Projective hyperbolicity

Def. An invariant set Λ is *projectively hyperbolic* if for every $x \in \Lambda$ there is a decomposition $T_xM = E_x^1 \oplus E_x^2$ satisfying

1. (invariance) $Df(x)E_x^* = E_{f(x)}^*$ for $* = 1, 2$
2. (domination) $\dfrac{\|Df^n(x)v_2\|}{\|v_2\|} \leq C\lambda^n \dfrac{\|Df^n(x)v_1\|}{\|v_1\|}$ for every $v_1 \in E_x^1$, $v_2 \in E_x^2$, and $n \geq 1$,

with $C > 0$ and $\lambda < 1$ independent of x.

Rmk. It is always assumed that the dimensions of E_x^1 and E_x^2 are constant over the invariant set.

More generally, a decomposition into $k \geq 2$ subspaces

$$T_xM = F_x^1 \oplus \cdots \oplus F_x^k$$

is called dominated if $E^1 = F^1 \oplus \cdots \oplus F^i$ dominates $E^2 = F^{i+1} \oplus \cdots \oplus F^k$ for every $1 \leq i < k$.

Basic properties:

• (transversality) If $T_\Lambda M = E^1 \oplus E^2$ is a dominated decomposition, the angles between E^1_x and E^2_x are bounded from zero.

• (uniqueness) Given the dimensions of the factors, a dominated decomposition is unique when it exists.

• (continuity) A dominated decomposition is continuous and extends uniquely to a dominated decomposition on the closure of the domain.

• (cone field) An invariant set Λ is projectively hyperbolic if and only if it admits a continuous invariant cone field.

• (extension) If Λ admits a dominated decomposition, so does any invariant set inside a small neighborhood U of Λ (with the same dimensions of the factors).

• (robustness) Given any g in a C^1 neighborhood of f, every g-invariant set contained in U has a dominated decomposition (with the same dimensions of the factors).

Partial hyperbolicity

Def. An invariant set Λ is *partially hyperbolic* if for every $x \in \Lambda$ there is a decomposition $T_xM = E^1_x \oplus E^2_x$ satisfying

1. E^1 and E^2 are invariant
2. E^1 dominates E^2
3. either E^1 is uniformly expanding or E^2 is uniformly contracting.

In the first case we write $E^1 = E^u$ and $E^2 = E^{cs}$, in the second one $E^1 = E^{cu}$ and $E^2 = E^s$.

• (cone field) An invariant set Λ is partially hyperbolic if and only if it admits a continuous unstable or stable cone field.

unstable cone field: $f(C_x) \subset C_{f(x)}$ and there are $c > 0$, $\sigma > 1$ such that

$$\|Df^n(x)v\| \geq c\sigma^n \|v\|$$

for $v \in C_x$ and $n \geq 0$.

• (integrability) If $E^1 = E^u$ is uniformly expanding it has a unique integral foliation $\mathcal{F}^u$. Analogously if $E^2 = E^s$ is uniformly contracting.

Generic dynamical systems

We consider C^1 dynamical systems, $\mathrm{Diff}^1(M)$ or $\mathcal{X}^1(M)$, endowed with the C^1 topology. The goal is to obtain a very general dynamical decomposition theorem.

Two fundamental tools:

C^1 **closing lemma (Pugh):** if $f^n(x)$ accumulates on x when $n \to \infty$, there exists a C^1 small perturbation g of f for which x is periodic.

C^1 **connecting lemma (Hayashi):** if $W^u(p, f)$ and $W^u(q, f)$ accumulate on some non-periodic point x, there exists a C^1 small perturbation g of f for which $W^u(p, g)$ intersects $W^s(p, g)$.

Kupka, Smale, Pugh:

Thm. *There exists a residual subset $\mathcal{R}_0$ of $\mathrm{Diff}^1(M)$ so that for every $f \in \mathcal{R}_0$*

1. *all periodic points are hyperbolic*
2. *all their stable and unstable manifolds are transverse*
3. *periodic points are dense in the non-wandering set*

Maximal transitive sets and homoclinic classes

An invariant set Λ is *transitive* $\Leftrightarrow$ the forward orbit of some $z \in \Lambda$ is dense in Λ $\Leftrightarrow$ for any open sets $A, B \subset \Lambda$ there exists $n \geq 1$ such that $f^n(A)$ intersects B.

We call it *maximal* if Λ is not properly contained in another transitive set.

The *homoclinic class* of a periodic point p of f is the closure $H(p, f)$ of the transverse intersections between the stable manifold and the unstable manifold of the orbit of p.

The elementary pieces of a uniformly hyperbolic system are homoclinic classes, and maximal transitive sets.

In general:

- Homoclinic classes are transitive sets. Every transitive set is contained in a maximal transitive set (Zorn).
- Maximal transitive sets and homoclinic classes need not be 2-by-2 disjoint.
- There may be infinitely many disjoint maximal transitive sets and infinitely many disjoint homoclinic classes.

For a residual subset $\mathcal{R} \subset \mathcal{R}_0$ of $\text{Diff}^1(M)$:

• (Bonatti, Díaz) Two periodic points are in the same transitive set if and only if their homoclinic classes coincide.

• (Arnaud) Any transitive set containing a periodic point p is contained in $H(p, f)$. Hence, homoclinic classes are maximal transitive sets.

A transitive set is *saturated* if every transitive set that intersects it is contained in it.

• (Carballo, Morales, Pacifico) Homoclinic classes are saturated transitive sets. Hence, two homoclinic classes either are disjoint or they coincide.

• (Abdenur) The number of homoclinic classes is either infinite or constant in a neighborhood of f.

Ideas of the proofs of (Arnaud) and (CMP):

Homoclinic classes and closures of stable and unstable manifolds of a periodic orbit vary semi-continuously with the dynamics. Hence, there is a residual subset $\mathcal{R}$ of points of continuity. Consider $f \in \mathcal{R}$:

1. $\overline{W^u(p,f)} \cap \overline{W^s(p,f)} = H(p,f)$.

Suppose x is in the intersection of the closures. The connecting lemma gives g arbitrarily close to f such that $x \in W^u(p,g) \cap W^s(p,g)$ (the periodic case requires a separate argument). Make the intersection transverse, then $x \in H(p,g)$. By continuity, $x \in H(p,f)$.

2. $\overline{W^u(p,f)}$ is Lyapunov stable for f.

Let V be a neighborhood of the closure. Suppose there are points arbitrarily close to some $x \in \overline{W^u(p,f)}$ whose forward orbits approach some point $z \notin V$. Using the connecting lemma, there are small perturbations g such that the closure of $W^u(p,g)$ contains z. This contradicts the continuity.

3. If K is a transitive set intersecting $H(p,f)$ then $K \subset H(p,f)$.

Because Lyapunov stability implies that K is contained in $\overline{W^u(p,f)}$ and in $\overline{W^s(p,f)}$.

So, homoclinic classes are maximal transitive sets and saturated transitive sets for every $f \in \mathcal{R}$.

We used a refined version of Hayashi's connecting lemma, by Arnaud, Hayashi, and Wen, Xia:

Thm. *Let* $f : M \to M$ *be a diffeomorphism,* x *be a non periodic point, and* $\mathcal{U}$ *be a* C^1 *neighborhood of* f*.*

There exists $N \geq 1$ *such that for every neighborhood* V *of* x *there exists a neighborhood* $W \subset V$ *such that, given any points* y *and* z *that are outside*

$$V_N = \bigcup_{j=0}^{N-1} f^j(V)$$

and such that W *contains some forward iterate of* y *and some backward iterate of* z*, there exists* $g \in \mathcal{U}$ *coinciding with* f *outside* V_N *and for which* z *is in the forward orbit of* y*.*

A dynamical decomposition theorem

A diffeomorphism $f \in \mathcal{R}$ is *tame* if it has finitely many homoclinic classes, and *wild* otherwise.

Thm (Abdenur). *Every tame diffeomorphism admits a dynamical decomposition into finitely many transitive sets, with no cycles. Moreover, this decomposition is robust restricted to the residual subset* $\mathcal{R}$.

More precisely, there exists

- a *filtration* $M_0 \subset M_1 \subset \cdots \subset M_N = M$,
- periodic points $p_i \in M_i \setminus M_{i-1}$, $1 \le i \le N$,
- and a C^1 neighborhood $\mathcal{U}$ of f,

such that for every $g \in \mathcal{U} \cap \mathcal{R}$ the continuation $p_i(g)$ of p_i is defined and its homoclinic class $\Lambda_i(g) = H(p_i(g), g)$ is the maximal invariant set of g in $M_i \setminus M_{i-1}$. Then

$$\Omega(g) = \Lambda_1(g) \cup \cdots \cup \Lambda_N(g).$$

Moreover, the elementary pieces $\Lambda_i(g)$ are projectively hyperbolic (hyperbolic if $\dim M = 2$).

Λ is *robustly transitive* if it is the maximal f-invariant set in a neighborhood U, and the maximal g-invariant set

$$\Lambda(g) = \bigcap_{n \in \mathbb{Z}} g^n(U)$$

in U is transitive for every g in a C^1 neighborhood $\mathcal{U}$ of f.

Λ is *generically transitive* if the latter holds for a residual subset of $\mathcal{U}$.

Mañé, Bonatti, Díaz, Pujals, Ures:

Robustly transitive $\Rightarrow$ Projectively hyperbolic.

The arguments extend to the elementary pieces in the theorem, which are generically transitive (at least).

Wild dynamics

The first example were Newhouse's C^2 diffeomorphisms with infinitely many periodic attractors coexisting.

For $\dim M \geq 3$ coexistence of periodic attractors occurs also in the C^1 setting (Bonatti, Díaz).

Moreover,

Carballo, Morales, Bonatti, Díaz:

Thm. *If* $\dim M \geq 3$, *there exist open sets* $\mathcal{U} \subset \mathrm{Diff}^1(M)$ *and residual subsets* $\mathcal{R}_{\mathcal{U}} \subset \mathcal{U}$ *such that every* $f \in \mathcal{R}_{\mathcal{U}}$ *has infinitely many non-trivial disjoint homoclinic classes, and also infinitely many saturated transitive sets without periodic points.*

Summary of Lecture #2

• Projectively hyperbolic set Λ: for each x there is a decomposition $T_xM = E^1_x \oplus E^2_x$ satisfying

1. $Df(x)E^1_x = E^1_{f(x)}$ and $Df(x)E^2_x = E^2_{f(x)}$
2. $\frac{\|Df^n(x)v_2\|}{\|v_2\|} \leq C\lambda^n \frac{\|Df^n(x)v_1\|}{\|v_1\|}$ (domination)

for every $x \in \Lambda$, $v_1 \in E^1_x$, $v_2 \in E^2_x$, and $n \geq 1$, and uniform constants $C > 0$ and $\lambda < 1$.

• Maximal transitive sets, saturated transitive sets, and homoclinic class: candidates to elementary dynamical pieces.

• For generic C^1 diffeomorphisms, homoclinic classes are maximal transitive sets and saturated transitive sets. In particular, they are 2-by-2 disjoint.

• If the number of disjoint homoclinic classes is finite (tame dynamics), there is a dynamical decomposition into finitely many pieces, transitive and projectively hyperbolic. Moreover, there are no cycles, and the decomposition is generically robust.