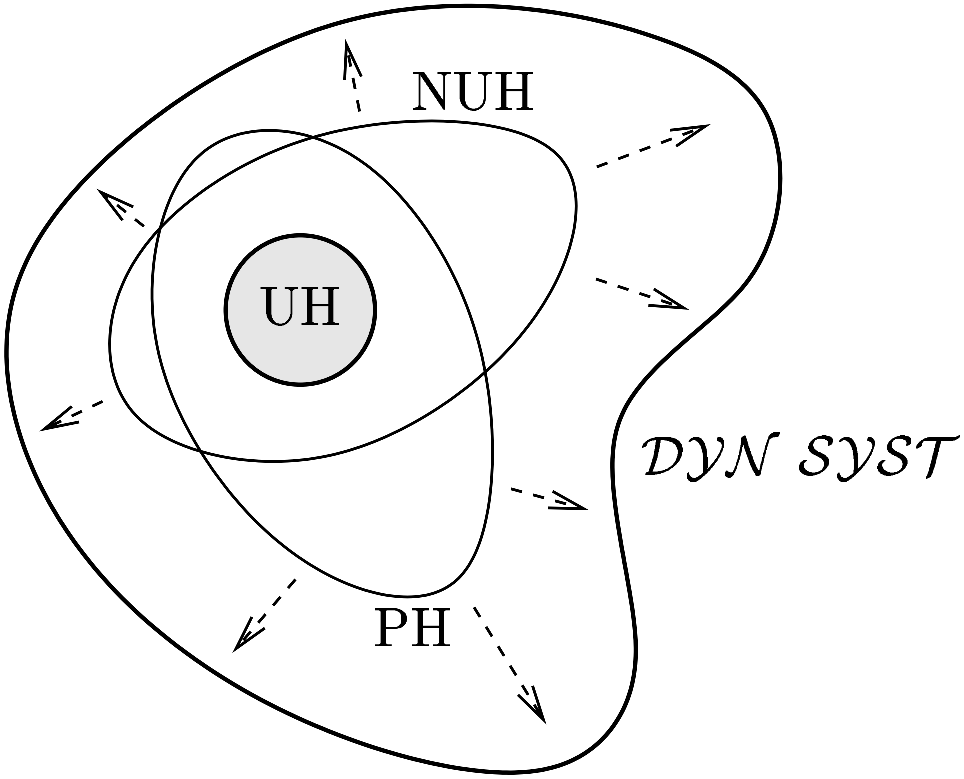
Dynamics : beyond uniform hyperbolicity

Marcelo Viana
IMPA, Rio de Janeiro

Collège de France, March 2002



Uniform Hyperbolicity: uniform expansion and contraction in complementary directions

Non-Uniform Hyperbolicity: expansion and contraction (not uniform) at almost every point

Partial and Projective Hyperbolicity: some directions neutral, with uniform bounds (everywhere)

Dynamics beyond uniform hyperbolicity: a geometric and probabilistic approach, Bonatti, Díaz, Viana

`www.impa.br/∼viana/outgoing/dbuh.ps`

1. Hyperbolic systems and beyond

Definitions. Dynamical decomposition. Dynamics near elementary pieces. Stability.

Mechanisms of robust non-hyperbolicity: Heteroclinic cycles. Homoclinic tangencies. Singular sets of flows. Robustly transitive systems.

2. Partial hyperbolicity and robust transitivity

Partial and projective hyperbolicity. Transitive sets and homoclinic classes. A decomposition theorem for tame systems. Wild systems. A conjecture on finiteness of attractors.

3. Statistics of projectively hyperbolic systems

SRB measures and Gibbs u-states. Existence of u-Gibbs states. Mostly contracting central direction. Hyperbolic times and cu-Gibbs states. A theorem of existence and finiteness of physical measures.

4. Prevalence of non-uniform hyperbolicity

A dichotomy for generic conservative systems. Deterministic products of matrices. Prevalence of non-zero Lyapunov exponents.

Definitions

We consider diffeomorphisms $f : M \to M$ and smooth flows $f^t : M \to M$, $t \in \mathbb{R}$, on a compact manifold M.

An invariant set Λ is *hyperbolic* $\Leftrightarrow$ for every $x \in \Lambda$ there is a decomposition $T_xM = E^u_x \oplus E^s_x$ satisfying

1. (invariance) $Df(x)E^*_x = E^*_{f(x)}$ for $* = u$ and $* = s$
2. (contraction) $\|Df^n(x)E^s_x\| \leq C\lambda^n$ for all $n \geq 1$
3. (expansion) $\|Df^{-n}(x)E^u_x\| \leq C\lambda^n$ for all $n \geq 1$

with $C > 0$ and $\lambda < 1$ independent of x.

Rmk. For flows take $T_xM = E^u_x \oplus E^X_x \oplus E^s_x$ with E^X generated by the vector field.

Def. A diffeomorphism $f : M \to M$ is *uniformly hyperbolic* if

- the non-wandering set $\Omega(f)$ is hyperbolic and
- periodic points are dense in $\Omega(f)$.

$x \in \Omega(f) \Leftrightarrow$ for any neighborhood U of z there exists $n \geq 1$ such that $f^n(U)$ intersects U.

Dynamical decomposition

An invariant set Λ is *transitive* if it contains some dense forward orbit $\{f^n(z) : n \geq 0\}$.

An invariant set Λ is *isolated* if it admits a neighborhood U such that the set of points whose orbits remain in U for all times coincides with Λ.

Thm (Smale). *If $f : M \to M$ is uniformly hyperbolic, the non-wandering set splits into a finite disjoint union*

$$\Omega(f) = \Lambda_1 \cup \cdots \cup \Lambda_N$$

of compact invariant sets Λ_i isolated and transitive. The α-limit set of every orbit is contained in some Λ_i, and analogously for the ω-limit set.

Λ_i is a (hyperbolic) *attractor* if the *basin of attraction*

$$B(\Lambda) = \{x \in M : \omega(x) \subset \Lambda_i\}$$

has positive Lebesgue measure.

Assuming Df is Hölder, a piece Λ_i is an attractor if and only if it has a neighborhood U such that $f(U) \Subset U$ and

$$\Lambda_i = \cap_{n=0}^{\infty} f^n(U).$$

Dynamics near elementary pieces

Let $f : M \to M$ be uniformly hyperbolic and $\Lambda = \Lambda_i$ be any of the elementary pieces of the dynamics.

Thm. *There exists a sub-shift of finite type $\sigma : \Sigma_T \to \Sigma_T$ and a continuous surjective map $\pi : \Sigma_T \to \Lambda$ such that*

$$f \circ \pi = \pi \circ \sigma$$

and π is injective on an open dense subset.

Rmk. But the topology and the geometry of hyperbolic elementary pieces are poorly understood when $\dim M > 2$.

Assume the derivative Df is Hölder continuous.

Thm (Sinai, Ruelle, Bowen). *Every attractor of f has a unique invariant probability measure μ such that for Lebesgue almost every point $x \in B(\Lambda)$,*

$$\frac{1}{n}\sum_{j=0}^{n-1} \delta_{f^j(x)} \to \mu \quad \textit{as } n \to +\infty.$$

That is, given any subset $V \subset M$ with $\mu(\partial V) = 0$,

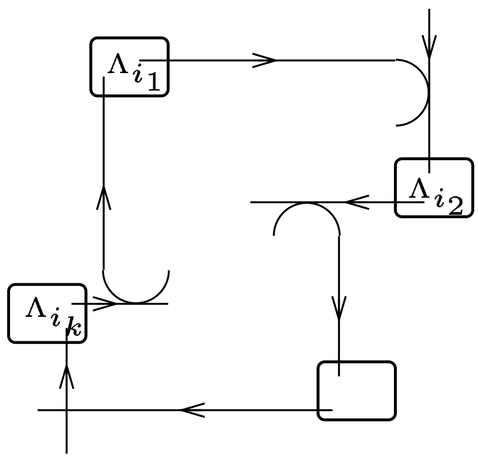
$$\mu(V) = \text{ fraction of time the orbit of } x \text{ spends in } V.$$

Stability

f uniformly hyperbolic + transversality property

$\Updownarrow$ (Robbin, de Melo, Robinson, Mañé)

for every g in a C^1 neighborhood there exists a homeomorphism $h_g : M \to M$ with $g \circ h_g = h_g \circ f$



Ω-stability

f uniformly hyperbolic + no-cycles

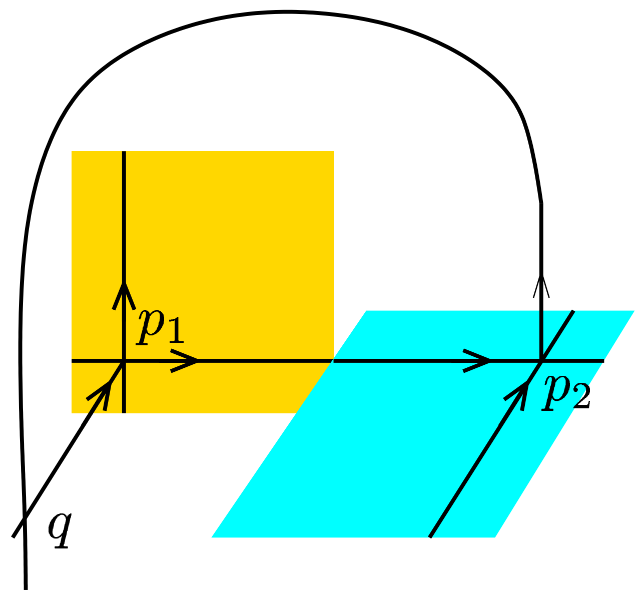
$\Updownarrow$ (Smale, Palis, Mañé)

for every g in a C^1 neighborhood there exists a homeomorphism $h_g : \Omega(f) \to \Omega(g)$ with $g \circ h_g = h_g \circ f$

Robinson, Hayashi: corresponding results for flows.

Heterodimensional cycles

Cycles involving periodic saddles with different stable dimensions:

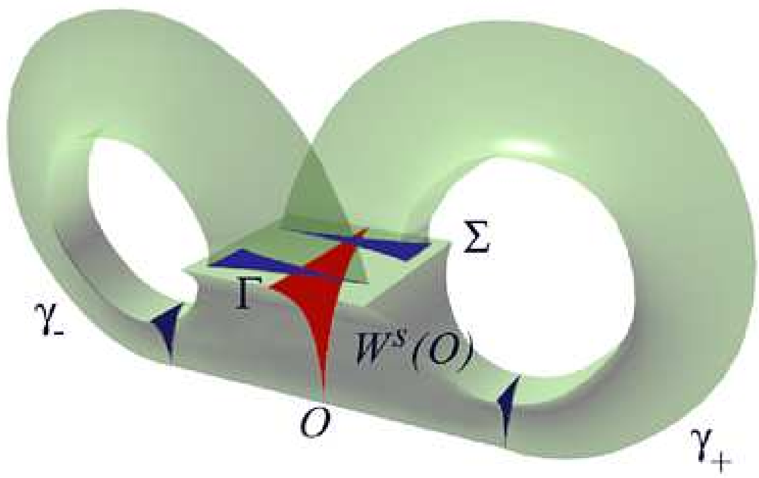


Abraham, Smale, Shub, Mañé:

Thm. *For $d \geq 3$ there are open sets $\mathcal{U} \subset \mathrm{Diff}^1(\mathbb{T}^d)$ such that every $f \in \mathcal{U}$ is transitive and has periodic saddles with different stable dimensions; in particular, f is not uniformly hyperbolic.*

Singular attractors of flows

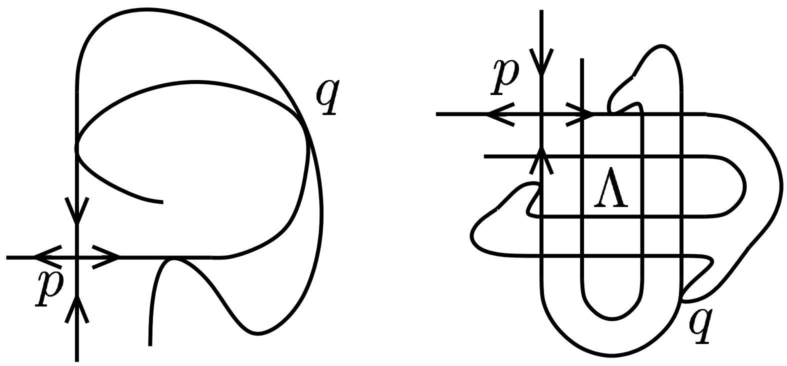
In the case of flows, heterodimensionality may arise from equilibrium points accumulated by regular orbits:



Afraimovich, Bykov, Shilnikov, Guckenheimer, Williams:
Thm. *If* $\dim M \geq 3$ *there are open sets* $\mathcal{V} \subset \mathcal{X}^1(M)$ *such that every* $X \in \mathcal{V}$ *has a (transitive) attractor* Λ_X *that contains equilibrium points and regular orbits.*

Λ_X is not hyperbolic: the decomposition $E^u \oplus E^X \oplus E^s$ can not extend continuously to the equilibrium points.

Homoclinic tangencies



Newhouse, Palis, Viana, Romero:

Thm. *Let* $\dim M \geq 2$. *Close to any* $f : M \to M$ *with a homoclinic tangency of a saddle* p, *there are open sets* $\mathcal{U} \subset \mathrm{Diff}^2(M)$ *such that:*

- *Every* $g \in \mathcal{U}$ *is approximated by a diffeomorphism with a tangency associated to the continuation of* p.

- *If* p *is sectionally dissipative, there exists a residual set* $\mathcal{R} \subset \mathcal{U}$ *such that every* $g \in \mathcal{R}$ *has infinitely many periodic attractors.*

A periodic point p is *sectionally dissipative* $\Leftrightarrow$ the product of any two eigenvalues has norm less than 1.

A robustly transitive map

The following construction is due to Mañé:

1. Start with a linear map $f_0 : M \to M$ of $M = \mathbb{T}^3$, with eigenvalues $\sigma_1 > 3$ and $\sigma_2 > 1 > \sigma_1 > 0$. Let

$$TM = E_0^1 \oplus E_0^2 \oplus E_0^3$$

be the decomposition into eigenspaces and $\mathcal{F}^2$ be the linear foliation tangent to E_0^2. Let $p = 0$ be the fixed point and $W \Supset V \ni p$ be small neighborhoods.

2. Consider a perturbation f of f_0 such that

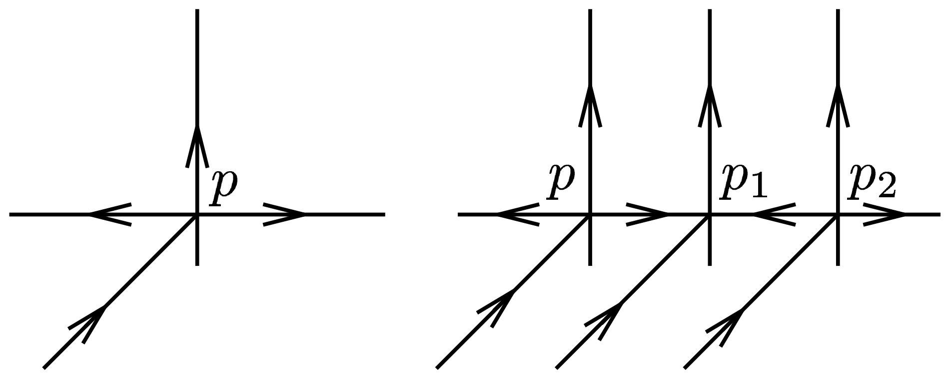
- f preserves $\mathcal{F}^2$ and coincides with f_0 outside V
- there is a Df-invariant decomposition

$$TM = E^1 \oplus E^2 \oplus E^3$$

such that E^1 is expanding, by a factor > 3, E^3 is contracting, and E^2 is "in between".

Let $\mathcal{F}^1$ be the strong-unstable foliation, tangent to E^1.

3. f may be taken with two saddle points with different stable dimensions inside V, for instance, via a saddle-node bifurcation:



4. There exists $L > 0$ such that for any strong-unstable segment γ_1 with length $|\gamma_1| > L$ the image $f(\gamma_1)$ contains some segment with length $> L$ outside W.

Consequently, every strong-unstable segment contains a point $\bar{z}$ whose forward orbit intersects W only finitely many times.

5. Let $\bar{z}$ be such a point and γ_2 be a segment of $\mathcal{F}^2$ through $\bar{z}$. The length of $f^n(\gamma_2)$ goes exponentially to infinity when $n \to +\infty$:

- If $|f^n(\gamma_2)|$ is smaller than $\operatorname{dist}(V, W^c)$ then $f^n(\gamma_2)$ is disjoint from V and so it is expanded by f.

- In any case most of $f^n(\gamma_2)$ is outside V, assuming the diameter of V is much smaller than $\operatorname{dist}(V, W^c)$.

6. Given non-empty open sets $A, B \subset M$ there is $n \geq 1$ such that $f^n(A)$ intersects B ($\Rightarrow f$ is transitive):

Take strong-unstable segment $\gamma_1 \subset A$, point $\bar{z} \in \gamma_1$, and segment γ_2 through $\bar{z}$ as before. Use the fact that the leaves of the foliation $\mathcal{F}^2$ are dense:

For every open set B there exists $K > 0$ such that any $\mathcal{F}^2$-segment with length $> K$ intersects B.

7. The argument works for any perturbation g of f: The main point is that foliation $\mathcal{F}^2$ is stable, because it is normally hyperbolic. This means that g has an invariant foliation $\mathcal{F}^2_g$ and there exists a homeomorphism close to the identity that sends leaves of $\mathcal{F}^2$ to leaves of $\mathcal{F}^2_g$. So the leaves of $\mathcal{F}^2_g$ are also dense in M.

Bonatti, Viana:

Thm. *There exist opens sets $\mathcal{U} \subset \mathrm{Diff}^1(\mathbb{T}^4)$ such that every $f \in \mathcal{U}$ is transitive and admits neither uniformly expanding nor uniformly contracting invariant subbundle.*

The proof is a variation of Mañé's argument. It extends to $\mathbb{T}^d$ for any $d \geq 4$.

1. Start with a linear map $f_0 : \mathbb{T}^4 \to \mathbb{T}^4$ with four real eigenvalues $\sigma_1 > \sigma_2 > 1 > \sigma_3 > \sigma_4$. Replace f_0 by an iterate to ensure that $\sigma_2 > 3$, $\sigma_3 < 1/3$ and there are at least two fixed points p_1 and p_2. Let

$$TM = E_0^u \oplus E_0^s$$

be the hyperbolic decomposition. Fix thin invariant cone fields $\mathcal{C}^u$ and $\mathcal{C}^s$ around E_0^u and E_0^s.

2. Let $W_i \supset V_i \ni p_i$ be small neighborhoods, for $i = 1, 2$. Consider a perturbation f of f_0 such that

- f is C^1 close to f_0 outside $V_1 \cup V_2$
- Df preserves $\mathcal{C}^u$ and uniformly expands area inside it, and Df^{-1} preserves $\mathcal{C}^s$ and uniformly expands area in it
- Df uniformly expands $\mathcal{C}^u$ outside V_1 and Df^{-1} uniformly expands $\mathcal{C}^s$ outside V_2.

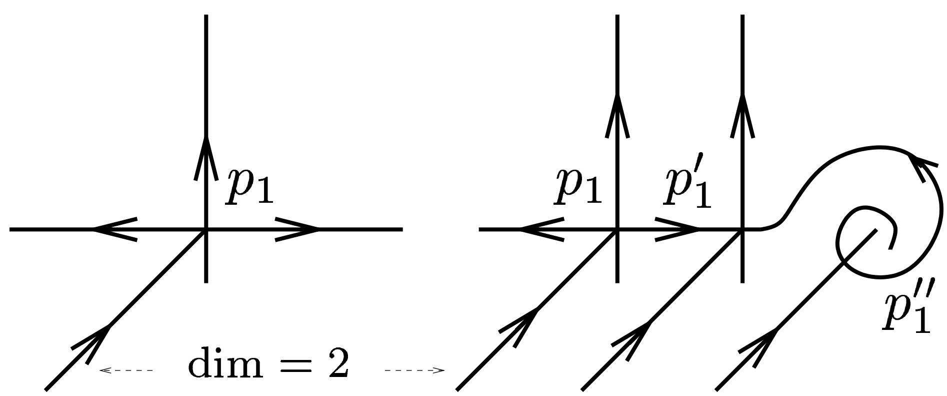
Invariant cone field $\Rightarrow$ Invariant decomposition $TM = E \oplus F$.

3. Take f with two more fixed points p_1', p_1'' inside V_1, besides p_1, and two more fixed points p_2', p_2'' in V_2, besides p_2, satisfying

p_1' has three real contracting and one real expanding eigenvalues

p_1'' has two complex expanding and two real contracting eigenvalues

similarly for p_2', p_2'' reversing the roles of expansion and contraction.



So, the bundles E and F are not hyperbolic, and they do not have invariant subbundles.

4. Taking the cone fields thin enough, there is $L > 0$ such that any centre-unstable disk of radius L intersects any centre-stable disk of radius L.

Using expansion of area (in the place of expansion of norm) inside $\mathcal{C}^u$ we show that every centre-unstable disk contains a point $\bar{z}$ whose forward orbit intersects V_2 finitely many times only.

It follows, as before, that every centre-unstable disk around $\bar{z}$ has an iterate containing a disk of radius L.

5. Given non-empty open sets $A, B \subset M$ consider centre-unstable disk $D^{cu} \subset A$ and centre-stable disk $D^{cs} \subset B$. By the previous step there exists $n \geq 1$ such that $f^n(D^{cu})$ intersects D^{cs}, and so $f^n(A)$ intersects B. This proves f is transitive.

Summary of Lecture # 1

• Hyperbolic systems admit a decomposition into finitely many invariant and indecomposable (transitive) pieces.

• The dynamics on each elementary piece and the statistics of orbits in the basins are well-understood.

• Hyperbolicity is the key ingredient for structural stability of the system.

• There are open subsets of non-hyperbolic systems in every $\mathrm{Diff}^r(M^d)$ except, possibly, for $r = 1$ and $d = 2$.

  • coexistence of infinitely many periodic attractors

  • robustly indecomposable sets that are not hyperbolic (even without any invariant contracting or expanding subbundles).

• There are two known mechanisms generating robustly non-hyperbolic systems: homoclinic tangencies and heterodimensional cycles. In the case of flows the latter may arise from the presence of singularities.