Flat Surfaces

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Main Reference

Interval Exchange Transformations and Teichmüller Flows
www.impa.br/ viana/out/ietf.pdf

Rauzy, Keane, Masur, Veech, Hubbard, Kerckhoff, Smillie, Kontsevich, Zorich, Eskin, Nogueira, Rudolph, Forni, Avila, . . .

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Outline

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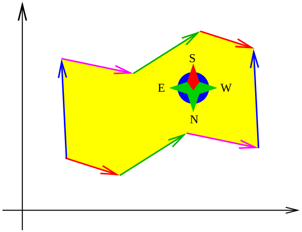
Translation surfaces

Consider any planar polygon with even number of sides, organized in pairs of parallel sides with the same length. Identify sides in the same pair, by translation.



Two translation surfaces are the same if they are isometric.

Structure of translation surfaces

- Riemann surface with a translation atlas
- holomorphic complex differential 1-form $\alpha_z = dz$
- flat Riemann metric with a unit parallel vector

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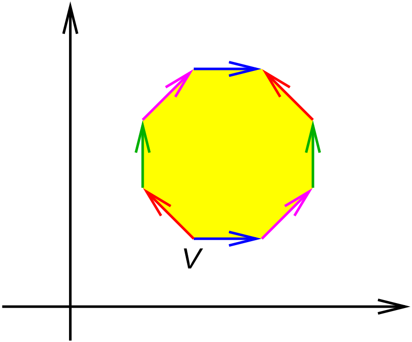
Singularities

Points of the surface arising from the vertices of the polygon may correspond to (conical) singularities of the metric.



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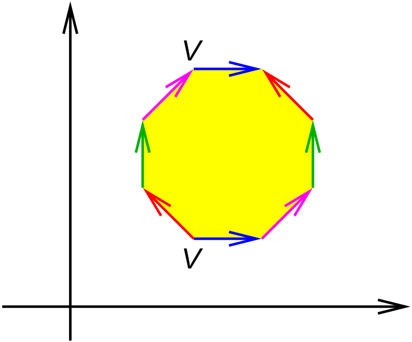

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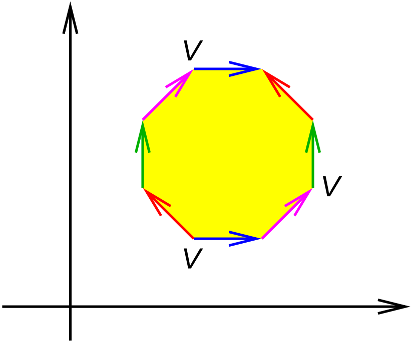

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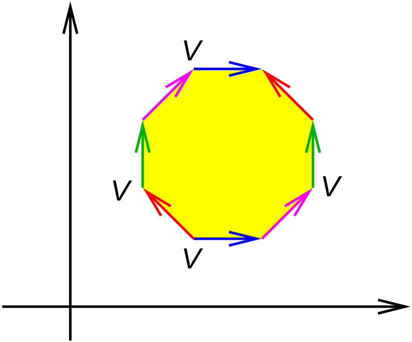

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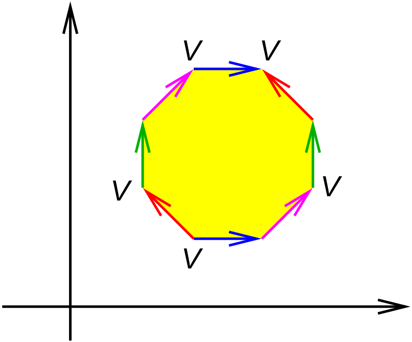

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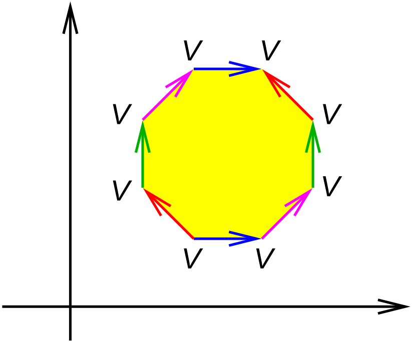
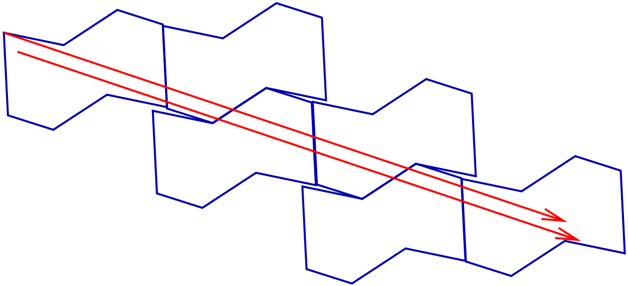


Singularities

In this example the neighborhood of V corresponds to gluing 8 copies of the angular sector of angle $3\pi/4$.

Thus, the total angle of the metric at V is 6π. This singularity corresponds to a zero of order 3 of the complex 1-form α.

Geodesics



We want to understand the behavior of geodesics with a given direction. In particular,

- When are the geodesics closed ?
- When are they dense in the surface ?

Motivation

A *quadratic differential* on a Riemann surface assigns to each point a complex quadratic form from the tangent space, depending holomorphically on the point.

In local coordinates z, it is given by some $\varphi(z)dz^2$ where $\varphi(z)$ is a holomorphic function. The expression $\psi(w)dw^2$ relative to another local coordinate w satisfies

$$\psi(w) = \varphi(z)\left(\frac{dz}{dw}\right)^2$$

on the intersection of the domains.

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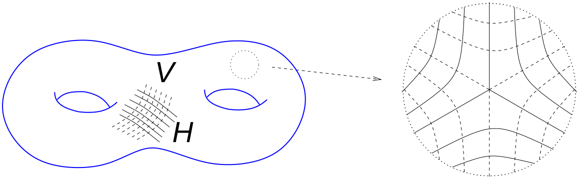
Motivation

Quadratic differentials play a central role in the theory of Riemann surfaces, in connection with understanding the deformations of the holomorphic structure.



A vector v is vertical if $q_z(v) > 0$ and it is horizontal if $q_z(v) < 0$.

Motivation

The square of a holomorphic complex 1-form $\alpha_z = \phi(z)dz$ is always a quadratic differential.

Every quadratic differential q is locally the square of a complex 1-form α. Moreover, we may find a two-to-one covering $\tilde{S} \to S$, ramified over some singularities, such that the lift of q to $\tilde{S}$ is the square of a complex 1-form α, globally.

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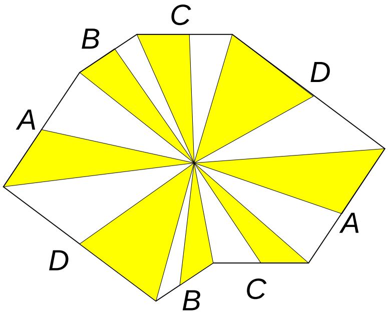
Calculating the genus



$$2 - 2g = F - A + V$$

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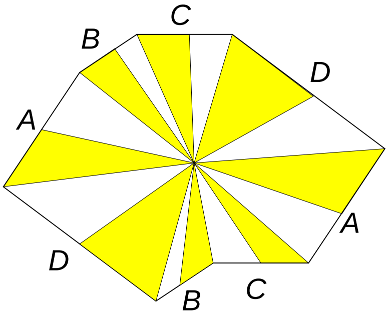

$$2 - 2g = 4d - A + V, \qquad 2d = \# \text{ sides}$$

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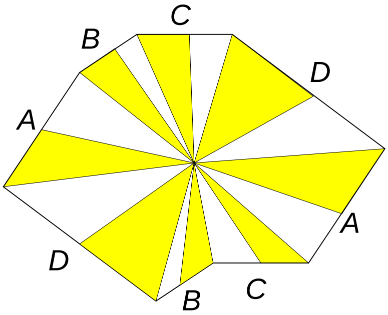

$$2 - 2g = 4d - 6d + V, \qquad 2d = \# \text{ sides}$$

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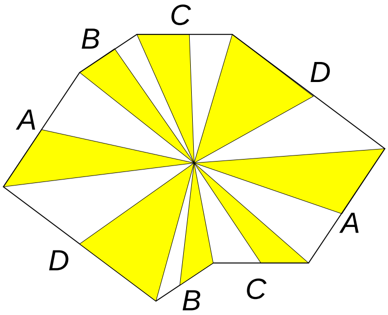
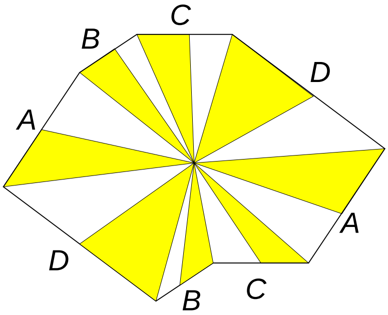


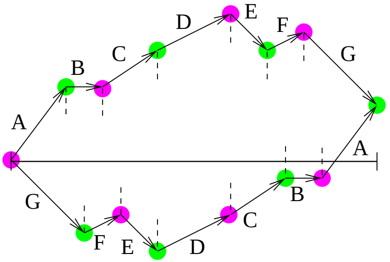
$$2 - 2g = 4d - 6d + (d + \kappa + 1), \qquad \kappa = \#\ \text{singularities}$$

Calculating the genus



$2g = d - \kappa + 1$, where $2d = \#$ sides and $\kappa = \#$ singularities.

Computing the singularities



Let $m+1 = 1/2$ the number of associated "interior" vertices:

$2\pi(m+1) =$ conical angle $\quad m =$ multiplicity of the singularity

Strata

Let $m_1, \dots, m_\kappa$ be the multiplicities of the singularities:

$$(m_1 + 1) + \cdots + (m_\kappa + 1) = d - 1 = 2g + \kappa - 2.$$

Gauss-Bonnet formula

$m_1 + \ldots + m_\kappa = 2g - 2$

$\mathcal{A}_g(m_1, \dots, m_\kappa)$ denotes the space of all translation surfaces with κ singularities, having multiplicities m_i.

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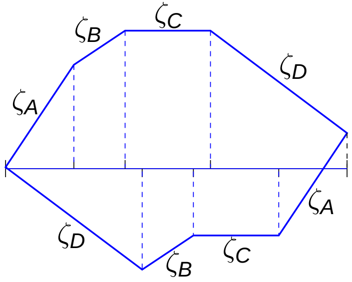
Strata

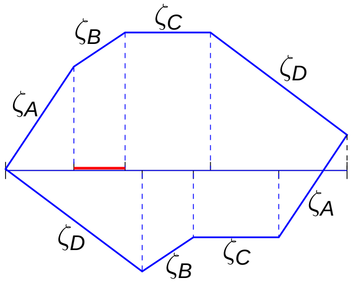
Each stratum $\mathcal{A}_g(m_1, \ldots, m_\kappa)$ is an orbifold of dimension $2d = 4g + 2\kappa - 2$. Local coordinates:



$\zeta_\alpha = (\lambda_\alpha, \tau_\alpha)$ together with $\pi =$ combinatorics of pairs of sides

Outline

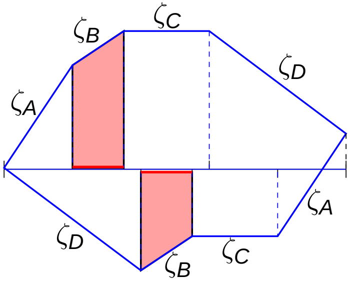
Cross-sections to vertical flow



The return map of the vertical geodesic flow to some cross-section is an interval exchange transformation.

To analyze the behavior of longer and longer geodesics, we consider return maps to shorter and shorter cross-sections.

Cross-sections to vertical flow



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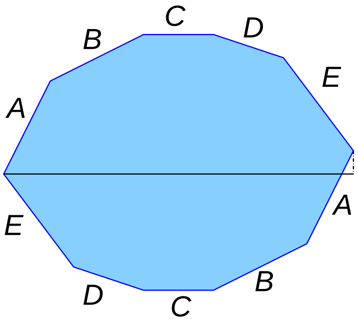
Shortening the cross-section



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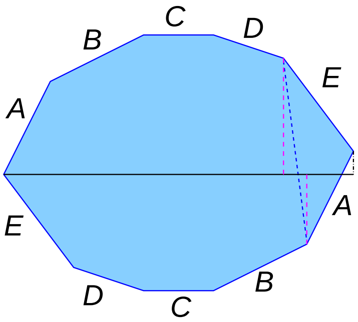

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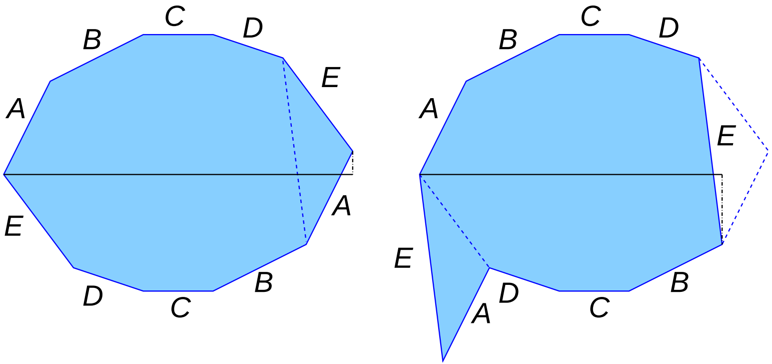


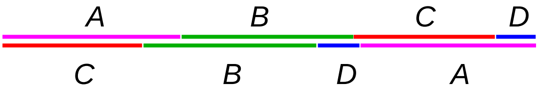
Interval exchanges

Interval exchanges are described by combinatorial data

$$\pi = \begin{pmatrix} A & B & C & D & E \\ E & D & C & B & A \end{pmatrix}$$

and metric data $\lambda = (\lambda_A, \lambda_B, \lambda_C, \lambda_D, \lambda_E)$.

Rauzy induction



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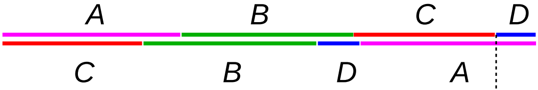

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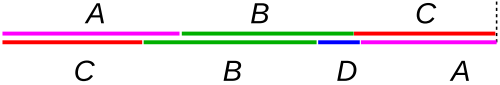
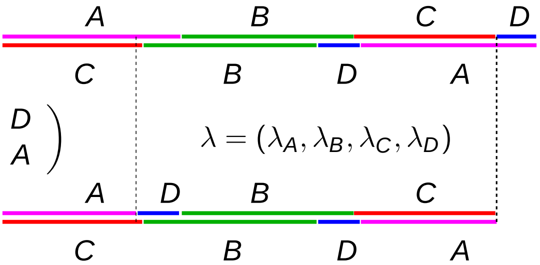


This is a “bottom” case: of the two rightmost intervals, the bottom one is longest.

The Rauzy Algorithm



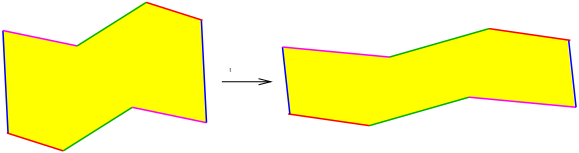
$$\pi = \begin{pmatrix} A & B & C & D \\ C & B & D & A \end{pmatrix} \qquad \lambda = (\lambda_A, \lambda_B, \lambda_C, \lambda_D)$$

$$\pi' = \begin{pmatrix} A & D & B & C \\ C & B & D & A \end{pmatrix} \qquad \lambda' = ({\color{red}\lambda_A - \lambda_D}, \lambda_B, \lambda_C, \lambda_D)$$

Teichm ¨uller fbw

The Teichmüller flow is the action induced on the stratum by the diagonal subgroup of $SL(2,\mathbb{R})$. In coordinates:

$$(\pi, \lambda, \tau) \mapsto (\pi, e^t \lambda, e^{-t} \tau)$$



Both the Teichmüller flow and the induction operator preserve the area. We always suppose area 1.

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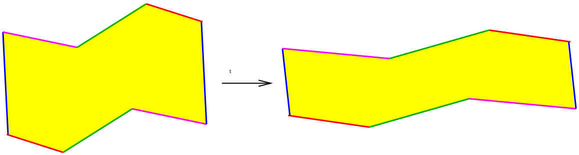

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Rauzy renormalization

The Rauzy renormalization operator is the composition of

- the Rauzy induction (this reduces the width of the surface/length of the cross-section)
- the Teichmüller (the right time to restore the width of the surface/length of the cross-section back to the initial value)

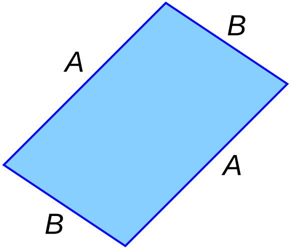
It is defined both on the space of translation surfaces

$$\mathcal{R} : (\pi, \lambda, \tau) \mapsto (\pi'', \lambda'', \tau'')$$

and on the space of interval exchange transformations

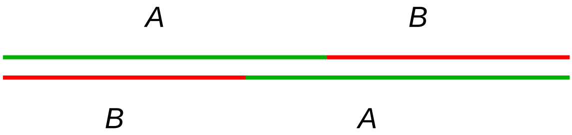
$$R : (\pi, \lambda) \mapsto (\pi'', \lambda'').$$

Flat torus



$$d = 2 \quad \kappa = 1 \quad m = 0 \text{ (removable!)} \quad g = 1$$

Exchanges of two intervals



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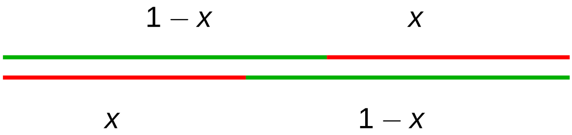

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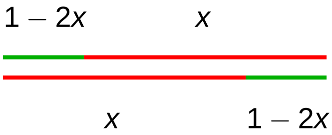

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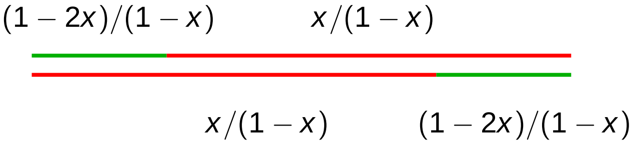
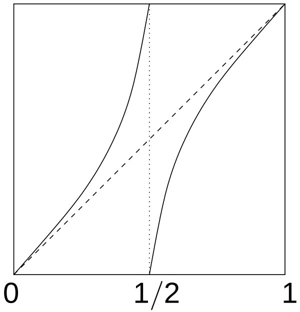


Renormalization for $d = 2$

$$R(x) = \begin{cases} x/(1-x) & \text{for } x \in (0, 1/2) \\ 2 - 1/x & \text{for } x \in (1/2, 1). \end{cases}$$



Outline

Invariant measures

Theorem (Masur, Veech, 1982)

There is a natural finite volume measure on each stratum, invariant under the Teichmüller flow. This measure is ergodic (restricted to the hypersurface of surfaces with unit area).

Ergodic means that almost every orbit spends in each subset of the stratum a fraction of the time equal to the volume of the subset.

In particular, almost every orbit is dense in the stratum.

Invariant measures

In coordinates, this invariant volume measure is given by $d\pi d\lambda d\tau$ where $d\pi$ is the counting measure

Theorem (Veech, 1982)

The renormalization operator R admits an absolutely continuous invariant measure absolutely continuous with respect to $d\lambda$. This measure is ergodic and unique up to rescaling.

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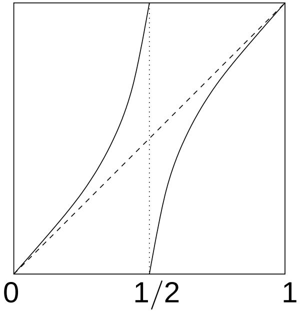
An example

For $d = 2$, the operator R is given by



The absolutely continuous invariant measure is infinite!

Minimality

Theorem (Keane, 1975)

If λ is rationally independent then the interval exchange defined by (π, λ) is minimal: every orbit is dense.

Keane conjecture

Theorem (Masur, Veech, 1982)

For every π and almost every λ, the interval exchange defined by (π, λ) is uniquely ergodic: it admits a unique invariant probability.

This result is a consequence of the ergodicity of the renormalization operator.

Unique ergodicity

Theorem (Kerckhoff, Masur, Smillie, 1986)

For every polygon, and almost every direction, the (translation) geodesic flow in that direction is uniquely ergodic.

Linear cocycles

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