## Dynamics of projectively hyperbolic systems

Next, we study the dynamics of projectively hyperbolic sets and attractors. Two warnings:

- Most results are for $C^2$ diffeomorphisms.
- Projective hyperbolicity is a very weak property (e.g. it may coexist with homoclinic tangencies), except when all the subspaces in the decomposition have dimension 1.

**Thm (Pujals, Sambarino).** *Let $\Lambda$ be a projectively hyperbolic set of a surface diffeomorphism such that all periodic points contained in it are hyperbolic saddles. Then $\Lambda$ is the union of a hyperbolic set and a finite number of smooth invariant circles supporting irrational rotations.*

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## Attractors and physical measures

A transitive invariant set $\Lambda$ is an *attractor* if the basin of attraction (topological)

$$B(\Lambda) := \{x \in M : \omega(x) \subset \Lambda\}$$

has positive Lebesgue measure.

*Topological attractor:* if the basin is a neighborhood of the attractor (e.g. tame systems).

An ergodic invariant probability $\mu$ is a *physical measure*, or *SRB measure,* if the basin of attraction (ergodic)

$$B(\mu) := \{x \in M : \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j(x)} = \mu\}$$

has positive Lebesgue measure.

**Conj (Palis).** For a dense subset of $\mathrm{Diff}^r(M)$, there are only finitely many attractors and physical measures, and Lebesgue almost every point is in the unions of their basins of attraction (topological and ergodic).

**Prob.** For generic diffeomorphisms, the union of the basins of all topological attractors is open and dense in $M$ ?

# An existence and finiteness theorem

**Thm (Alves, Bonatti, Viana).** *Let $\Lambda$ be a projectively hyperbolic attractor of a $C^2$ diffeomorphism $f : M \to M$, with decomposition $T_x M = E_x^{cu} \oplus E_x^{cs}$, $x \in \Lambda$. Assume*

1. *$E_z^{cu}$ is non-uniformly expanding and $E_z^{cs}$ is non-uniformly contracting*
2. *$z$ has simultaneous cu- and cs-hyperbolic times*

*for Lebesgue almost every point $z \in B(\Lambda)$.*

*There are finitely many SRB measures supported in $\Lambda$, and the union of the basins contains Lebesgue almost every point in $B(\Lambda)$.*

(Extend $E^{cu} \oplus E^{cs}$ continuously to the basin.)

Non-uniformly expanding:
$\limsup_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \log \|Df^{-1} \mid E^{cu}_{f^j(z)}\| < -c < 0$

Non-uniformly contracting:
$\limsup_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df \mid E^{cs}_{f^j(z)}\| < -c < 0$

Condition 2 is automatic if $E^{cu}$ is uniformly expanding or $E^{cs}$ is uniformly contracting.

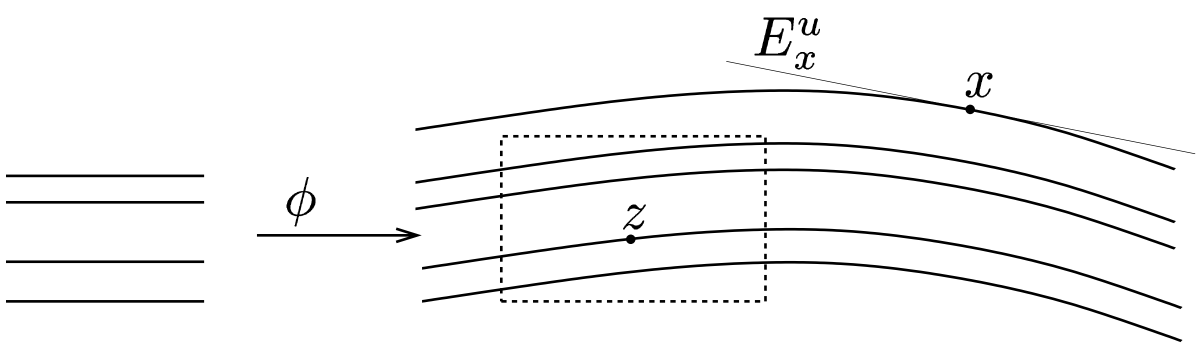
## Partially hyperbolic attractors

Let $f : M \to M$ be $C^2$. Let $\Lambda$ be an invariant set with dominated decomposition

$$T_x M = E^u_x \oplus E^c_x \oplus E^s_x, \quad x \in \Lambda,$$

with $E^u$ uniformly expanding, $E^s$ uniformly contracting, and $\dim E^u > 0$.

Then there exists a unique *strong-unstable* foliation $\mathcal{F}^u$ tangent to $E^u_x$ at every point $x \in \Lambda$. Assume the leaves are entirely contained in $\Lambda$.




*Foliated neighborhood at $z \in \Lambda$:* homeomorphism $\phi : B \times \Sigma \to \Lambda$ onto a neighborhood of $z$ inside $\Lambda$, with

- $B$ = unit disk of dimension $\dim E^u$, and $\Sigma$ compact
- each $\phi(\cdot, \eta)$ a diffeomorphism to a strong-unstable disk.

## Gibbs $u$-states

An invariant probability $\mu$ on $\Lambda$ is a *Gibbs $u$-state* if every $z \in \Lambda$ has some foliated neighborhood $\phi$ such that $\mu$ is equivalent to a product measure

$$\mu \approx \phi_*\big(\text{Lebesgue} \times \nu\big).$$

restricted to the image of $\phi$.

**Thm (Pesin, Sinai).** *Let $m_D$ be normalized Lebesgue measure along a disk $D$ transverse to $E^c \oplus E^s$. Every limit measure of*

$$\frac{1}{n}\sum_{j=0}^{n-1} f_*^j(m_D)$$

*is a Gibbs $u$-state (density bounded from zero and $\infty$).*

**Thm (Bonatti, Viana).** *Suppose $\Lambda$ is a topological attractor. For Lebesgue almost every point $x \in B(\Lambda)$, every limit measure of*

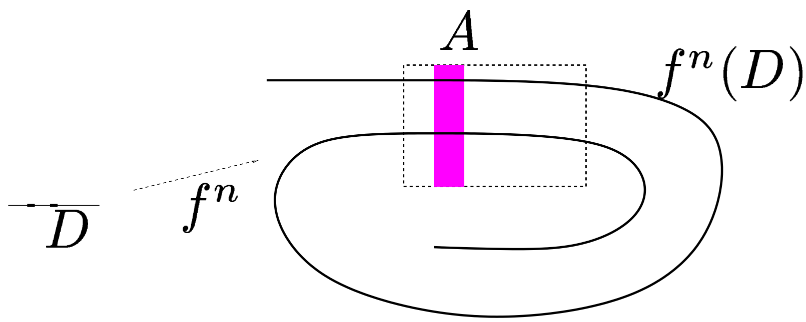
$$\frac{1}{n}\sum_{j=0}^{n-1} \delta_{f^j(x)}$$

*is a Gibbs $u$-state (density bounded from zero and $\infty$).*

So, SRB measures must be Gibbs $u$-states if they exist.

1. Let $D \subset B(\Lambda)$ be any disk transverse to $E^c \oplus E^s$. Using curvature and distortion control,

$$m_D\big(\{f^n(x) \in A\}\big) \leq \text{const}\,|A| \quad \text{for all } n.$$




2. Events $f^j(x) \in A$ and $f^k(x) \in A$ are "independent" if $|j-k|$ is big, because iterates of $D$ are expanded.

By a large deviations argument, the $m_D$-probability of

$$\frac{1}{n}\#\{0 \leq j \leq n-1 : f^j(x) \in A\} > \text{const}\,|A|$$

decays exponentially with $n$. So, for $m_D$-almost every $x$,

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j(x)}(A) \leq \text{const}\,|A|\,.$$

## Mostly contracting central direction

Conversely: if $\mu$ is an ergodic Gibbs $u$-state such that the central direction is mostly contracting

$$\lambda^c(x) := \limsup_{n\to+\infty} \frac{1}{n} \log \|Df^n \mid E^c_x\| < 0 \quad \mu - \text{a.e.}$$

then $\mu$ is an SRB measure.

**Thm (Bonatti, Viana).** *Let $\lambda^c(x) < 0$ on a positive Lebesgue measure subset of every strong-unstable disk. Then $\Lambda$ supports finitely many SRB measures and the union of their basins contains Lebesgue almost every point in $B(\Lambda)$.*

*If the strong-unstable foliation $\mathcal{F}^u$ is minimal, the SRB measure is unique.*

Minimal foliation: all leaves dense in $\Lambda$.

## Gibbs *cu*-states

Now let $\Lambda$ be projectively hyperbolic with decomposition

$$T_x M = E^{cu}_x \oplus E^{cs}_x, \quad x \in \Lambda.$$

**Thm (Alves, Bonatti, Viana).** *Assume that $E^{cu}$ is non-uniformly expanding on a positive Lebesgue measure subset of $B(\Lambda)$. Then there exist ergodic Gibbs cu-states supported in $\Lambda$.*

Non-uniformly expanding:

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \log \|Df^{-1} \mid E^{cu}_{f^j(z)}\| < -c < 0$$

*Gibbs cu-state:* invariant probability measure with $\dim E^{cu}$ expanding directions (positive Lyapunov exponents) and absolutely continuous conditional measures along the corresponding unstable manifolds.

## Hyperbolic times

The key tool in the proof is the following notion: $n$ is a *cu-hyperbolic time* for $z$ if

$$\|Df^{-k} \mid E^{cu}_{f^n(z)}\| \le e^{-ck/2} \quad \text{for all } 1 \le k \le n.$$

If $E^{cu}$ non-uniformly expanding at $z$ then $z$ has positive frequency of $cu$-hyperbolic times:

$$\#\{cu\text{-hyperbolic times } \le n\} > \theta(c)n \quad \text{for all } n$$

with $\theta(c) > 0$.

**Rmk.** If $w = \lim_i f^{n_i}(z_i)$ where each $n_i$ is a hyperbolic time of $z_i$ and $n_i \to \infty$, then $w$ has unstable manifold of dimension $\dim E^{cu}$ and size $> \delta(c) > 0$.

To construct Gibbs $cu$-states one starts with any disk $D$ transverse to $E^{cs}$ such that $E^{cu}$ is non-uniformly expanding on a positive Lebesgue measure set $D_0 \subset D$, and considers accumulation points of

$$\frac{1}{n}\sum_{j=0}^{n-1} f_*^j\big(m_D \mid \{z : j \text{ is a } cu\text{-hyperbolic time for } z\}\big).$$

Any ergodic Gibbs $cu$-state $\mu$ that has $\dim E^{cs}$ negative Lyapunov exponents is an SRB measure.

That is the case for Gibbs $cu$-states obtained before, if on the positive Lebesgue measure set $D_0 \subset D$

• $E^{cs}$ is non-uniformly contracting

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df \mid E^{cs}_{f^j(z)}\| < -c < 0$$

• there is positive frequency of simultaneous $cu$- and $cs$-hyperbolic times

$n$ is a *$cs$-hyperbolic time* for $z$ if

$$\|Df^k \mid E^{cs}_{f^{n-k}(z)}\| \le e^{-ck/2} \quad \text{for all } 1 \le k \le n.$$

In this way one constructs SRB measures supported in $\Lambda$.

Using the existence of stable and unstable manifolds with size $> \delta(c)$ we get that the SRB measures are finitely many.

To prove that their basins cover Lebesgue almost every point in $\Lambda$: if not, we could use the exceptional positive Lebesgue set to construct one more SRB measure.

## Summary of Lecture # 3

• Definitions of attractor and physical (SRB) measure.

• Gibbs $u$-states of partially hyperbolic attractors with expanding subbundle. Gibbs $u$-states exist, and every SRB measure is an ergodic Gibbs $u$-state.

• Existence and finiteness of SRB measures when the central direction is mostly contracting.

• Gibbs $cu$-states of projectively hyperbolic attractors with non-uniformly expanding subbundle. Positive frequency of $cu$-hyperbolic times yields Gibbs $cu$-states.

• Existence and finiteness of SRB measures for projectively hyperbolic attractors non-uniformly hyperbolic and with simultaneous hyperbolic times.