Non-uniform hyperbolicity

Let $f : M \to M$ be a C^r diffeomorphism, $r \geq 1$, on a compact manifold M and μ an f-invariant probability measure.

Oseledets: μ-almost every point admits a splitting

$$T_xM = E_x^1 \oplus \cdots \oplus E_x^k\,, \quad k = k(x),$$

and real numbers $\lambda_1(f,x) > \cdots > \lambda_k(f,x)$ such that

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \|Df^n(x)v_i\| = \lambda_i(f,x)$$

for every non-zero $v_i \in E_x^i$.

non-uniform hyperbolicity $\Leftrightarrow$ all Lyapunov exponents $\lambda_i(f,x)$ non-zero μ-almost everywhere

Prob. Are most systems non-uniformly hyperbolic ?

Fix μ (e.g. Lebesgue measure) and consider

$$\mathrm{Diff}^r_\mu(M) = \{C^r \text{ diffeomorphisms preserving } \mu\}.$$

A dichotomy for conservative systems

Let μ be Lebesgue measure and $r = 1$.

Thm (Bochi). *If* $\dim M = 2$*, there is a residual subset* $\mathcal{R}$ *of* $\mathrm{Diff}^1_\mu(M)$ *such that for every* $f \in \mathcal{R}$

- *either* $\lambda_i(f, x) = 0$ *at* μ*-almost every* $x \in M$
- *or* f *is Anosov (and then* $M = \mathbb{T}^2$*).*

Thm (Bochi, Viana). *In any dimension, there is a residual subset* $\mathcal{R}$ *of* $\mathrm{Diff}^1_\mu(M)$ *such that for* $f \in \mathcal{R}$ *and* μ*-almost every* $x \in M$*,*

- *either all* $\lambda_i(f, x) = 0$ *or*
- *the Oseledets splitting is dominated on the orbit of* x

The latter implies that

• the angles between the subspaces E^i are bounded from zero over the orbit of x

• and the splitting extends continuously to the closure of the orbit.

Ex (Bonatti, Viana, Tahzibi). For $M = \mathbb{T}^4$ and $\mu =$Lebesgue measure, there exists an open subset $\mathcal{U}$ of $\mathrm{Diff}^1_\mu(M)$ such that, for every $f \in \mathcal{U}$,

- f admits a dominated splitting $TM = E \oplus F$ with $\dim E = \dim F = 2$
- these are the only continuous invariant subbundles;
- f is transitive and, for a residual subset $\mathcal{R}_0$, ergodic;
- E is not expanding and F is not contracting.

Cor. *There exists a residual subset $\mathcal{S}$ of $\mathcal{U}$ such that for every $f \in \mathcal{S}$ the Oseledets splitting of f is $E^1_x \oplus E^2_x$ with $\dim E^1_x = \dim E^2_x = 2$.*

1. Take $\mathcal{S} = \mathcal{R} \cap \mathcal{R}_0$, where $\mathcal{R}$ is the residual set in the theorem. For $f \in \mathcal{S}$,

2. The Lyapunov exponents are not all zero, by the first property. So, by the theorem and using ergodicity, the Oseledets splitting extends to a dominated splitting on the whole M.

3. Since f has no other continuous invariant subbundles, this extension must coincide with $E \oplus F$.

Deterministic products of matrices

Let $f : M \to M$ be a transformation on a compact metric space M. A *linear cocycle* over f is a skew-product

$$F : M \times \mathbb{R}^d \to M \times \mathbb{R}^d, \quad F(x,v) = (f(x), A(x)v)$$

where $A : M \to \mathrm{SL}(d,\mathbb{R})$ (or $\mathrm{GL}(d,\mathbb{R})$).

Then $F^n(x) = (f^n(x), A^n(x)v)$ with

$$A^n(x) = A(f^{n-1}(x)) \cdots A(f(x))\, A(x).$$

Oseledets: Let μ be any f-invariant probability. For μ-almost every point there is a filtration

$$\{x\} \times \mathbb{R}^d = F_x^1 > \cdots > F_x^k > \{0\}, \quad k = k(x),$$

and real numbers $\lambda_1(A,x) > \cdots > \lambda_k(A,x)$ such that

$$\lim_{n\to+\infty} \frac{1}{n} \log \|A^n(x) v_i\| = \lambda_i(A,x)$$

for every $v_i \in F_x^i \setminus F_x^{i+1}$. If f is invertible we even have a splitting of $\{x\} \times \mathbb{R}^d$ (better than a filtration).

Let μ be any ergodic f-invariant probability.

Thm (Bochi, Viana). *There is a residual subset $\mathcal{R}$ of all continuous maps $M \to \mathrm{SL}(d, \mathbb{R})$ such that for $A \in \mathcal{R}$*

- *either all $\lambda_i(A, x) = 0$ at μ-almost every point*
- *or the Oseledets splitting extends to a dominated splitting on the whole support of μ.*

Ex. Suppose for every $1 \leq i < d$ there exists a periodic point p_i of f in the support of μ, with period κ_i , such that the eigenvalues $\{\beta_j^i : 1 \leq j \leq d\}$ of $A^{\kappa_i}(p_i)$ satisfy

$$|\beta_1^i| \geq \cdots \geq |\beta_{i-1}^i| > |\beta_{i-1}^i| = |\beta_i^i| > |\beta_{i+1}^i| \geq \cdots \geq |\beta_d^i|$$

and β_i^i, β_{i+1}^i are complex conjugate (not real).

This obstructs the existence of a dominated splitting, for any map $B : M \to \mathrm{SL}(d, \mathbb{R})$ in a C^0 neighborhood $\mathcal{U}$ of A. Therefore, all Lyapunov exponents are zero, for B in a residual subset of $\mathcal{U}$.

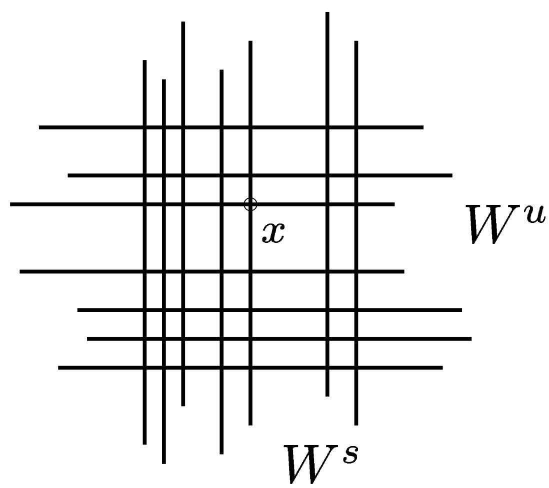
Prevalence of non-zero exponents

Suppose $f : M \to M$ is a C^1 diffeomorphism with derivative Hölder continuous, and

• μ is (non-uniformly) *hyperbolic:* $\lambda_i(f, x) \neq 0$ for all i and μ-almost every point;

• μ has *local product structure:* for μ-almost every point x there is a product "neighborhood" V such that

$$\mu \mid V \approx \mu^u \times \mu^s.$$



Rmk. Lebesgue measure has local product structure if it is hyperbolic ($\Leftarrow$ absolute continuity of foliations).

Same for hyperbolic invariant measures with conditional measures along unstable manifolds absolutely continuous with respect to Lebesgue measure.

Let (f, μ) be non-uniformly hyperbolic with local product structure.

Thm. *For any $0 < r \leq \infty$, there exists an open and dense subset $\mathcal{O}$ of all C^r maps $A : M \to \mathrm{SL}(d, \mathbb{R})$ such that every $A \in \mathcal{O}$ has non-zero Lyapunov exponents at μ-almost every point. The complement of $\mathcal{O}$ has ∞ codimension in $C^r(M, \mathrm{SL}(d, \mathbb{R}))$.*

∞ *codimension* $\Leftrightarrow$ contained in finite unions of closed submanifolds with arbitrarily large codimension.

Suppose $f : M \to M$ is uniformly hyperbolic.

Thm. *For any $0 < r \leq \infty$, there exists an open and dense subset $\mathcal{O}$ of all C^r maps $A : M \to \mathrm{SL}(d, \mathbb{R})$, whose complement has ∞ codimension, such that every $A \in \mathcal{O}$ has non-zero Lyapunov exponents at μ-almost every point, for every invariant measure with local product structure.*

Probably, all Lyapunov exponents have multiplicity 1.

Bonatti, Gomez-Mont, Viana: a particular case of the second theorem, assuming a property of domination.

An application

Consider $d = 2$ and $f : M \to M$ uniformly expanding.

Def. $A : M \to \mathrm{SL}(2, \mathbb{R})$ is *bundle-free* if, for any $\eta \geq 1$, there is no $C^{\min(r,\mathrm{Lip})}$ map $\psi : x \mapsto \{\psi_1(x), \ldots, \psi_\eta(x)\}$ assigning to each $x \in M$ a subset of $\mathbb{RP}^1$ with exactly η elements, invariant under the cocycle

$$A(x)\big(\{\psi_1(x), \ldots, \psi_\eta(x)\}\big) = \{\psi_1(f(x)), \ldots, \psi_\eta(f(x))\}$$

for all $x \in M$.

Thm. *Suppose $A \in C^r$, $r > 0$ is bundle-free and there exists some periodic point $p \in M$ of f such that A is hyperbolic over the orbit of p. Then $\lambda_1(A, x) > 0$ at μ-almost every point, for every f-invariant ergodic measure with local product structure.*

The condition on the existence of some periodic point over which the cocycle is hyperbolic, is mild (open and dense subset whose complement has ∞ codimension).

Ex. Let $f : S^1 \to S^1$ be expanding, μ be the absolutely continuous invariant measure, and $A : S^1 \to \mathrm{SL}(2, \mathbb{R})$ be of the form

$$A(x) = R_{\alpha(x)} A_0$$

- A_0 is some hyperbolic matrix,
- $\alpha : S^1 \to S^1$ is a continuous function with $\alpha(0) = 0$,
- $R_{\alpha(x)}$ is the rotation of angle $\alpha(x)$.

Assume that $2 \deg(\alpha)$ is *not* a multiple of $\deg(f) - 1$.

Cor. *There exists a C^0 neighbourhood $\mathcal{U}$ of A such that*

1. $\lambda_1(B, \mu) = 0$ *for B in a residual subset $\mathcal{R} \cap \mathcal{U}$;*
2. $\lambda_1(B, \mu) > 0$ *for all $B \in \mathcal{U} \cap C^r$, any $r > 0$.*

First, let $\mathcal{U}_0$ be the isotopy class of A in the space of continuous maps from M to $\mathrm{SL}(2, \mathbb{R})$.

Claim: Given $B \in \mathcal{U}_0$ there is no B-invariant continuous map

$$\psi : M \ni x \mapsto \{\psi_1(x), \ldots, \psi_\eta(x)\}$$

assigning a constant number $\eta \geq 1$ of elements of $\mathbb{RP}^1$ to each point $x \in M$.

The proof of the claim is by contradiction. Suppose there exists such a map. The graph

$$G = \{(x, \psi_i(x)) \in S^1 \times \mathbb{RP}^1 : x \in S^1 \text{ and } 1 \leq i \leq \eta\}$$

represents an element (η, ζ) of the fundamental group $\pi_1(S^1 \times \mathbb{RP}^1) = \mathbb{Z} \oplus \mathbb{Z}$ (if it is connected, otherwise consider connected components).

Because B is isotopic to A, the image of G must represent

$$(\eta \deg(f), \zeta + 2\deg(\alpha)) \in \pi_1(S^1 \times \mathbb{RP}^1).$$

The factor 2 comes from the fact that S^1 is the 2-fold covering of $\mathbb{RP}^1$.

By the invariance of ψ

$$\zeta + 2\deg(\alpha) = \deg(f)\zeta$$

which contradicts the hypothesis that $\deg(f) - 1$ does not divide $2\deg(\alpha)$.

Now we can prove the Corollary:

1. By Bochi there is a residual subset $\mathcal{R}$ of continuous cocycles which either are uniformly hyperbolic or have both Lyapunov exponents equal to zero.

The claim rules out the first case, for all $B \in \mathcal{R} \cap \mathcal{U}_0$. Hence, B has both Lyapunov exponents equal to zero almost everywhere.

2. The claim implies that every $B \in \mathcal{U}_0 \cap C^r$, $r > 0$ is bundle-free. Condition $\alpha(0) = 0$ ensures that $p = 0$ is a hyperbolic fixed point, for B in a neighborhood $\mathcal{U} \subset \mathcal{U}_0$.

Using the theorem, $\lambda_1(B, \mu) > 0$ for all $B \in \mathcal{U} \cap C^r$.

Summary of Lecture # 4

• A dichotomy for generic conservative diffeomorphisms: projective hyperbolicity or no hyperbolicity at all (every Lyapunov exponent equal to zero), at Lebesgue almost every orbit.

• This extends to generic continuous linear cocycles over any transformation.

• The conclusion is radically different for C^r cocycles, $r > 0$, over a non-uniformly hyperbolic transformation with local product structure: the overwhelming majority of C^r cocycles have non-zero Lyapunov exponents.