcal{O}(p_0)$, of period $t_0 \geq 1$ say, such that $Df^{t_0}(p_0)$ has a unique eigenvalue in the unit circle and this is equal to 1. In what follows we always take saddle-nodes to be normally contracting: all the other eigenvalues are strictly less than 1 in norm. Thus the tangent space to M at p_0 admits a $Df^{t_0}(p_0)$-invariant splitting $T_{p_0}M = E^c \oplus E^{ss}$ with $\dim E^c = 1$, $Df^{t_0}(p_0)|E^c =$id and $Df^{t_0}(p_0)|E^{ss}$ a contraction. According to invariant manifold theory, [HPS], there exist 1-dimensional *center manifolds* W_0^c, tangent to E^c at p_0 and locally invariant under f^{t_0} in the sense that $f^{t_0}(W_0^c)$ contains a neighbourhood of p_0 in W_0^c. We also assume all saddle-nodes to be nondegenerate – or *quadratic* – meaning that $(f^{t_0}|W_0^c)''(p_0) \neq 0$ for some choice of a (twice differentiable) center manifold W_0^c; actually, in this case one may take W_0^c to be C^∞, see e.g. [Sh].

We say that the diffeomorphism f exhibits a *saddle-node ℓ-cycle*, $\ell \geq 1$, if it has periodic orbits $\mathcal{O}(p_0)$, $\mathcal{O}(p_1)$, ..., $\mathcal{O}(p_{\ell-1})$, where $\mathcal{O}(p_0)$ is a saddle-node and $\mathcal{O}(p_i)$, $1 \leq i \leq \ell - 1$, are hyperbolic saddles, satisfying

- $W^u(\mathcal{O}(p_{i-1}))$ has some transverse intersection with $W^s(\mathcal{O}(p_i))$, for every $1 \leq i \leq \ell - 1$;
- $W^u(\mathcal{O}(p_{\ell-1}))$ intersects the interior of $W^s(\mathcal{O}(p_0))$.

For $\ell = 1$ we require $W^u(\mathcal{O}(p_0))$ to be contained in the interior of $W^s(\mathcal{O}(p_0))$. Note that $W^u(\mathcal{O}(p_0)) = \cup_{i=0}^{t_0-1} f^i(W^u(p_0))$ and $W^u(p_0)$ is (strictly) contained in W^c. On the other hand, $W^s(\mathcal{O}(p_0))$ is the union of the $f^i(W^s(p_0))$, $0 \leq i \leq t_0 - 1$, and $W^s(p_0) \subset M$ is a (codimension zero) submanifold with boundary: $\partial W^s(p_0) = W^{ss}$ is the *strong-stable manifold* of p_0, i.e. the unique f^{t_0}-invariant submanifold satisfying $T_{p_0}W^{ss} = E^{ss}$, see [HPS].

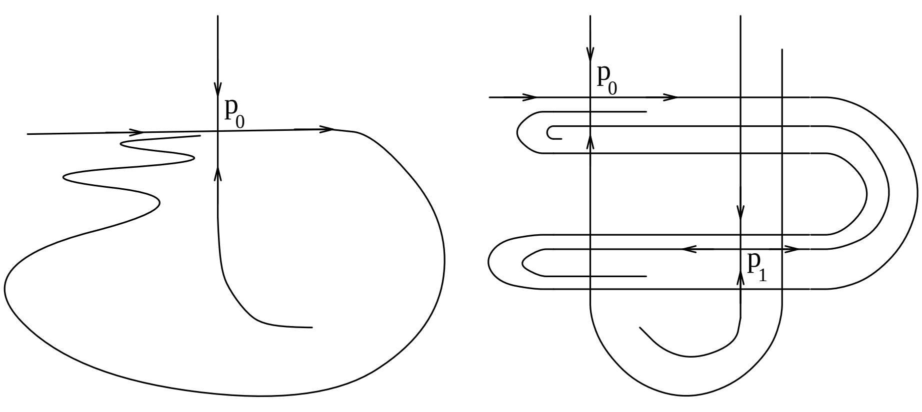

Figure 1:

Following [NPT], we say that a saddle-node cycle is *critical* if $W^u(\mathcal{O}(p_{\ell-1}))$ has some point χ of nontransverse intersection with a leaf F of $\mathcal{F}^{ss}$. Here $\mathcal{F}^{ss}$ denotes the *strong-stable foliation* of p_0, i.e. the unique $f_0^{t_0}$-invariant foliation of $W^s(p_0)$ by codimension-1 submanifolds, having W^{ss} as a leaf, see [NPT, Theorem II.3.4]. We always assume the criticality to be generic: $W^u(p_1)$ and F have a *quadratic* contact at χ. Two examples of critical cycles – a 1-cycle and a 2-cycle – are depicted in Figure 1. Note that this 2-cycle may be thought of as the result of collapsing a periodic attractor with a saddle point contained in a horseshoe; clearly, such an example also exhibits ℓ-cycles for every $\ell \geq 2$.

Now let $(f_\mu)_{\mu \in \mathrm{R}}$ be a smooth family of diffeomorphisms with $f_0 = f$. Then, [HPS], there exist two-dimensional submanifolds W^c of $\mathrm{R} \times M$, tangent to $\mathrm{R} \times E^c$ at $(0, p_0)$ and locally invariant under $\mathrm{R} \times M \ni (\mu, z) \mapsto (\mu, f_\mu^{t_0}(z)) \in \mathrm{R} \times M$. For each such W^c we have $W_0^c = W^c \cap (\{0\} \times M)$ a center manifold for f_0. We say that $(f_\mu)_\mu$ *unfolds* the saddle-node – and the saddle-node cycle – *generically* if one may take W^c so that $\partial_\mu(\varphi|W^c)(0, p_0) \neq 0$, where $\varphi(\mu, z) = f_\mu^{t_0}(z)$. In more precise terms: there are local coordinates (μ, x) on W^c, with respect to which $\varphi|W^c$ is given by

$$\varphi(\mu, x) = x + v\mu + \alpha x^2 + \beta \mu x + \gamma \mu^2 + O(|\mu|^3 + |x|^3),$$

with $v \neq 0$ (and $\alpha \neq 0$, recall above). Such coordinates (and parametrized center manifold W^c) may be taken with an arbitrarily high degree of smoothness, up to restricting the parameter range to a sufficiently small neighbourhood of $\mu = 0$. Here, for the sake of technical simplicity, we take them to be C^∞ but a finite amount of smoothness is sufficient for all our arguments.

Families of diffeomorphisms bifurcating *via* saddle-node cycles, both critical and non-critical, were studied in [NPT] from the point of view of structural stability. Their analysis shows that the presence of a criticality leads to a much richer and unstable unfolding. Indeed, [NPT, Theorem III.4.1], any family of diffeomorphisms going through a critical saddle-node cycle exhibits homoclinic tangencies, at parameter values arbitrarily close to zero. As a consequence, it must display a wide variety of phenomena which are known to be involved in the unfolding of such tangencies: cascades of period-doubling bifurcations [YA], [Mr]; frequency of hyperbolicity *vs.* bifurcations determined by fractal dimensions [PT1], [PY]; persistence of tangencies, coexistence of infinitely many sinks [N2], [Rb], [PV]; Hénon-like strange attractors [BC], [MV], [Vi]. In particular, the results in [MV], [Vi], in view of the previous conclusion from [NPT], imply that strange attractors are persistent in the unfolding of any critical saddle-node cycle. Our first result here asserts a much stronger fact: strange attractors even constitute a prevalent phenomenon.

Theorem A *Let $(f_\mu)_\mu$ be a generic smooth family of diffeomorphisms on a manifold M, unfolding a critical saddle-node cycle. Then the set $\mathcal{S}$ of values of μ for which f_μ exhibits Hénon-like strange attractors satisfies*

$$\liminf_{\varepsilon \to 0} \frac{m\,(\mathcal{S} \cap (-\varepsilon, \varepsilon))}{2\varepsilon} > 0.$$

By a *strange attractor* of a transformation $g\colon M \longrightarrow M$ we mean a compact g-invariant set $A \subset M$ such that

- the stable set $W^s(A) = \{z \in M \colon \lim_{n\to+\infty} \text{dist}\,(g^n(z), A) = 0\}$ has nonempty interior;
- there exists $\bar{z} \in A$ such that the orbit $\{g^n(\bar{z})\colon n \geq 0\}$ is dense in A and there exist $c > 0$, $\sigma > 1$ and $v \in T_{\bar{z}}M$ such that $\|Dg^n(\bar{z})v\| \geq c\sigma^n$ for every $n \geq 0$.

The strange attractors we encounter here have some additional properties, namely

- A coincides with the closure of $W^u(\mathcal{O}(q))$ for some hyperbolic saddle point q;
- A is not (uniformly) hyperbolic: it contains *critical points*, admitting tangent directions which are contracted by both positive and negative iteration under Dg.

We call them *Hénon-like* strange attractors, cf. [BC], [MV]. See also [BY] for important ergodic features of this kind of attractors.

Let us point out that a converse to [NPT, Theorem III.4.1] is also true, as observed by L. Mora: critical saddle-node cycles are formed whenever a homoclinic tangency is generically unfolded, see [PT2, page 150]. On the other hand, it remains an interesting open question whether strange attractors can be a prevalent phenomenon in the unfolding of a homoclinic tangency. It follows from [PT1] that this can not always be the case: in a large (open) class of examples *hyperbolicity* is fully prevalent, in the sense that the limit set of f_μ, see below, is hyperbolic for a set of values of μ with Lebesgue density 1 at $\mu = 0$. On the other hand, it may be that prevalence of strange attractors does hold in some relevant situations in this setting, e.g. for homoclinic tangencies occurring in the Hénon family $(x, y) \mapsto (1-ax^2+y, bx)$ at parameter values (a, b) close to $(2, 0)$.

In a somewhat opposite direction, we observe that hyperbolicity of the dynamics related to the cycle may also be a prevalent feature, at least for an open set of parametrized families $(f_\mu)_\mu$ unfolding a critical saddle-node cycle. In order to illustrate this fact, we consider two different situations, corresponding to the two types of cycles described in Figure 1. In both cases we fix a compact set $V \subset M$ such that all the periodic orbits involved in the cycle are contained in $\text{int}\,(V)$ and, moreover, $f_0(V) \subset \text{int}\,(V)$. This last condition remains true for f_μ, any μ close to zero, and then $L(f_\mu|V) = L(f_\mu) \cap V$. Recall that the *limit set* of f_μ is defined by

$$L(f_\mu) = \overline{\bigcup_{x\in M} \alpha(x)} \cup \overline{\bigcup_{x\in M} \omega(x)},$$

where the α-limit $\alpha(x)$, resp. the ω-limit $\omega(x)$, is the set of accumulation points of $f_\mu^n(x)$ as $n \to -\infty$, resp. $n \to +\infty$. Moreover, $L(f_\mu|V)$ is defined in a similar way, restricting to those orbits which are contained in V. Then we denote by $\mathcal{H}$ the set of values of μ for which $L(f_\mu|V)$ is a hyperbolic set of f_μ (and so the dynamics of f_μ restricted to V is stable [N1]). Note that $\mathcal{H}$ is disjoint from $\mathcal{S}$ since Hénon-like strange attractors are, by definition, nonhyperbolic.

Theorem B *There exist open sets of smooth families of diffeomorphisms* $(f_\mu)_{\mu\in\mathrm{R}}$ *unfolding a critical saddle-node cycle, for which* $\mathcal{H}$ *satisfies*

$$\liminf_{\varepsilon\to 0}\frac{m\,(\mathcal{H}\cap(-\varepsilon,\varepsilon))}{2\varepsilon}>0.$$

For the proof of Theorem A we combine the results in [MV], [Vi] with a careful analysis of the distribution of parameter values corresponding to homoclinic tangencies. A main ingredient is the notion of transition maps, which we borrow from [NPT] and further develop in Section 2. This construction yields attractors which are related to the dynamics in the vicinity of homoclinic orbits and thus have just a semi-global character. Under additional hypotheses, we can further push our arguments to obtain, for the first time, prevalent Hénon-like attractors of a truly global nature. For that we consider the setting of 1-cycles. Given any diffeomorphism $f_0\colon M\longrightarrow M$ exhibiting a saddle-node 1-cycle, one may construct a compact subset $V\subset M$ as above, such that $f_0(V)\subset \operatorname{int}(V)$ and the closure of $W^u(\mathcal{O}(p_0))$ is contained in $\operatorname{int}(V)$; see Section 4.1. Then we also have $f_\mu(V)\subset\operatorname{int}(V)$ for every small μ, which means that all the asymptotic dynamics of f_μ near the cycle is concentrated on $A_\mu=\cap_{n\geq 0}f_\mu^n(V)$. It follows from Theorem A that A_μ contains strange attractors for a large set of parameter values. Actually, a stronger statement holds, at least in an open set of cases:

Theorem C *For an open class* $\mathcal{A}_0$ *of smooth families of diffeomorphisms* $(f_\mu)_{\mu\in\mathrm{R}}$ *unfolding a critical saddle-node 1-cycle,* A_μ *is a Hénon-like attractor for a set* $\mathcal{G}$ *of parameter values with*

$$\liminf_{\varepsilon\to 0}\frac{m\,(\mathcal{G}\cap(-\varepsilon,\varepsilon))}{2\varepsilon}>0.$$

Observe that, by construction, the basin $W^s(A_\mu)$ of A_μ contains V, which depends only on the initial map f_0.

Acknowledgements. We are grateful to J. Palis and F. Takens for several useful conversations and to the hospitality of IMPA where most of this work was done. M. Viana acknowledges the hospitality of the University of Groningen and financial support from NWO/Holland. L. J. Díaz is partially supported by CNPq/Brazil. This work was also supported by JNICT-PBICT/C/CEN/1020/92.

2 Saddle-node local dynamics: transition maps

2.1 The one-dimensional case

Throughout this section $\varphi=(\varphi_\mu)_\mu$ denotes a *saddle-node arc* of 1-dimensional maps. By this we mean a C^∞ map $\varphi\colon(-\delta_0,\delta_0)\times(-\delta_0,\delta_0)\longrightarrow\mathrm{R}$, $(\mu,x)\mapsto\varphi(\mu,x)=\varphi_\mu(x)$, satisfying

$$\varphi(0,0)=0,\ \partial_x\varphi(0,0)=1,\ \partial_x^2\varphi(0,0)\neq 0\ \text{ and }\ \partial_\mu\varphi(0,0)\neq 0.$$

Up to performing simple (affine) changes of the coordinate x and of the parameter μ we may, and do from now on, assume that $2\alpha = \partial_x^2\varphi(0,0) > 0$ and $\partial_\mu\varphi(0,0) = 1$. Then

$$\varphi_\mu(x) = x + \mu + \alpha x^2 + \beta x\mu + \gamma\mu^2 + O(|\mu|^3 + |x|^3). \tag{1}$$

The local dynamics of these arcs may be analysed by considering certain related families of real vector fields of the form

$$X(\mu, x) = \mu + \alpha x^2 + \beta x\mu + \gamma\mu^2 + O(|\mu|^3 + |x|^3) \quad \text{with } \alpha > 0, \tag{2}$$

so-called *saddle-node arcs of vector fields.* One way is through (an extension) of the notion of adapted arc of vector fields in [NPT], see also [Le], [T1], [T2]. Here we just take the arc $(\varphi_\mu)_\mu$ to embed as the time-1 of an arc $(X(\mu,\cdot))_\mu$ as in (2). This alternative approach implies no restriction in our setting since, by [IY, Section 3], such an embbeding always exists restricted to closed sectors containing $\{(x,\mu) : \mu \geq 0\}$, and this is all we need here.

Now let φ be a saddle-node arc and fix X a saddle-node arc of vector fields with $\varphi = X_1$ on $\{(x,\mu) : \mu \geq 0\}$. Let moreover $a < 0 < b$ be fixed, close enough to zero so that $X(0,x) > 0$ for all $x \in [\varphi_0^{-2}(a), \varphi_0^2(b)]\backslash\{0\}$. For $k \in \mathrm{N}$ and small $\mu > 0$ we define $\sigma_k(\mu)$ by the relation

$$X_{k+\sigma_k(\mu)}(\mu, a) = b,$$

where $X_s(\mu,\cdot)$ denotes the time-s map of $X(\mu,\cdot)$.

Proposition 2.1 *For sufficiently large $k \geq 1$ there is a unique $\mu_k^* > 0$ satisfying $\sigma_k(\mu_k^*) = 0$. Moreover, $\sigma_k\colon [\mu_{k+1}^*, \mu_k^*] \longrightarrow [0,1]$ is a C^∞ (decreasing) diffeomorphism onto $[0,1]$.*

We also get that the *reparametrization maps* $\mu_k = (\sigma_k|\,[\mu_{k+1}^*, \mu_k^*])^{-1} : [0,1] \longrightarrow [\mu_{k+1}^*, \mu_k^*]$ have uniformly bounded metric distorsion:

Proposition 2.2 *Given $\varepsilon > 0$ there is $k_0 \in \mathrm{N}$ such that for every $k \geq k_0$ and every Borel subset A of $[0,1]$ we have*

$$(1-\varepsilon)\, m(A) \leq \frac{m(\mu_k(A))}{m([\mu_{k+1}^*, \mu_k^*])} \leq (1+\varepsilon)\, m(A) \quad (m = \text{ Lebesgue measure}).$$

For $\varphi_0^{-1}(a) \leq x \leq \varphi_0(a)$ and $|\mu|$ small we define $t(\mu,x)$ by $x = X_{t(\mu,x)}(\mu,a)$ and then the main result of this section is

Theorem 2.3 *The sequence $T_k\colon [0,1] \times [\varphi_0^{-1}(a), \varphi_0(a)] \longrightarrow \mathrm{R}$, $T_k(\sigma,x) = \varphi_{\mu_k(\sigma)}^k(x)$, converges in the C^∞ topology to the* transition map $T_\infty\colon [0,1] \times [\varphi_0^{-1}(a), \varphi_0(a)] \longrightarrow \mathrm{R}$ *defined by $T_\infty(\sigma,x) = X_{t(0,x)-\sigma}(0,b)$.*

Starting the proof of these statements, we define for $\mu > 0$ small

$$f(\mu) = \int_a^b \frac{dx}{X(\mu,x)}.$$

Then $b = X_{f(\mu)}(\mu,a)$, that is $f(\mu) = k + \sigma_k(\mu)$ for any $k \in \mathrm{N}$.

Lemma 2.4 *For every $i \geq 0$,*

$$\lim_{\mu\to 0^+} \mu^{i+1/2} f^{(i)}(\mu) = \tilde{C}_i = \frac{(-1)^i i!}{\sqrt{\alpha}} \int_{-\pi/2}^{\pi/2} \cos^{2i} x\, dx.$$

Proof: A direct calculation gives

$$f^{(i)}(\mu) = (-1)^i i! \int_a^b \frac{(\partial_\mu X)^i(\mu,x)}{X(\mu,x)^{i+1}}\, dx + \sum_{j=1}^{i} \int_a^b \frac{F_j(\mu,x)}{X(\mu,x)^j}\, dx, \tag{3}$$

where each F_j is a polynomial function of the partial derivatives of X, $(\partial_\mu^r X)$, $r \leq i$. We claim that

$$\lim_{\mu\to 0^+} \mu^{i+1/2} \int_a^b \frac{dx}{X(\mu,x)^{i+1}} = \frac{1}{\sqrt{\alpha}} \int_{-\pi/2}^{\pi/2} \cos^{2i} x\, dx \qquad \text{for } i \geq 0 \tag{4}$$

$$\lim_{\mu\to 0^+} \mu^{i+1/2} \int_a^b \frac{|x|\, dx}{X(\mu,x)^{i+1}} = 0 \qquad \text{for } i \geq 1. \tag{5}$$

First we explain how the lemma follows from these claims. Observe that case $i = 0$ is a direct consequence of claim (4). From now on we consider $i \geq 1$. Returning to (3), we get for every $1 \leq j \leq i$

$$|\mu^{i+1/2} \int_a^b \frac{F_j}{X^j}\, dx| \leq M\mu^{i-j+1} |\mu^{j-1/2} \int_a^b \frac{dx}{X^j}| \leq M\mu |\mu^{(j-1)+1/2} \int_a^b \frac{dx}{X^j}|,$$

where $M = \sup\{|F_j(\mu,x)| \colon 1 \leq j \leq i,\ a \leq x \leq b,\ |\mu| \leq \delta_0/2\}$. Then claim (4) implies

$$\lim_{\mu\to 0^+} \mu^{i+1/2} \int_a^b \frac{F_j}{X^j}\, dx = 0, \text{ for every } 1 \leq j \leq i. \tag{6}$$

Now, in order to prove that

$$\lim_{\mu\to 0^+} \mu^{i+1/2} \int_a^b \left(\frac{(\partial_\mu X)^i}{X^{i+1}} - \frac{1}{X^{i+1}} \right) dx = 0 \text{ for each } i \geq 1, \tag{7}$$

we use (2) and then

$$(\partial_\mu X)^i - 1 = (1 + \beta x + 2\gamma\mu + O(\mu^2 + x^2))^i - 1 = A_i(\mu,x)(\beta x + 2\gamma\mu + O(\mu^2 + x^2)),$$

where $A_i(\mu,x) = \sum_{j=0}^{i-1} (1 + \beta x + 2\gamma\mu + O(\mu^2 + x^2))^j$. Note also that for some $c > 0$ we have $X(\mu,x) \geq c(\mu^2 + x^2)$ for all small x and $\mu > 0$. Hence,

$$\int_a^b \frac{|(\partial_\mu X)^i - 1|}{X^{i+1}}\, dx \leq K \left(|\beta| \int_a^b \frac{|x|\, dx}{X^{i+1}} + 2|\gamma|\mu \int_a^b \frac{dx}{X^{i+1}} + K \int_a^b \frac{dx}{X^i} \right)$$

for some $K = K_i > 0$. Now, it is clear that the claims imply (7) and, on their turn, (6), (7) and claim (4) imply the conclusion of the lemma. Now we come to the proof of the claims. First we note that for the particular arc $\tilde{X}(\mu,x) = \mu + \alpha x^2$, (4) and (5) follow, easily, from an explicit calculation of the integrals. On the other hand, for a general saddle-node arc of vector fields $X(\mu, x)$ as in (2),

$$1 - O(|\mu| + |x|) \le \frac{X(\mu, x)}{\tilde{X}(\mu, x)} \le 1 + O(|\mu| + |x|).$$

Hence, given any $\varepsilon > 0$ we may fix $\delta > 0$ and $\nu > 0$ such that

$$1 - \varepsilon \le \frac{X(\mu, x)}{\tilde{X}(\mu, x)} \le 1 + \varepsilon \quad \text{for every} \quad (\mu, x) \in (0, \nu) \times [-\delta, \delta].$$

Observing that

$$\limsup_{\mu \to 0^+} \mu^{i+1/2} \int_a^b \frac{dx}{Y^{i+1}} = \limsup_{\mu \to 0^+} \mu^{i+1/2} \int_{-\delta}^{\delta} \frac{dx}{Y^{i+1}}$$

for both $Y = X$ and $Y = \tilde{X}$, we obtain

$$\limsup_{\mu \to 0^+} \mu^{i+1/2} \int_a^b \frac{dx}{X^{i+1}} \le \frac{1}{(1-\varepsilon)^{i+1}} \lim_{\mu \to 0^+} \mu^{i+1/2} \int_a^b \frac{dx}{\tilde{X}^{i+1}}$$

and, analogously,

$$\liminf_{\mu \to 0^+} \mu^{i+1/2} \int_a^b \frac{dx}{X^{i+1}} \ge \frac{1}{(1+\varepsilon)^{i+1}} \lim_{\mu \to 0^+} \mu^{i+1/2} \int_a^b \frac{dx}{\tilde{X}^{i+1}}.$$

Since $\varepsilon > 0$ is arbitrary this proves that

$$\lim_{\mu \to 0^+} \mu^{i+1/2} \int_a^b \frac{dx}{\tilde{X}^{i+1}} = \lim_{\mu \to 0^+} \mu^{i+1/2} \int_a^b \frac{dx}{X^{i+1}}$$

implying that claim (4) holds for X. Claim (5) is proved in the same way. □

In particular, $\lim_{\mu \to 0^+} f(\mu) = +\infty$ and $\lim_{\mu \to 0^+} f'(\mu) = -\infty$. Hence, for each sufficiently large $k \in \mathrm{N}$ there is a unique $\mu = \mu_k^*$ such that $f(\mu) = k$, i.e. $\sigma_k(\mu_k^*) = 0$. Note that the sequence $(\mu_k^*)_k \to 0$: in fact

(8)
$$\lim_{k \to \infty} k\sqrt{\mu_k^*} = \tilde{C}_0 = \frac{\pi}{\sqrt{\alpha}}$$

The relation $\sigma_k(\mu_{k+1}^*) = 1$ is obvious and so the proof of Proposition 2.1 is complete.

Lemma 2.5 *For each $i \in \mathrm{N}$ there exists $C_i \in \mathrm{R}$ such that*

$$\lim_{k \to \infty} \frac{\mu_k^{(i)}(\sigma)}{\mu_k(\sigma)^{1+i/2}} = C_i, \ \textit{uniformly on } \sigma \in [0, 1].$$

Proof: From $f(\mu_k(\sigma)) = k + \sigma$ we get $f'(\mu_k)\mu_k' = 1$. Then, by Lemma 2.4,

$$\lim_{k\to\infty} \frac{\mu_k'}{\mu_k^{1+1/2}} = \lim_{k\to\infty} \frac{1}{\mu_k^{1+1/2} f'(\mu_k)} = \frac{1}{\tilde{C}_1}.$$

Now we proceed by induction on i. A simple inductive argument shows that for every $i \geq 2$

$$f'(\mu_k)\mu_k^{(i)} + \sum_{j=1}^{n(i)} G_j = 0 \tag{9}$$

with $n(i) \geq 1$ and each G_j, $1 \leq j \leq n(i)$, having the form $f^{(l)}(\mu_k) \cdot (\mu_k')^{m_1} \cdots (\mu_k^{(i-1)})^{m_{i-1}}$ with $m_1, \ldots, m_{i-1} \geq 0$ satisfying $1 \leq l = \sum_{s=1}^{i-1} m_s \leq \sum_{s=1}^{i-1} s m_s = i$. Then, using this last property, we find

$$\lim_{k\to\infty} \frac{G_j}{f'(\mu_k)\mu_k^{1+i/2}} = \lim_{k\to\infty} \frac{f^{(l)}(\mu_k)\mu_k^{l+1/2}}{f'(\mu_k)\mu_k^{1+1/2}} \prod_{s=1}^{i-1} \left(\frac{\mu_k^{(s)}}{\mu_k^{1+s/2}} \right)^{m_s} = \frac{\tilde{C}_l}{\tilde{C}_1} \prod_{s=1}^{i-1} C_s^{m_s}$$

for every $1 \leq j \leq n(i)$. In view of (9), this proves that $\mu_k^{(i)}/\mu_k^{1+i/2}$ converges as $k \to \infty$ and it is also clear from the argument that this convergence is uniform. □

Proposition 2.2 is a direct consequence of (8) and Lemma 2.5. Indeed, let $\varepsilon > 0$; if k is large enough then for every $\sigma_1, \sigma_2 \in [0, 1]$

$$\frac{\mu_k'(\sigma_1)}{\mu_k'(\sigma_2)} \leq (1 + \frac{\varepsilon}{3}) \left(\frac{\mu_k(\sigma_1)}{\mu_k(\sigma_2)} \right)^{3/2} \leq (1 + \frac{2\varepsilon}{3})(\frac{k+1}{k})^3 \leq 1 + \varepsilon.$$

In particular,

$$1 - \varepsilon \leq \frac{\mu_k'(\sigma)}{m([\mu_{k+1}^*, \mu_k^*])} \leq 1 + \varepsilon \quad \text{for every } \sigma \in [0, 1]$$

and the proposition follows immediately.

On the other hand, (8) and Lemma 2.5 imply that $(\mu_k)_k$ converges in the C^∞ topology to the null function. This completes the proof of Theorem 2.3.

2.2 The general setting

Now we prove a higher-dimensional version of Theorem 2.3. Let M be a smooth manifold, $m = \dim M \geq 2$, and let $(f_\mu)_{\mu\in\mathrm{R}}$ be a smooth arc of diffeomorphisms unfolding a saddle-node bifurcation on M, recall the Introduction. Let the point $(0, p_0) \in \mathrm{R} \times M$ to correspond to the saddle-node. Then there are smooth coordinates $(\mu, x, Y) \in \mathrm{R} \times \mathrm{R} \times \mathrm{R}^{m-1}$ on a neighbourhood U of $(0, p_0)$, such that $f_\mu(x, Y) = (\varphi_\mu(x), \Psi(\mu, x, Y))$ with

(a) φ a 1-dimensional saddle-node arc and

(b) $\Psi(\mu, x, 0) = 0$ and $\|\partial_Y \Psi(\mu, x, Y)\| \leq \lambda < 1$ for every $(\mu, x, Y) \in U$.

As in the previous section, we denote by μ_k the reparametrization maps of φ and by T_∞ its transition map. We fix V a small neighbourhood of $0 \in \mathrm{R}^{m-1}$, and using the coordinates above we define $\hat{T}_k, \hat{T}_\infty$: $[0,1] \times [\varphi_0^{-1}(a), \varphi_0(a)] \times V \longrightarrow \mathrm{R} \times \mathrm{R}^{m-1}$ by

$$\hat{T}_k(\sigma, x, Y) = f^k_{\mu_k(\sigma)}(x, Y) \quad \text{and} \quad \hat{T}_\infty(\sigma, x, Y) = (T_\infty(\sigma, x), 0).$$

Theorem 2.6 *The sequence $(\hat{T}_k)_k$ converges to the map $\hat{T}_\infty$ in the C^∞ topology.*

Proof: We have $\hat{T}_k(\sigma, x, Y) = (\varphi^k_{\mu_k(\sigma)}(x), Y_k(\mu_k(\sigma), x, Y)$, where the Y_k are defined inductively by $Y_0(\mu, x, Y) = Y$ and $Y_{j+1}(\mu, x, Y) = \Psi(\mu, x_j, Y_j)$, $x_j = \varphi^j_\mu(x)$. We begin with the following auxiliary statement: given u, $v \geq 0$ there are $E > 0$ and $m \geq 0$ such that

(10) $\quad |\partial^u_\mu \partial^v_x \varphi^j_\mu(x)| \leq E k^m$ for all $x \in [\varphi_0^{-1}(a), \varphi_0(a)]$, $\mu^*_{k+1} < \mu \leq \mu^*_k$ and $0 \leq j \leq k$.

In order to prove (10) we note that, given any smooth function $\psi_\mu(x) = \psi(\mu, x)$, we have $\partial^u_\mu \partial^v_x(\varphi_\mu(\psi_\mu(x))) = \partial_x \varphi_\mu(\psi_\mu(x)) \cdot \partial^u_\mu \partial^v_x \psi_\mu(x) + \sum_{i=1}^{m(u,v)} I_i$, where $m(u,v) \geq 0$ and each I_i has the form

$$\partial^a_\mu \partial^b_x \varphi_\mu(\psi_\mu(x)) \prod_{t=1}^{s} \partial^{c_t}_\mu \partial^{d_t}_x \psi_\mu(x)$$

with a, b, s, c_t, $d_t \geq 0$, $(a+b) \leq (u+v)$, $s \leq (u+v)$ and $(c_t + d_t) \leq (u+v-1)$. Then, by induction on j

$$\partial^u_\mu \partial^v_x \varphi^j_\mu(x) = \sum_{i=1}^{n(u,v,j)} J_i \tag{11}$$

where $n(u,v,j) \leq j m(u,v) + 1$ and each J_i can be written

$$\partial_x \varphi^{j-e-1}_\mu(\varphi^e_\mu(x)) \cdot \partial^a_\mu \partial^b_x \varphi_\mu(\varphi^e_\mu(x)) \cdot \prod_{t=1}^{s} \partial^{c_t}_\mu \partial^{d_t}_x \varphi^e_\mu(x) \tag{12}$$

with a, b, s, c_t, d_t, $e \geq 0$, $(a+b) \leq (u+v)$, $s \leq (u+v)$, $(c_t + d_t) \leq (u+v-1)$ and $e \leq j$. On the other hand, there is $E_1 > 0$ such that for any large k

$$\partial_x \varphi^i_\mu(\varphi^e_\mu(x)) = \frac{X(\mu, \varphi^{i+e}_\mu(x))}{X(\mu, \varphi^e_\mu(x))} \leq \frac{E_1}{\mu} \leq \frac{2\alpha E_1}{\pi^2} k^2, \tag{13}$$

for every $(\mu, x) \in (\mu^*_{k+1}, \mu^*_k] \times [\varphi_0^{-1}(a), \varphi_0(a)]$ and e, $i \geq 0$ with $e + i \leq k$. Here $(X(\mu, \cdot))_\mu$ is any arc of vector fields having $X_1(\mu, \cdot) = \varphi_\mu$ on $\{(\mu, x): \mu \geq 0\}$. Case $u, v = 0$ of (10) is trivial. For general (u,v), (11), (12) and (13) give, by induction on $(u+v)$,

$$|\partial^u_\mu \partial^v_x \varphi^j_\mu(x)| \leq \sum_{i=1}^{n(u,v,j)} E_2 k^2 \prod_{t=1}^{s} E(c_t, d_t) k^{m(c_t, d_t)}$$

with $E_2 = (2\alpha E_1/\pi^2) \sup\{|\partial^i_\mu \partial^j_x \varphi_\mu(z)|: (i+j) \leq (u+v), (\mu, z) \in (0, \delta] \times [\varphi_0^{-1}(a), \varphi_0(b)]\}$. Therefore, it is sufficient to take $m(u,v) = 3 + (u+v) \sup\{m(c,d): (c+d) \leq (u+v-1)\}$

and $E(u,v) = (m(u,v)+1)E_2(\sup\{E(c,d)\colon (c+d) \leq (u+v-1)\})^{u+v}$. This completes the proof of (10).

Now we claim that given any $p,q,r \geq 0$ there are $C_0 > 0$ and $m_0 \geq 0$ such that for every $1 \leq j \leq k$

(14) $\sup\{\|\partial_\mu^p \partial_x^q \partial_Y^r Y_j(\mu,x,Y)\|\colon (\mu,x,Y) \in [\mu_{k+1}^*, \mu_k^*] \times [\varphi_0^{-1}(a), \varphi_0(a)] \times V\} \leq C_0 j \lambda^j k^{m_0}$.

In particular, $\sup\{\|\partial_\mu^p \partial_x^q \partial_Y^r Y_k\|\} \leq C_0 \lambda^k k^{m_0+1} \to 0$ which, together with Lemma 2.5, implies that $(Y_k)_k$ converges to zero in the C^∞ topology. In view of Theorem 2.3, this proves our statement. Hence, we are left to prove the claim and we do this by induction on $p+q+r$. First, we note that (b) implies $\|Y_j(\mu,x,Y)\| \leq \lambda^j \|Y\|$ and so in case $p,q,r=0$ it is sufficient to take $C_0 = 1$ and $m_0 = 0$. Now, given any $p,q,r \geq 0$ we assume that (14) holds for every derivative of order strictly less than $p+q+r$ and we conclude that it also holds for $\partial_\mu^p \partial_x^q \partial_Y^r Y_j$, any $1 \leq j \leq k$. Case $j=1$ is easy: clearly, $\|\partial_\mu^p \partial_x^q \partial_Y^r Y_1\| \leq C_1$, where $C_1 = \sup \|\partial_\mu^p \partial_x^q \partial_Y^r \Psi\|$, and so we only have to take $C_0 \geq C_1/\lambda$. Observe now that successive derivation of $Y_{j+1}(\mu,x,Y) = \Psi(\mu,x_j,Y_j)$ leads to

$$\partial_\mu^p \partial_x^q \partial_Y^r Y_{j+1}(\mu,x,Y) = \partial_Y \Psi(\mu,x_j,Y_j) \partial_\mu^p \partial_x^q \partial_Y^r Y_j(\mu,x,Y) + \sum_{l=1}^{l_0} H_l \tag{15}$$

where $l_0 = l_0(p,q,r) \geq 0$ and each H_l has the form

$$\partial_\mu^a \partial_x^b \partial_Y^c \Psi(\mu,x_j,Y_j) \prod_{s=1}^{s_0} \partial_\mu^{\alpha_s} \partial_x^{\beta_s} x_j \prod_{t=1}^{t_0} \partial_\mu^{u_t} \partial_x^{v_t} \partial_Y^{w_t} Y_j$$

with $a,b,c,s_0,\alpha_s,\beta_s,t_0,u_t,v_t,w_t \geq 0$ depending only on p,q,r,l and satisfying $a+b+c \leq p+q+r$, $\sum_s(\alpha_s+\beta_s)+\sum_t(u_t+v_t+w_t) \leq p+q+r$ and $c \leq \sum_t(u_t+v_t+w_t) \leq p+q+r-1$. By (10), we have $|\partial_\mu^{\alpha_s} \partial_x^{\beta_s} x_j| \leq C_{2,s} k^{m_{2,s}}$ for some $C_{2,s} > 0$ and $m_{2,s} \geq 0$ depending only on α_s, β_s. Moreover, (b) above implies $\|\partial_\mu^a \partial_x^b \Psi(\mu,x,Y)\| \leq C_3 \|Y\|$ for some $C_3 = C_3(a,b)$. Finally, we denote $C_4 = \sup \|\partial_\mu^a \partial_x^b \partial_Y^c \Psi\|$, where the supremum is taken over all $a+b+c \leq p+q+r$. In order to estimate $|H_l|$ we distinguish two cases. Suppose first that $c \geq 1$. Then it must be $t_0 \geq 1$ (because $c \leq \sum_1^{t_0}(u_t+v_t+w_t)$) and, by induction, we get

$$|H_l| \leq C_4 \prod_{s=1}^{s_0} (C_{2,s} k^{m_{2,s}}) \prod_{t=1}^{t_0} (C_{0,t} j \lambda^j k^{m_{0,t}}) \leq C_5 \lambda^j k^{m_5}$$

where $C_{0,t} = C_0(u_t,v_t,w_t)$, $m_{0,t} = m_0(u_t,v_t,w_t)$ and $C_5 > 0$, $m_5 \geq 1$, depend only on p,q,r,l. On the other hand, if $c = 0$ we have

$$|H_l| \leq C_3 \lambda^j \prod_{s=1}^{s_0} (C_{2,s} k^{m_{2,s}}) \prod_{t=1}^{t_0} (C_{0,t} j \lambda^j k^{m_{0,t}}) \leq C_5 \lambda^j k^{m_5}$$

(for possibly larger C_5, m_5). Replacing these estimates in (15) we get

$$\|\partial_\mu^p \partial_x^q \partial_Y^r Y_{j+1}\| \leq \lambda \|\partial_\mu^p \partial_x^q \partial_Y^r Y_j\| + C_6 \lambda^j k^{m_6},$$

where C_6, m_6 are determined by p, q, r. Thus, by induction on j,

$$\|\partial_\mu^p \partial_x^q \partial_Y^r Y_{j+1}\| \leq C_0 j \lambda^{j+1} k^{m_0} + C_6 \lambda^j k^{m_6} \leq C_0 (j+1) \lambda^{j+1} k^{m_0},$$

as long as we fix $C_0 \geq C_6/\lambda$ and $m_0 \geq m_6$. This finishes the proof of the claim (14) and so our argument is complete. □

3 Proof of Theorem A

As we said before, the strange attractors we exhibit for the proof of Theorem A are associated to homoclinic tangencies occurring in the unfolding of any critical saddle-node cycle. The existence of such tangencies is a particularly subtle fact in the setting of 1-cycles, see [NPT], and so we treat first the relatively simpler case of ℓ-cycles, $\ell \geq 2$.

3.1 Saddle-node ℓ-cycles, $\ell \geq 2$

Let $f_\mu : M \longrightarrow M$, $\mu \in \mathrm{R}$, be a smooth arc of diffeomorphisms unfolding a critical saddle-node ℓ-cycle, $\ell \geq 2$. For the sake of clearness we treat first the case when $\ell = 2$ and the periodic orbits involved in the cycle $\mathcal{O}(p_0), \mathcal{O}(p_1)$, both consist of fixed points. On the other hand, our arguments extend easily to the general case, as we shall explain afterwards.

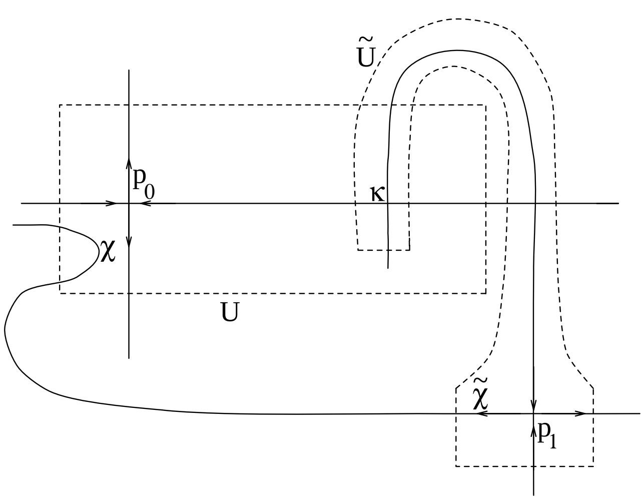

Figure 2:

Let (x, Y) be smooth coordinates on a small neighbourhood U of p_0 as in Section 2.2: $f_\mu(x, Y) = (\varphi_\mu(x), \Psi_\mu(x, Y))$ with φ a saddle-node arc on the real line, $\Psi(\mu, x, 0) = 0$ and $\|\partial_Y \Psi(\mu, x, Y)\| \leq \lambda < 1$ for every (μ, x, Y). For simplicity we assume the diffeomorphism f_0 to be C^N-linearizable near p_1: there exist a neighbourhood $\tilde{U}$ of p_1 and C^N coordinates $(\tilde{x}, \tilde{Y})$ on $\tilde{U}$ such that $f_0(\tilde{x}, \tilde{Y}) = (\rho \tilde{x}, \Lambda \tilde{Y})$, where $\rho \in \mathrm{R}$ and $\Lambda \in \mathcal{GL}(\mathrm{R}^{m-1})$ satisfy $\|\Lambda\| < 1 < |\rho|$. Here N is a fixed integer; for all our purposes it is sufficient to take $N \geq 3$.

Let $\chi = (a, Y_0) \in U$, $a < 0$, be a critical point, i.e. a point of tangency of $W^u(p_1)$ with some strong-stable leaf of p_0. We fix $l \geq 1$ such that $\tilde{\chi} = f_0^{-l}(\chi) \in \tilde{U}$; up to rescaling the coordinate $\tilde{x}$, we may suppose $\tilde{\chi} = (-1, 0)$. We assume the criticality to be quadratic and then for $(\tilde{x}, \tilde{Y})$ near $(-1, 0)$

$$f_0^l(\tilde{x}, \tilde{Y}) = (a + \beta_1(\tilde{x}+1)^2 + h_1(\tilde{x}+1, \tilde{Y}), Y_0 + H_1(\tilde{x}+1, \tilde{Y}))$$

with $\beta_1 \neq 0$ and $h_1 = \partial_{\tilde{x}} h_1 = \partial_{\tilde{x}}^2 h_1 = 0 = H_1$ at $(0,0)$. We also fix $\kappa = (q, 0) \in U$, a point of transverse intersection of $W^u(p_0)$ and $W^s(p_1)$ close to p_0. Extending $\tilde{U}$ by negative iteration if necessary, we may suppose that $\kappa \in \tilde{U}$, see Figure 2. Then we let $\tilde{Y} = \Gamma(\tilde{x})$ be the expression of $W^u(p_0)$ in $(\tilde{x}, \tilde{Y})$ coordinates, near κ. On the other hand, we take X to be a saddle-node arc of vector fields with $X_1 = \varphi$ on $\{(x, \mu) \colon \mu \geq 0\}$ and, using a above and $b = X_{1/2}(0, q)$, we define the transition maps $T_\infty(\sigma, x)$ of φ and $\hat{T}_\infty(\sigma, x, Y) = (T_\infty(\sigma, x), 0)$ of f, recall Section 2. Then $\hat{T}_\infty(1/2, \chi) = \kappa$ and so there is a neighbourhood V of $(1/2, \chi)$ such that $\hat{T}_\infty(V) \subset \tilde{U}$. It is clear from the definition $T_\infty(\sigma, x) = X_{t(0,x)-\sigma}(0, b)$ that $\partial_\sigma T_\infty$ and $\partial_x T_\infty$ are nonzero. Hence we may write $\hat{T}_\infty \colon V \longrightarrow \tilde{U}$ as

$$\hat{T}_\infty(\sigma, x, Y) = (\tilde{x}, \Gamma(\tilde{x})), \qquad \tilde{x} = \beta_2(\sigma - 1/2) + \beta_3(x - a) + h_2(\sigma - 1/2, x - a),$$

with $\beta_2\beta_3 \neq 0$ and $h_2 = \partial_\sigma h_2 = \partial_x h_2 = 0$ at the origin.

Now we introduce n-dependent parameter $\theta = \tau_n(\sigma)$ and coordinates $(\xi, \Theta) = \phi_n(\tilde{x}, \tilde{Y})$ as follows. Let $x_n, \sigma_n \in \mathrm{R}$, close to zero, be given implicitly by

(a) $2\beta_1 x_n + \partial_{\tilde{x}} h_1(w_n) = 0, \quad w_n = (x_n, \Lambda^n \Gamma(\rho^{-n}(-1 + x_n)))$

(b) $\beta_2 \sigma_n + \rho^{-n} + \beta_1\beta_3 x_n^2 + \beta_3 h_1(w_n) + h_2(z_n) = 0, \quad z_n = (\sigma_n, \beta_1 x_n^2 + h_1(w_n))$.

Note that $\partial_{\tilde{x}} h_1(0,0) = \partial_\sigma h_2(0,0) = 0 \neq \beta_1\beta_2$, together with $\|\Lambda\| < 1$, assures the existence of x_n and σ_n for every large n. Then we define

$$\theta = \rho^{2n}(\sigma - 1/2 - \sigma_n) \qquad \xi = \rho^n(\tilde{x} + 1 - x_n) \qquad \Theta = \rho^{3n}(\tilde{Y} - \Lambda^n \Gamma(\rho^{-n}\tilde{x}))$$

and we let ψ_n be the expression of $f_0^n \circ \hat{T}_\infty \circ f_0^l$ with respect to (θ, ξ, Θ), that is

$$\psi_n(\theta, \xi, \Theta) = \phi_n \circ f_0^n \circ \hat{T}_\infty(\tau_n^{-1}(\theta), \cdot) \circ f_0^l \circ \phi_n^{-1}(\xi, \Theta).$$

Proposition 3.1 *Given any compact $K \subset \mathrm{R}^{m+1}$, ψ_n is defined on K for every sufficiently large n. Moreover, $(\psi_n | K)_n$ converges in the C^N topology to the map $\psi \colon K \longrightarrow \mathrm{R}^m$ given by $(\theta, \xi, \Theta) \mapsto (\beta_2\theta + \beta_1\beta_3\xi^2, 0)$.*

Proof: The first part is an easy consequence of the fact that $(\tau_n^{-1}, \phi_n^{-1})(K) \to (1/2, \tilde{\chi})$ as $n \to \infty$. This, on its turn, follows immediately from the definitions above: note, in particular, that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} \sigma_n = 0$. As to the second part, a direct calculation gives

$$\psi_n(\theta, \xi, \Theta) = (\beta_2\theta + \beta_2\rho^{2n}\sigma_n + \rho^n + \beta_1\beta_3\rho^{2n}(x_n + \rho^{-n}\xi)^2 + \beta_3\rho^{2n}h_1(w) + \rho^{2n}h_2(z), \ 0),$$

where

$$w=(x_n+\rho^{-n}\xi,\Lambda^n\Gamma(\rho^{-n}(-1+x_n+\rho^{-n}\xi))+\rho^{-3n}\Theta),\ z=(\sigma_n+\rho^{-2n}\theta,\beta_1(x_n+\rho^{-n}\xi)^2+h_1(w)).$$

We set

$$F_n(\xi,\Theta)=\beta_1\rho^{2n}(x_n+\rho^{-n}\xi)^2+\rho^{2n}h_1(w)\quad\text{and}\quad G_n(\theta,\xi,\Theta)=\rho^{2n}h_2(z)$$

and then, in view of our definition of σ_n,

$$\beta_2\rho^{2n}\sigma_n+\rho^n+\beta_3F_n(0,0)+G_n(0,0)=0.$$

Let us estimate the partial derivatives of F_n. Let $p,q\geq 0$ be such that $1\leq p+q\leq N$. It is easy to see (e.g. by induction on $p+q$) that $\partial_\xi^p\partial_\Theta^q(h_1(w))$ may be written as $\sum_{i=1}^{u(p,q)}H_i$, each H_i having the form

$$\partial_{\tilde{x}}^k\partial_{\tilde{Y}}^{q+l}h_1(w)\rho^{-kn-3qn}\prod_{j=1}^{l}\left(\Lambda^n\Gamma^{(s_j)}(\rho^{-n}(-1+x_n+\rho^{-n}\xi))\rho^{-2s_jn}\right),$$

with $0\leq k,l\leq p$ and $1\leq s_1,\dots,s_l\leq p$ depending only on p,q,i and satisfying $k+\sum_1^l s_j=p$. As a direct consequence,

$$\|\partial_\xi^p\partial_\Theta^qF_n(\xi,\Theta)\|\leq\ \text{const}\,\rho^{(2-(p+3q))n}\tag{1}$$

and so $\partial_\xi^p\partial_\Theta^qF_n$ converges to zero, for all $p+3q>2$. Moreover, due to our definition of x_n,

$$\partial_\xi F_n(0,0)=\partial_{\tilde{Y}}h_1(w_n)\Lambda^n\Gamma'(y_n),\quad y_n=\rho^{-n}(-1+x_n),$$

and so $\partial_\xi F_n(0,0)\to 0$. On the other hand,

$$\partial_\xi^2F_n(0,0)=2\beta_1+\partial_{\tilde{x}}^2h_1(w_n)+\partial_{\tilde{x}}\partial_{\tilde{Y}}h_1(w_n)\Lambda^n\Gamma'(y_n)\rho^{-n}+$$

$$+\partial_{\tilde{Y}}^2h_1(w_n)\left(\Lambda^n\Gamma'(y_n)\right)^2\rho^{-2n}+\partial_{\tilde{Y}}h_1(w_n)\Lambda^n\Gamma''(y_n)\rho^{-2n},$$

which, together with $\partial_{\tilde{x}}^2h_1(0,0)=0$, proves that $\partial_\xi^2F_n(0,0)\to 2\beta_1$. Now we show that all partial derivatives of G_n converge to zero. Observe that $z=(\sigma_n+\rho^{-2n}\theta,\rho^{-2n}F_n(\xi,\Theta))$ and so, for every $1\leq p+q+r\leq N$, we may write $\partial_\xi^p\partial_\Theta^q\partial_\theta^r(h_2(z))=\sum_{i=1}^{v(p,q,r)}K_i$, each K_i having the form

$$\rho^{-2rn}\partial_\sigma^r\partial_x^kh_2(z)\prod_{j=1}^{k}\partial_\xi^{s_j}\partial_\Theta^{t_j}\left(\rho^{-2n}F_n(\xi,\Theta)\right)$$

with $0\leq k\leq(p+q)$, $1\leq(s_j+t_j)\leq(p+q)$, and $\sum_1^k s_j=p$, $\sum_1^k t_j=q$. Using (1) to bound the $\partial_\xi^{s_j}\partial_\Theta^{t_j}(\rho^{-2n}F_n)$ we conclude that

$$\|\partial_\xi^p\partial_\Theta^q\partial_\theta^rG_n(\theta,\xi,\Theta)\|\leq\rho^{(2-(2r+p+3q))n}$$

and so $\partial_\xi^p \partial_\Theta^q \partial_\theta^r G_n \to 0$ whenever $2r+p+3q>2$. This leaves out three cases, which we treat separately. For $(p,q,r)=(0,0,1)$ we have

$$\partial_\theta G_n(0,0,0) = \partial_\sigma h_2(z_n) \to 0,$$

because $\partial_\sigma h_2(0,0)=0$. For $(p,q,r)=(1,0,0)$ we recall that $\partial_x h_2(0,0)=0$ and then

$$\partial_\xi G_n(0,0,0) = \partial_x h_2(z_n) \partial_\xi F_n(0,0) \to 0,$$

since both factors converge to zero. Finally, for $(p,q,r)=(2,0,0)$

$$\partial_\xi^2 G_n(0,0,0) = \partial_x h_2(z_n) \partial_\xi^2 F_n(0,0) + \rho^{-2n} \partial_x^2 h_2(z_n) \left(\partial_\xi F_n(0,0)\right)^2 \to 0$$

where we use once more the fact that $\partial_x h_2(0,0)=0$. Altogether this shows that

$$\beta_2\theta + \beta_2\rho^{2n}\sigma_n + \rho^n + \beta_3 F_n(\xi,\Theta) + G_n(\theta,\xi,\Theta) \to \beta_2\theta + \beta_1\beta_3\xi^2$$

in the C^N topology as $n\to\infty$ and so the proof of the proposition is complete. $\square$

We set $(\hat\theta,\hat\xi,\hat\Theta) = (\hat\tau_n(\sigma), \hat\phi_n(\tilde x,\tilde Y)) = (-\beta_1\beta_2\beta_3\theta, (\beta_2\theta)^{-1}\xi, \Theta)$ and then the expression $\hat\psi_n$ of $f_0^n \circ \hat T_\infty \circ f_0^l$ with respect to $(\hat\theta,\hat\xi,\hat\Theta)$ converges to the map $\hat\psi(\hat\theta,\hat\xi,\hat\Theta) = (1-\hat\theta\hat\xi^2, 0)$ as $n\to\infty$. We take $n \gg 1$ to be fixed from now on. On the other hand, by Theorem 2.6,

$$\hat\psi_{n,k}(\hat\theta,\hat\xi,\hat\Theta) = \hat\phi_n \circ f_\mu^{n+k+l} \circ \hat\phi_n^{-1}(\xi,\Theta), \quad \mu = \mu_k(\hat\tau_n^{-1}(\hat\theta)),$$

converges to $\hat\psi_n$ as $k\to\infty$. Hence, for k large $\hat\psi_{n,k}$ is a Hénon- (or quadratic-) like family, in the sense that it is C^N-close to the family of quadratic maps $\hat\psi$, some $N\ge 3$. According to [MV], [Vi], any such family has Hénon-like strange attractors for a positive Lebesgue measure set of parameter values. Actually, the proof of this fact even provides a uniform lower bound for the measure of these sets on a neighbourhood of the family $\hat\psi$. We conclude that there are $c_1>0$ and $k_1\ge 1$ such that for every $k\ge k_1$ the set Σ_k of values of $\hat\theta$ for which the diffeomorphism $\hat\psi_{n,k}(\hat\theta,\cdot)$ exhibits Hénon-like attractors has Lebesgue measure $m(\Sigma_k)\ge c_1$. Now we denote $S_k = \mu_k(\hat\tau_n^{-1}(\Sigma_k)) \subset [\mu_{k+1}^*, \mu_k^*]$. Then, by construction, the dynamics of f_μ contains Hénon-like strange attractors for every $\mu\in S_k$ and $k\ge k_1$. Moreover, using also Proposition 2.2, $m(S_k)\ge c_2(\mu_k^* - \mu_{k+1}^*)$ for every $k\ge k_1$ and some $c_2>0$ independent of k. Recalling, in addition, that $\mu_{k+1}^*/\mu_k^* \to 1$ we conclude that

$$\frac{m((\cup_{k\ge k_1} S_k)\cap[-\varepsilon,\varepsilon])}{2\varepsilon} \ge \frac{c_2}{3} \quad \text{for every small } \varepsilon>0$$

and the statement of the theorem follows immediately.

Finally, we discuss the general case of saddle-node ℓ-cycles, $\ell\ge 2$. Let $\mathcal{O}(p_0)$, $\mathcal{O}(p_1)$, ..., $\mathcal{O}(p_{\ell-1})$ be the periodic orbits involved in the cycle and let $t_0, t_1, \ldots, t_{\ell-1}$ be the corresponding periods. We fix a point p_0 in the saddle-node orbit. On the other hand, we assume $f_0^{t_0 t_i}$ to be C^N-linearizable on a neighbourhood of $\mathcal{O}(p_i)$ for some $1\le i\le \ell-1$ and we fix such an i from now on. Note also that, by the inclination lemma, we may take the point p_i and some $0\le t<t_0$ satisfying

- $W^u(p_i)$ has a critical intersection χ with $W^s(p_0)$;
- $W^u(f_0^t(p_0))$ has a transverse intersection $\tilde{\kappa}$ with $W^s(p_i)$.

We choose neighbourhoods U of p_0 and $\tilde{U}$ of p_i as before and we may suppose $\chi = (a, Y_0) \in U$ and $\tilde{\kappa} \in \tilde{U}$. We fix $l \geq 1$ such that $\tilde{\chi} = f_0^{lt_0t_i} = (-1, 0) \in \tilde{U}$ and we also denote $\kappa = f_0^{-t}(\tilde{\kappa}) = (q, 0) \in U$. Then we define the transition map $\hat{T}_\infty$ of $f_0^{t_0t_i}$ in U. From this point on the argument proceeds in just the same way as before, with $f_0^{nt_0t_i} \circ (f_0^t \circ \hat{T}_\infty) \circ f_0^{lt_0t_i}$, resp. $f_\mu^{(n+k+l)t_0t_i+t}$, in the place of $f_0^n \circ \hat{T}_\infty \circ f_0^l$, resp f_μ^{n+k+l}.

3.2 Saddle-node 1-cycles

Now we prove Theorem A for critical cycles involving a unique (saddle-node) periodic orbit $\mathcal{O}(p_0)$. For simplicity we take p_0 to be a fixed point, the general case following from this in just the same way as in the previous section. Once more, let U be a small open neighbourhood of p_0 and (x, Y) be coordinates on U such that $f_\mu(x, Y) = (\varphi_\mu(x), \Psi_\mu(x, Y))$, where φ is a 1-dimensional saddle-node arc, $\Psi(\mu, x, 0) = 0$ and $\|\partial_Y \Psi(\mu, x, Y)\| \leq \lambda < 1$. We also continue to denote by X some saddle-node arc of vector fields having φ as its time-1.

Let $\sim_\mu$ be the equivalence relation on U spanned by

$$z \sim_\mu f_\mu(z) \quad \text{for every} \quad z \in U \cap f_\mu^{-1}(U)$$

and denote by $\pi_\mu: U \longrightarrow (U/\sim_\mu)$ the canonical projection associated to it. For $a^- < 0 < a^+$ close to zero we let $D_\mu^\pm = \{(x, 0) \in U: a^\pm \leq x \leq \varphi_\mu(a^\pm)\}$ and $\tilde{D}_\mu^\pm = \pi_\mu(D_\mu^\pm)$. Then, as long as $|\mu| \ll |a^\pm|$, each $\tilde{D}_\mu^\pm$ is a smooth circle and diffeomorphisms $\phi_\mu^\pm: \tilde{D}_\mu^\pm \longrightarrow S^1$ may be taken depending smoothly on μ. In addition, up to replacing $(\phi_\mu^\pm)_\mu$ by $(\tau^\pm \circ (\phi_0^\pm)^{-1} \circ \phi_\mu^\pm)_\mu$ with $\tau^\pm(\pi_0(x, 0)) = \int_{a^\pm}^x X(0, s)^{-1} ds \bmod 1$, we may suppose that $\theta_\mu^\pm = \phi_\mu^\pm \circ \pi_\mu$ satisfies

$$\theta_0^\pm(X_t(0, a^\pm)) = t \bmod 1, \quad \text{for every } t. \tag{2}$$

Next, we let $C_\mu^\pm = \{(x, Y) \in U: a^\pm \leq x \leq \varphi_\mu(a^\pm)\}$ and endow $\tilde{C}_\mu^\pm = \pi_\mu(C_\mu^\pm)$ with the quotient manifold structure. Clearly, every $\tilde{C}_\mu^\pm$ is diffeomorphic to the solid m-torus $S^1 \times B$, B =unit ball in R^{m-1}, and we may choose C^∞ diffeomorphisms

$$\Phi_\mu^\pm: \tilde{C}_\mu^\pm \longrightarrow S^1 \times B, \quad \Phi_\mu^\pm(\pi_\mu(x, Y)) = (\theta_\mu^\pm(x), \Theta_\mu^\pm(x, Y))$$

depending smoothly on μ, with $\theta_\mu^\pm$ as above and $\Theta_\mu^\pm(x, 0) = 0$ for every μ and x.

Now we construct smooth maps $\psi, \psi_k: [0, 1] \times S^1 \times B \longrightarrow S^1 \times B$ in the following way. Up to taking $|a^\pm|$ small enough, we may fix a (large) integer l such that $f_0^l(D_0^+)$ is contained in $\{(x, Y) \in U: x < \varphi_0(a^-)\}$. Then f_0^l induces a smooth map $h_l: \tilde{D}_0^+ \longrightarrow \tilde{C}_0^-$: $h_l \circ \pi_0 = \pi_0 \circ f_0^l$. Let $\hat{T}_\infty = (T_\infty, 0)$ be the transition map of f, with $a = a^-$, $b = a^+$, recall Section 2. Given $0 \leq \sigma \leq 1$, we denote by $\tau_\sigma: \tilde{C}_0^- \longrightarrow \tilde{D}_0^+ \subset \tilde{C}_0^+$ the map induced by $\hat{T}_\infty(\sigma, \cdot)$. Then we define $\psi(\sigma, \cdot) = \Phi_0^- \circ h_l \circ \tau_\sigma \circ (\Phi_0^-)^{-1}$. Note that $\Phi_0^+ \circ \tau_\sigma \circ (\Phi_0^-)^{-1}(\theta, \Theta) = (\theta - \sigma \bmod 1, 0)$, in view of (2) and the definition of $T_\infty, \hat{T}_\infty$. We write $\Phi_0^- \circ h_l \circ (\Phi_0^+)^{-1}(\theta, 0) = (g(\theta), G(\theta))$ and then

$$\psi(\sigma, \theta, \Theta) = (g \circ R_{-\sigma}(\theta), G \circ R_{-\sigma}(\theta)), \quad R_{-\sigma}(\theta) = \theta - \sigma \bmod 1.$$

On the other hand, we set $\psi_k(\sigma,\cdot) = \Phi_\mu^- \circ h_{k,l,\sigma} \circ (\Phi_\mu^-)^{-1}$, where $h_{k,l,\sigma}\colon \tilde{C}_\mu^- \longrightarrow \tilde{C}_\mu^-$ denotes the map induced by f_μ^{l+k} and $\mu = \mu_k(\sigma)$. By Theorem 2.6, this is well defined for every large k and $(\psi_k)_k$ converges to ψ in the C^∞ topology.

By [NPT, Theorem 3.7], the family of circle maps $(g_\sigma = g \circ R_{-\sigma})_{\sigma\in[0,1]}$ unfolds generically homoclinic tangencies associated to some hyperbolic periodic point. In other words, there are $I \subset [0,1]$ an open interval, $\tilde{\sigma} \in I$ and smooth maps $I \ni \sigma \mapsto p_\sigma, c_\sigma$ with p_σ a repelling periodic point and c_σ a critical point of g_σ, such that $c_{\tilde{\sigma}} \in W^u(p_{\tilde{\sigma}})$, $g_{\tilde{\sigma}}^j(c_{\tilde{\sigma}}) = p_{\tilde{\sigma}}$ and $\partial_\sigma\left(g_\sigma^j(c_\sigma) - p_\sigma\right)(\tilde{\sigma}) \neq 0$ for some $j \geq 1$. We assume g to be a Morse function and then such tangencies are quadratic. As a consequence, for every large k the family $\psi_k = (\psi_k(\sigma,\cdot))_{\sigma\in[0,1]}$ of diffeomorphisms of $S^1 \times B$ unfolds generically quadratic homoclinic tangencies associated to some hyperbolic saddle point. Hence, we may conclude from [MV], [Vi], that $\psi_k(\sigma,\cdot)$ has Hénon-like strange attractors for a set Σ_k of values of σ with positive Lebesgue measure (smooth linearizability near the saddle is not necessary for this conclusion, see [Rm].) Moreover, the fact that the limit family ψ itself undergoes generic quadratic tangencies permits to apply the renormalization scheme in [MV], [Vi], uniformly to the sequence $(\psi_k)_k$, to conclude that $m(\Sigma_k)$ is uniformly bounded away from zero. Let us explain this in more detail. First one constructs for ψ a sequence $(\phi_n)_n$ of local coordinates such that, up to an appropriate n-dependent reparametrization $\sigma = \tau_n(\nu)$, the expression of ψ^n in these coordinates converges to $\hat{\psi}\colon (\nu, x, Y) \mapsto (1 - \nu x^2, 0)$ as $n \to \infty$. More precisely: $(\phi_n \circ \psi(\tau_n(\nu),\cdot) \circ \phi_n^{-1})(x,Y) \to \hat{\psi}(\nu,x,Y)$ in the C^N topology, for some fixed $N \geq 3$. Thus, for every n and k sufficiently large $(\phi_n \circ \psi_k(\tau_n(\nu),\cdot) \circ \phi_n^{-1})_\nu$ is a Hénon-like family and so it exhibits strange attractors for a set of values of the parameter ν whose Lebesgue measure is positive and even uniformly bounded away from zero. We fix n large and conclude in this way that there are $c_1 > 0$ and $k_1 \geq 1$ such that $m(\Sigma_k) \geq c_1$ for every $k \geq k_1$. Now we let $S_k = \mu_k(\Sigma_k) \subset [\mu_{k+1}^*, \mu_k^*]$ and then precisely the same argument as in the previous case shows that $(\cup_{k\geq k_1} S_k)$ has positive density at $\mu = 0$. Finally, it is immediate to check that f_μ has Hénon-like strange attractors for every $\mu \in (\cup_{k\geq k_1} S_k)$. The proof of Theorem A is complete.

4 Proof of Theorem B

Here we construct open classes of families of diffeomorphisms bifurcating through critical saddle-node cycles, for which hyperbolicity of the dynamics near the cycle is also a prevalent feature. We exhibit two such classes $\mathcal{A}_1$, $\mathcal{A}_2$, corresponding, respectively, to 1- and 2-cycles, cf. the Introduction. The conditions in the definition of $\mathcal{A}_1$, $\mathcal{A}_2$, are just meant as simple sets of assumptions assuring hyperbolic behaviour and no effort was made to optimize them.

4.1 Hyperbolicity for 1-cycles

In the present section $(f_\mu)_\mu$ always refers to a smooth family of diffeomorphisms unfolding a critical saddle-node 1-cycle of f_0. We keep the notations of Section 3.2.

First, we let $h: [-1/2, 1/2] \longrightarrow [-1/2, 1/2]$ be given by $h(x) = x(-a + 4(1+a)x^2)$, where $a \in (0,1)$ is fixed. Note that h has three fixed points $-1/2, 0, 1/2$. Since $h'(-1/2) = h'(1/2)$, we may identify h with a C^1 endomorphism of $S^1 = [-1/2, 1/2]/(-1/2 \sim 1/2)$ and we do so in the sequel. On the other hand, the origin is an attractor, $|h'(0)| = a < 1$, and its basin $W^s(0)$ contains both critical points of h. This last statement follows, for instance, from the fact that h has negative schwarzian derivative, [Si], using also that it is an odd function. As a consequence, [Ma], $K = S^1 \backslash W^s(0)$ is a hyperbolic set for h: there are $N \geq 1$ and $\rho > 1$ such that $|(h^N)'(\theta)| \geq \rho$ for all $\theta \in K$. Moreover, these dynamical features are robust under small perturbations of the map: there exists a neighbourhood $\mathcal{N}$ of h in $C^1(S^1, S^1)$ such that, given any $\tilde{h} \in \mathcal{N}$, $\tilde{h}$ has a fixed attractor $x(\tilde{h})$ close to zero and $K(\tilde{h}) = S^1 \backslash W^s(x(\tilde{h}))$ is a hyperbolic set for $\tilde{h}$.

Now we define $\mathcal{A}_1$, simply, by the condition (recall Section 3.2)

$$(g \circ R_{-\frac{1}{2}}) \in \mathcal{N}. \tag{1}$$

Let us show that the conclusion of Theorem B is indeed satisfied by every family $(f_\mu)_\mu$ as above. This requires a few preliminary considerations. We introduce

$$U_0 = \{(x, Y) \in U : c \leq x \leq \varphi_0(a^+), \|Y\| \leq \xi(x)\} \quad \text{and} \quad \hat{C} = U_0 \cap C_0^+.$$

Here $c < a^-$ and ξ are chosen in such a way that $f_0^l(\hat{C}) \subset \operatorname{int}(U_0)$ and $f_0(U_0 \backslash \hat{C}) \subset \operatorname{int}(U_0)$. Then there are compact sets $U_{l-1}, \ldots, U_1$ such that

- $f_0(\hat{C}) \subset \operatorname{int}(U_1)$, $f_0(U_j) \subset \operatorname{int}(U_{j+1})$ for $1 \leq j \leq l-2$ and $f_0(U_{l-1}) \subset \operatorname{int}(U_0)$.

As a consequence, $V = U_0 \cup U_1 \cup \cdots \cup U_{l-1}$ satisfies $f_0(V) \subset \operatorname{int}(V)$. Now our goal is to show that $L(f_\mu|V)$ is a hyperbolic set of f_μ, for a set of values of μ with positive Lebesgue density at $\mu = 0$.

In order to do this we consider the maps $\psi, \psi_k: [0,1] \times S^1 \times B \longrightarrow S^1 \times B$ introduced in Section 3.2. Recall that $\psi(\sigma, \theta, \Theta) = (g \circ R_{-\sigma}(\theta), G \circ R_{-\sigma}(\theta))$ and $(\psi_k)_k \to \psi$ as $k \to \infty$. Condition (1) implies that $g_\sigma = g \circ R_{-\sigma} \in \mathcal{N}$ for σ in some interval $I = [1/2 - \delta, 1/2 + \delta]$; from now on we always consider $\sigma \in I$. Then $\psi(\sigma, \cdot): S^1 \times B \longrightarrow S^1 \times B$ has a fixed attractor z_σ close to $\{\theta = 0\}$ and $W^s(z_\sigma) = (K(g_\sigma) \times B)^c$. We fix $N \geq 1$, $\rho_1 > 1$ and W an open neighbourhood of $K(g_\sigma)$ such that $|(g_\sigma^N)'(\theta)| \geq \rho_1$ for every $\theta \in W$. Then, for k sufficiently large, $\psi_k(\sigma, \cdot)$ has a fixed attractor $z_{k,\sigma}$ satisfying $(z_{k,\sigma})_k \to z_\sigma$ as $k \to \infty$ and $W^s(z_{k,\sigma}) \supset (W \times B)^c$. Moreover,

$$\Lambda_{k,\sigma} = (S^1 \times B) \backslash W^s(z_{k,\sigma}) = \{z \in S^1 \times B : \psi_k^n(\sigma, z) \in W \times B \text{ for every integer } n\}$$

is a hyperbolic set of the diffeomorphism $\psi_k(\sigma, \cdot)$. Indeed, invariant stable and unstable cone fields for $\psi_k(\sigma, \cdot)$ on $W \times B$ may be constructed in a fairly straightforward way, see for instance [PT2, Chapter 6.3], where a similar situation is treated.

Now we go back to analysing the limit set of f_μ in V. We consider $\mu = \mu_k(\sigma)$, with $\sigma \in I$ and k large. Let w be any point in $L(f_\mu|V)$. In particular, the orbit of w is contained in V

and it follows from our construction that there are $n_\pm = n_\pm(\mu, w)$ with $n_- < 0 \le n_+$ such that both points $w_\pm = f_\mu^{n_\pm}(w)$ belong to C_μ^-. We take $n_\pm$ minima (in absolute value) with that property and then they satisfy

$$|n_\pm(\mu, w)| \le k + \text{const} \tag{2}$$

where const is independent of μ, w, k (basically, it is determined by the choice of $a^\pm$). Moreover, $\psi_k(\sigma, \tilde{w}_-) = \tilde{w}_+$, where $\tilde{w}_\pm = \pi_\mu(w_\pm)$. At this point we distinguish two cases. Suppose first that $\tilde{w}_\pm \in W^s(z_{k,\sigma})$. Since $\tilde{w}_\pm$ belong to the limit set of $\psi_k(\sigma, \cdot)$, we must have $\tilde{w}_+ = \tilde{w}_- = z_{k,\sigma}$ and so $w \in \mathcal{O}(\hat{z}_\mu)$, where $\hat{z}_\mu$ denotes any point in $\pi_\mu^{-1}(z_{k,\sigma})$. Clearly, $\mathcal{O}(\hat{z}_\mu)$ is a periodic attractor for f_μ. Now assume that $\tilde{w}_\pm \in \Lambda_{k,\sigma}$ and let $\tilde{E}_\pm^s \oplus \tilde{E}_\pm^u$ be the corresponding splitting of the tangent space of $(S^1 \times B)$ at $\tilde{w}_\pm$. We take $T_w M = E_w^s \oplus E_w^u$ to be given by $D\pi_\mu \cdot Df_\mu^{n_+}(E_w^*) = \tilde{E}_+^*$ for $* = s, u$. We set this for every $w \in \hat{\Lambda}_\mu = L(f_\mu|V) \backslash W^s(\hat{z}_\mu)$ and then the splitting $E^s \oplus E^u$ of the tangent bundle over $\hat{\Lambda}_\mu$ defined in this way is invariant under Df_μ. Moreover, the uniform bound (2) assures that this is a hyperbolic splitting: there are $\lambda < 1$ and $L \ge 1$ (depending on k) such that $\|Df_\mu^L|E^s\|$, $\|Df_\mu^{-L}|E^u\| \le \lambda$.

Altogether, we have proved that $L(f_\mu|V)$ is hyperbolic for every $\mu \in \cup_k \mu_k(I)$: more precisely, it may be written as $\mathcal{O}(\hat{z}_\mu) \cup \hat{\Lambda}_\mu$, where $\mathcal{O}(\hat{z}_\mu)$ is a hyperbolic periodic attractor and $\hat{\Lambda}_\mu$ is a hyperbolic set of saddle-type. On the other hand, $\cup_k \mu_k(I)$ has positive Lebesgue density at $\mu = 0$, as a consequence of Proposition 2.2. Our argument is complete.

4.2 Hyperbolicity for 2-cycles

For the construction of $\mathcal{A}_2$ we start with a (linear) horseshoe map $\psi: R \longrightarrow R$ as described in Figure 3. More precisely, we let R be a compact rectangle in $\mathrm{R}^m = \mathrm{R} \times \mathrm{R}^{m-1}$, $m \ge 2$,

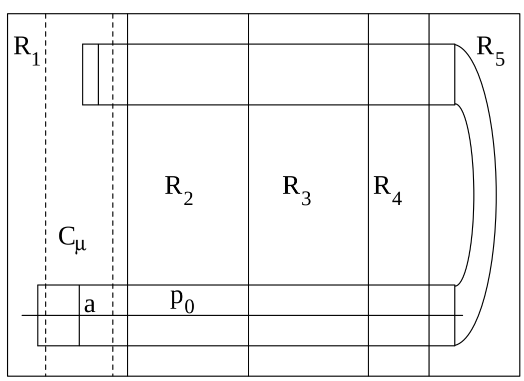

Figure 3:

and we write $R = R_1 \cup \cdots \cup R_5$, satisfying $\psi(R_1 \cup R_5) \subset \operatorname{int}(R_1)$, $\psi(R_3) \subset \operatorname{int}(R_5)$ and

$$D\psi|R_1 = \begin{pmatrix} \xi & 0 \\ 0 & \Lambda \end{pmatrix}, \quad D\psi|R_2 = \begin{pmatrix} \rho_1 & 0 \\ 0 & \Lambda \end{pmatrix}, \quad D\psi|(R_4 \cup R_5) = \begin{pmatrix} -\rho_2 & 0 \\ 0 & -\Lambda \end{pmatrix},$$

where $\rho_1, \rho_2 > 1$, Λ is a contraction on R^{m-1}, and ξ is some smooth function defined on R. For the sake of notational simplicity we take the fixed saddle point p_0 of ψ in R_2 to coincide with the origin. Moreover, ψ has a fixed sink s_0 in R_1, with $s_0 = (s, 0)$ for some $s \in \mathrm{R}$.

Now we modify ψ in order to collapse p_0 and s_0 to a saddle-node point, in the following way. We fix compact intervals W_0, W in $\mathrm{R} \times \{0\} \subset \mathrm{R} \times \mathrm{R}^{m-1}$ such that

$$\{p_0, s_0\} \subset \operatorname{int}(W_0), \quad W_0 \subset \operatorname{int}(W) \quad \text{and} \quad \psi(W) \subset \operatorname{int}(R_1 \cup \cdots \cup R_4).$$

Then we let $\varphi_\mu \colon W \longrightarrow \mathrm{R}$, $\mu \in [-1, 1]$, be a C^∞ saddle-node arc, in the sense of Section 2.1, such that

- $\varphi_{-1} = \pi_1 \circ \psi | W$ and $\varphi_\mu | (W \backslash W_0) = \pi_1 \circ \psi | (W \backslash W_0)$ for every μ;
- the map φ_0 has a fixed saddle-node point at the origin;
- the derivative φ_μ' is a monotone function, for every small μ.

Here $\pi_1 \colon \mathrm{R} \times \mathrm{R}^{m-1} \longrightarrow \mathrm{R}$ and $\pi_2 \colon \mathrm{R} \times \mathrm{R}^{m-1} \longrightarrow \mathrm{R}^{m-1}$ are the canonical projections. We also consider $\phi \colon \mathrm{R}^{m-1} \to \mathrm{R}$ to be a C^∞ bump function with support contained in $\pi_2(R)$, satisfying $\phi | \pi_2(\psi(R_2)) = 1$ and $\phi | \pi_2(\psi(R_4)) = 0$. Then we set

$$f_\mu(x, Y) = \psi(x, Y) + \phi(Y)(\varphi_\mu(x) - \pi_1 \circ \psi(x, Y), 0)$$

if $x \in W$ and $f_\mu(x, Y) = \psi(x, Y)$ otherwise. This defines a family of smooth transformations of R into itself, unfolding a critical saddle-node cycle and, clearly, given any m-dimensional manifold M one may extend $(f_\mu)_\mu$ to a family of diffeomorphisms on M. Now we show that ψ and $(\varphi_\mu)_\mu$ may be chosen in such a way that this family satisfies the conclusion of Theorem B, with $V = R$.

In order to do this we fix $a < 0 < b$ such that $(a, 0) \in \operatorname{int}(R_1)$ and $(b, 0) \in \operatorname{int}(R_4)$ and we let $\mu_k(\sigma)$, $T_k(\sigma, x) = \varphi^k_{\mu_k(\sigma)}(x)$, $\hat{T}_k(\sigma, x, Y) = f^k_{\mu_k(\sigma)}(x, Y)$ be as in Section 2. It follows from Theorem 2.3 that there is $\tau > 0$ such that $|(\varphi^k_{\mu_k(\sigma)})'(x)| > \tau$, for every $x \in [\varphi_0^{-1}(a), a]$, $\sigma \in [0, 1]$, and k large enough. From now on we assume that $\rho_2 \tau > 1$ and $\psi(R_5) \subset \operatorname{int}(C_\mu)$ for all small μ, where $C_\mu = \{(x, Y) \in R \colon x \in [\varphi_\mu^{-1}(a), a]\}$. These properties may be obtained just by supposing that $\pi_1(R_4)$ and $\pi_1(R_5)$ have sufficiently small diameters. Actually, by further reducing the diameter of $\pi_1(R_5)$ if necessary, we may assume that $\hat{T}_\infty(\sigma, \psi(R_5)) \subset \operatorname{int}(R_3)$, for some interval J of values of σ. As a consequence, $f^{k+2}_{\mu_k(\sigma)}(R_5) \subset R_5$ for every $\sigma \in J$ and k sufficiently large; moreover, it is easy to see that this construction may be performed in such a way that $\|Df^{k+2}_{\mu_k(\sigma)} | R_5\| < 1$. Hence, every f_μ, $\mu \in \mu_k(J)$, has an attracting $(k+2)$-periodic orbit $\mathcal{O}(s_\mu)$ whose basin contains R_5 and R_3. From now on we restrict to values of $\mu \in \cup_{k > k_0} \mu_k(J)$, for some large k_0, and we prove that $L(f_\mu | R)$ is a hyperbolic set for f_μ.

First we note that $L(f_\mu | R) \subset H_\mu \cup \mathcal{O}(s_\mu)$, where H_μ is the maximal invariant set of f_μ in $R_1 \cup R_2 \cup R_4$, and so we only have to show that H_μ is a hyperbolic set. We introduce the (constant) cone fields on $U = \operatorname{int}((R_1 \cup R_2 \cup R_4) \cap \psi(R))$ defined by

$$C^u(z) = \{w = (u, V) \in T_z R : |u| \geq \|V\|\} \quad \text{and} \quad C^s(z) = \{w = (u, V) \in T_z R : |u| \leq \|V\|\}.$$

In the sequel we always take $\|w\| = \sup\{|u|, \|V\|\}$. Note that $f_\mu(R) \subset \psi(R)$ and so $H_\mu \subset U$; moreover, $Df_\mu|U$ has the form

$$\begin{pmatrix} \eta_\mu & 0 \\ 0 & \pm\Lambda \end{pmatrix}.$$

Clearly, we may assume, right from the start, that $\|\Lambda\|$ is small enough so as to assure that $\|\Lambda\eta_\mu^{-1}\| < 1$ on U, for every small μ. This implies the invariance of the two cone fields above

$$Df_\mu(C^u(z)) \subset C^u(f_\mu(z)) \quad \text{and} \quad Df_\mu^{-1}(C^s(z)) \subset C^s(f_\mu^{-1}(z))$$

and it also follows that vectors in $C^s(z)$ are geometrically expanded by Df_μ^{-1}. Now we claim that there is $\theta > 1$ such that, given any $z \in H_\mu$, there exists $j = j(z) \leq k+2$ for which

$$\|Df_\mu^j(z)w\| \geq \theta\|w\| \text{ for all } w \in C^u(z)$$

(k is determined by the condition $\mu \in \mu_k(J)$). Note that hyperbolicity of H_μ is an immediate consequence of this claim together with the previous arguments. On the other hand, the claim may be justified as follows. Given $z \in H_\mu$, let $l \geq 0$ be minimum such that $f_\mu^l(z) \in R_4$; then, by construction, $l \leq k+1$ (at most k iterates are spent in $R_1 \cup R_2$, near the origin). We take $j = l+1$ and then

$$\|Df_\mu^j(z)w\| = \|Df_\mu(f_\mu^l(z))) \cdot Df_\mu^l(z)w\| \geq \rho_2\tau\|w\|,$$

(here we make use of the monotonicity of φ'_μ). This proves the claim with $\theta = \rho_2\tau$.

Observe now that $\cup_{k>k_0}\mu_k(J)$ has positive Lebesgue density at zero, as a consequence of Proposition 2.2. Therefore, we have proved that the conclusion of the theorem holds for the family $(f_\mu)_\mu$. Finally, it is clear that the previous arguments and conclusions carry on to every one-parameter family close enough to $(f_\mu)_\mu$ and so we may take $\mathcal{A}_2$ to be any sufficiently small neighbourhood of $(f_\mu)_\mu$.

As a concluding remark, let us note that appropriate choices of ψ, $R_1, \ldots, R_5$, and φ_μ above lead to $L(f_0|R)$ having Hausdorff dimension arbitrarily close to $m \geq 2$. In particular, this shows that in the present context prevalence of hyperbolicity is compatible with large Hausdorf dimension of the limit set, compare [PT1], [PY].

5 Proof of Theorem C

Our proof is based on two main results which we state in Sections 5.1 and 5.2. The definition of the open class $\mathcal{A}_0$ in the statement is given in Section 5.5, involving these results. In Section 5.5 we also put together all the ingredients to conclude the proof. From Theorem 5.1 we deduce that, for any family $(f_\mu)_\mu \in \mathcal{A}_0$ and for a large set of values of μ, the maximal invariant set A_μ coincides with the closure of the unstable manifold of some periodic saddle. Then we also need an extension of the arguments in [BC] and [MV] for more general families of dissipative diffeomorphisms, which may viewed as perturbations of multimodal maps of the interval. In Section 5.2 we describe a class of such "multimodal Hénon-like families" suitable for our purposes. Then Sections 5.3–5.4 are devoted to proving that Hénon-like attractors are a persistent phenomenon in any such family (Theorem 5.2). At this point we assume that the reader is familiar with the methods in [BC], [MV]. Indeed, most steps follow closely those papers and, as a rule, we do not reproduce them here. Instead, we sketch the global structure of the argument and detail the changes one has to perform to adapt it to the present situation. For our presentation we take M to be a surface but the general case $m = \dim M \geq 2$ follows directly from combining the ideas in this paper with those in [Vi].

5.1 Maps of the annulus with global attractors

Let $\xi_a: \mathrm{R} \longrightarrow \mathrm{R}$, with $a > 1/2\pi$, be given by $\xi_a(x) = x - a\sin 2\pi x$. Clearly, ξ_a is a 1-periodic odd function with negative schwarzian derivative. Its fixed points have the form $n/2$ and the critical points are $\pm\chi_a + n$, with n an arbitrary integer and $\chi_a = (1/2\pi)\arccos(1/a2\pi) \in (0, 1/4)$. Moreover, $a \mapsto \xi_a(-\chi_a) = -\xi_a(\chi_a)$ is strictly increasing, converging to infinity as $a \to \infty$ and to zero as $a \to 1/2\pi$. Hence, there is a unique $\bar{a} > 1/2\pi$ satisfying $\xi_{\bar{a}}(\pm\chi_{\bar{a}}) = \mp 1$. In what follows we denote $\xi = \xi_{\bar{a}}$ and $\chi = \chi_{\bar{a}}$ and let $x_i \in \mathrm{R}$, $1 \leq i \leq 4$, be given by

$$0 < x_1 < x_3 < \chi < x_4 < x_2 < 1/2, \quad \xi(x_1) = \xi(x_2) = -1/2 \ \text{ and } \ \xi(x_3) = \xi(x_4) = x_1 - 1.$$

Now we consider the endomorphism $h: S^1 \longrightarrow S^1$ induced by ξ on $S^1 = \mathrm{R}/(x \sim x+1)$. Let $\pi_1: \mathrm{R} \longrightarrow S^1$ be the canonical projection and consider in S^1 the orientation induced by the usual order of R. Then h is a degree-1 map with two fixed points $p = \pi_1(0)$, $q = \pi_1(\pm 1/2)$, and two critical points $c^\pm = \pi_1(\pm\chi)$. Both fixed points are repelling since $h(c^\pm) = p$, see [Si]. Moreover, h is surjective (and monotone) on each of the intervals $[c^-, c^+]$ and $[c^+, c^-]$. As a consequence, both unstable sets $W^u(p)$ and $W^u(q)$ fill-in the whole S^1.

We also introduce $\Delta = \mathrm{R}^2/((x,y) \sim (x+1, y+1))$ and let $H: \Delta \longrightarrow \Delta$ be the map induced on Δ by $(x,y) \mapsto (\xi(x), x)$. Clearly, H has the same dynamics as h (up to semiconjugacy). In particular, it has exactly two fixed points $P = \pi_2(0,0)$, $Q = \pi_2(\pm 1/2, \pm 1/2)$, and these are hyperbolic saddles (with one zero eigenvalue). We fix some $r > \bar{a}$ and denote by $\Delta_r \subset \Delta$ the image of $\{(x,y): |x| \leq 1/2, |y - x| \leq r\}$ under the canonical projection $\pi_2: \mathrm{R}^2 \longrightarrow \Delta$. Note that $r > \bar{a}$ ensures $H(\Delta_r) \subset \operatorname{int}(\Delta_r)$

The following topological result plays a key role in the proof of Theorem C.

Theorem 5.1 *There is an open neighbourhood $\mathcal{N}$ of $(H|\Delta_r)$ in $C^1(\Delta_r, \Delta_r)$ such that for every diffeomorphism (onto its image) $F \in \mathcal{N}$*

$$\bigcap_{n=0}^{\infty} F^n(\Delta_r) = \mathit{closure}\,(W^u(P_F)),$$

where P_F is the continuation for F of the fixed point P of H.

Proof: Let $q_i^\pm = \pi_1(\pm x_i) \in S^1$, for $1 \leq i \leq 4$. Then we have $h(q_1^\pm) = h(q_2^\pm) = q$, $h(q_3^-) = h(q_4^-) = q_1^-$ and $h(q_3^+) = h(q_4^+) = q_1^+$. Denote $\hat{K} = (q_4^-, q_3^-) \cup (q_3^+, q_4^+) \subset S^1$ and let $\hat{\Lambda} = \{x \in S^1 \colon h^n(x) \notin \hat{K}$ for all $n \geq 0\}$. Then $\hat{\Lambda}$ is an invariant hyperbolic set for h: hyperbolicity follows, for instance, from the fact that $\operatorname{int}(\hat{K})$ contains all the critical points of h, see [Ma]. Moreover, the subintervals of $S^1 \backslash \hat{K}$ bounded by the points q, $q_i^{\pm}$, $1 \leq i \leq 4$, form a Markov partition for $h|\hat{\Lambda}$. It is immediate to check that the corresponding subshift of finite type is topologically mixing. We also consider parallel objects for H. Given $1 \leq i \leq 4$, we let $Q_i^{\pm} = \pi_2(\pm x_i, \mp y_i)$, where $y_i \in (0, \chi)$ is defined by $\xi(y_i) = -x_i$. The local stable set of Q is the vertical line Γ through it. Then $W^s(Q)$ also contains $H^{-1}(\Gamma) \supset \Gamma_1^+ \cup \Gamma_1^- \cup \Gamma_2^+ \cup \Gamma_2^-$ and $H^{-1}(\Gamma_1^{\pm}) \supset \Gamma_3^{\pm} \cup \Gamma_4^{\pm}$, where $\Gamma_i^{\pm}$ is the vertical line passing through $Q_i^{\pm}$. Furthermore, $W^u(P) = W^u(Q) = H(\Delta)$. On the other hand, we let $K = \pi_2\left((-x_4, -x_3) \cup (x_3, x_4) \times \mathrm{R}\right)$ and $\Lambda = \{z \in H(\Delta) \colon H^n(z) \notin K$ for all $n \geq 0\}$. The previous statements give that Λ is an invariant hyperbolic set for H and that $H|\Lambda$ admits a topologically mixing Markov partition.

Now let $F \colon \Delta_r \longrightarrow \Delta_r$ be a diffeomorphism onto its image, close to $(H|\Delta_r)$. We take F to preserve orientation: the opposite case is treated in the same way and we do not detail it here. As a first step, we extend it into an area-dissipative surjective diffeomorphism $\Delta \longrightarrow \Delta$, which we continue to denote by F (this leads to no ambiguity, as neither $\cap_{n=0}^{\infty} F^n(\Delta_r)$ nor $\operatorname{closure}(W^u(P_F))$ depend on the choice of the extension). In the sequel we invoke a number of well-known facts, see [HPS], concerning persistence and continuous variation of hyperbolic objects under small C^1 perturbations of the dynamical system. At each stage, we assume $(F|\Delta_r)$ to be close enough to $(H|\Delta_r)$ so that the corresponding perturbation statement holds; these conditions are our definition of $\mathcal{N}$. We observe that these statements remain valid in a noninvertible situation such as ours, see e.g. [PT2, Appendices 1, 4].

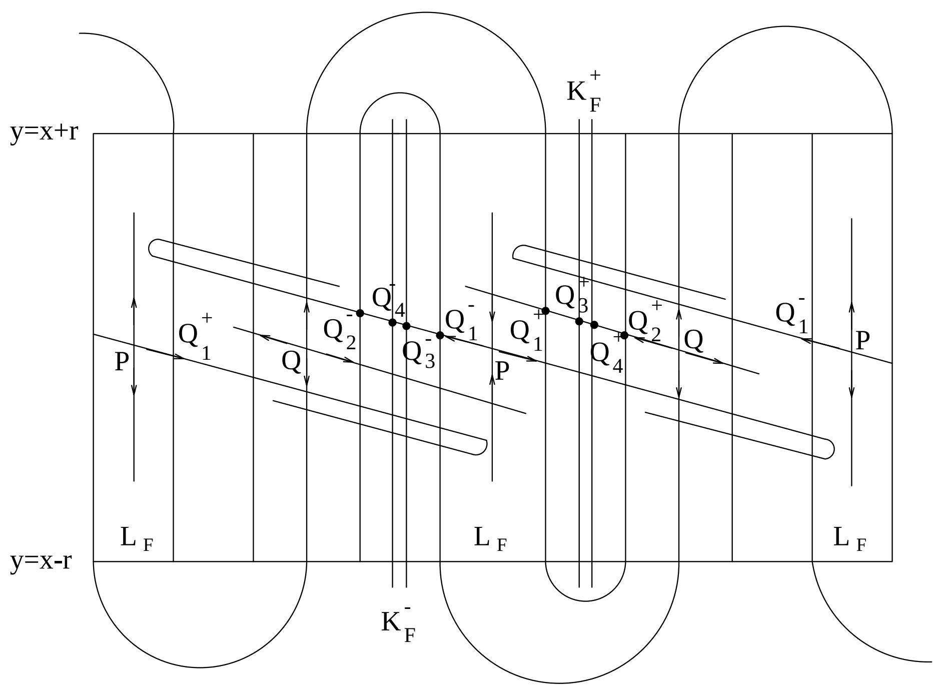

Figure 4:

Let P_F, Q_F, be the continuations for F of the fixed points P, Q. One may view the $Q_i^{\pm}$ as transverse intersections between $W^u(P)$ and $W^s(Q)$ and we let $Q_{i,F}^{\pm} \in W^u(P_F) \cap W^s(Q_F)$ be the corresponding continuations for F. By continuity of local stable manifolds with respect to the map, $W^s(Q_F)$ must contain a nearly straight and vertical segment Γ_F passing through Q_F and crossing both connected components of $\partial\Delta_r$. Iterating twice under $(F|\Delta_r)^{-1}$ we conclude that $W^s(Q_F)$ also contains segments $\Gamma_{i,F}^{\pm}$ through each $Q_{i,F}^{\pm}$, $1 \le i \le 4$, with similar properties. On the other hand, $W^s(Q_F)$ is diffeomorphic to R, since F is a diffeomorphism, and so these Γ_F, $\Gamma_{i,F}^{\pm}$, must be connected to each other in Δ. The connections between Γ_F and $\Gamma_{1,F}^{\pm}$, $\Gamma_{2,F}^{\pm} \subset F^{-1}(\Gamma_F)$ are shown in Figure 4 and can be justified in the following way. Both eigenvalues of $DF(P_F)$, resp. $DF(Q_F)$ are negative, resp. positive, since F preserves orientation. In particular, $Q_{1,F}^{+}$ and $Q_{2,F}^{+}$ belong to the same separatrix of $W^s(Q_F)$ as their own images. Moreover, $F(Q_{1,F}^{+})$ separates $F(Q_{2,F}^{+})$ from Q_F inside $W^s(Q_F)$ and so the same must hold for $Q_{1,F}^{+}$ and $Q_{2,F}^{+}$, respectively. See Figure 5. Finally, Γ_F has exactly two intersections with $F(\partial\Delta_r)$ in between $F(Q_{1,F}^{+})$ and $F(Q_{2,F}^{+})$, hence the segment of $W^s(Q_F)$ in between $Q_{1,F}^{+}$ and $Q_{2,F}^{+}$ intersects $\partial\Delta_r$ exactly twice. Of course, the same reasoning applies for $Q_{1,F}^{-}$, $Q_{2,F}^{-}$.

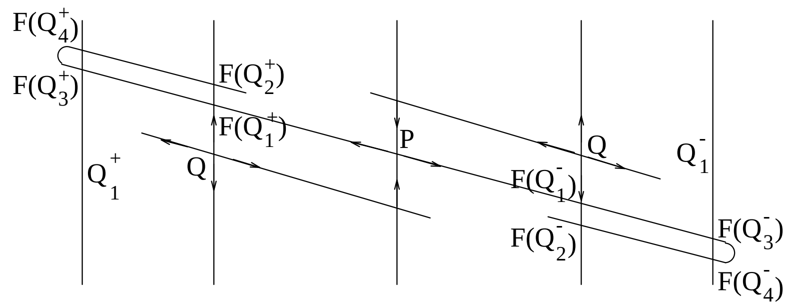

Figure 5:

The hyperbolic set Λ also admits a continuation Λ_F, which may be characterized as follows. Let $K_F = K_F^- \cup K_F^+$, where $K_F^{\pm} \subset \Delta_r$ is the domain bounded by $\Gamma_3^{\pm}$, $\Gamma_4^{\pm}$, and $\partial\Delta_r$. Then $\Lambda_F = \{z \in \Delta_r : F^n(z) \notin K_F \text{ for all } -\infty < n < +\infty\}$ and a topologically mixing Markov partition for $F|\Lambda_F$ is provided by the set of subdomains of $\Delta_r \backslash K_F$ bounded by $\partial\Delta_r$, Γ_F and the $\Gamma_i^{\pm}$, $1 \le i \le 4$. We call L_F the subdomain containing P_F, which is to play some role below. Moreover, there is a foliation $\mathcal{F}_F$ of $\Delta_r \backslash K_F$ by nearly vertical curves joining the two connected components of $\partial\Delta_r$ which is invariant and contracting under F: for some $\theta < 1$ and any $z \in \Delta_r \backslash K_F$ with $F(z) \in \Delta_r \backslash K_F$, we

$$(1) \qquad w \in \mathcal{F}_F(z) \quad \Longrightarrow \quad F(w) \in \mathcal{F}_F(F(z)) \text{ and } \operatorname{dist}(F(w), F(z)) \le \theta \operatorname{dist}(w, z).$$

Such a foliation may be constructed following well-known arguments from [HPS] (as the fixed point of a convenient graph-transform operator).

Now let $z \in \Delta_r$ be such that $F^{-n}(z) \in \Delta_r$ for every $n \ge 0$. We want to show that $z \in \operatorname{closure}(W^u(P_F))$ and the proof is divided into two cases.

Suppose first that $F^{-n_k}(z) \in K_F$ for some sequence $n_k \to \infty$. By construction, $F(K_F) \subset L_F$ and so $F^{-n_k+1}(z) \in L_F$. Now, see Figure 4, $L_F \subset \tilde{L}_F$ where $\tilde{L}_F$ is a domain in Δ bounded by two segments of $W^u(P_F)$ and two segments of $W^s(Q_F)$. Note that $\text{area}\,(F^n(\tilde{L}_F)) \to 0$, because F is area-dissipative. Hence, the fact that $z \in F^{n_k-1}(\tilde{L}_F)$ for each k implies $\text{dist}\,(z, \partial F^{n_k-1}(\tilde{L}_F)) \to 0$. On the other hand, the boundary of $F^{n_k-1}(\tilde{L}_F)$ consists of segments of $W^u(P_F)$ and $W^s(Q_F)$ and the lengths of the latter ones go to zero as $k \to \infty$. This yields $\text{dist}\,(z, W^u(P_F)) = 0$, as we claimed.

Now assume that $F^{-n}(z) \notin K_F$ for every $n \geq n_0$, some $n_0 \geq 0$. It is no restriction to assume $n_0 = 0$ and we do so from now on. By construction, the leaf $\mathcal{F}_F(F^{-n}(z))$ intersects $W^u(P_F)$ in some point w_n. By (1)

$$\text{dist}\,(z, W^u(P_F)) \leq \text{dist}\,(z, F^n(w_n)) \leq \theta^n \,\text{dist}\,(F^{-n}(z), w_n) \leq \text{const}\,\theta^n.$$

Since n is arbitrary this gives $\text{dist}\,(z, W^u(P)) = 0$. □

Note also that the map h defined above is topologically mixing: given any open interval $J \subset S^1$ there is $n \geq 1$ such that $h^n(J) = S^1$. This can be seen as follows. We claim that J must intersect the negative orbit of $\{c^-, c^+\}$. Indeed, otherwise there is a maximal interval $\tilde{J} \supset J$ such that $h^j|\tilde{J}$ is a homeomorphism for every $j \geq 1$. Since h has no wandering intervals, see [MS], it must be $h^j(\tilde{J}) \cap \tilde{J} \neq \emptyset$ for some $j \geq 1$ and then the maximality of $\tilde{J}$ yields $h^j(\tilde{J}) \subset \tilde{J}$. Then, necessarily, h has some periodic attractor or semi-attractor in $\tilde{J}$. On the other hand, such an attractor or semi-attractor can not exist, by [Si]. This contradiction proves our claim. Now, this claim means that there is $m \geq 1$ such that $h^m(J)$ contains a neighbourhood of $c^{\pm}$. Then $h^{m+1}(J)$ contains at least a half neighbourhood of P. Finally, it is easy to check that both separatrices of $W^u(P)$ fill-in the whole S^1 and so it must be $h^n(J) = S^1$ for some $n > m$.

A direct consequence is that all the periodic points of H are heteroclinically related (mutual transverse intersections between their stable and unstable sets). Thus, given any pair of periodic points, their continuations for nearby diffeomorphisms remain heteroclinically related and so the corresponding unstable manifolds are mutually dense in each other. It follows that the conclusion of the theorem holds for the continuation of any one of these periodic points of H (the neighbourhood $\mathcal{N}$ depends on the periodic point, though).

5.2 Nonsingular perturbations of multimodal maps

Let N denote either S^1 or a compact interval of R. For each fixed $d \geq 1$, $k \geq 3$, $\delta > 0$, we consider the class of all C^k families of maps $\phi_\nu\colon N \longrightarrow N$, with $\nu \in [-\delta, \delta]$, satisfying

1. (invariance) $\phi_0(N) \subset \text{int}\,(N)$ (this is automatic if $N = S^1$);

2. (nondegenerate critical points) ϕ_0 has exactly d critical points $c_1, \ldots, c_d$ and $\phi_0''(c_i) \neq 0$ for all $i = 1, \ldots, d$; moreover, $\phi_0(c_i) \neq c_j$ for all $1 \leq i, j \leq d$;

3. (negative schwarzian derivative) $S\phi_0(x) < 0$ for every $x \neq c_1, \ldots, c_d$;

4. (topological mixing) for any open intervals $J_1, J_2 \subset N$ there is $n_0 = n_0(J_1, J_2) \geq 1$ such that $\phi_0^n(J_1) \cap J_2 \neq \emptyset$ for every $n \geq n_0$;

5. (preperiodicity) for each $1 \leq i \leq d$ there is $k_i \geq 1$ such that $p_i = \phi_0^{k_i}(c_i)$ is a (repelling) periodic point of ϕ_0;

6. (generic unfolding) $\partial_\nu \left(\phi_\nu^{k_i}(c_i(\nu)) - p_i(\nu)\right) \neq 0$ at $\nu = 0$, where $c_i(\nu)$, $p_i(\nu)$ are the continuations for ϕ_ν, small ν, of c_i, p_i, respectively.

We refer to the elements of this class, simply, as *d-modal families*. We fix some $\rho > 0$ and denote $B_\rho = [-\rho, \rho]$. Given $b > 0$, a C^k family of diffeomorphisms $\varphi_\nu \colon N \times B_\rho \longrightarrow N \times B_\rho$ is a *nonsingular b-perturbation* of the d-modal family $(\phi_\nu)_\nu$ if $\|\varphi - \hat{\phi}\| \leq b$, where we write $\varphi(\nu, x, y) = \varphi_\nu(x, y)$ and $\hat{\phi}(\nu, x, y) = \hat{\phi}_\nu(x, y) = (\phi_\nu(x), 0)$, and $\|\cdot\|$ is the C^k-norm over $(\nu, x, y) \in [-\delta, \delta] \times N \times B_\rho$.

Theorem 5.2 *Let $(\phi_\nu)_\nu$ be a d-modal family and P be a periodic point of ϕ_0. Then there are $b > 0$ and $\chi > 0$ such that given any nonsingular b-perturbation $(\varphi_\nu)_\nu$ of $(\phi_\nu)_\nu$ there exists $S \subset [-\delta, \delta]$ with $m(S) \geq \chi$ and (P_ν denotes the continuation for φ_ν of the periodic point $(P, 0)$ of $\hat{\phi}_0$) for every $\nu \in S$ there is some $\bar{z} \in W^u(P_\nu)$ satisfying*

i) the orbit $\{\varphi_\nu^n(\bar{z}) \colon n \geq 0\}$ of $\bar{z}$ is dense in closure$(W^u(P_\nu))$;

ii) φ_ν has a positive Lyapunov exponent at $\bar{z}$, that is, there are $c > 0$, $\sigma > 1$, and $v \neq 0$ such that $\|D\varphi_\nu^n(\bar{z})v\| \geq c\sigma^n$ for all $n \geq 0$;

iii) there is $w \neq 0$ such that $\|D\varphi_\nu^n(\bar{z})w\| \to 0$ as both $n \to \pm\infty$.

Let us briefly comment on the strategy to prove this result. Diffeomorphisms φ_ν as above combine two rather distinct types of dynamics: away from $\hat{\mathcal{C}} = \cup_{i=1}^d \{x = c_i\}$ they behave in an essentially hyperbolic way, namely they admit invariant stable and unstable cone fields (see Corollary 5.4 below); on the other hand, the "foldings" occurring near $\hat{\mathcal{C}}$ prevent such invariant cone fields from extending to the whole dynamical plane. In order to deal with this combination of hyperbolic and critical (folding) behaviour we follow the approach of [BC]. This is based on constructing, for a positive measure set of parameters, a sequence of *critical sets* $(\mathcal{C}_k)_{k\geq 1}$ as follows. Each $\mathcal{C}_k = \mathcal{C}_k(\nu)$ is a finite subset of $W^u(P_\nu)$ located close to $\hat{\mathcal{C}}$. Each element $z_0^{(k)} \in \mathcal{C}_k$ is a critical approximation (of order k), meaning that the tangent direction to $W^u(P_\nu)$ at $z_1^{(k)} = \varphi_\nu(z_0^{(k)})$ is contracted by the first k iterates of $D\varphi_\nu$. Moreover, $z_1^{(k)}$ expands during k iterates: $\|D\varphi_\nu^j(z_1^{(k)})\| \geq \text{const}\,\sigma^j$ for some fixed $\sigma > 1$ and all $1 \leq j \leq k$. The point $\bar{z}$ in the statement is found as the limit of a convenient sequence $z_1^{(k)}$ as $k \to \infty$. One shows that, up to an additional (unimportant) restriction on the parameter, its orbit is dense in the closure of $W^u(P_\nu)$.

In the sequel we concentrate on the case when $N = S^1$ and the maps ϕ_ν have nonzero degree, which is the one we actually use here and allows for a slightly more elegant treatment. The general case follows in basically the same way. In fact, these arguments apply also to d-modal families of maps in compact 1-dimensional branched manifolds (nonsingular perturbations are diffeomorphisms on tubular neighbourhoods of the manifold), providing other examples of Hénon-like attractors with rich topology. We also note that our particular choice of assumptions 1–6 above is somewhat arbitrary. Further extension of Theorem 5.2, for instance in the spirit of [TTY], can also be carried out along similar lines, combining the arguments in that paper with the present ones.

5.3 Critical approximations

In this section we describe the initial steps in the construction of the critical sets $\mathcal{C}_k$. We fix a d-modal family $(\phi_\nu)_\nu$ and a periodic point P of ϕ_0, and take $(\varphi_\nu)_\nu$ to be a nonsingular b-perturbation, for sufficiently small b. First we recall a few notions and facts from [BC], [MV]. For the time being, the parameter ν is fixed, close to zero.

A point z_1 is *λ-expanding up to time m* if $\|D\varphi_\nu^k(z_1)(1,0)\| \geq \lambda^k$ for all $1 \leq k \leq m$. We always suppose $b \ll \lambda$. For such a point and $1 \leq k \leq m$, let $e^{(k)}$ and $f^{(k)}$ denote unit vectors corresponding, respectively, to the maximal contraction and the maximal expansion of $\|D\varphi_\nu^k(z_1)\|$. Then $e^{(k)}$ and $f^{(k)}$ are orthogonal and the same holds for $D\varphi_\nu^k(z_1)e^{(k)}$ and $D\varphi_\nu^k(z_1)f^{(k)}$. Since φ_ν is a b-perturbtion of ϕ_ν, we have $|\det D\varphi_\nu| \leq \text{const}\, b$. Then the expansiveness assumption gives

$$\|D\varphi_\nu^k(z_1)f^{(k)}\| \geq \lambda^k, \quad \text{hence} \quad \|D\varphi_\nu^k(z_1)e^{(k)}\| \leq \left(\frac{\text{const}\, b}{\lambda}\right)^k.$$

The properties of such *contracting approximations* $e^{(k)}$ were studied in [MV, Section 6] and [Vi, Section 4] in a fairly abstract setting and all the results obtained there apply directly in the present situation. In particular, we have

$$(2) \qquad \|e^{(j)} - e^{(k)}\| \leq (\text{const}\, b)^j \quad \text{and} \quad \|D_{(\nu,x,y)}e^{(k)}\| = \|D_{(\nu,x,y)}f^{(k)}\| \leq \text{const}\, b,$$

for $1 \leq j \leq k \leq m$, where $D_{(\nu,x,y)}$ is derivation with respect to all three variables (ν, x, y) and the constants depend only on λ and the family $(\phi_\nu)_\nu$.

We say that a curve $\gamma \subset W^u(P_\nu)$ is *b-flat* if it is the graph of a C^2 function $y = y(x)$ with $|y'| \leq b^{\frac{1}{2}}$ and $|y''| \leq b^{\frac{1}{2}}$. A point $z_0 \in W^u(P_\nu)$ is a *k-th critical approximation* if $z_1 = \varphi_\nu(z_0)$ is λ-expanding up to time k, the vector $e^{(k)}(z_1)$ is tangent to $W^u(P_\nu)$ at z_1, and the r^k-neighbourhood $\gamma(z_0, r^k)$ of z_0 in $W^u(P_\nu)$ is a b-flat curve. The constant $r > 0$ is fixed (see next section) and we suppose $b \ll r$. We denote a kth critical approximation by $z_0^{(k)}$ and let $z_i^{(k)} = \varphi_\nu^i(z_0^{(k)})$ for $i \geq 1$.

Lemma 5.3 *Let $(\phi_\nu)_\nu$ be a d-modal family. There exist $\sigma_0 > 1$, $\delta_0 > 0$, and $C_0 > 0$ such that for all $0 < \delta < \delta_0$ there are $\nu_0(\delta) > 0$ and $C(\delta) > 0$ satisfying*

i) $|(\phi_\nu^n)'(x)| \geq C(\delta)\sigma_0^n$;

ii) if $\phi_\nu^n(x) \in B_\delta(\mathcal{C})$ *then* $|(\phi_\nu^n)'(x)| \geq C_0\sigma_0^k$;

for all $|\nu| < \nu_0(\delta)$ *and every* x *with* $x, \ldots \phi_\nu^{n-1}(x) \notin B_\delta(\mathcal{C})$.

Here $B_\delta(\mathcal{C})$ denotes the δ-neighbourhood of the critical set $\mathcal{C} = \{c_1, \ldots, c_d\}$ of ϕ_0. We postpone the proof of this result to the end of the section. From similar arguments we also get the following higher-dimensional version.

Corollary 5.4 *Reducing* $\sigma_0 > 1$, $\delta_0 > 0$, $C_0 > 0$, $\nu_0(\delta) > 0$, $C(\delta) > 0$, *if necessary, we have, for any nonsingular b-perturbation* $(\varphi_\nu)_\nu$ *of* $(\phi_\nu)_\nu$ *with* $b \ll \delta$,

i) $|slope|\,(D\varphi_\nu^j(z_0)v) \ll 1$ *for all* $1 \le j \le n$;

ii) $\|D\varphi_\nu^n(z_0)v\| \ge C(\delta)\sigma_0^n$;

iii) if $x_n \in B_\delta(\mathcal{C})$ *then* $\|D\varphi_\nu^n(z_0)v\| \ge C_0\sigma_0^n$;

for any trajectory $z_j = (x_j, y_j) = \varphi_\nu^j(z_0)$ *having* $x_j \notin B_\delta(\mathcal{C})$ *for* $0 \le j \le n-1$, *and any tangent vector* v *at* z_0 *with* $|slope|\,(v) \le 1$.

Recall that we are treating the case when $N = S^1$ and the degree of ϕ_ν is nonzero, in particular ϕ_ν is surjective. As $\phi_0(c_i) \ne c_j$ for every i and j, we may take compact intervals $U_1, \ldots, U_d \subset N \backslash \mathcal{C}$ and fix $\delta > 0$ small enough so that

a) each U_i is contained in a different connected component of $N \backslash B_\delta(\mathcal{C})$;

b) the intervals $\phi_0(U_i)$, $1 \le i \le d$, cover the manifold N;

c) if $\phi_0(U_i)$ intersects $B_\delta(c_j)$ then it actually contains $B_\delta(c_j)$.

Recall that $W^u(P) = N$. Then, by Lipschitz dependence of unstable manifolds on the dynamics (see e.g. [MV, Proposition 7.1]), $W^u(P_\nu)$ contains segments $U_i(\nu)$, $1 \le i \le d$, such that each $U_i(\nu)$ is const b-close to $U_i \times \{0\}$ in the C^2 topology. In particular, each $V_i(\nu) = \varphi_\nu(U_i(\nu))$ is a b-flat curve and the length of $\pi_1(V_i(\nu)) \subset N$ is uniformly bounded from below. Moreover, we may take analogs of properties b) and c) to hold for the $V_i(\nu)$:

b') the intervals $\pi_1(V_i(\nu))$, $1 \le i \le d$, cover N;

c') if $\pi_1(V_i(\nu)) \cap B_\delta(c_j) \ne \emptyset$ then $\pi_1(V_i(\nu)) \supset B_\delta(c_j)$.

Let $H_0(\nu) = \cup_{i=1}^d V_i(\nu)$. For each $j \in \{1, \ldots, d\}$ there is at least one segment $V_i(\nu)$ whose projection contains $B_\delta(c_j)$. We choose such a segment once and for all, and denote by $z_{0,j}^{(0)}$ the unique point in $V_i(\nu)$ such that $\pi_1(z_{0,j}^{(0)}) = c_j$. By Corollary 5.4 the point $z_{1,j}^{(0)} = \varphi_\nu(z_{0,j}^{(0)})$ is λ-expanding up to some time $M_j \ge 1$, where $\lambda = C(\delta)\sigma_0 \gg b$. Note that M_j can be made arbitrarily large by taking $\delta > 0$, $b > 0$, and ν sufficiently small. Consider a parametrization $z(s) = (c_j + s, y(s))$ of $V_i(\nu)$ with $z(0) = z_{0,j}^{(0)}$. Denote by $t(s)$ the tangent vector to $W^u(P_\nu)$ at $\varphi_\nu(z(s))$ given by $t(s) = D\varphi_\nu(z(s))(1, y'(s))$. Then

$$|t(0) \cdot f^{(1)}(z_1(0))| \le \text{const}\, b \quad \text{and} \quad |D_s(t(s) \cdot e^{(1)}(z_1(s)))| \ge \text{const} > 0.$$

Indeed, the first inequality follows from the fact that $V_i(\nu)$ is const b-close to the horizontal, recall above, and that $f^{(1)}$ is also nearly horizontal ($|\text{slope}|\,(f^{(1)}) \le$ const b) away from $\{x \in \mathcal{C}\}$. The second one is a consequence of (2) and the quadratic nature of the critical point c_j. See [MV, Section 7A]. We conclude that there is a unique s_1, with $|s_1| \le$ const b, such that $t(s_1) \cdot f^{(1)}(z_1(s_1)) = 0$, in other words, such that $z_{0,j}^{(1)} = z(s_1)$ is a 1st critical approximation.

Now a similar argument permits to construct, by recurrence, a sequence $z_{0,j}^{(k)} \in V_i(\nu)$, $1 \le k \le M_j$, with $|\pi_1(z_{0,j}^{(k+1)}) - \pi_1(z_{0,j}^{(k)})| \le (\text{const}\, b)^k$, such that each $z_{0,j}^{(k)}$ is a kth critical approximation, and each $z_{1,j}^{(k)} = \varphi_\nu(z_{0,j}^{(k)})$ is λ-expanding up to time M_j. Observe that $|\pi_1(z_{0,j}^{(k)}) - c_j| < \delta$ for all k if b is small. We let $M = \min\{M_1, \ldots, M_d\}$ and for $1 \le k < M$ we define the kth critical set $\mathcal{C}_k$ of φ_ν by $\mathcal{C}_k = \{z_{0,1}^{(k)}, \cdots, z_{0,d}^{(k)}\}$. The construction of the critical sets of order $k \ge M$ requires parameter exclusions and will be sketched in the next section.

Proof of Lemma 5.3: First we note that, see [MS, Theorem III.3.3], given any $\delta_1 > 0$ there are $m \ge 1$ and $\sigma_1 > 1$ such that $|(\phi_0^m)'(x)| \ge \sigma_1^m$ if $x, \phi_0(x), \cdots, \phi_0^{m-1}(x) \notin B_{\delta_1}(\mathcal{C})$. We shall fix $0 < \delta_1 < L$ in b) below, depending only on ϕ_0. Then, by continuity and reducing $\sigma_1 > 1$ if necessary, there is $\nu_1 > 0$ such that the same holds for all ϕ_ν with $|\nu| \le \nu_1$:

$$|(\phi_\nu^m)'(x)| \ge \sigma_1^m \quad \text{whenever} \quad x, \phi_\nu(x), \cdots, \phi_\nu^{m-1}(x) \notin B_{\delta_1}(\mathcal{C}). \tag{3}$$

Let $\mathcal{C}^+ = \{\phi_0^j(c_1) \colon j \ge 0,\ 1 \le i \le d\}$. By condition 5 in the definition of d-modal family, $\mathcal{C}^+$ is finite, hence $L = \frac{1}{2}\inf\{|z - w| \colon z \in \mathcal{C}^+, w \in \mathcal{C}^+, z \ne w\}$ is strictly positive. We claim

a) There are $C_2 > 0$, $\sigma_2 > 1$, $0 < \delta_2 < L$, and $\nu_2 > 0$ such that, given any $1 \le \ell \le m$ and $|\nu| \le \nu_2$, if $\phi_\nu^\ell(x) \in B_{\delta_2}(\mathcal{C})$ then $|(\phi_\nu^\ell)'(x)| \ge C_2\sigma_2^\ell$. Moreover, $C_2 > 0$ may be taken independent of m.

Clearly, it suffices to consider the case $\nu = 0$, the general statement then following by continuity. We begin by noting that there is ℓ_0 such that if I_k is a monotonicity interval for ϕ_0^k with $k \ge \ell_0$ then $|I_k| \le L/2$ (even more, the length of these I_k must go to uniformly to zero as $k \to \infty$). Otherwise, by considering a convenient sequence I_{k_n} with $k_n \to \infty$ and passing to the limit, one would get a nondegenerate interval I with $\phi_0^k|I$ monotone for every $k \ge 1$. In particular, each $\phi_0^k(I)$ would be contained in some connected component of $N \backslash \mathcal{C}$, contradicting the assumption of topological mixing. We fix $0 < \delta_2 < L$ small enough so that $C_2 = \frac{1}{2}\inf\{|(\phi_0^k)'(y)| \colon 1 \le k \le \ell_0,\ \phi_0^k(y) \in B_{\delta_2}(\mathcal{C})\}$ be strictly positive (using preperiodicity once more). We also take $\sigma_2 = 2^{\frac{1}{m}} > 1$. Let ℓ and x be as in the statement of a). If $\ell \le \ell_0$ then $|(\phi_0^\ell)'(x)| \ge 2C_2 \ge C_2\sigma_2^\ell$, by definition. Now, suppose $\ell > \ell_0$ and let $I_x = [a_x, b_x]$ be the maximal interval containing x on which ϕ_0^ℓ is monotone. The maximality of I_x gives that $\phi_0^\ell(a_x)$ and $\phi_0^\ell(b_x)$ are in $\mathcal{C}^+ \backslash \mathcal{C}$, hence (since $\delta_2 < L$) $|\phi_0^\ell([a_x, x])| \ge L$ and $|\phi_0^\ell([x, b_x])| \ge L$, where $|\cdot|$ denotes length. As a consequence, there is $x_1 \in (a_x, x)$ such that

$$|(\phi_0^\ell)'(x_1)| = \frac{|\phi_0^\ell([a_x, x])|}{|x - a_x|} \ge \frac{L}{L/2} = 2$$

and, in the same way, there is $x_2 \in (x, b_x)$ such that $|(\phi_0^\ell)'(x_2)| \geq 2$. Recall that we are assuming ϕ_0 to have negative schwarzian derivative. Since I_x does not contain critical points of ϕ_0^ℓ, the minimum principle (see [MS]) yields $|(\phi_0^\ell)'(x)| \geq 2 \geq \sigma_2^\ell$, proving a).

In the sequel we shall also need

b) There is $\sigma_3 > 1$ such that, if $\delta_1 > 0$ is fixed small enough (depending only on ϕ_0), then for each $x \in B_{\delta_1}(\mathcal{C}) \backslash \mathcal{C}$ there is $k(x) \geq 1$ such that $|(\phi_0^{k(x)})'(x)| \geq \frac{1}{C_2}\sigma_3^{k(x)}$ and $\phi_0^j(x) \notin B_{\delta_1}(\mathcal{C})$ for every $1 \leq j \leq k(x)$.

In order to prove b), suppose that $x \in B_{\delta_1}(c_i) \backslash \{c_i\}$. Take $k_i \geq 1$ minimum such that $p_i = \phi_0^{k_i}(c_i)$ is a periodic point of ϕ_0. Let $s_i \geq 1$ be the period of p_i and fix $\rho_i > 1$ and small numbers $0 < \varepsilon_1 < \varepsilon_2 < L$ (depending only on ϕ_0), such that

$$(\rho_i - \varepsilon_2)^{s_i} \leq |(\phi_0^{s_i})'(y)| \leq (\rho_i + \varepsilon_2)^{s_i} \quad \text{whenever } |y - p_i| \leq \varepsilon_1.$$

Then we take $1 < \sigma_3 < ((\rho_i - \varepsilon_2)/\sqrt{\rho_i + \varepsilon_2})$ (assuming ε_2 small enough). Since the critical point c_i is quadratic, there are constants $0 < k_1 < k_2$, depending only on ϕ_0, such that

$$k_1|x - c_i|^2 \leq |\phi_0^{k_i}(x) - p_i| \leq k_2|x - c_i|^2. \tag{4}$$

We suppose $\delta_1 > 0$ small enough so that $k_2\delta_1^2 \leq \varepsilon_1$ and then we let $\varsigma(x) \geq 0$ be the maximum integer such that $(\rho_i + \varepsilon_2)^{s_i\varsigma(x)}|\phi_0^{k_i}(x) - p_i| \leq \varepsilon_1$. Since we assume $\delta_1, \varepsilon_1 < L$, this ensures that $\phi_0^j(x) \notin B_{\delta_1}(\mathcal{C})$ for all $1 \leq j \leq k(x)$, where $k(x) = s_i\varsigma(x) + k_i$. Moreover, using (4) and the maximality of $\varsigma(x)$,

$$|(\phi_0^{k(x)})'(x)| \geq k_3|x - c_i|(\rho_i - \varepsilon_2)^{s_i\varsigma(x)} \geq k_3\sqrt{\frac{|\phi_0^{k_i}(x) - p_i|}{k_2}}(\rho_i - \varepsilon_2)^{s_i\varsigma(x)} \geq k_4\left(\frac{\rho_i - \varepsilon_2}{\sqrt{\rho_i + \varepsilon_2}}\right)^{s_i\varsigma(x)}$$

where $k_3, k_4 > 0$ depend only on ϕ_0. Observing that $\varsigma(x)$ can be made arbitrarily large by fixing $\delta_1 > 0$ sufficiently small, we deduce $|(\phi_0^{k(x)})'(x)| \geq \frac{1}{C_2}\sigma_3^{k(x)}$, which completes the proof of claim b).

The relation (4) also implies that $\varsigma(x)$ is bounded from above if $|x - c_i|$ is bounded from below. This remark, together with a continuity argument, yields

c) There is $\sigma_3 > 1$ and for each $0 < \delta \leq \delta_1$ there is $\nu_3(\delta) > 0$ such that given $|\nu| \leq \nu_3(\delta)$ and $x \in B_{\delta_1}(\mathcal{C}) \backslash B_\delta(\mathcal{C})$ we have $|(\phi_\nu^{k(x)})'(x)| \geq \frac{1}{C_2}\sigma_3^{k(x)}$ and $\phi_\nu^j(x) \notin B_{\delta_1}(\mathcal{C})$ for every $1 \leq j \leq k(x)$.

Now we derive the conclusion of the lemma. Take $m \geq 1$ as above, $\sigma_0 = \min\{\sigma_1, \sigma_2, \sigma_3\} > 1$, $\delta_0 = \min\{\delta_1, \delta_2\}$, $C_0 = C_2$, $\nu_0(\delta) = \min\{\nu_1, \nu_2, \nu_3(\delta)\}$, and $C(\delta) = C_0\,(\inf|\phi_\nu'(y)|/\sigma_0)^m$, where the infimum is over $|\nu| \leq \nu_0(\delta)$ and $y \notin B_\delta(\mathcal{C})$. For any x with $x, \ldots, \phi_\nu^{n-1}(x) \notin B_\delta(\mathcal{C})$, we list $0 \leq i_1 < \ldots < i_s \leq n$ the integers with $\phi_\nu^{i_j}(x) \in B_{\delta_1}(\mathcal{C})$. The last statement in c) implies that $i_{j+1} \geq i_j + k_j$, where $k_j = k(\phi_\nu^{i_j}(x))$. For each $1 \leq j \leq s$ we write $i_{j+1} - i_j = qm + r + k_j$ with $0 \leq r \leq m-1$. Then, by (3), a), and c),

$$(5) \qquad |(\phi_\nu^{i_{j+1}-i_j})'(\phi_\nu^{i_j}(x))| \geq \sigma_1^{qm} C_2 \sigma_2^r \frac{1}{C_2} \sigma_3^{k_j} \geq \sigma_0^{i_{j+1}-i_j}.$$

If $i_1 > 0$ we also write $i_1 = qm + r$, $0 \leq r \leq m-1$, and then (3) and a) give

$$(6) \qquad |(\phi_\nu^{i_1})'(x)| \geq \sigma_1^{qm} C_2 \sigma_2^r \geq C_0 \sigma_0^{i_1}.$$

Consider first case ii), that is $\phi_\nu^n(x) \in B_\delta(\mathcal{C})$. Writing $n - i_s = mq + r + k_s$ we get, in the same way as in (5),

$$(7) \qquad |(\phi_\nu^{n-i_s})'(\phi_\nu^{i_s}(x))| \geq \sigma_0^{n-i_s}.$$

On the other hand, in case i) we may write $n - i_s = qm + r$ with $0 \leq r \leq m-1$ and then, using a) and the definition of $C(\delta)$,

$$(8) \qquad |(\phi_\nu^{n-i_s})'(\phi_\nu^{i_s}(x))| \geq |(\phi_\nu^r)'(\phi_\nu^{i_s}(x))| \, \sigma_1^{qm} \geq \frac{C(\delta)}{C_0} \sigma_0^{n-i_s}$$

The lemma is a direct consequence of (5)–(8) □

Now Corollary 5.4 can be easily deduced using the following remarks. Let (x, y) be such that $x \notin B_\delta(\mathcal{C})$, and v be a vector with $|\text{slope}|\,(v) \leq 1$. Then $|\text{slope}|\,(D\varphi_\nu(x,y)v) \ll 1$, in the sense that the bound goes to zero when $b \to 0$. Moreover, $\|D\varphi_\nu(x,y)v\|/\|v\| \approx \phi_\nu'(x)$, up to a multiplicative constant which goes to 1 when $b \to 0$. Both statements follow directly from the fact that φ_ν is b-close to $\hat{\phi}_\nu$ in the C^2 sense.

5.4 The induction

In this section we outline the inductive procedure defining the critical sets $\mathcal{C}_k \subset W^u(P_\nu)$ of order k, for $k \geq M$. This is similar to the constructions in [BC], [MV], [Vi], to which we refer the reader for details.

At each step n we assume that all sets $\mathcal{C}_k$ with $k \leq n-1$ were already defined, satisfying a number of conditions which we list in (P1)–(P4) below, and we explain how to construct $\mathcal{C}_n$. This requires two conditions on the parameter, stated in (CP1)–(CP2), which are also supposed to hold for all previous steps.

We let $\sigma_0 > 1$ be as in Corollary 5.4 and fix $1 < \sigma < \sigma_0$ and small constants $\tau > 0$, $\rho > 0$ (see below). Our first property is

(P1) Each element $z_0^{(k)}$ of $\mathcal{C}_k$ is a critical approximation. In fact, (i) $z_1^{(k)} = \varphi_\nu(z_0^{(k)})$ has $\|D\varphi_\nu^j(z_1^{(k)})(1,0)\| \geq \text{const}\,\sigma^j$ for all $1 \leq j \leq k+1$, (ii) $e^{(k)}(z_1^{(k)})$ is tangent to $W^u(P_\nu)$ at $z_1^{(k)}$, and (iii) the $\tau\rho^{\theta k}$-neighbourhood $\gamma(z_0^{(k)}, \tau\rho^{\theta k})$ of $z_0^{(k)}$ in $W^u(P_\nu)$ is a b-flat curve.

We take $\tau > 0$ to be a lower bound for the distance from each critical approximation $z_{0,j}^{(k)} \in H_0(\nu)$, $k < M$, to the boundary of the corresponding segment $V_i(\nu)$, recall the previous section. Then (P1) holds for all $\mathcal{C}_k$ with $k < M$. Now we define $H_j(\nu) = \varphi_\nu^j(H_0(\nu))$, for $j \geq 1$, and further assume

(P2) (i) $\mathcal{C}_k \subset \bigcup_{0 \le j \le \theta k} H_j(\nu)$, where $\theta = \theta(b) = \text{const} / \log \frac{1}{b}$, and (ii) if $z_0^{(k)} \in H_j(\nu)$ with $j \ge 1$ then $D\varphi_\nu^{j-1}$ expands every tangent vector of $\varphi_\nu^{1-j}(\gamma(z_0^{(k)}, \tau\rho^{\theta k}))$.

By construction $\mathcal{C}_k \subset H_0(\nu)$ for all $k < M$, so that (P2) holds for these first critical sets.

Clearly, the total length of $H_j(\nu)$ grows, at most, exponentially as a function of j. Moreover, it is not difficult to see that each b-flat curve $\gamma(z_0^{(k)}, \tau\rho^{\theta k})$ contains at most one critical approximation. Hence (P2)(i) and (P1)(iii) yield the following bound on the number of elements of $\mathcal{C}_k$:

(9) $$\#\mathcal{C}_k \le (\text{ const })^{\theta k} d$$

The definition of $\mathcal{C}_n$ involves two different mechanisms:

a) Given $z_0^{(n-1)} \in \mathcal{C}_{n-1}$ there exists a unique nth approximation $z_0^{(n)}$ in $\gamma(z_0^{(n-1)}, \tau\rho^{\theta(n-1)})$ (this makes sense because we take $z_1^{(n-1)}$ to be expanding up to time n). Moreover, $\text{dist}\,(z_0^{(n)}, z_0^{(n-1)}) \le \text{const}\, b^n$, hence $\gamma(z_0^{(n)}, \tau\rho^{\theta n}) \subset \gamma(z_0^{(n-1)}, \tau\rho^{\theta(n-1)})$. We let $\mathcal{C}'_n$ be the set of all $z_0^{(n)}$ found in this way. By construction they satisfy (P1)–(P2).

b) Let $z_0^{(n-1)} \in \mathcal{C}_{n-1}$ and $\zeta_0 \in H_m(\nu)$ with $\theta(n-1) < m < \theta n$. Let $\gamma(\zeta_0, \rho^{\theta(n-1)})$ be b-flat and both $\text{dist}\,(\zeta_0, z_0^{(n-1)})$ and the angle between the tangents to $W^u(P_\nu)$ at $z_0^{(n-1)}$ and ζ_0 be less than $\frac{1}{2}\tau\rho^k$. Then there exists a unique nth approximation $\zeta_0^{(n)}$ in $\gamma(\zeta_0, \tau\rho^{\theta(n-1)})$ and, actually, $\gamma(\zeta_0^{(n)}, \tau\rho^{\theta n}) \subset \gamma(\zeta_0, \tau\rho^{\theta(n-1)})$. We take $\mathcal{C}''_n$ to be the set of all $\zeta_0^{(n)}$ obtained in this way which satisfy (P2) (property (P1) follows from the construction) and $\text{dist}\,(\zeta_0^{(n)}, z_0^{(n-1)}) < \text{const}\, b^{m/10}$.

Then we take $\mathcal{C}_n = \mathcal{C}'_n \cup \mathcal{C}''_n$.

The fact that all points $z_0^{(n)} \in \mathcal{C}_n$ are taken very close to $\mathcal{C}_{n-1}$ ensures that the corresponding $z_1^{(n)}$ are expanding up to time n. In order to prove that

(10) $$\|D\varphi_\nu^{n+1}(z_1^{(n)})(1,0)\| \ge \text{const}\, \sigma^{n+1}$$

we introduce the notions of *return* (roughly, an iterate j for which $\pi_1(z_j^{(n)}) \in B_\delta(\mathcal{C})$), *free period*, *binding period*, *folding period*, and *binding point*. These are defined in the same way as for Hénon-like families, we only make a few simple comments concerning the *capture procedure* leading to the construction of binding points (cf. [MV, Section 7C]). Let a point z be λ-expanding up to time k and such that $\pi_1(z) \notin B_\delta(\mathcal{C})$. By integrating the kth contracting vector field $e^{(k)}$ one obtains a curve $\Gamma^{(k)}$ which (i) is exponentially contracted by φ_ν^j for all $1 \le j \le k$, and (ii) is nearly vertical and crosses the region $\{(x,y) \colon |y| \le \sqrt{b}\}$. In particular, $\Gamma^{(k)}$ intersects $H_0(\nu)$, recall that $\pi_1(H_0(\nu)) = N$ and $H_0(\nu)$ is $\text{const}\, b$-close to $N \times \{0\}$. Even more, in view of our definitions, we may always choose some b-flat segment $V_i(\nu) \subset H_0(\nu)$ intersecting $\Gamma^{(k)}$ at a point w, in such a way that $\text{dist}\,(w, \partial V_i(\nu))$ remains bounded away from zero (independent of z, k, or ν). We also take $\tau > 0$ to be a lower bound for this distance.

Now, suppose that n is a (free) return for a point $z_0 = z_0^{(n)} \in \mathcal{C}_n$ and that a corresponding binding point $\zeta_0 = \zeta_0^{(n)}$ has been found. For the estimates to appear in (P3) we need z_n and ζ_0 to be in *tangential position*, meaning that $\operatorname{dist}(z_n, \gamma) << \operatorname{dist}(z_n, \zeta_0)$, where γ is some b-flat curve in $W^u(P_\nu)$ containing ζ_0. This is ensured by the following assumption on the parameter ν:

(CP1) $d_n(z_0) = \operatorname{dist}(z_n, \zeta_0) \geq e^{-\alpha n}$.

Here α is a small positive constant (in particular, we want $1 < \sigma < \sigma_0^{1-\alpha}$). Note that (CP1) (as well as (P3), (P4) below) is void for $n < M$: if ν and b are small then the points in $\mathcal{C}_k$, $k < M$, have no returns prior to time M (recall that the critical points of ϕ_0 are preperiodic).

The statement of the inductive hypotheses (P3)–(P4) involves the following splitting algorithm. Given $z_0 \in \mathcal{C}_n$, let $w_0(z_1) = \omega_0(z_1) = (1,0)$ and $\sigma_0(z_1) = (0,0)$. Then write $w_\mu(z_1) = D\varphi_\nu^\mu(z_1)(1,0) = \omega_\mu(z_1) + \sigma_\mu(z_1)$, where $\omega_\mu(z_1)$ is defined inductively in the following way. For each $\mu \geq 1$, let $\tilde{\omega}_\mu(z_1) = D\varphi_\nu(z_\mu)\omega_{\mu-1}(z_1)$. Then

i) If μ is a return for z_0, with folding period $[\mu+1, \mu+\ell]$, write $\tilde{\omega}_\mu(z_1) = \beta_\mu(z_1)(1,0) + \alpha_\mu(z_1)e_\mu$, where $e_\mu = e^{(\ell)}(z_{\mu+1})$. Then let $\omega_\mu(z_1) = \beta_\mu(z_1)(1,0)$.

ii) If μ is the end of some folding period of z_0, let $\mu_1 < \mu_2 < \ldots < \mu_s < \mu$ be all the returns with $\mu_i + \ell_i = \mu$ ($[\mu_i+1, \mu_i+\ell_i]$ the folding period corresponding to μ_i). Then, define

$$\omega_\mu(z_1) = \tilde{\omega}_\mu(z_1) + \sum_{i=1}^{s} \alpha_{\mu_i}(z_1) D\varphi_\nu^{\ell_i}(z_{\mu_i+1}) e_{\mu_i}.$$

iii) If μ is neither a return nor the end of a folding period, let $\omega_\mu(z_1) = \tilde{\omega}_\mu(z_1)$.

This algorithm is designed in such a way that $\omega_\mu(z_1)$ is almost horizontal for all $\mu \geq 0$. Now we may state the remaining induction assumptions.

(P3) Let $z_0 \in \mathcal{C}_k$, $k \leq n$, and $\mu \leq k$ be a return for z_0. Suppose that $\pi_1(z_\mu) \in B_\delta(c_j)$ and denote $a_j = |\phi_0''(c_j)| > 0$. Then

$$\frac{1}{2} a_i d_\mu(z_0) \leq \frac{|\beta_\mu(z_1)|}{\|\omega_{\mu-1}(z_1)\|} = \frac{\|\omega_\nu(z_1)\|}{\|\omega_{\nu-1}(z_1)\|} \leq 2 a_i d_\mu(z_0),$$

and

$$|\alpha_\mu(z_1)| \leq \operatorname{const} b \|\omega_{\mu-1}(z_1)\|.$$

(P4) Moreover, the binding point ζ_0 of z_μ and the binding period $[\mu+1, \mu+p]$ satisfy $p \leq \operatorname{const} \alpha\mu < \mu$ and

$$\frac{1}{A_1} \leq \frac{\|\omega_{\mu+j}(z_1)\|}{|\beta_\mu(z_1)| \|\omega_j(\zeta_1)\|} \leq A_1 \quad \text{and} \quad \frac{\|\omega_{\mu+p}(z_1)\|}{\|\omega_\mu(z_1)\|} d_\mu(z_0) \geq A_2 \sigma^{(p+1)/3} > 1.$$

for fixed positive constants A_1, A_2.

In order to prove the expansion property (10) one combines the last inequality with Corollary 5.4, but we also need a second condition on the parameter. Given $z_0 \in \mathcal{C}_n$ we take $M \leq \mu_1 < \ldots < \mu_s \leq n$ to be its (free) returns and let $p_1, \ldots, p_s$ be the length of the corresponding binding periods. Set the *total binding time* of z_0 to be $B_n(z_0) = \sum_{i=1}^{s}(1+p_i)$. By definition, this is zero if $n < M$. We exclude the parameters that do not satisfy

(CP2) $B_n(z_0) \leq \alpha n$.

Given any $z_0 \in \mathcal{C}_n$, the set of parameters values for which (CP1) or (CP2) do not hold has Lebesgue measure bounded by $\text{const}\, e^{-2\gamma n}$, some $\gamma > 0$ independent of b. Combining this with (9) we get that, if b is small enough, the total measure of the exclusions at time n decreases exponentially fast: it is bounded by $\text{const}\, e^{-\gamma n}$. In this way one gets that a positive measure set $\tilde{S}$ of parameter values satisfy both conditions at all times $n \geq 1$.

Then, for every parameter $\nu \in \tilde{S}$ we choose a sequence $(z_0^{(n)})_n$ such that each $z_0^{(n)}$ is obtained from $z_0^{(n-1)}$ by mechanism a) in the definition of $\mathcal{C}_n$ and we define $\bar{z}(\nu) = \lim z_0^{(n)}$. By construction,

$$\|D\varphi_\nu^n(\bar{z}(\nu))(1,0)\| \geq \text{const}\, \sigma^n \quad \text{for all } n \geq 1.$$

Moreover, there is a full measure subset S of $\tilde{S}$ such that the forward orbit of $\bar{z}(\nu)$ is dense in $W^u(P_\nu)$ for all $\nu \in S$. This follows from precisely the same argument as in the last section of [BC], observing that, since we take ϕ_0 to be topologically mixing, the preorbit of any of its periodic points is dense in N.

5.5 Conclusion of the proof

Now we are in a position to complete the proof of Theorem C. First, we describe a suitable C^3-open set $\mathcal{A}_0$ of parametrized families of diffeomorphisms unfolding a critical saddle-node 1-cycle. Then we show, using the analysis in the previous sections, that such families satisfy the conclusion of the theorem. We keep the notations of Sections 3.2, 4.2, 5.1 and 5.2.

Let $\tilde{R}_{-\nu}\colon \Delta_r \longrightarrow \Delta_r$ be induced by $(x,y) \mapsto (x-\nu, y-\nu)$ and denote $H_\nu = H \circ \tilde{R}_{-\nu}$. Let $\mathcal{N} \subset C^1(\Delta_r, \Delta_r)$ be as in Theorem 5.1. We fix $\delta > 0$ small enough so that $H_\nu \in \mathcal{N}$ for every $|\nu| \leq \delta$. We also let $\phi_\nu = h \circ R_{-\nu}$, where $R_{-\nu}$ is induced by $x \mapsto x - \nu$. Then $(\phi_\nu)_{\nu \in [-\delta,\delta]}$ is a 2-modal family, recall Sections 5.1 and 5.2. We take $b > 0$ as given by Theorem 5.2 for this family. In the sequel we identify Δ_r with $S^1 \times B$, where $B = [-1,1]$, via the diffeomorphism $\eta\colon \Delta \longrightarrow S^1 \times \mathrm{R}$, $\eta(\pi_2(x,y)) = (\pi_1(x), (y-x)/r)$. Now, given $(f_\mu)_\mu$ a family of diffeomorphisms unfolding a critical saddle-node 1-cycle, we let $\psi(\sigma,\cdot) = \Phi_0^- \circ h_l \circ \tau_\sigma \circ (\Phi_0^-)^{-1} = (g \circ R_{-\sigma}, G \circ R_{-\sigma})$ be as in Section 3.2 and we also write $\gamma_\nu = (g \circ R_{-(1/2+\nu)})$. Then we define $\mathcal{A}_0$ by

$$(11) \qquad (f_\mu)_\mu \in \mathcal{A}_0 \iff \begin{cases} \psi(\sigma,\cdot) \in \mathcal{N} & \text{for all } \sigma \in J = [1/2-\delta, 1/2+\delta] \text{ and} \\ \|\gamma - \phi\| < b/2 & \text{with } \|\cdot\| = C^3\text{-norm over } [-\delta,\delta] \times S^1. \end{cases}$$

The first condition in (11) ensures that, given any $(f_\mu)_\mu \in \mathcal{A}_0$, there is $k_1 \geq 1$ such that $\psi_k(\sigma,\cdot) \in \mathcal{N}$ for every $\sigma \in J$ and $k \geq k_1$. Recall that $\psi_k(\sigma,\cdot) = \Phi_\mu^- \circ h_{k,l,\sigma} \circ (\Phi_\mu^-)^{-1}$, with $h_{k,l,\sigma}$ = map induced by f_μ^{k+l} on $\tilde{C}_\mu^-$, and that $\psi_k \to \psi$ as $k \to \infty$. Here and in what follows we always consider $\mu = \mu_k(\sigma)$, with $\sigma \in J$ and $k \geq k_1$. Let $P_{k,\sigma}$ be the fixed point of $\psi_k(\sigma,\cdot)$ close to $P = (0,0)$. By Theorem 5.1 applied to $\psi_k(\sigma,\cdot)$, given any $z \in \Delta_r \approx S^1 \times B$ such that $\psi_k^{-n}(\sigma, z) \in \Delta_r$ for all $n \geq 0$, there is $(z_i)_i \to z$ with $z_i \in W^u(P_{k,\sigma})$ for all $i \geq 0$. Write $\tilde{\pi}_\mu = \Phi_\mu^- \circ \pi_\mu \colon C_\mu^- \longrightarrow S^1 \times B$, recall Section 3.2, and take $P_\mu = \tilde{\pi}_\mu^{-1}(P_{k,\sigma})$. Clearly, P_μ is a periodic point of f_μ. We claim that $W^u(P_\mu)$ is dense in $A_\mu = \cap_{n\geq 0} f_\mu^n(V)$. In order to justify this claim, we begin by observing that $W^u(P_\mu)$ intersects $W^s(f(P_\mu))$ transversely: this corresponds to transverse homoclinic intersections associated to $P_{k,\sigma}$. As a consequence, $\operatorname{closure}(W^u(P_\mu)) = \operatorname{closure}(W^u(f_\mu^i(P_\mu)))$ for all i. On the other hand, recall the construction of V in Section 4.2,

a) given any $\xi \in V$ there is $p_+ > 0$ such that $f_\mu^{p_+}(\xi) \in C_\mu^-$; if $\xi \in A_\mu$ then there is also $p_- > 0$ such that $f_\mu^{-p_-}(\xi) \in C_\mu^-$;

b) if $\xi \in A_\mu \cap C_\mu^-$ and $p_\pm$ are the minima integers as above, then $p_\pm < \bar{p}$ for some $\bar{p}$ depending only on k and l (not on ξ) and $\tilde{\pi}_\mu(f_\mu^{\pm p_\pm}(\xi)) = \tilde{\pi}_\mu(f_\mu^{\pm(k+l)}(\xi)) = \psi^{\pm 1}(\sigma, \tilde{\pi}_\mu(\xi))$.

In particular, the claim will follow if we show that every $\xi \in A_\mu \cap C_\mu^-$ is accumulated by points ξ_i in the unstable manifold of some iterate of P_μ. We write $z = \tilde{\pi}_\mu(\xi)$ and then take $(z_i)_i \to z$ with $z_i \in W^u(P_{k,\sigma})$, and $\xi_i = \tilde{\pi}_\mu^{-1}(z_i) \in C_\mu^-$. Let i be fixed and, for each $j \geq 0$, denote $w_j = \psi^{-j}(\sigma, z_i)$ and $\eta_j = \tilde{\pi}_\mu^{-1}(w_j)$. The fact that $w_j \to P_{k,\sigma}$ implies that $\eta_j \to P_\mu$ and so the α-limit set of ξ_i contains P_μ. On the other hand, by b), $\eta_j = f_\mu^{p_j}(\eta_{j+1})$ with $0 < p_j < \bar{p}$ for all $j \geq 0$. It follows that $\alpha(\xi)$ consists only of periodic points with period less than $\bar{p}$. Since P_μ is isolated in the set of such points (because it is hyperbolic), we conclude that $\alpha(\xi)$ coincides with the orbit of P_μ. This completes the proof of the claim.

Now we introduce the families $\varphi(\sigma,\cdot)$ and $\varphi_k(\sigma,\cdot)$ of maps of $S^1 \times B_\rho$ given by

$$\varphi(\sigma,\cdot) = \Phi_0^+ \circ \tau_\sigma \circ h_l \circ (\Phi_0^+)^{-1} \quad \text{and} \quad \varphi_k(\sigma,\cdot) = \Phi_\mu^+ \circ h_{k,l,\sigma} \circ (\Phi_\mu^+)^{-1}.$$

Here τ_σ, h_l, and $h_{k,l,\sigma}$ have the same meaning as in Section 3.2 and $\rho > 0$ is fixed small enough so that these maps are well defined. Clearly, $\varphi_k(\sigma,\cdot)$ is conjugate to $\psi_k(\sigma,\cdot)$ by the map $\xi_{k,\sigma} = \Phi_\mu^- \circ \eta_{l,\mu} \circ (\Phi_\mu^+)^{-1}$, where $\eta_{l,\mu} \colon C_\mu^+ \longrightarrow C_\mu^-$ is induced by f_μ^l. The same arguments as in Section 3.2 give that $\varphi(\sigma, x, y) = (g \circ R_{-\sigma}(x), 0) = (\gamma_{\sigma-1/2}(x), 0)$ and $\varphi_k \to \varphi$ as $k \to \infty$. Therefore, the second condition in (11) implies that there is $k_2 \geq 1$ such that $(\varphi_k(1/2+\nu,\cdot))_\nu$ is a nonsingular b-perturbation of $(\phi_\nu)_\nu$ for all $k \geq k_2$. From now on we suppose $k \geq k_0 = \max\{k_1, k_2\}$. We denote $Q_{k,\sigma} = \xi_{k,\sigma}^{-1}(P_{k,\sigma})$, which is the continuation for $\varphi_k(\sigma,\cdot)$ of the fixed point $(0,0)$ of $\hat{\phi}_0$. Now, by Theorem 5.2, there is $S_k \subset [-\delta, \delta]$ with $m(S_k) \geq \chi$ such that, given any $\sigma \in (1/2 + S_k) \subset J$, there is $\bar{z} \in W^u(Q_{k,\sigma})$ a critical point for $\varphi_k(\sigma,\cdot)$ whose orbit is dense in $\operatorname{closure}(W^u(Q_{k,\sigma}))$ and which exhibits exponential growth of the derivative. Clearly, these three features are preserved by C^1 conjugacies and so $\bar{w} = \xi_{k,\sigma}(\bar{w})$ has the analogous properties with respect to $\psi_k(\sigma,\cdot)$ and $\operatorname{closure}(W^u(P_{k,\sigma}))$. Let $\bar{\xi} = \tilde{\pi}_\mu^{-1}(\bar{w}) \in C_\mu^-$. It is easy to check, using a) above together with the arguments in the previous paragraph, that the trajectory of $\bar{\xi}$ is dense in $\operatorname{closure}(W^u(P_\mu))$. Moreover, using b) we also get that $\bar{\xi}$ is a critical point of f_μ and exhibits exponential growth of the derivative.

Altogether, these two last paragraphs show that, given any family $(f_\mu)_\mu \in \mathcal{A}_0$, the maximal invariant set A_μ is a Hénon-like attractor for every μ in $\mathcal{G} = \cup_{k \geq k_0} \mu_k(1/2 + S_k)$. The fact that $\mathcal{G}$ has positive Lebesgue density at $\mu = 0$ is a direct consequence of Proposition 2.2. The proof of Theorem C is now complete.

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L. J. Díaz (lodiaz@mat.puc-rio.br)
Dep. Matemática PUC-RJ, Marquês de S. Vicente 225, 22453-900 Rio de Janeiro, Brazil

J. Rocha (jrocha@fc.up.pt)
Departamento de Matemática Pura, Faculdade de Ciências, 4000 Porto, Portugal

M. Viana (viana@impa.br)
IMPA, Est. D. Castorina 110, Jardim Botânico, 22460-320 Rio de Janeiro, Brazil