Lyapunov exponents of Teichmüller flows

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Lecture # 1

Geodesic flows on translation surfaces

Abelian differentials

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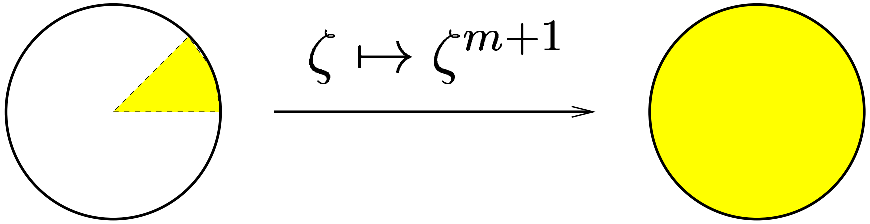
Adapted coordinates form a translation atlas: coordinate changes near any regular point have the form

$$\zeta' = \zeta + \text{constant}$$

Translation surfaces

The translation atlas defines

- a flat metric with a finite number of conical singularities:

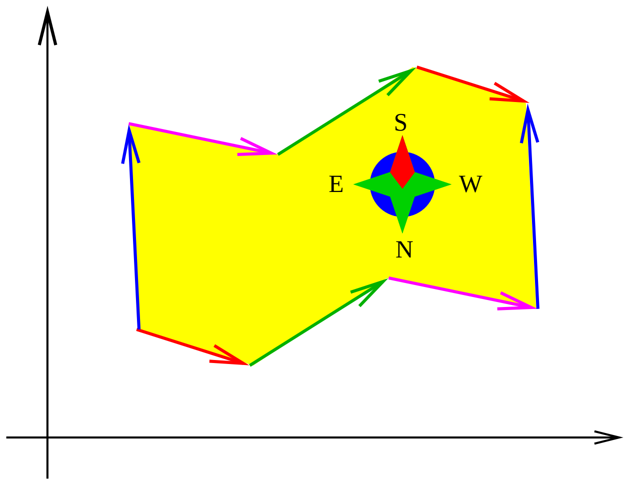


- a parallel unit vector field (the “upward” direction) on the complement of the singularities.

Conversely, the flat metric and the parallel unit vector field characterize the translation structure completely.

Geometric representation

Consider any planar polygon with even number of sides, organized in pairs of parallel sides with the same length.



Identifying sides in the same pair, by translation, yields a translation surface. Singularities arise from the vertices.

Every translation surface can be represented in this way, but not uniquely.

Moduli spaces

$\mathcal{A}_g =$ moduli space of Abelian differentials on Riemann surfaces of genus $g \geq 2$

- a complex orbifold of dimension $\dim_{\mathbb{C}} \mathcal{A}_g = 4g - 3$
- a fiber bundle over the moduli space $\mathcal{M}_g$ of Riemann surfaces of genus $g \geq 2$

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$\mathcal{A}_g(m_1, \ldots, m_\kappa) =$ stratum of Abelian differentials having κ zeroes, with multiplicities $m_1, \ldots, m_\kappa$

$$\sum_{i=1}^{\kappa} m_i = 2g - 2.$$

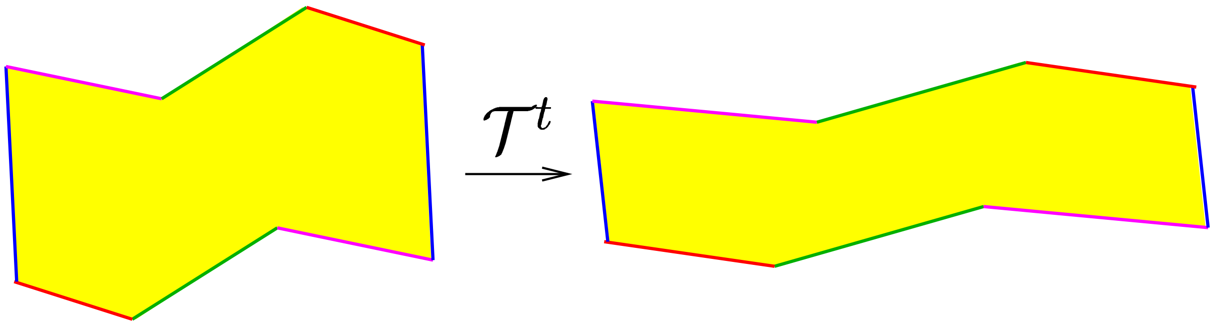
- complex orbifold with $\dim_{\mathbb{C}} \mathcal{A}_g(m_1, \ldots, m_\kappa) = 2g + \kappa - 1$

Teichmüller geodesic flow

The Teichmüller flow is the natural action $\mathcal{T}^t$ on the fiber bundle $\mathcal{A}_g$ by the diagonal subgroup of $\mathrm{SL}(2,\mathbb{R})$:

$$(\mathcal{T}^t\alpha)_z = e^t(\Re\alpha_z) + ie^{-t}(\Im\alpha_z)$$

Geometrically:



The area of the surface S is constant on orbits of this flow.

Ergodicity

Masur, Veech:

Each stratum of $\mathcal{A}_g$ carries a canonical volume measure, which is finite and invariant under the Teichmüller flow.

The Teichmüller flow $\mathcal{T}^t$ is ergodic on every connected component of every stratum, restricted to any hypersurface of constant area.

Kontsevich, Zorich:

Classified the connected components of all strata. Each stratum has at most 3 connected components.

Eskin, Okounkov, Pandharipande

Computed the volumes of all the strata.

Lyapunov exponents

Fix a connected component of a stratum. The Lyapunov spectrum of the Teichmüller flow has the form

$$2 \geq 1+\nu_2 \geq \cdots \geq 1+\nu_g \geq 1 = \cdots = 1 \geq 1-\nu_g \geq \cdots \geq 1-\nu_2 \geq 0 \geq$$
$$\geq -1+\nu_2 \geq \cdots \geq -1+\nu_g \geq -1 = \cdots = -1 \geq -1-\nu_g \geq \cdots \geq -1-\nu_2 \geq -2$$

Theorem (Avila, Viana). $1 > \nu_2 > \cdots > \nu_g > 0.$

Conjectured by Zorich, Kontsevich.

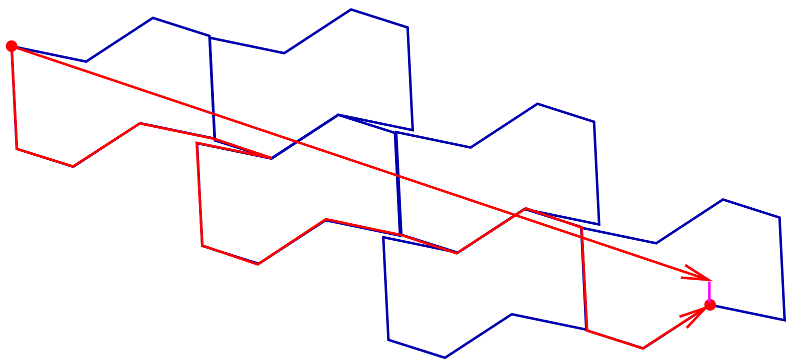
Veech proved $\nu_2 < 1$ (flow is non-uniformly hyperbolic)

Forni proved $\nu_g > 0$ (exactly $2\kappa - 2$ exponents $= \pm 1$)

Before discussing the proof, let us describe an application.

Asymptotic cycles

Given any long geodesic segment γ in a given direction, "close" it to get an element $h(\gamma)$ of $H_1(S, \mathbb{Z})$:



Kerckhoff, Masur, Smillie: The geodesic flow in almost every direction is uniquely ergodic.

Then $h(\gamma)/|\gamma|$ converges uniformly to some $c_1 \in H_1(S, \mathbb{R})$ when the length $|\gamma| \to \infty$, where the asymptotic cycle c_1 depends only on the surface and the direction.

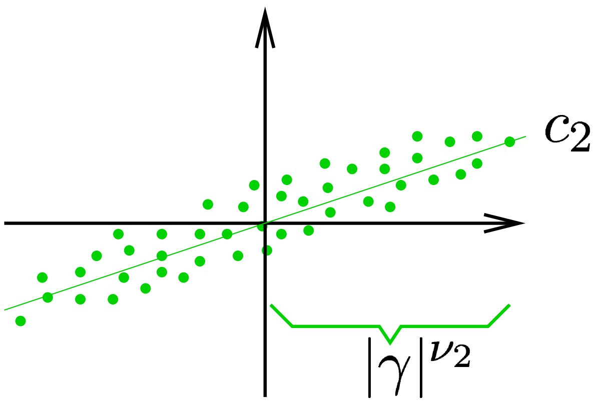
Asymptotic flag in homology

Corollary (Zorich). *There are subspaces* $L_1 \subset L_2 \subset \cdots \subset L_g$ *of* $H_1(S,\mathbb{R}) \simeq \mathbb{R}^{2g}$ *with* $\dim L_i = i$*, such that*

- *the deviation of* $h(\gamma)$ *from* L_i *has amplitude* $|\gamma|^{\nu_{i+1}}$ *for all* $i < g$

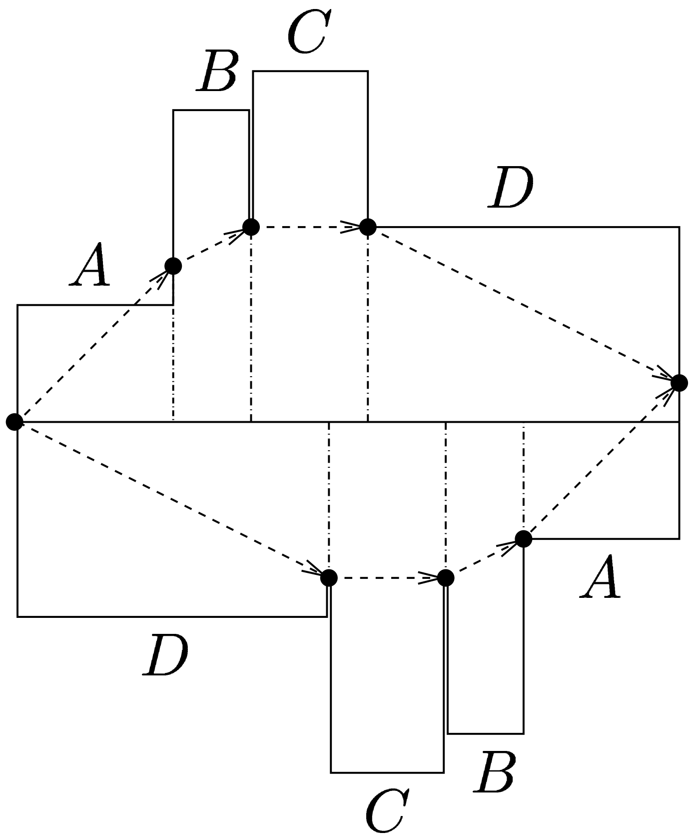
$$\limsup_{|\gamma|\to\infty} \frac{\log \operatorname{dist}(h(\gamma), L_i)}{\log|\gamma|} = \nu_{i+1}$$

- *the deviation of* $h(\gamma)$ *from* L_g *is bounded* *(g = genus of S).*



Zippered rectangles

Almost every translation surface may be represented in the form of zippered rectangles (minimal number of rectangles required is $d = 2g - \kappa - 1$):



Coordinates in the stratum

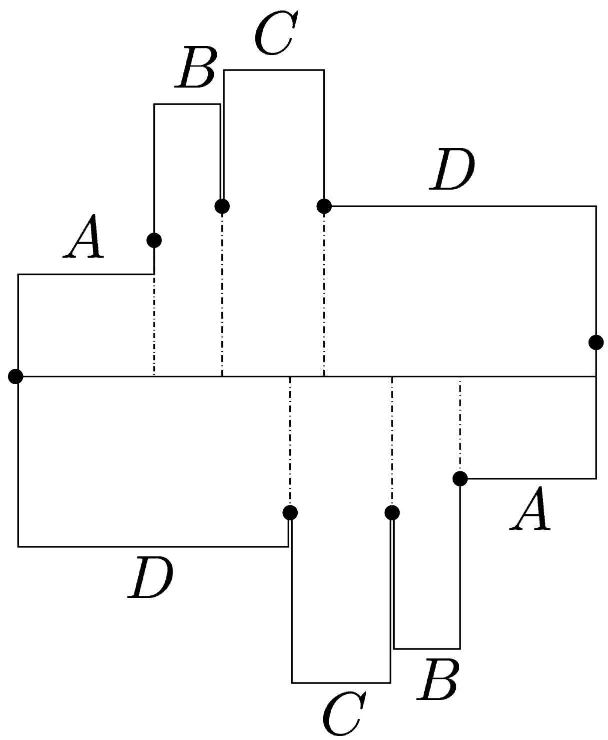
This defines local coordinates (π, λ, τ, h) in the stratum:

- $\pi = \begin{pmatrix} A & B & C & D \\ D & C & B & A \end{pmatrix}$ describes the combinatorics of the associated interval exchange map
- $\lambda = (\lambda_A, \lambda_B, \lambda_C, \lambda_D)$ are the horizontal coordinates of the saddle-connections (= widths of the rectangles)
- $\tau = (\tau_A, \tau_B, \tau_C, \tau_D)$ are the vertical components of the saddle-connections

  $\lambda + i\tau$ has complex dimension $d = 2g + \kappa - 1$
- The heights $h = (h_A, h_B, h_C, h_D)$ of the rectangles are linear functions of τ

Poincaré return map - 1st step

We consider the return map of the Teichmüller flow to the cross section $\{\sum_\alpha \lambda_\alpha = 1\}$.



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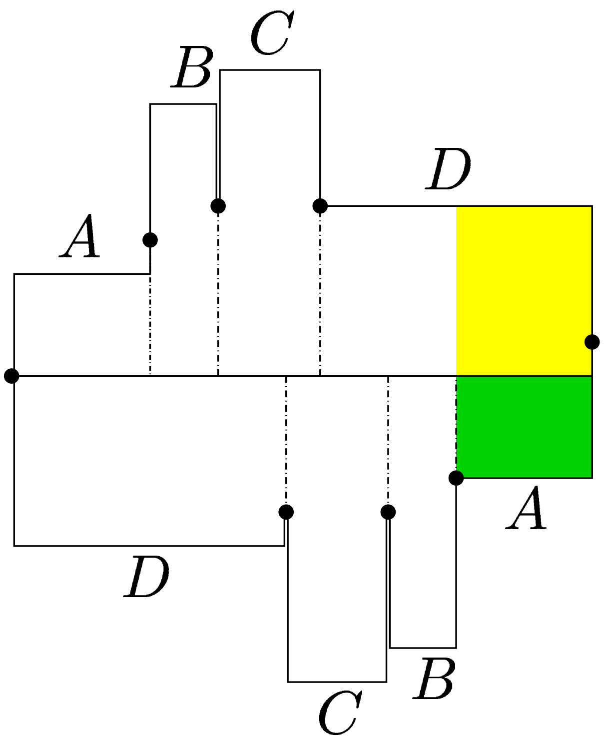

In this example: the top rectangle is the widest of the two.

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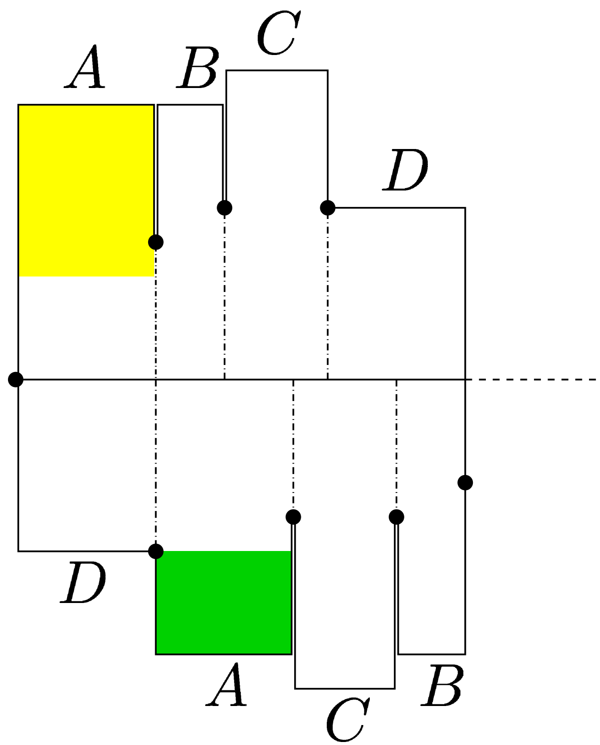


Rauzy-Veech induction

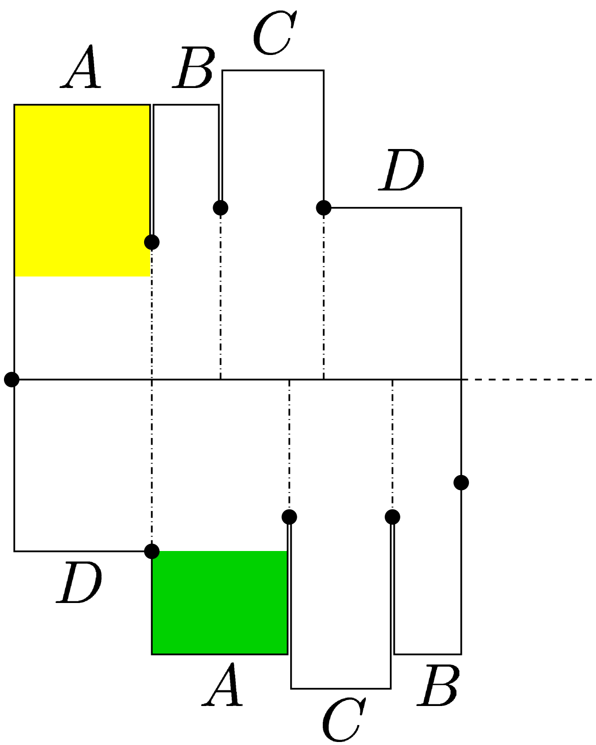
This 1st step corresponds to $(\pi, \lambda, \tau, h) \mapsto (\pi', \lambda', \tau', h')$, with

- $\pi' = \begin{pmatrix} A & B & C & D \\ D & A & C & B \end{pmatrix}$ (top case)
- $\lambda'_\alpha = \lambda_\alpha$ except that $\lambda'_D = \lambda_D - \lambda_A$
- same for τ
- $h'_\alpha = h_\alpha$ except that $h'_A = h_A + h_D$

Write $h' = \Theta(h)$ where $\Theta = \Theta_{\pi,\lambda}$ is the linear operator defined in this way. Then $\lambda = \Theta^*(\lambda')$ and analogously for τ.

Poincaré return map - 2nd step

We consider the return map of the Teichmüller flow to the cross section $\{\sum_\alpha \lambda_\alpha = 1\}$.



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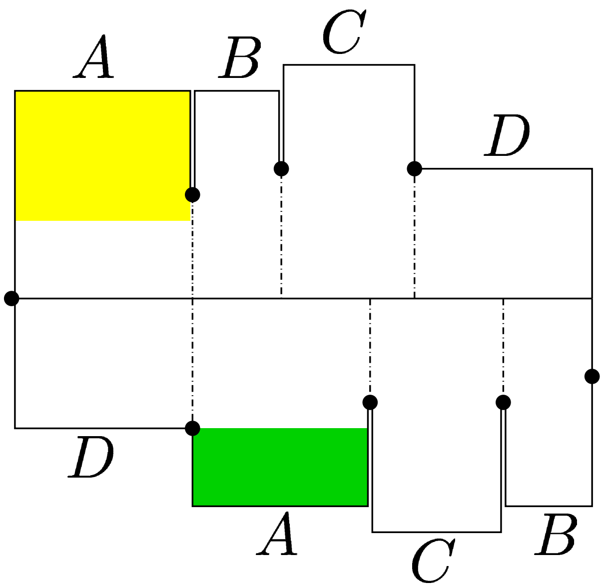

Rauzy-Veech renormalization

This 2nd step corresponds to $(\pi', \lambda', \tau', h') \mapsto (\pi'', \lambda'', \tau'', h'')$, with

- $\pi'' = \pi'$
- $\lambda'' = \lambda'/c$
- $\tau'' = c\tau'$
- and $h'' = ch'$

where $c = \sum_\alpha \lambda'_\alpha$ is the normalizing factor.

The Poincaré return map $\mathcal{R} : (\pi, \lambda, \tau) \mapsto (\pi'', \lambda'', \tau'')$ is called (invertible) Rauzy-Veech renormalization.

Rauzy-Veech cocycle and long geodesics

Consider the linear cocycle over the map $\mathcal{R}$ defined by

$$F_{\mathcal{R}} : (\pi, \lambda, \tau, h) \mapsto (\mathcal{R}(\pi, \lambda, \tau), \Theta(h)), \qquad \Theta = \Theta_{\pi,\lambda}.$$

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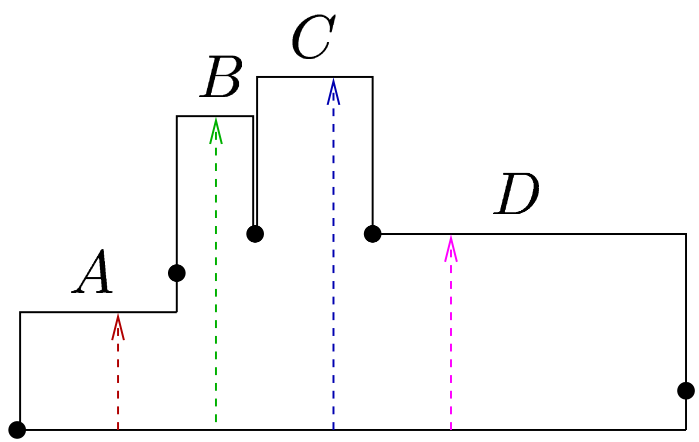

Consider a geodesic segment crossing each rectangle α vertically. “Close” it by joining the endpoints to some fixed point in the cross-section. This defines some $v_\alpha \in H_1(S, \mathbb{Z})$.

Rauzy-Veech cocycle and long geodesics

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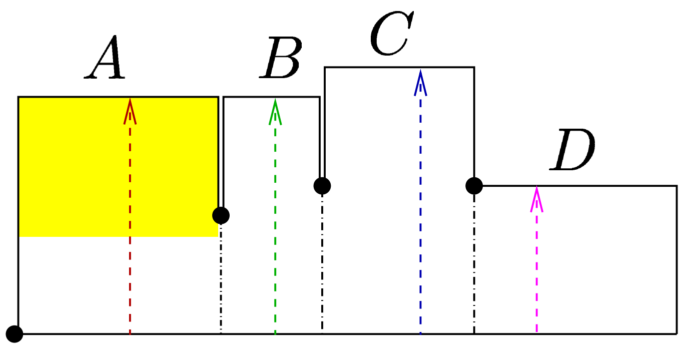

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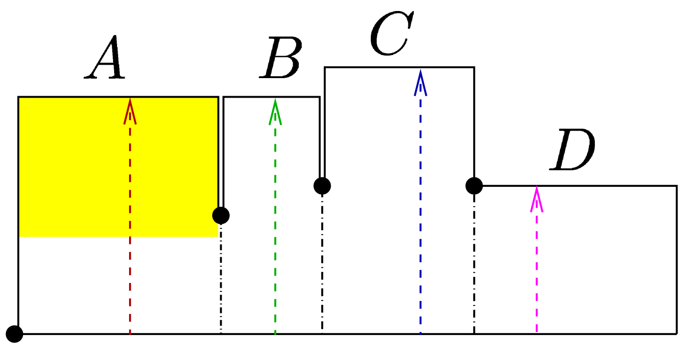

- $v'_\alpha = v_\alpha$ except that $v'_A = v_A + v_D$

In other words, $v' = \Theta(v)$. $F_{\mathcal{R}}$ "builds" long geodesics.

Lyapunov exponents and asymptotic flag

So, it seems likely that

- the behavior of long geodesics be described by the asymptotics of the Rauzy-Veech cocycle, the flag $L_1 \subset L_2 \subset \cdots \subset L_g$ being defined by the Oseledets subspaces of $F_{\mathcal{R}}$ with positive Lyapunov exponents.
- the Lyapunov exponents of $F_{\mathcal{R}}$ (for an appropriate invariant measure) be related to the Lyapunov exponents of the Teichmüller flow.

This suggests we should try to prove that $F_{\mathcal{R}}$ has g positive Lyapunov exponents and they are all distinct. There is one technical difficulty, however:

Zorich cocycle

The Rauzy-Veech renormalization does have a natural invariant measure (on each stratum) related to the invariant volume of the Teichmüller flow. But this measure is infinite.

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Zorich introduced an accelerated renormalization and an accelerated cocycle

$$\mathcal{Z}(\pi,\lambda,\tau)=\mathcal{R}^n(\pi,\lambda,\tau) \quad \text{and} \quad F_{\mathcal{Z}}(\pi,\lambda,\tau,h)=F^n_{\mathcal{R}}(\pi,\lambda,\tau,h)$$

where $n=n(\pi,\lambda)$ is smallest such that the Rauzy iteration changes from “top” to “bottom” or vice-versa.

This map $\mathcal{Z}$ has a natural invariant volume probability (on each stratum) and this probability is ergodic.

Zorich cocycle

The Lyapunov spectrum of the Zorich cocycle $F_{\mathcal{Z}}$ has the form

$$\theta_1 \geq \cdots \geq \theta_g \geq 0 = \cdots = 0 \geq -\theta_g \geq \cdots \geq -\theta_1$$

and it is related to the spectrum of the Teichmüller flow by

$$\nu_i = \frac{\theta_i}{\theta_1} \quad i = 2, \ldots, g.$$

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Theorem (Avila, Viana). $\theta_1 > \theta_2 > \cdots > \theta_g > 0.$

Forni: proved $\theta_g > 0$ (cocycle is non-uniformly hyperbolic)

Lecture # 2

Simplicity criterium for Lyapunov spectra

Linear cocycles

A linear cocycle over a map $f : M \to M$ is an extension

$$F : M \times \mathbb{R}^d \to M \times \mathbb{R}^d, \quad F(x, v) = \big(f(x), A(x)v\big)$$

where $A : M \to \mathrm{GL}(d, \mathbb{R})$. Note $F^n(x, v) = (f^n(x), A^n(x)v)$ with $A^n(x) = A(f^{n-1}(x)) \cdots A(f(x)) \cdot A(x)$.

Let μ be an f-invariant ergodic probability on M such that $\log \|A^{\pm 1}\| \in L^1(\mu)$. Then the Oseledets theorem applies.

We say F has simple Lyapunov spectrum if all subspaces in the Oseledets decomposition are 1-dimensional.

Hypotheses

Assume that

- $f : \Sigma \to \Sigma$ is the shift map on $\Sigma = \Lambda^{\mathbb{Z}}$, where the alphabet Λ is at most countable

Denote

$$\Sigma^+ = \Lambda^{\{n \geq 0\}} \qquad W^s_{loc}(x) = \{y \in \Sigma : y_n = x_n \text{ for all } n \geq 0\}$$

$$\Sigma^- = \Lambda^{\{n < 0\}} \qquad W^u_{loc}(x) = \{y \in \Sigma : y_n = x_n \text{ for all } n < 0\}$$

Denote by $[i_m, \ldots, i_{-1} : i_0, \ldots, i_n]$ the cylinder of sequences $x \in \Sigma$ such that $x_j = i_j$ for all $j = m, \ldots, -1, 0, \ldots, n$.

Hypotheses

Assume that

- the ergodic probability μ has bounded distortion: μ is positive on cylinders and there exists $C > 0$ such that

$$\frac{1}{C} \leq \frac{\mu([i_m, \ldots, i_{-1} : i_0, i_1, \ldots, i_n])}{\mu([i_m, \ldots, i_{-1}])\mu([i_0, i_1, \ldots, i_n])} \leq C$$

for every $i_m, \ldots, i_0, \ldots, i_n$ and every $m < 0 \leq n$.

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  for every $i_m, \ldots, i_0, \ldots, i_n$ and every $m < 0 \leq n$.

- A is locally constant: $A(\ldots, i_{-1}, i_0, i_1, \ldots) = A(i_0)$

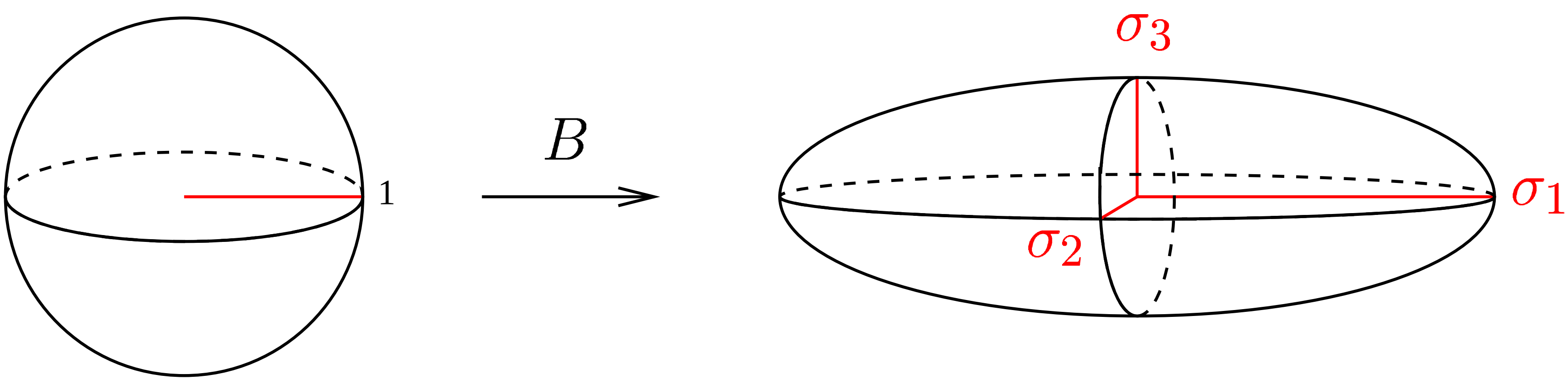
  More generally: A is continuous and admits stable and unstable holonomies.

Pinching

Let $\mathcal{B} \subset \mathrm{GL}(d, \mathbb{R})$ be the monoid generated by the $A(i)$, $i \in \Lambda$. We call the cocycle F

- (pinching) if for every $C > 0$ there exists some $B \in \mathcal{B}$ such that the ℓ-eccentricity

$$\mathrm{Ecc}(\ell, B) := \frac{\sigma_\ell}{\sigma_{\ell+1}} > C \quad \text{for every } 1 \leq \ell < d.$$



Twisting

Let $\mathcal{B} \subset \mathrm{GL}(d, \mathbb{R})$ be the monoid generated by the $A(i)$, $i \in \Lambda$. We call the cocycle F

- twisting if for any $F \in \mathrm{Grass}(\ell, \mathbb{R}^d)$ and any finite family $G_1, \ldots, G_N \in \mathrm{Grass}(d - \ell, \mathbb{R}^d)$ there exists some $B \in \mathcal{B}$ such that $B(F) \cap G_i = \{0\}$ for all $j = 1, \ldots, N$.

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- simple if F is both pinching and twisting.

Lemma. *$\mathcal{B}$ is simple if and only if $\mathcal{B}^{-1} = \{B^{-1} : B \in \mathcal{B}\}$ is simple.*

Criterium for simplicity

Theorem 1 (Bonatti-V, Avila-V). *If the cocycle F is simple then its Lyapunov spectrum is simple.*

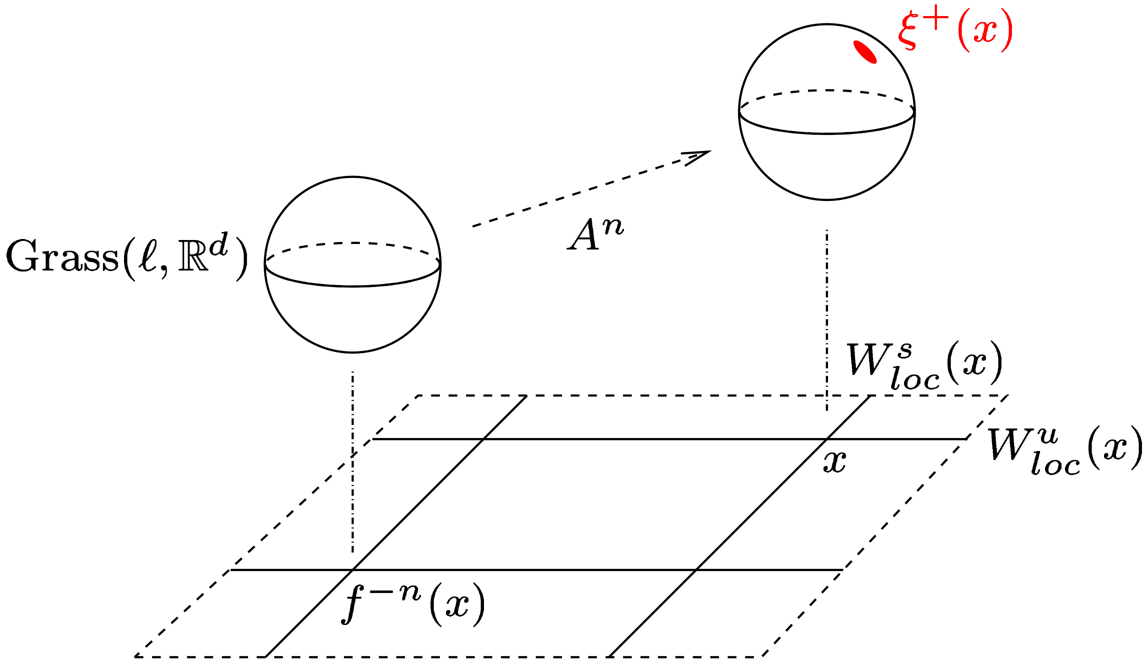
Guivarc'h, Raugi obtained a simplicity condition for products of independent random matrices, and this was improved by Gol'dsheid, Margulis.

Later we shall apply this criterium to the Zorich cocycles.

Right now let us describe some main ideas in the proof.

Comments on the strategy

If the first ℓ exponents are strictly larger, the corresponding subbundle $\{(x, \xi^+(x)) : x \in \Sigma\}$ is an "attractor" for the action of F on the Grassmannian bundle $\Sigma \times \mathrm{Grass}(\ell, \mathbb{R}^d)$:

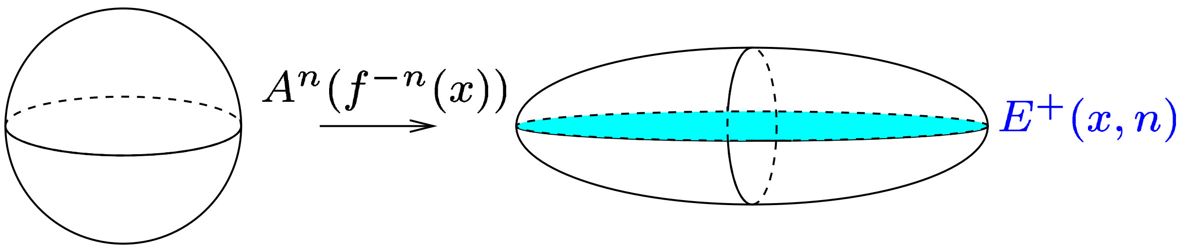


Notice that $\xi^+(x)$ is constant on local unstable sets $W^u_{loc}(x)$.

Invariant section

Proposition 1. *Fix ℓ. Assume the cocycle F is simple. Then there exists a measurable section $\xi^+ : \Sigma \to \mathrm{Grass}(\ell, \mathbb{R}^d)$ such that*

1. *ξ^+ is constant on local unstable sets and F-invariant, that is, $A(x)\xi^+(x) = \xi^+(f(x))$ at μ-almost every point.*
2. *$\mathrm{Ecc}(\ell, A^n(f^{-n}(x))) \to \infty$ and the image $E^+(x,n)$ of the most expanded ℓ-subspace converges to $\xi^+(x)$*
3. *for any $V \in \mathrm{Grass}(d-\ell, \mathbb{R}^d)$, we have that $\xi^+(x)$ is transverse to V at μ-almost every point.*



Proof of Theorem 1

- Applying the proposition to the inverse cocycle and to the dimension $d - \ell$, we find another invariant section $\xi^- : \Sigma \to \mathrm{Grass}(d - \ell, \mathbb{R}^d)$ with similar properties for F^{-1}.

  Recall $\mathcal{B}$ is simple if and only if $\mathcal{B}^{-1}$ is simple.

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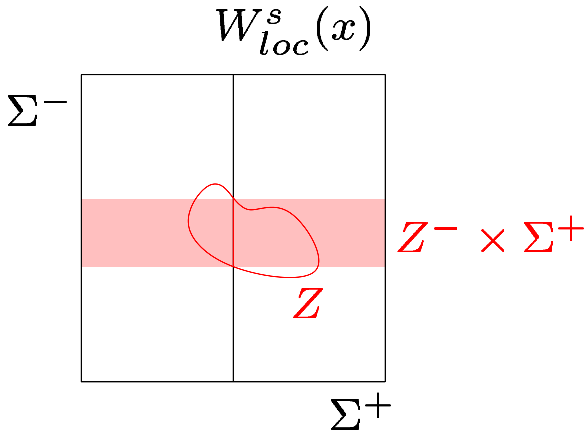
  Recall $\mathcal{B}$ is simple if and only if $\mathcal{B}^{-1}$ is simple.

- Keeping in mind that ξ^+ is constant on unstable sets and ξ^- is constant on stable sets, part 3 implies that $\xi^+(x) \oplus \xi^-(x) = \mathbb{R}^d$ at μ-almost every point.

Proof of $\xi^+(x) \oplus \xi^-(x) = \mathbb{R}^d$



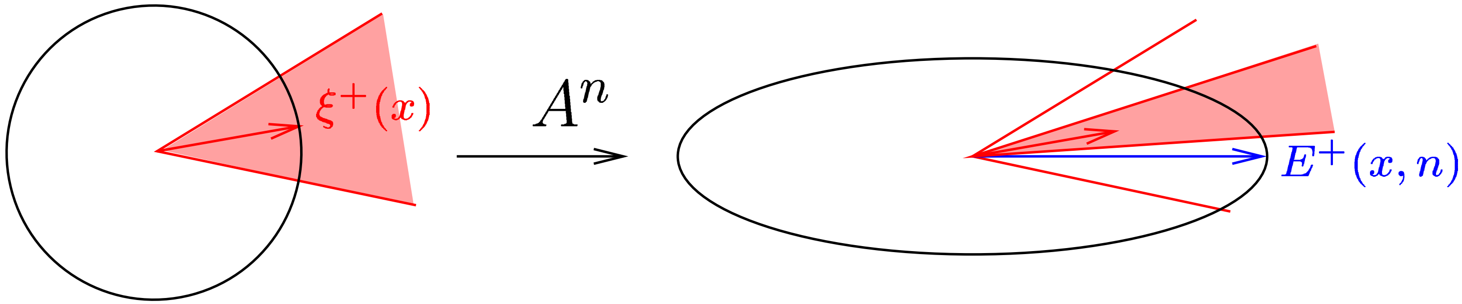
Suppose the claim fails on a set $Z \subset \Sigma$ with $\mu(Z) > 0$.

By local product structure, there exist points $x \in \Sigma$ such that $Z^- \times \Sigma^+$ has positive μ-measure, where $Z^- = W^s_{loc}(x) \cap Z$.

Define $V = \xi^-(x)$. Then $\xi^-(y) = V$ for all $y \in W^s_{loc}(x)$ and ξ^+ is not transverse to V on $Z^- \times \Sigma^+$. This contradicts part 3.

Proof of Theorem 1

- From part 2 we get that the Lyapunov exponents of $F \mid \xi^+$ are strictly larger than the ones of $F \mid \xi^-$.



Using that $E^+(x,n)$ is close to $\xi(x)$ for large n, one finds an invariant cone field around ξ^+ for the cocycle induced by F on some convenient subset (with large return times).

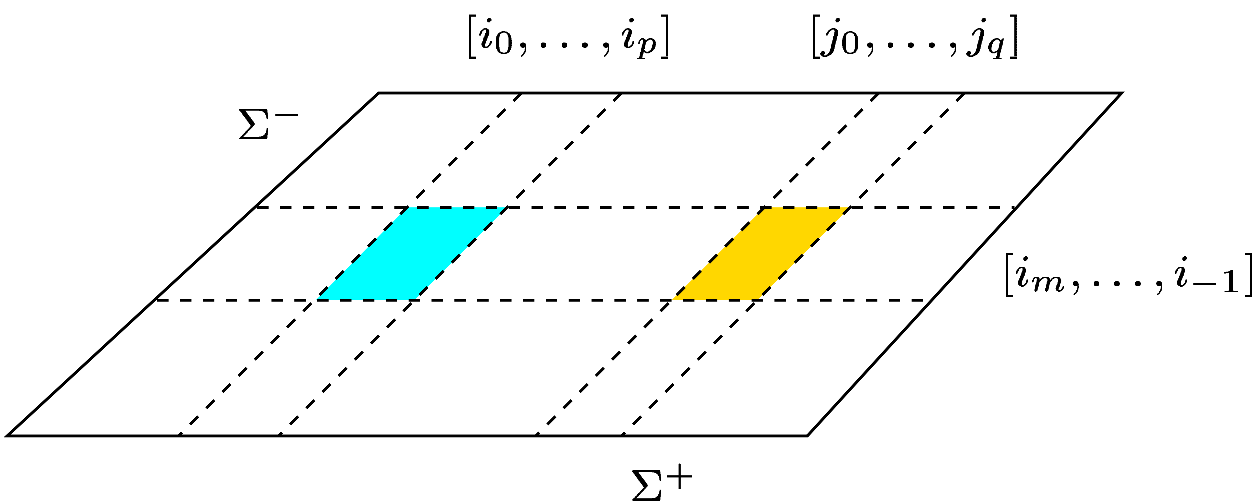
Hence, there are ℓ Lyapunov exponents (for either cocycle) which are larger than the other ones. Theorem 1 follows.

u-states

A probability m on $\Sigma \times \mathrm{Grass}(\ell, \mathbb{R}^d)$ is a u-state if it projects down to μ and there is $C > 0$ such that

$$\frac{m([i_m, \ldots, i_{-1} : i_0, \ldots, i_p] \times X)}{\mu([i_0, \ldots, i_p])} \leq C \frac{m([i_m, \ldots, i_{-1} : j_0, \ldots, j_q] \times X)}{\mu([j_0, \ldots, j_q])}$$

for every $i_m, \ldots, i_0, \ldots, i_p, j_0, \ldots, j_q$ and $X \subset \mathrm{Grass}(\ell, \mathbb{R}^d)$.



u-states

Equivalently, m is a u-state iff it may be disintegrated as

$$m = \int_\Sigma m_x \, d\mu(x), \qquad m_x \text{ on } \operatorname{Grass}(\ell, \mathbb{R}^d),$$

where m_x is equivalent to m_y whenever $x \in W^u_{loc}(y)$, with derivative uniformly bounded by C.

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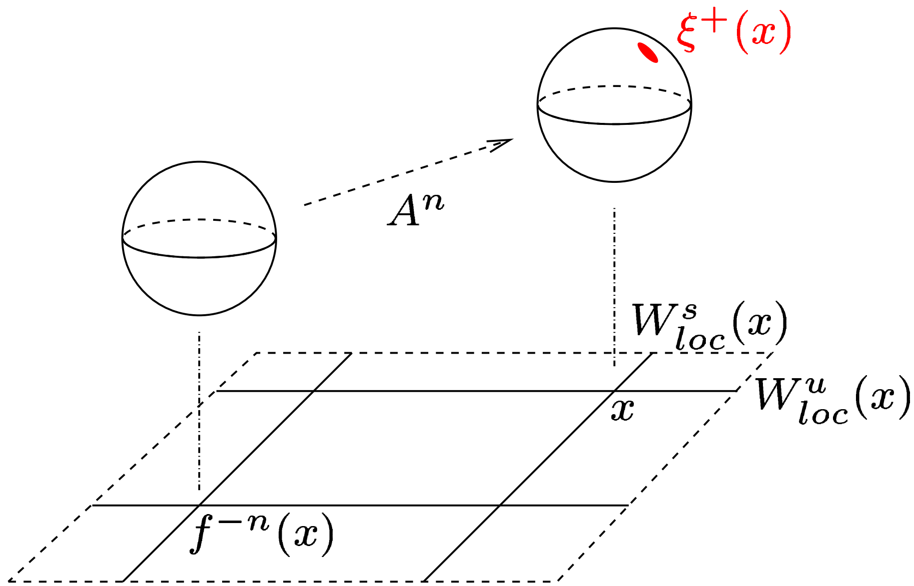
- u-states always exist: for instance, $m = \mu \times \nu$ for any probability ν in the Grassmannian.
- any Cesaro limit of iterates of a u-state is invariant under the cocycle $\Sigma \times \mathrm{Grass}(\ell, \mathbb{R}^d) \to \Sigma \times \mathrm{Grass}(\ell, \mathbb{R}^d)$.

Proof of Proposition 1

Proposition 2. *Let m be an invariant u-state and ν be its projection to $\mathrm{Grass}(\ell, \mathbb{R}^d)$. Then*

- *the support of ν is not contained in any hyperplane section of the Grassmannian*
- *the push-forwards $\nu^n(x)$ of ν under $A^n(f^{-n}(x))$ converge almost surely to a Dirac measure at some point $\xi^+(x)$.*



Proof of Proposition 1

Proposition 1 follows from Proposition 2 together with

Lemma. *Let $L_n : \mathbb{R}^d \to \mathbb{R}^d$ be a sequence of linear isomorphisms and ρ be a probability measure on $\mathrm{Grass}(\ell, \mathbb{R}^d)$ which is not supported in a hyperplane section*

$$\{E \in \mathrm{Grass}(\ell, \mathbb{R}^d) : E \cap V \neq 0\}, \qquad V \in \mathrm{Grass}(d-\ell, \mathbb{R}^d).$$

If the push-forwards $(L_n)_*\rho$ converge to a Dirac measure δ_ξ then the eccentricity $\mathrm{Ecc}(\ell, L_n) \to \infty$ and the images $E^+(L_n)$ of the most expanded ℓ-subspace converge to ξ.

Proof of Proposition 2

- Part 1: ν is not supported in any hyperplane section.

  This follows from the twisting hypothesis.

- Part 2: the sequence ν^n converges to a Dirac measure.

  The proof has a few steps. The most delicate is to show that some subsequence converges to a Dirac measure:

Proof of Proposition 2

- For almost every x there exist $n_j \to \infty$ such that $\nu^{n_j}(x)$ converges to a Dirac measure.

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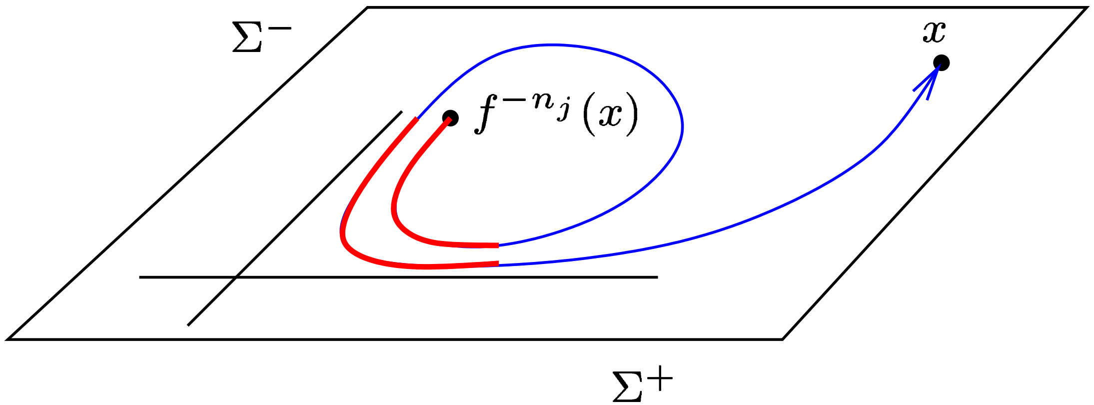
- For almost every x there exist $n_j \to \infty$ such that $\nu^{n_j}(x)$ converges to a Dirac measure.

By hypothesis, there is some strongly pinching $B_0 \in \mathcal{B}$. The n_j are chosen so that $A^{n_j}(f^{-n_j}(x)) = B_1 B_0^p B_2 B_0^q$ for some convenient B_2 and large p, q. This uses ergodicity.



Proof of Proposition 2

Let $m^{(n)}(x)$ denote the projection to the Grassmannian of the normalized restriction of m to the cylinder $[i_{-n}, \dots, i_{-1}]$ that contains x.

- The sequence $m^{(n)}(x)$ converges almost surely to some probability $m(x)$ on $\mathrm{Grass}(\ell, \mathbb{R}^d)$.

  This follows from a simple martingale argument.

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- Moreover, $\frac{1}{C} \le \frac{\nu^n(x)}{m^{(n)}(x)} \le C$ for all x, with $C = C(m)$.

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Thus, $\lim \nu^{(n)}(x)$ exists and is a Dirac measure, as stated.

Lecture # 3

Zorich cocycles are pinching and twisting

Strata and Rauzy classes

On each stratum $\mathcal{A}_g(m_1, \ldots, m_\kappa)$ we introduced a system of local coordinates (π, λ, τ), where

- $\pi = \begin{pmatrix} \pi_0 \\ \pi_1 \end{pmatrix}$ is a pair of permutations of the alphabet
- λ and τ live in convenient subsets of $\mathbb{R}^d$, $d = 2g + \kappa - 1$.

The pair π determines the stratum and even its connected component.

An (extended) Rauzy class is the set of all π corresponding to a given connected component of a stratum (there is also an intrinsic combinatorial definition).

Zorich cocycle

Fix some (irreducible) Rauzy class $\mathcal{C}$ and consider $\pi \in \mathcal{C}$.

Previously, we introduced the Zorich renormalization and the Zorich cocycle for translation surfaces

- $\mathcal{Z} = \mathcal{R}^n : (\pi, \lambda, \tau) \mapsto (\pi', \lambda', \tau')$
- $F_{\mathcal{Z}} = F_{\mathcal{R}}^n : (\pi, \lambda, \tau, h) \mapsto (\pi', \lambda', \tau', h')$, with $n = n(\pi, \lambda)$

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Dropping τ, we find the Zorich renormalization and Zorich cocycle for interval exchange maps

- $Z : (\pi, \lambda) \mapsto (\pi', \lambda')$
- $F_Z : (\pi, \lambda, h) \mapsto (\pi', \lambda', h')$

For our purposes, $\mathcal{Z}$ is equivalent to the inverse limit of Z.

Checking the hypotheses

- Z is a Markov map: There exists a finite partition $\mathcal{D}_0$ and a countable partition $\mathcal{D} \succ \mathcal{D}_0$ such that $Z(\Delta) \in \mathcal{D}_0$ for every $\Delta \in \mathcal{D}$.

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- There exists a Z-invariant probability μ which is ergodic and equivalent to volume $d\lambda$. Moreover, μ has bounded distortion.
- The cocycle F_Z is constant on every atom of $\mathcal{D}$, and it is integrable with respect to μ.

Zorich cocycle

There is an F_Z-invariant subbundle $H = \{(\pi, \lambda) \times H_{\pi,\lambda}\}$ of the vector bundle $\{(\pi, \lambda)\} \times \mathbb{R}^d$, such that the

- fibers have dimension $2g$ and they depend only on the permutation pair: $H_{\pi,\lambda} = H_\pi$
- Lyapunov exponents of the cocycle F_Z in the directions transverse to H_π vanish (in a trivial fashion)
- restricted Zorich cocycle $F_R \mid H$ preserves a symplectic form $\omega = \{\omega_\pi\}$ on this subbundle

Thus, the Lyapunov spectrum of the Zorich cocycle has the form

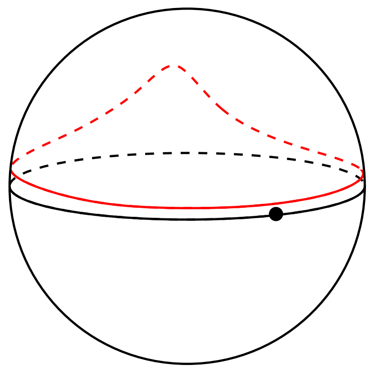
$$\theta_1 \geq \theta_2 \geq \cdots \geq \theta_g \geq 0 = \cdots = 0 \geq -\theta_g \geq \cdots \geq -\theta_2 \geq -\theta_1 .$$

Zorich cocycles are simple

The goal is to prove that the θ_i are all distinct and positive. By Theorem 1 and previous observations, all we need is

Theorem 2. *Every restricted Zorich cocycle is twisting and pinching.*

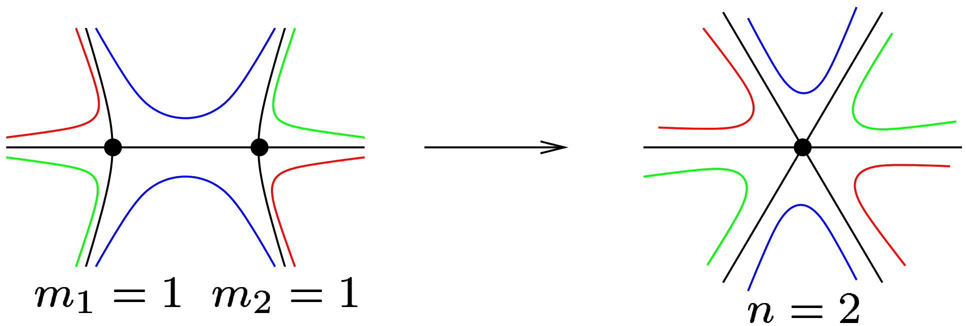
The proof is by induction on the complexity ($\kappa =$ number of singularities, $g =$ genus of the surface) of the stratum:



we look for orbits of $\mathcal{T}^t$ that spend a long time close to the boundary of each stratum (hence, close to “simpler” strata).

Hierarchy of strata

Let $\mathcal{A} = \mathcal{A}_g(m_1, \ldots, m_\kappa)$ be any stratum. Collapsing two or more singularities of an $\alpha \in \mathcal{A}$ together (multiplicities add) one obtains an Abelian differential in another stratum, on the boundary of $\mathcal{A}_g(m_1, \ldots, m_\kappa)$.



We write $\mathcal{A}_g(n_1, \ldots, n_\sigma) \prec \mathcal{A}_g(m_1, \ldots, m_\kappa)$ if $(n_1, \ldots, n_\sigma)$ can be attained from $(m_1, \ldots, m_\kappa)$ by collapsing singularities.

$\mathcal{A}_6(3, 5, 2) \prec \mathcal{A}_6(3, 3, 2, 1, 1)$ but $\mathcal{A}_4(3, 3) \not\prec \mathcal{A}_4(2, 2, 2)$.

Starting the induction

The case $g = 1$, $\kappa = 0$, $d = 2$ is easy:

- There is only one permutation pair $\pi = \begin{pmatrix} A & B \\ B & A \end{pmatrix}$.
- Top case is $\lambda_A < \lambda_B$ and bottom case is $\lambda_B < \lambda_A$, with

$$\Theta_{top} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \Theta_{bot} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

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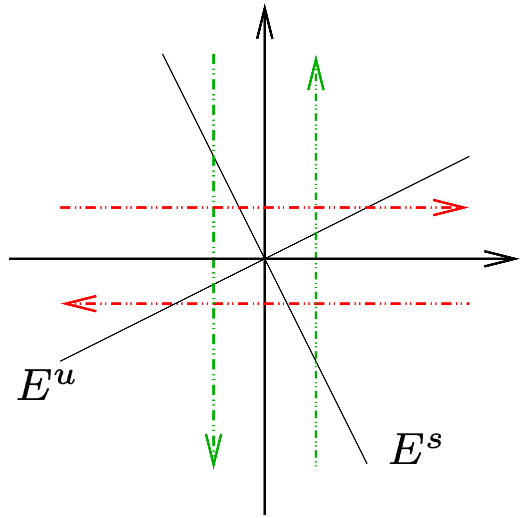
$B = \Theta_{top}\Theta_{bot} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ is hyperbolic and so its powers have arbitrarily large eccentricity. This proves pinching.

Starting the induction

To prove twisting, consider F and $G_1, \ldots, G_N \in \mathrm{Grass}(1, \mathbb{R}^2)$.



Fix k large enough so that no $\Theta_{top}^{-k}(G_i)$ coincides with any of the eigenspaces of B.

Then $B^n(F) \cap \Theta_{top}^{-k}(G_i) = \{0\}$, that is, $\Theta_{top}^{k} B^n(F) \cap G_i = \{0\}$ for all i and any sufficiently large n.

Inductive step

Fix any pair $\pi \in \mathcal{C}$ and denote by $\mathcal{B}_\pi$ the submonoid of $\mathcal{B}$ corresponding to orbit segments $(\pi^0, \lambda^0), \ldots, (\pi^k, \lambda^k)$ such that $\pi^0 = \pi = \pi^k$.

It suffices to prove that the action of $\mathcal{B}_\pi$ on the space H_π is simple. The proof is by induction on the complexity of the stratum, with respect to the order $\prec$ and to the genus.

This is easier to implement at the level of interval exchange transformations. In that setting, approaching the boundary corresponds to making some coefficient λ_α very small. Then it remains small for a long time, under iteration by the renormalization operator.

Reduction and extension

We consider operations of simple reduction/extension:

$$\pi = \begin{pmatrix} a_1 & \cdots & a_{i-1} & c & a_{i+1} & \cdots & \cdots & \cdots & a_d \\ b_1 & \cdots & & \cdots\cdots & b_{j-1} & c & b_{j+1} & \cdots & b_d \end{pmatrix}$$

$$\updownarrow$$

$$\pi' = \begin{pmatrix} a_1 & \cdots & a_{i-1} & a_{i+1} & \cdots & \cdots & a_d \\ b_1 & \cdots & \cdots & b_{j-1} & b_{j+1} & \cdots & b_d \end{pmatrix}$$

extension: inserted letter can not be last in either row and can not be fi rst in both rows simultaneously

Lemma. *Given any π there exists π' such that π is a simple extension of π'. Moreover, either $g(\pi) = g(\pi')$ or $g(\pi) = g(\pi') + 1$.*

Proof of Theorem 2

For proving twsting and pinching, we take advantage of the symplectic structure:

Proposition 3. *The action of $\mathcal{B}_\pi$ twists isotropic subspaces of H_π.*

That is, the twisting property holds for the subspaces F on which the symplectic form vanishes identically.

To compensate for this weaker twisting statement, we prove a stronger form of pinching:

Proof of Theorem 2

Proposition 4. *The action of $\mathcal{B}_\pi$ on H_π is strongly pinching.*

That is, given any $C > 0$ there exist B in the monoid for which $\log \sigma_g > C$ and $\dfrac{\log \sigma_j}{\log \sigma_{j+1}} > C$ for all $1 \leq j < g$.

For symplectic actions in $d = 2$, twisting = isotropic twisting and pinching = strong pinching. In any dimension,

Lemma. *isotropic twisting & strong pinching $\Rightarrow$ twisting & pinching.*

This reduces Theorem 2 to proving the two propositions.

Proof of Proposition 3

Take π' such that π is a simple extension of π'.

- If $g(\pi) = g(\pi')$ then there is a symplectic isomorphism $H_{\pi'} \to H_\pi$ that conjugates the action of $\mathcal{B}_{\pi'}$ on $H_{\pi'}$ to the action of $\mathcal{B}_\pi$ on H_π.

Therefore, by induction, $\mathcal{B}_\pi$ does twist isotropic subspaces.

Proof of Proposition 3

- If $g(\pi) = g(\pi') + 1$, there is some symplectic reduction H_π^v of H_π and some symplectic isomorphism $H_{\pi'} \to H_\pi^v$ that conjugates the action of $\mathcal{B}_{\pi'}$ on $H_{\pi'}$ to the action induced by $\mathcal{B}_\pi$ on H_π^v.

symplectic reduction: $H^v =$ quotient by v of the symplectic orthogonal of v, where $v \in \mathbb{P}(H)$. $\dim H^v = \dim H - 2$

induced action: the action on H^v of the stabilizer of v, that is, the submonoid of elements of $\mathcal{B}$ that preserve v.

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Lemma. *If the action of $\mathcal{B}$ on $\mathbb{P}(H)$ is minimal and its induced action on H^v twists isotropic subspaces, then $\mathcal{B}$ twists isotropic subspaces of H.*

The action of $\mathcal{B}_\pi$ on the projective space $\mathbb{P}(H_\pi)$ is indeed minimal, that is, all its orbits are dense.

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