Dynamics : beyond uniform hyperbolicity

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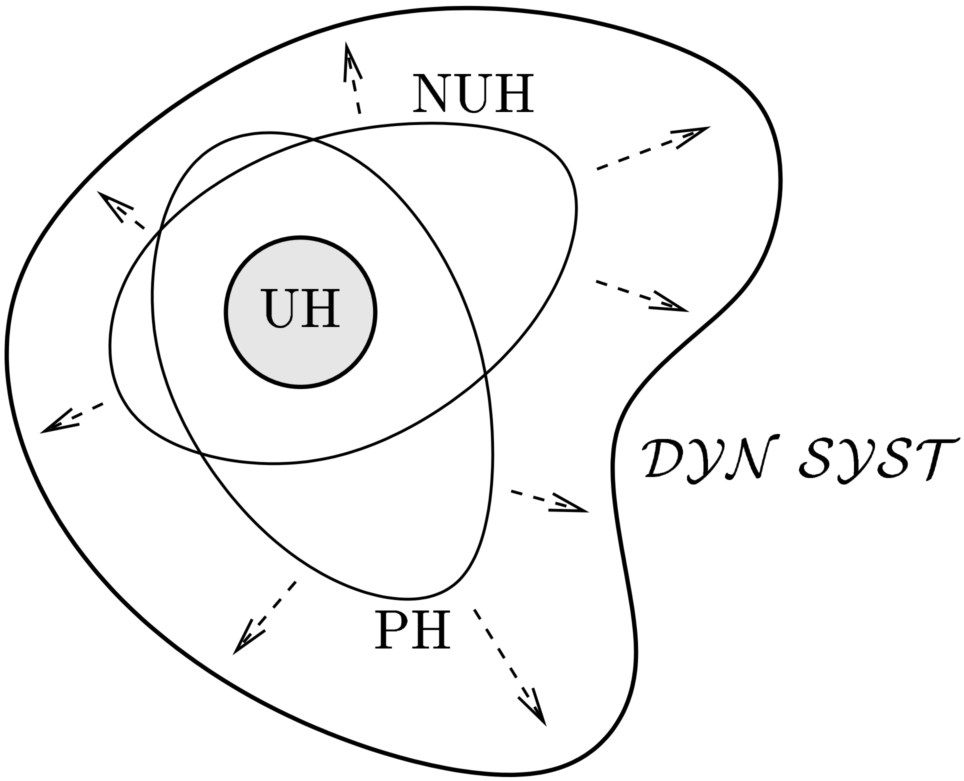


Uniform Hyperbolicity: uniform expansion and contraction in complementary directions

Non-Uniform Hyperbolicity: expansion and contraction (not uniform) at almost every point

Partial and Projective Hyperbolicity: some directions neutral, with uniform bounds (everywhere)

Dynamics beyond uniform hyperbolicity: a geometric and probabilistic approach, Bonatti, Díaz, Viana

`www.impa.br/~viana/outgoing/dbuh.ps`

1. Hyperbolic systems and beyond

Definitions. Dynamical decomposition. Dynamics near elementary pieces. Stability.

Mechanisms of robust non-hyperbolicity: Heteroclinic cycles. Homoclinic tangencies. Singular sets of flows. Robustly transitive systems.

2. Partial hyperbolicity and robust transitivity

Partial and projective hyperbolicity. Transitive sets and homoclinic classes. A decomposition theorem for tame systems. Wild systems. A conjecture on finiteness of attractors.

3. Statistics of projectively hyperbolic systems

SRB measures and Gibbs u-states. Existence of u-Gibbs states. Mostly contracting central direction. Hyperbolic times and cu-Gibbs states. A theorem of existence and finiteness of physical measures.

4. Prevalence of non-uniform hyperbolicity

A dichotomy for generic conservative systems. Deterministic products of matrices. Prevalence of non-zero Lyapunov exponents.

Definitions

We consider diffeomorphisms $f : M \to M$ and smooth flows $f^t : M \to M$, $t \in \mathbb{R}$, on a compact manifold M.

An invariant set Λ is *hyperbolic* $\Leftrightarrow$ for every $x \in \Lambda$ there is a decomposition $T_x M = E^u_x \oplus E^s_x$ satisfying

1. (invariance) $Df(x)E^*_x = E^*_{f(x)}$ for $* = u$ and $* = s$
2. (contraction) $\|Df^n(x)E^s_x\| \leq C\lambda^n$ for all $n \geq 1$
3. (expansion) $\|Df^{-n}(x)E^u_x\| \leq C\lambda^n$ for all $n \geq 1$

with $C > 0$ and $\lambda < 1$ independent of x.

Rmk. For flows take $T_x M = E^u_x \oplus E^X_x \oplus E^s_x$ with E^X generated by the vector field.

Def. A diffeomorphism $f : M \to M$ is *uniformly hyperbolic* if

- the non-wandering set $\Omega(f)$ is hyperbolic and
- periodic points are dense in $\Omega(f)$.

$x \in \Omega(f) \Leftrightarrow$ for any neighborhood U of z there exists $n \geq 1$ such that $f^n(U)$ intersects U.

Dynamical decomposition

An invariant set Λ is *transitive* if it contains some dense forward orbit $\{f^n(z) : n \geq 0\}$.

An invariant set Λ is *isolated* if it admits a neighborhood U such that the set of points whose orbits remain in U for all times coincides with Λ.

Thm (Smale). *If $f : M \to M$ is uniformly hyperbolic, the non-wandering set splits into a finite disjoint union*

$$\Omega(f) = \Lambda_1 \cup \cdots \cup \Lambda_N$$

of compact invariant sets Λ_i isolated and transitive. The α-limit set of every orbit is contained in some Λ_i, and analogously for the ω-limit set.

Λ_i is a (hyperbolic) *attractor* if the *basin of attraction*

$$B(\Lambda) = \{x \in M : \omega(x) \subset \Lambda_i\}$$

has positive Lebesgue measure.

Assuming Df is Hölder, a piece Λ_i is an attractor if and only if it has a neighborhood U such that $f(U) \Subset U$ and

$$\Lambda_i = \cap_{n=0}^{\infty} f^n(U).$$

Dynamics near elementary pieces

Let $f : M \to M$ be uniformly hyperbolic and $\Lambda = \Lambda_i$ be any of the elementary pieces of the dynamics.

Thm. *There exists a sub-shift of finite type* $\sigma : \Sigma_T \to \Sigma_T$ *and a continuous surjective map* $\pi : \Sigma_T \to \Lambda$ *such that*

$$f \circ \pi = \pi \circ \sigma$$

and π *is injective on an open dense subset.*

Rmk. But the topology and the geometry of hyperbolic elementary pieces are poorly understood when $\dim M > 2$.

Assume the derivative Df is Hölder continuous.

Thm (Sinai, Ruelle, Bowen). *Every attractor of* f *has a unique invariant probability measure* μ *such that for Lebesgue almost every point* $x \in B(\Lambda)$,

$$\frac{1}{n}\sum_{j=0}^{n-1} \delta_{f^j(x)} \to \mu \quad \textit{as } n \to +\infty.$$

That is, given any subset $V \subset M$ with $\mu(\partial V) = 0$,

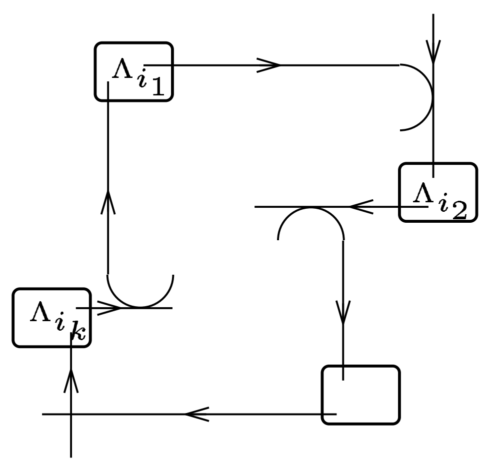
$$\mu(V) = \text{ fraction of time the orbit of } x \text{ spends in } V.$$

Stability

f uniformly hyperbolic + transversality property

$\Updownarrow$ (Robbin, de Melo, Robinson, Mañé)

for every g in a C^1 neighborhood there exists a homeomorphism $h_g : M \to M$ with $g \circ h_g = h_g \circ f$



Ω-stability

f uniformly hyperbolic + no-cycles

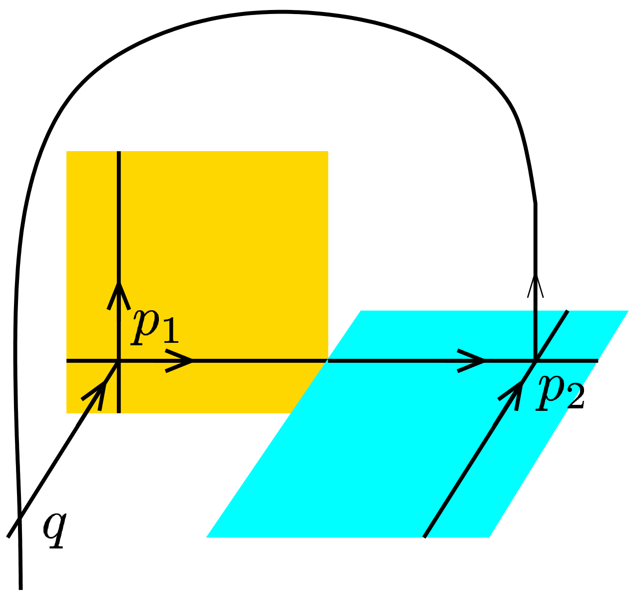
$\Updownarrow$ (Smale, Palis, Mañé)

for every g in a C^1 neighborhood there exists a homeomorphism $h_g : \Omega(f) \to \Omega(g)$ with $g \circ h_g = h_g \circ f$

Robinson, Hayashi: corresponding results for flows.

Heterodimensional cycles

Cycles involving periodic saddles with different stable dimensions:

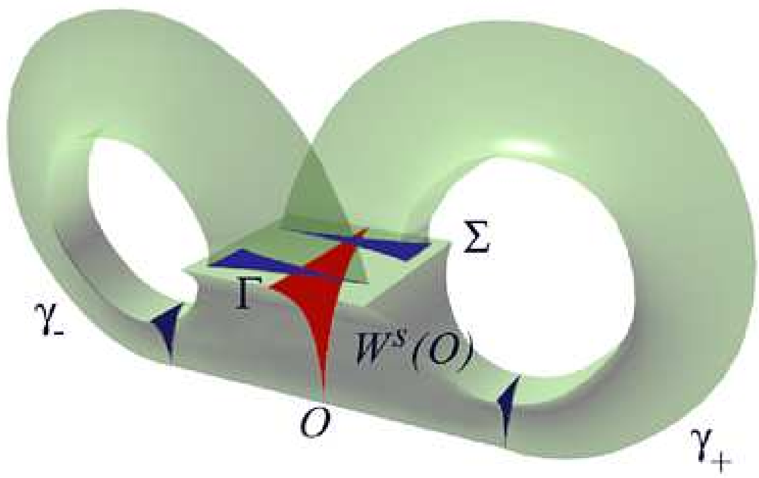


Abraham, Smale, Shub, Mañé:

Thm. *For $d \geq 3$ there are open sets $\mathcal{U} \subset \mathrm{Diff}^1(\mathbb{T}^d)$ such that every $f \in \mathcal{U}$ is transitive and has periodic saddles with different stable dimensions; in particular, f is not uniformly hyperbolic.*

Singular attractors of flows

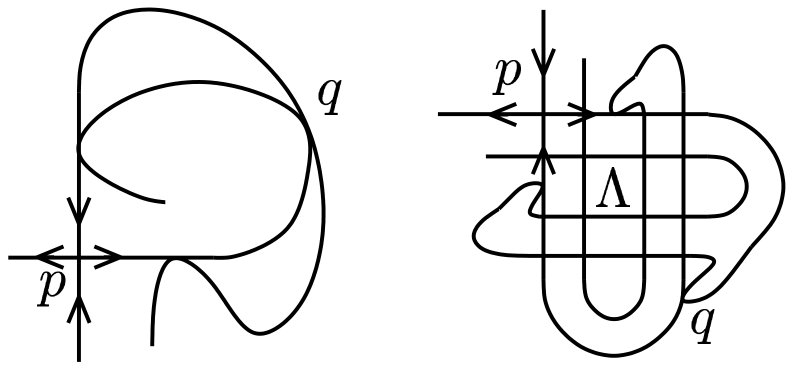
In the case of flows, heterodimensionality may arise from equilibrium points accumulated by regular orbits:



Afraimovich, Bykov, Shilnikov, Guckenheimer, Williams:
Thm. *If* $\dim M \geq 3$ *there are open sets* $\mathcal{V} \subset \mathcal{X}^1(M)$ *such that every* $X \in \mathcal{V}$ *has a (transitive) attractor* Λ_X *that contains equilibrium points and regular orbits.*

Λ_X is not hyperbolic: the decomposition $E^u \oplus E^X \oplus E^s$ can not extend continuously to the equilibrium points.

Homoclinic tangencies



Newhouse, Palis, Viana, Romero:

Thm. *Let* $\dim M \geq 2$*. Close to any* $f : M \to M$ *with a homoclinic tangency of a saddle* p*, there are open sets* $\mathcal{U} \subset \mathrm{Diff}^2(M)$ *such that:*

- *Every* $g \in \mathcal{U}$ *is approximated by a diffeomorphism with a tangency associated to the continuation of* p*.*

- *If* p *is sectionally dissipative, there exists a residual set* $\mathcal{R} \subset \mathcal{U}$ *such that every* $g \in \mathcal{R}$ *has infinitely many periodic attractors.*

A periodic point p is *sectionally dissipative* $\Leftrightarrow$ the product of any two eigenvalues has norm less than 1.

A robustly transitive map

The following construction is due to Mañé:

1. Start with a linear map $f_0 : M \to M$ of $M = \mathbb{T}^3$, with eigenvalues $\sigma_1 > 3$ and $\sigma_2 > 1 > \sigma_1 > 0$. Let

$$TM = E_0^1 \oplus E_0^2 \oplus E_0^3$$

be the decomposition into eigenspaces and $\mathcal{F}^2$ be the linear foliation tangent to E_0^2. Let $p = 0$ be the fixed point and $W \supset V \ni p$ be small neighborhoods.

2. Consider a perturbation f of f_0 such that

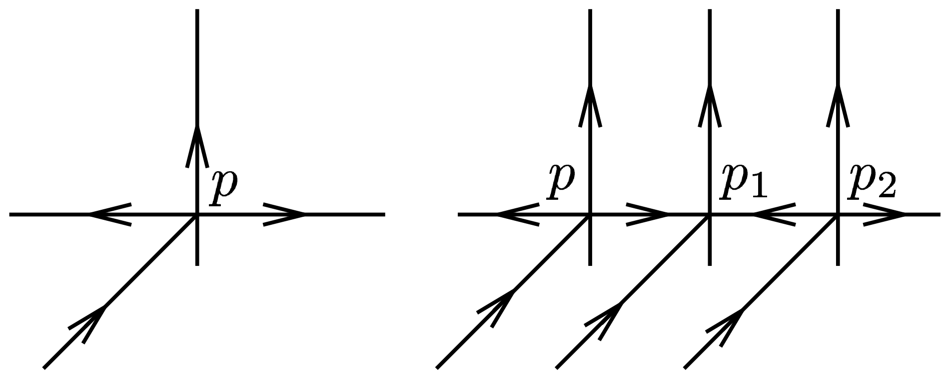
- f preserves $\mathcal{F}^2$ and coincides with f_0 outside V
- there is a Df-invariant decomposition

$$TM = E^1 \oplus E^2 \oplus E^3$$

such that E^1 is expanding, by a factor > 3, E^3 is contracting, and E^2 is "in between".

Let $\mathcal{F}^1$ be the strong-unstable foliation, tangent to E^1.

3. f may be taken with two saddle points with different stable dimensions inside V, for instance, via a saddle-node bifurcation:



4. There exists $L > 0$ such that for any strong-unstable segment γ_1 with length $|\gamma_1| > L$ the image $f(\gamma_1)$ contains some segment with length $> L$ outside W.

Consequently, every strong-unstable segment contains a point $\bar{z}$ whose forward orbit intersects W only finitely many times.

5. Let $\bar{z}$ be such a point and γ_2 be a segment of $\mathcal{F}^2$ through $\bar{z}$. The length of $f^n(\gamma_2)$ goes exponentially to infinity when $n \to +\infty$:

- If $|f^n(\gamma_2)|$ is smaller than $\operatorname{dist}(V, W^c)$ then $f^n(\gamma_2)$ is disjoint from V and so it is expanded by f.

- In any case most of $f^n(\gamma_2)$ is outside V, assuming the diameter of V is much smaller than $\operatorname{dist}(V, W^c)$.

6. Given non-empty open sets $A, B \subset M$ there is $n \geq 1$ such that $f^n(A)$ intersects B ($\Rightarrow f$ is transitive):

Take strong-unstable segment $\gamma_1 \subset A$, point $\bar{z} \in \gamma_1$, and segment γ_2 through $\bar{z}$ as before. Use the fact that the leaves of the foliation $\mathcal{F}^2$ are dense:

For every open set B there exists $K > 0$ such that any $\mathcal{F}^2$-segment with length $> K$ intersects B.

7. The argument works for any perturbation g of f: The main point is that foliation $\mathcal{F}^2$ is stable, because it is normally hyperbolic. This means that g has an invariant foliation $\mathcal{F}^2_g$ and there exists a homeomorphism close to the identity that sends leaves of $\mathcal{F}^2$ to leaves of $\mathcal{F}^2_g$. So the leaves of $\mathcal{F}^2_g$ are also dense in M.

Bonatti, Viana:

Thm. *There exist opens sets $\mathcal{U} \subset \mathrm{Diff}^1(\mathbb{T}^4)$ such that every $f \in \mathcal{U}$ is transitive and admits neither uniformly expanding nor uniformly contracting invariant subbundle.*

The proof is a variation of Mañé's argument. It extends to $\mathbb{T}^d$ for any $d \geq 4$.

1. Start with a linear map $f_0 : \mathbb{T}^4 \to \mathbb{T}^4$ with four real eigenvalues $\sigma_1 > \sigma_2 > 1 > \sigma_3 > \sigma_4$. Replace f_0 by an iterate to ensure that $\sigma_2 > 3$, $\sigma_3 < 1/3$ and there are at least two fixed points p_1 and p_2. Let

$$TM = E_0^u \oplus E_0^s$$

be the hyperbolic decomposition. Fix thin invariant cone fields $\mathcal{C}^u$ and $\mathcal{C}^s$ around E_0^u and E_0^s.

2. Let $W_i \Supset V_i \ni p_i$ be small neighborhoods, for $i = 1, 2$. Consider a perturbation f of f_0 such that

- f is C^1 close to f_0 outside $V_1 \cup V_2$
- Df preserves $\mathcal{C}^u$ and uniformly expands area inside it, and Df^{-1} preserves $\mathcal{C}^s$ and uniformly expands area in it
- Df uniformly expands $\mathcal{C}^u$ outside V_1 and Df^{-1} uniformly expands $\mathcal{C}^s$ outside V_2.

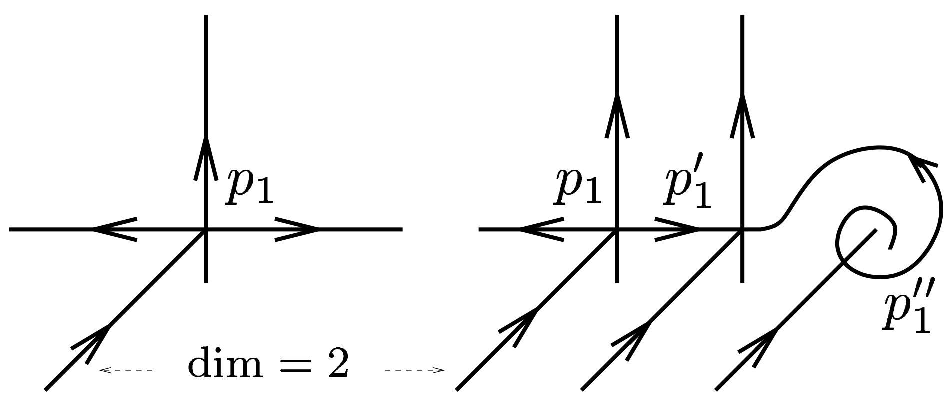
Invariant cone field $\Rightarrow$ Invariant decomposition $TM = E \oplus F$.

3. Take f with two more fixed points p_1', p_1'' inside V_1, besides p_1, and two more fixed points p_2', p_2'' in V_2, besides p_2, satisfying

p_1' has three real contracting and one real expanding eigenvalues

p_1'' has two complex expanding and two real contracting eigenvalues

similarly for p_2', p_2'' reversing the roles of expansion and contraction.



So, the bundles E and F are not hyperbolic, and they do not have invariant subbundles.

4. Taking the cone fields thin enough, there is $L > 0$ such that any centre-unstable disk of radius L intersects any centre-stable disk of radius L.

Using expansion of area (in the place of expansion of norm) inside $\mathcal{C}^u$ we show that every centre-unstable disk contains a point $\bar{z}$ whose forward orbit intersects V_2 finitely many times only.

It follows, as before, that every centre-unstable disk around $\bar{z}$ has an iterate containing a disk of radius L.

5. Given non-empty open sets $A, B \subset M$ consider centre-unstable disk $D^{cu} \subset A$ and centre-stable disk $D^{cs} \subset B$. By the previous step there exists $n \geq 1$ such that $f^n(D^{cu})$ intersects D^{cs}, and so $f^n(A)$ intersects B. This proves f is transitive.

Summary of Lecture # 1

• Hyperbolic systems admit a decomposition into finitely many invariant and indecomposable (transitive) pieces.

• The dynamics on each elementary piece and the statistics of orbits in the basins are well-understood.

• Hyperbolicity is the key ingredient for structural stability of the system.

• There are open subsets of non-hyperbolic systems in every $\mathrm{Diff}^r(M^d)$ except, possibly, for $r = 1$ and $d = 2$.

  • coexistence of infinitely many periodic attractors

  • robustly indecomposable sets that are not hyperbolic (even without any invariant contracting or expanding subbundles).

• There are two known mechanisms generating robustly non-hyperbolic systems: homoclinic tangencies and heterodimensional cycles. In the case of flows the latter may arise from the presence of singularities.

Projective hyperbolicity

Def. An invariant set Λ is *projectively hyperbolic* if for every $x \in \Lambda$ there is a decomposition $T_xM = E^1_x \oplus E^2_x$ satisfying

1. (invariance) $Df(x)E^*_x = E^*_{f(x)}$ for $* = 1, 2$
2. (domination) $\dfrac{\|Df^n(x)v_2\|}{\|v_2\|} \le C\lambda^n \dfrac{\|Df^n(x)v_1\|}{\|v_1\|}$ for every $v_1 \in E^1_x$, $v_2 \in E^2_x$, and $n \ge 1$,

with $C > 0$ and $\lambda < 1$ independent of x.

Rmk. It is always assumed that the dimensions of E^1_x and E^2_x are constant over the invariant set.

More generally, a decomposition into $k \ge 2$ subspaces

$$T_xM = F^1_x \oplus \cdots \oplus F^k_x$$

is called dominated if $E^1 = F^1 \oplus \cdots \oplus F^i$ dominates $E^2 = F^{i+1} \oplus \cdots \oplus F^k$ for every $1 \le i < k$.

Basic properties:

• (transversality) If $T_\Lambda M = E^1 \oplus E^2$ is a dominated decomposition, the angles between E^1_x and E^2_x are bounded from zero.

• (uniqueness) Given the dimensions of the factors, a dominated decomposition is unique when it exists.

• (continuity) A dominated decomposition is continuous and extends uniquely to a dominated decomposition on the closure of the domain.

• (cone field) An invariant set Λ is projectively hyperbolic if and only if it admits a continuous invariant cone field.

• (extension) If Λ admits a dominated decomposition, so does any invariant set inside a small neighborhood U of Λ (with the same dimensions of the factors).

• (robustness) Given any g in a C^1 neighborhood of f, every g-invariant set contained in U has a dominated decomposition (with the same dimensions of the factors).

Partial hyperbolicity

Def. An invariant set Λ is *partially hyperbolic* if for every $x \in \Lambda$ there is a decomposition $T_xM = E^1_x \oplus E^2_x$ satisfying

1. E^1 and E^2 are invariant
2. E^1 dominates E^2
3. either E^1 is uniformly expanding or E^2 is uniformly contracting.

In the first case we write $E^1 = E^u$ and $E^2 = E^{cs}$, in the second one $E^1 = E^{cu}$ and $E^2 = E^s$.

• (cone field) An invariant set Λ is partially hyperbolic if and only if it admits a continuous unstable or stable cone field.

unstable cone field: $f(C_x) \subset C_{f(x)}$ and there are $c > 0$, $\sigma > 1$ such that

$$\|Df^n(x)v\| \geq c\sigma^n \|v\|$$

for $v \in C_x$ and $n \geq 0$.

• (integrability) If $E^1 = E^u$ is uniformly expanding it has a unique integral foliation $\mathcal{F}^u$. Analogously if $E^2 = E^s$ is uniformly contracting.

Generic dynamical systems

We consider C^1 dynamical systems, $\text{Diff}^1(M)$ or $\mathcal{X}^1(M)$, endowed with the C^1 topology. The goal is to obtain a very general dynamical decomposition theorem.

Two fundamental tools:

C^1 **closing lemma (Pugh):** if $f^n(x)$ accumulates on x when $n \to \infty$, there exists a C^1 small perturbation g of f for which x is periodic.

C^1 **connecting lemma (Hayashi):** if $W^u(p, f)$ and $W^u(q, f)$ accumulate on some non-periodic point x, there exists a C^1 small perturbation g of f for which $W^u(p, g)$ intersects $W^s(p, g)$.

Kupka, Smale, Pugh:

Thm. *There exists a residual subset* $\mathcal{R}_0$ *of* $\text{Diff}^1(M)$ *so that for every* $f \in \mathcal{R}_0$

1. *all periodic points are hyperbolic*
2. *all their stable and unstable manifolds are transverse*
3. *periodic points are dense in the non-wandering set*

Maximal transitive sets and homoclinic classes

An invariant set Λ is *transitive* $\Leftrightarrow$ the forward orbit of some $z \in \Lambda$ is dense in Λ $\Leftrightarrow$ for any open sets $A, B \subset \Lambda$ there exists $n \geq 1$ such that $f^n(A)$ intersects B.

We call it *maximal* if Λ is not properly contained in another transitive set.

The *homoclinic class* of a periodic point p of f is the closure $H(p, f)$ of the transverse intersections between the stable manifold and the unstable manifold of the orbit of p.

The elementary pieces of a uniformly hyperbolic system are homoclinic classes, and maximal transitive sets.

In general:

- Homoclinic classes are transitive sets. Every transitive set is contained in a maximal transitive set (Zorn).
- Maximal transitive sets and homoclinic classes need not be 2-by-2 disjoint.
- There may be infinitely many disjoint maximal transitive sets and infinitely many disjoint homoclinic classes.

For a residual subset $\mathcal{R} \subset \mathcal{R}_0$ of $\mathrm{Diff}^1(M)$:

• (Bonatti, Díaz) Two periodic points are in the same transitive set if and only if their homoclinic classes coincide.

• (Arnaud) Any transitive set containing a periodic point p is contained in $H(p, f)$. Hence, homoclinic classes are maximal transitive sets.

A transitive set is *saturated* if every transitive set that intersects it is contained in it.

• (Carballo, Morales, Pacifico) Homoclinic classes are saturated transitive sets. Hence, two homoclinic classes either are disjoint or they coincide.

• (Abdenur) The number of homoclinic classes is either infinite or constant in a neighborhood of f.

Ideas of the proofs of (Arnaud) and (CMP):

Homoclinic classes and closures of stable and unstable manifolds of a periodic orbit vary semi-continuously with the dynamics. Hence, there is a residual subset $\mathcal{R}$ of points of continuity. Consider $f \in \mathcal{R}$:

1. $\overline{W^u(p,f)} \cap \overline{W^s(p,f)} = H(p,f)$.

Suppose x is in the intersection of the closures. The connecting lemma gives g arbitrarily close to f such that $x \in W^u(p,g) \cap W^s(p,g)$ (the periodic case requires a separate argument). Make the intersection transverse, then $x \in H(p,g)$. By continuity, $x \in H(p,f)$.

2. $\overline{W^u(p,f)}$ is Lyapunov stable for f.

Let V be a neighborhood of the closure. Suppose there are points arbitrarily close to some $x \in \overline{W^u(p,f)}$ whose forward orbits approach some point $z \notin V$. Using the connecting lemma, there are small perturbations g such that the closure of $W^u(p,g)$ contains z. This contradicts the continuity.

3. If K is a transitive set intersecting $H(p,f)$ then $K \subset H(p,f)$.

Because Lyapunov stability implies that K is contained in $\overline{W^u(p,f)}$ and in $\overline{W^s(p,f)}$.

So, homoclinic classes are maximal transitive sets and saturated transitive sets for every $f \in \mathcal{R}$.

We used a refined version of Hayashi's connecting lemma, by Arnaud, Hayashi, and Wen, Xia:

Thm. *Let $f : M \to M$ be a diffeomorphism, x be a non periodic point, and $\mathcal{U}$ be a C^1 neighborhood of f.*

There exists $N \geq 1$ such that for every neighborhood V of x there exists a neighborhood $W \subset V$ such that, given any points y and z that are outside

$$V_N = \bigcup_{j=0}^{N-1} f^j(V)$$

and such that W contains some forward iterate of y and some backward iterate of z, there exists $g \in \mathcal{U}$ coinciding with f outside V_N and for which z is in the forward orbit of y.

A dynamical decomposition theorem

A diffeomorphism $f \in \mathcal{R}$ is *tame* if it has finitely many homoclinic classes, and *wild* otherwise.

Thm (Abdenur). *Every tame diffeomorphism admits a dynamical decomposition into finitely many transitive sets, with no cycles. Moreover, this decomposition is robust restricted to the residual subset $\mathcal{R}$.*

More precisely, there exists

- a *filtration* $M_0 \subset M_1 \subset \cdots \subset M_N = M$,
- periodic points $p_i \in M_i \setminus M_{i-1}$, $1 \leq i \leq N$,
- and a C^1 neighborhood $\mathcal{U}$ of f,

such that for every $g \in \mathcal{U} \cap \mathcal{R}$ the continuation $p_i(g)$ of p_i is defined and its homoclinic class $\Lambda_i(g) = H(p_i(g), g)$ is the maximal invariant set of g in $M_i \setminus M_{i-1}$. Then

$$\Omega(g) = \Lambda_1(g) \cup \cdots \cup \Lambda_N(g).$$

Moreover, the elementary pieces $\Lambda_i(g)$ are projectively hyperbolic (hyperbolic if $\dim M = 2$).

Λ is *robustly transitive* if it is the maximal f-invariant set in a neighborhood U, and the maximal g-invariant set

$$\Lambda(g) = \bigcap_{n \in \mathbb{Z}} g^n(U)$$

in U is transitive for every g in a C^1 neighborhood $\mathcal{U}$ of f.

Λ is *generically transitive* if the latter holds for a residual subset of $\mathcal{U}$.

Mañé, Bonatti, Díaz, Pujals, Ures:

Robustly transitive $\Rightarrow$ Projectively hyperbolic.

The arguments extend to the elementary pieces in the theorem, which are generically transitive (at least).

Wild dynamics

The first example were Newhouse's C^2 diffeomorphisms with infinitely many periodic attractors coexisting.

For $\dim M \geq 3$ coexistence of periodic attractors occurs also in the C^1 setting (Bonatti, Díaz).

Moreover,

Carballo, Morales, Bonatti, Díaz:

Thm. *If* $\dim M \geq 3$, *there exist open sets* $\mathcal{U} \subset \mathrm{Diff}^1(M)$ *and residual subsets* $\mathcal{R}_{\mathcal{U}} \subset \mathcal{U}$ *such that every* $f \in \mathcal{R}_{\mathcal{U}}$ *has infinitely many non-trivial disjoint homoclinic classes, and also infinitely many saturated transitive sets without periodic points.*

Summary of Lecture #2

• Projectively hyperbolic set Λ: for each x there is a decomposition $T_x M = E_x^1 \oplus E_x^2$ satisfying

1. $Df(x)E_x^1 = E_{f(x)}^1$ and $Df(x)E_x^2 = E_{f(x)}^2$
2. $\dfrac{\|Df^n(x)v_2\|}{\|v_2\|} \leq C\lambda^n \dfrac{\|Df^n(x)v_1\|}{\|v_1\|}$ (domination)

for every $x \in \Lambda$, $v_1 \in E_x^1$, $v_2 \in E_x^2$, and $n \geq 1$, and uniform constants $C > 0$ and $\lambda < 1$.

• Maximal transitive sets, saturated transitive sets, and homoclinic class: candidates to elementary dynamical pieces.

• For generic C^1 diffeomorphisms, homoclinic classes are maximal transitive sets and saturated transitive sets. In particular, they are 2-by-2 disjoint.

• If the number of disjoint homoclinic classes is finite (tame dynamics), there is a dynamical decomposition into finitely many pieces, transitive and projectively hyperbolic. Moreover, there are no cycles, and the decomposition is generically robust.

Dynamics of projectively hyperbolic systems

Next, we study the dynamics of projectively hyperbolic sets and attractors. Two warnings:

- Most results are for C^2 diffeomorphisms.

- Projective hyperbolicity is a very weak property (e.g. it may coexist with homoclinic tangencies), except when all the subspaces in the decomposition have dimension 1.

Thm (Pujals, Sambarino). *Let Λ be a projectively hyperbolic set of a surface diffeomorphism such that all periodic points contained in it are hyperbolic saddles. Then Λ is the union of a hyperbolic set and a finite number of smooth invariant circles supporting irrational rotations.*

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Attractors and physical measures

A transitive invariant set Λ is an *attractor* if the basin of attraction (topological)

$$B(\Lambda) := \{x \in M : \omega(x) \subset \Lambda\}$$

has positive Lebesgue measure.

Topological attractor: if the basin is a neighborhood of the attractor (e.g. tame systems).

An ergodic invariant probability μ is a *physical measure*, or *SRB measure,* if the basin of attraction (ergodic)

$$B(\mu) := \{x \in M : \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j(x)} = \mu\}$$

has positive Lebesgue measure.

Conj (Palis). For a dense subset of $\mathrm{Diff}^r(M)$, there are only finitely many attractors and physical measures, and Lebesgue almost every point is in the unions of their basins of attraction (topological and ergodic).

Prob. For generic diffeomorphisms, the union of the basins of all topological attractors is open and dense in M ?

An existence and finiteness theorem

Thm (Alves, Bonatti, Viana). *Let Λ be a projectively hyperbolic attractor of a C^2 diffeomorphism $f : M \to M$, with decomposition $T_x M = E_x^{cu} \oplus E_x^{cs}$, $x \in \Lambda$. Assume*

1. *E_z^{cu} is non-uniformly expanding and E_z^{cs} is non-uniformly contracting*
2. *z has simultaneous cu- and cs-hyperbolic times*

for Lebesgue almost every point $z \in B(\Lambda)$.

There are finitely many SRB measures supported in Λ, and the union of the basins contains Lebesgue almost every point in $B(\Lambda)$.

(Extend $E^{cu} \oplus E^{cs}$ continuously to the basin.)

Non-uniformly expanding:
$\limsup_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \log \|Df^{-1} \mid E^{cu}_{f^j(z)}\| < -c < 0$

Non-uniformly contracting:
$\limsup_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df \mid E^{cs}_{f^j(z)}\| < -c < 0$

Condition 2 is automatic if E^{cu} is uniformly expanding or E^{cs} is uniformly contracting.

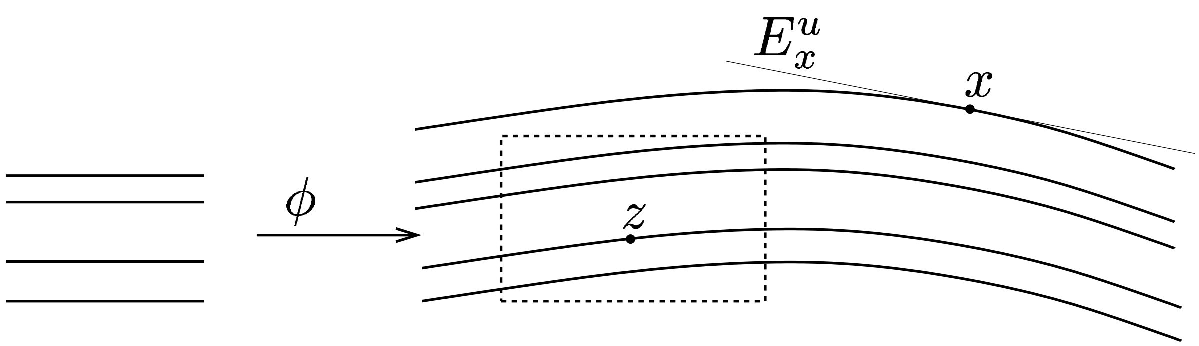
Partially hyperbolic attractors

Let $f : M \to M$ be C^2. Let Λ be an invariant set with dominated decomposition

$$T_x M = E^u_x \oplus E^c_x \oplus E^s_x, \quad x \in \Lambda,$$

with E^u uniformly expanding, E^s uniformly contracting, and $\dim E^u > 0$.

Then there exists a unique *strong-unstable* foliation $\mathcal{F}^u$ tangent to E^u_x at every point $x \in \Lambda$. Assume the leaves are entirely contained in Λ.



Foliated neighborhood at $z \in \Lambda$: homeomorphism $\phi : B \times \Sigma \to \Lambda$ onto a neighborhood of z inside Λ, with

- B = unit disk of dimension $\dim E^u$, and Σ compact
- each $\phi(\cdot, \eta)$ a diffeomorphism to a strong-unstable disk.

Gibbs u-states

An invariant probability μ on Λ is a *Gibbs u-state* if every $z \in \Lambda$ has some foliated neighborhood ϕ such that μ is equivalent to a product measure

$$\mu \approx \phi_*(\text{Lebesgue} \times \nu).$$

restricted to the image of ϕ.

Thm (Pesin, Sinai). *Let m_D be normalized Lebesgue measure along a disk D transverse to $E^c \oplus E^s$. Every limit measure of*

$$\frac{1}{n}\sum_{j=0}^{n-1} f_*^j(m_D)$$

is a Gibbs u-state (density bounded from zero and ∞).

Thm (Bonatti, Viana). *Suppose Λ is a topological attractor. For Lebesgue almost every point $x \in B(\Lambda)$, every limit measure of*

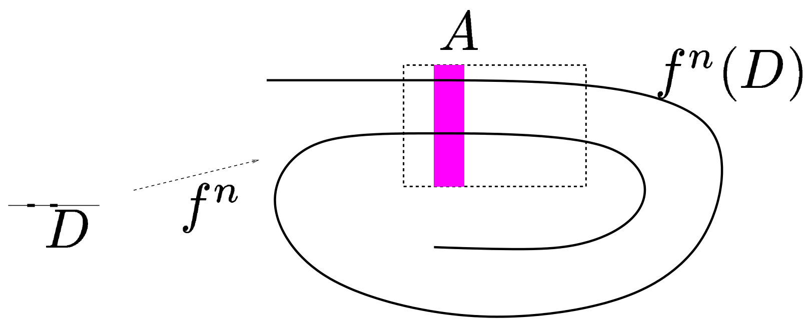
$$\frac{1}{n}\sum_{j=0}^{n-1} \delta_{f^j(x)}$$

is a Gibbs u-state (density bounded from zero and ∞).

So, SRB measures must be Gibbs u-states if they exist.

1. Let $D \subset B(\Lambda)$ be any disk transverse to $E^c \oplus E^s$. Using curvature and distortion control,

$$m_D\big(\{f^n(x) \in A\}\big) \leq \text{const}\,|A| \quad \text{for all } n.$$



2. Events $f^j(x) \in A$ and $f^k(x) \in A$ are "independent" if $|j-k|$ is big, because iterates of D are expanded.

By a large deviations argument, the m_D-probability of

$$\frac{1}{n}\#\{0 \leq j \leq n-1 : f^j(x) \in A\} > \text{const}\,|A|$$

decays exponentially with n. So, for m_D-almost every x,

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \delta_{f^j(x)}(A) \leq \text{const}\,|A|\,.$$

Mostly contracting central direction

Conversely: if μ is an ergodic Gibbs u-state such that the central direction is mostly contracting

$$\lambda^c(x) := \limsup_{n\to+\infty} \frac{1}{n} \log \|Df^n \mid E^c_x\| < 0 \quad \mu - \text{a.e.}$$

then μ is an SRB measure.

Thm (Bonatti, Viana). *Let* $\lambda^c(x) < 0$ *on a positive Lebesgue measure subset of every strong-unstable disk. Then* Λ *supports finitely many SRB measures and the union of their basins contains Lebesgue almost every point in* $B(\Lambda)$*.*

If the strong-unstable foliation $\mathcal{F}^u$ *is minimal, the SRB measure is unique.*

Minimal foliation: all leaves dense in Λ.

Gibbs *cu*-states

Now let Λ be projectively hyperbolic with decomposition

$$T_xM = E^{cu}_x \oplus E^{cs}_x, \quad x \in \Lambda.$$

Thm (Alves, Bonatti, Viana). *Assume that E^{cu} is non-uniformly expanding on a positive Lebesgue measure subset of $B(\Lambda)$. Then there exist ergodic Gibbs cu-states supported in Λ.*

Non-uniformly expanding:

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \log \|Df^{-1} \mid E^{cu}_{f^j(z)}\| < -c < 0$$

Gibbs cu-state: invariant probability measure with $\dim E^{cu}$ expanding directions (positive Lyapunov exponents) and absolutely continuous conditional measures along the corresponding unstable manifolds.

Hyperbolic times

The key tool in the proof is the following notion: n is a *cu-hyperbolic time* for z if

$$\|Df^{-k} \mid E^{cu}_{f^n(z)}\| \le e^{-ck/2} \quad \text{for all } 1 \le k \le n.$$

If E^{cu} non-uniformly expanding at z then z has positive frequency of cu-hyperbolic times:

$$\#\{cu\text{-hyperbolic times } \le n\} > \theta(c)n \quad \text{for all } n$$

with $\theta(c) > 0$.

Rmk. If $w = \lim_i f^{n_i}(z_i)$ where each n_i is a hyperbolic time of z_i and $n_i \to \infty$, then w has unstable manifold of dimension $\dim E^{cu}$ and size $> \delta(c) > 0$.

To construct Gibbs cu-states one starts with any disk D transverse to E^{cs} such that E^{cu} is non-uniformly expanding on a positive Lebesgue measure set $D_0 \subset D$, and considers accumulation points of

$$\frac{1}{n}\sum_{j=0}^{n-1} f_*^j\big(m_D \mid \{z : j \text{ is a } cu\text{-hyperbolic time for } z\}\big).$$

Any ergodic Gibbs cu-state μ that has $\dim E^{cs}$ negative Lyapunov exponents is an SRB measure.

That is the case for Gibbs cu-states obtained before, if on the positive Lebesgue measure set $D_0 \subset D$

• E^{cs} is non-uniformly contracting

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df \mid E^{cs}_{f^j(z)}\| < -c < 0$$

• there is positive frequency of simultaneous cu- and cs-hyperbolic times

n is a *cs-hyperbolic time* for z if

$$\|Df^k \mid E^{cs}_{f^{n-k}(z)}\| \le e^{-ck/2} \quad \text{for all } 1 \le k \le n.$$

In this way one constructs SRB measures supported in Λ.

Using the existence of stable and unstable manifolds with size $> \delta(c)$ we get that the SRB measures are finitely many.

To prove that their basins cover Lebesgue almost every point in Λ: if not, we could use the exceptional positive Lebesgue set to construct one more SRB measure.

Summary of Lecture # 3

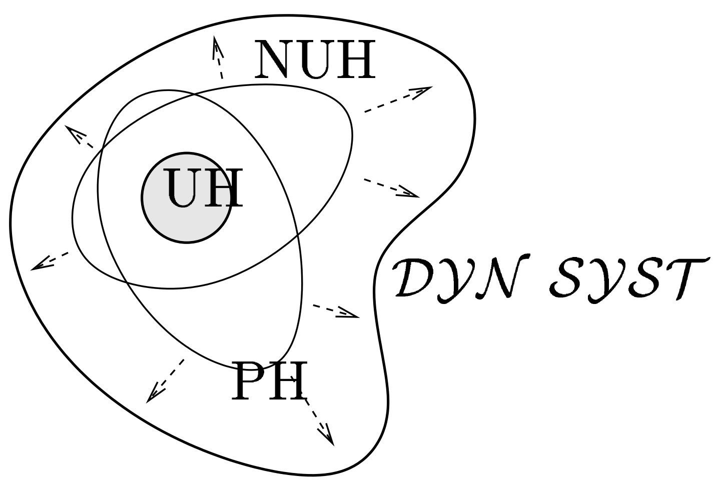
• Definitions of attractor and physical (SRB) measure.

• Gibbs u-states of partially hyperbolic attractors with expanding subbundle. Gibbs u-states exist, and every SRB measure is an ergodic Gibbs u-state.

• Existence and finiteness of SRB measures when the central direction is mostly contracting.

• Gibbs cu-states of projectively hyperbolic attractors with non-uniformly expanding subbundle. Positive frequency of cu-hyperbolic times yields Gibbs cu-states.

• Existence and finiteness of SRB measures for projectively hyperbolic attractors non-uniformly hyperbolic and with simultaneous hyperbolic times.



Non-uniform hyperbolicity

Let $f : M \to M$ be a C^r diffeomorphism, $r \geq 1$, on a compact manifold M and μ an f-invariant probability measure.

Oseledets: μ-almost every point admits a splitting

$$T_x M = E_x^1 \oplus \cdots \oplus E_x^k\,, \quad k = k(x),$$

and real numbers $\lambda_1(f,x) > \cdots > \lambda_k(f,x)$ such that

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \|Df^n(x) v_i\| = \lambda_i(f,x)$$

for every non-zero $v_i \in E_x^i$.

non-uniform hyperbolicity $\Leftrightarrow$ all Lyapunov exponents $\lambda_i(f,x)$ non-zero μ-almost everywhere

Prob. Are most systems non-uniformly hyperbolic ?

Fix μ (e.g. Lebesgue measure) and consider

$$\mathrm{Diff}^r_\mu(M) = \{C^r \text{ diffeomorphisms preserving } \mu\}.$$

A dichotomy for conservative systems

Let μ be Lebesgue measure and $r = 1$.

Thm (Bochi). *If* $\dim M = 2$, *there is a residual subset* $\mathcal{R}$ *of* $\mathrm{Diff}^1_\mu(M)$ *such that for every* $f \in \mathcal{R}$

- *either* $\lambda_i(f, x) = 0$ *at* μ*-almost every* $x \in M$
- *or* f *is Anosov (and then* $M = \mathbb{T}^2$*).*

Thm (Bochi, Viana). *In any dimension, there is a residual subset* $\mathcal{R}$ *of* $\mathrm{Diff}^1_\mu(M)$ *such that for* $f \in \mathcal{R}$ *and* μ*-almost every* $x \in M$,

- *either all* $\lambda_i(f, x) = 0$ *or*
- *the Oseledets splitting is dominated on the orbit of* x

The latter implies that

• the angles between the subspaces E^i are bounded from zero over the orbit of x

• and the splitting extends continuously to the closure of the orbit.

Ex (Bonatti, Viana, Tahzibi). For $M = \mathbb{T}^4$ and μ =Lebesgue measure, there exists an open subset $\mathcal{U}$ of $\mathrm{Diff}^1_\mu(M)$ such that, for every $f \in \mathcal{U}$,

- f admits a dominated splitting $TM = E \oplus F$ with $\dim E = \dim F = 2$
- these are the only continuous invariant subbundles;
- f is transitive and, for a residual subset $\mathcal{R}_0$, ergodic;
- E is not expanding and F is not contracting.

Cor. *There exists a residual subset $\mathcal{S}$ of $\mathcal{U}$ such that for every $f \in \mathcal{S}$ the Oseledets splitting of f is $E^1_x \oplus E^2_x$ with $\dim E^1_x = \dim E^2_x = 2$.*

1. Take $\mathcal{S} = \mathcal{R} \cap \mathcal{R}_0$, where $\mathcal{R}$ is the residual set in the theorem. For $f \in \mathcal{S}$,

2. The Lyapunov exponents are not all zero, by the first property. So, by the theorem and using ergodicity, the Oseledets splitting extends to a dominated splitting on the whole M.

3. Since f has no other continuous invariant subbundles, this extension must coincide with $E \oplus F$.

Deterministic products of matrices

Let $f : M \to M$ be a transformation on a compact metric space M. A *linear cocycle* over f is a skew-product

$$F : M \times \mathbb{R}^d \to M \times \mathbb{R}^d, \quad F(x,v) = (f(x), A(x)v)$$

where $A : M \to \mathrm{SL}(d,\mathbb{R})$ (or $\mathrm{GL}(d,\mathbb{R})$).

Then $F^n(x) = (f^n(x), A^n(x)v)$ with

$$A^n(x) = A(f^{n-1}(x)) \cdots A(f(x))\, A(x).$$

Oseledets: Let μ be any f-invariant probability. For μ-almost every point there is a filtration

$$\{x\} \times \mathbb{R}^d = F_x^1 > \cdots > F_x^k > \{0\}\,, \quad k = k(x),$$

and real numbers $\lambda_1(A,x) > \cdots > \lambda_k(A,x)$ such that

$$\lim_{n \to +\infty} \frac{1}{n} \log \|A^n(x) v_i\| = \lambda_i(A,x)$$

for every $v_i \in F_x^i \setminus F_x^{i+1}$. If f is invertible we even have a splitting of $\{x\} \times \mathbb{R}^d$ (better than a filtration).

Let μ be any ergodic f-invariant probability.

Thm (Bochi, Viana). *There is a residual subset $\mathcal{R}$ of all continuous maps $M \to \mathrm{SL}(d, \mathbb{R})$ such that for $A \in \mathcal{R}$*

- *either all $\lambda_i(A, x) = 0$ at μ-almost every point*
- *or the Oseledets splitting extends to a dominated splitting on the whole support of μ.*

Ex. Suppose for every $1 \leq i < d$ there exists a periodic point p_i of f in the support of μ, with period κ_i , such that the eigenvalues $\{\beta_j^i : 1 \leq j \leq d\}$ of $A^{\kappa_i}(p_i)$ satisfy

$$|\beta_1^i| \geq \cdots \geq |\beta_{i-1}^i| > |\beta_{i-1}^i| = |\beta_i^i| > |\beta_{i+1}^i| \geq \cdots \geq |\beta_d^i|$$

and β_i^i, β_{i+1}^i are complex conjugate (not real).

This obstructs the existence of a dominated splitting, for any map $B : M \to \mathrm{SL}(d, \mathbb{R})$ in a C^0 neighborhood $\mathcal{U}$ of A. Therefore, all Lyapunov exponents are zero, for B in a residual subset of $\mathcal{U}$.

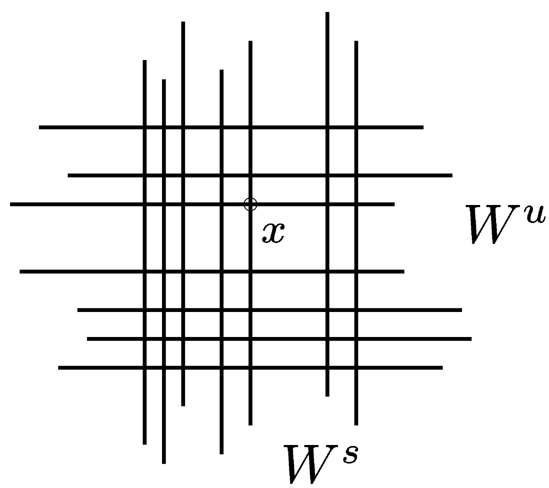
Prevalence of non-zero exponents

Suppose $f : M \to M$ is a C^1 diffeomorphism with derivative Hölder continuous, and

• μ is (non-uniformly) *hyperbolic:* $\lambda_i(f, x) \neq 0$ for all i and μ-almost every point;

• μ has *local product structure:* for μ-almost every point x there is a product "neighborhood" V such that

$$\mu \mid V \approx \mu^u \times \mu^s.$$



Rmk. Lebesgue measure has local product structure if it is hyperbolic ($\Leftarrow$ absolute continuity of foliations).

Same for hyperbolic invariant measures with conditional measures along unstable manifolds absolutely continuous with respect to Lebesgue measure.

Let (f, μ) be non-uniformly hyperbolic with local product structure.

Thm. *For any $0 < r \leq \infty$, there exists an open and dense subset $\mathcal{O}$ of all C^r maps $A : M \to \mathrm{SL}(d, \mathbb{R})$ such that every $A \in \mathcal{O}$ has non-zero Lyapunov exponents at μ-almost every point. The complement of $\mathcal{O}$ has ∞ codimension in $C^r(M, \mathrm{SL}(d, \mathbb{R}))$.*

∞ *codimension* $\Leftrightarrow$ contained in finite unions of closed submanifolds with arbitrarily large codimension.

Suppose $f : M \to M$ is uniformly hyperbolic.

Thm. *For any $0 < r \leq \infty$, there exists an open and dense subset $\mathcal{O}$ of all C^r maps $A : M \to \mathrm{SL}(d, \mathbb{R})$, whose complement has ∞ codimension, such that every $A \in \mathcal{O}$ has non-zero Lyapunov exponents at μ-almost every point, for every invariant measure with local product structure.*

Probably, all Lyapunov exponents have multiplicity 1.

Bonatti, Gomez-Mont, Viana: a particular case of the second theorem, assuming a property of domination.

An application

Consider $d = 2$ and $f : M \to M$ uniformly expanding.

Def. $A : M \to \mathrm{SL}(2, \mathbb{R})$ is *bundle-free* if, for any $\eta \geq 1$, there is no $C^{\min(r,\mathrm{Lip})}$ map $\psi : x \mapsto \{\psi_1(x), \ldots, \psi_\eta(x)\}$ assigning to each $x \in M$ a subset of $\mathbb{RP}^1$ with exactly η elements, invariant under the cocycle

$$A(x)\big(\{\psi_1(x), \ldots, \psi_\eta(x)\}\big) = \{\psi_1(f(x)), \ldots, \psi_\eta(f(x))\}$$

for all $x \in M$.

Thm. *Suppose $A \in C^r$, $r > 0$ is bundle-free and there exists some periodic point $p \in M$ of f such that A is hyperbolic over the orbit of p. Then $\lambda_1(A, x) > 0$ at μ-almost every point, for every f-invariant ergodic measure with local product structure.*

The condition on the existence of some periodic point over which the cocycle is hyperbolic, is mild (open and dense subset whose complement has ∞ codimension).

Ex. Let $f : S^1 \to S^1$ be expanding, μ be the absolutely continuous invariant measure, and $A : S^1 \to \mathrm{SL}(2, \mathbb{R})$ be of the form

$$A(x) = R_{\alpha(x)} A_0$$

- A_0 is some hyperbolic matrix,
- $\alpha : S^1 \to S^1$ is a continuous function with $\alpha(0) = 0$,
- $R_{\alpha(x)}$ is the rotation of angle $\alpha(x)$.

Assume that $2 \deg(\alpha)$ is *not* a multiple of $\deg(f) - 1$.

Cor. *There exists a C^0 neighbourhood $\mathcal{U}$ of A such that*

1. $\lambda_1(B, \mu) = 0$ *for B in a residual subset $\mathcal{R} \cap \mathcal{U}$;*
2. $\lambda_1(B, \mu) > 0$ *for all $B \in \mathcal{U} \cap C^r$, any $r > 0$.*

First, let $\mathcal{U}_0$ be the isotopy class of A in the space of continuous maps from M to $\mathrm{SL}(2, \mathbb{R})$.

Claim: Given $B \in \mathcal{U}_0$ there is no B-invariant continuous map

$$\psi : M \ni x \mapsto \{\psi_1(x), \ldots, \psi_\eta(x)\}$$

assigning a constant number $\eta \geq 1$ of elements of $\mathbb{RP}^1$ to each point $x \in M$.

The proof of the claim is by contradiction. Suppose there exists such a map. The graph

$$G = \{(x, \psi_i(x)) \in S^1 \times \mathbb{RP}^1 : x \in S^1 \text{ and } 1 \leq i \leq \eta\}$$

represents an element (η, ζ) of the fundamental group $\pi_1(S^1 \times \mathbb{RP}^1) = \mathbb{Z} \oplus \mathbb{Z}$ (if it is connected, otherwise consider connected components).

Because B is isotopic to A, the image of G must represent

$$(\eta \deg(f), \zeta + 2\deg(\alpha)) \in \pi_1(S^1 \times \mathbb{RP}^1).$$

The factor 2 comes from the fact that S^1 is the 2-fold covering of $\mathbb{RP}^1$.

By the invariance of ψ

$$\zeta + 2\deg(\alpha) = \deg(f)\zeta$$

which contradicts the hypothesis that $\deg(f) - 1$ does not divide $2\deg(\alpha)$.

Now we can prove the Corollary:

1. By Bochi there is a residual subset $\mathcal{R}$ of continuous cocycles which either are uniformly hyperbolic or have both Lyapunov exponents equal to zero.

The claim rules out the first case, for all $B \in \mathcal{R} \cap \mathcal{U}_0$. Hence, B has both Lyapunov exponents equal to zero almost everywhere.

2. The claim implies that every $B \in \mathcal{U}_0 \cap C^r$, $r > 0$ is bundle-free. Condition $\alpha(0) = 0$ ensures that $p = 0$ is a hyperbolic fixed point, for B in a neighborhood $\mathcal{U} \subset \mathcal{U}_0$.

Using the theorem, $\lambda_1(B, \mu) > 0$ for all $B \in \mathcal{U} \cap C^r$.

Summary of Lecture # 4

• A dichotomy for generic conservative diffeomorphisms: projective hyperbolicity or no hyperbolicity at all (every Lyapunov exponent equal to zero), at Lebesgue almost every orbit.

• This extends to generic continuous linear cocycles over any transformation.

• The conclusion is radically different for C^r cocycles, $r > 0$, over a non-uniformly hyperbolic transformation with local product structure: the overwhelming majority of C^r cocycles have non-zero Lyapunov exponents.