SRB measures for partially hyperbolic systems whose central direction is mostly contracting *

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Abstract

We consider partially hyperbolic diffeomorphisms preserving a splitting of the tangent bundle into a strong-unstable subbundle E^{uu} (uniformly expanding) and a subbundle E^c, dominated by E^{uu}.

We prove that if the central direction E^c is mostly contracting for the diffeomorphism (negative Lyapunov exponents), then the ergodic Gibbs u-states are the Sinai-Ruelle-Bowen measures, there are finitely many of them, and their basins cover a full measure subset. If the strong-unstable leaves are dense, there is a unique Sinai-Ruelle-Bowen measure.

We describe some applications of these results, and we also introduce a construction of robustly transitive diffeomorphisms in dimension larger than three, having no uniformly hyperbolic (neither contracting nor expanding) invariant subbundles.

1 Introduction

Uniformly hyperbolic systems [Sm] may present very rich and complicated dynamical features: even a small modification of the initial condition often leads to rather different behaviour of the orbit over long periods of time. This means that the position of individual points after a large number of iterations is essentially unpredictable. Because of this, such systems are sometimes considered “chaotic”.

Nevertheless, hyperbolic systems have very well-defined statistical properties. [Si], [Ru], [BoRu] showed that time-averages of any continuous function along almost every orbit converge to a limit as time goes to infinity. More precisely, if $f : M \to M$ is a hyperbolic diffeomorphism (similar results hold for flows) then there exist finitely many f-invariant probability measures $\mu_1, \ldots, \mu_k$ such that for any continuous function $\varphi : M \to \mathbb{R}$ and for Lebesgue almost every point $z \in M$

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(z)) = \int \varphi \, d\mu_i \tag{1}$$

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for some i. We call *basin of* μ_i the set $B(\mu_i)$ of points $z \in M$ for which (1) holds. More generally, an invariant probability measure of a general diffeomorphism is called an *SRB* (for *Sinai-Ruelle-Bowen*) *measure* if its basin $B(\mu)$ has positive Lebesgue measure. For hyperbolic diffeomorphisms f, the properties of the systems (f, μ_i) are now well-understood. In particular, they are exponentially mixing (exponential decay of correlation functions) [Bow], and stochastically stable [Ki], [Yo].

One would like to have such a satisfactory understanding of the dynamics for very general systems. On the other hand, several robust models that do not fit in the hyperbolic theory have been described since the sixties: Lorenz-like attractors [Lo], [ABS], [GuWi], Hénon-like attractors [He], [BeCa], partially hyperbolic diffeomorphisms [AbSm], [Sh1], [Ma1], [BoDí]. An important goal in Dynamics in recent years has been to enlarge the framework of hyperbolicity, in order to encompass such models in a global theory of "chaotic" dynamical systems.

A program towards such a global theory has been proposed a few years ago by J. Palis, see [Pa]. At its core is his conjecture that every dynamical system can be approximated by another having only finitely many attractors, all of which have good statistical properties (SRB measures, statistical stability).

The ergodic properties of these systems have been studied to some extent: see e.g. [Sp], [CoTr], [Pe2], [Sa] for the Lorenz-like attractors, and [BeCa], [BeYo1], [BeYo2], [BeVi1], [BeVi2] for the Hénon-like attractors. Partially hyperbolic systems are a rather large class and exhibit a very broad spectrum of dynamical behaviour. See for instance the example in [Ka] of partially hyperbolic diffeomorphisms with intertwined basins of attraction. Despite substantial progress, e.g. by [Al], [BrPe], [Car], [PeSi], [GPS], [Yo], their ergodic properties are still far from being completely understood.

In particular, it is not known in which generality such systems admit SRB measures, and this problem is a main motivation for the present work. We obtain results of existence and finitude of SRB measures, that we state in more detail below. These may be thought of as a positive step in Palis' program mentioned above.

1.1 Partially hyperbolic diffeomorphisms

Let M be a compact riemannian manifold and f be a C^1 diffeomorphism on M. Here we call f *partially hyperbolic* if there exists a continuous Df-invariant splitting

$$TM = E^{uu} \oplus E^{c} \tag{2}$$

of the tangent bundle of M, such that

$$\|(Df|E^{uu})^{-1}\| < 1 \quad \text{and} \quad \|Df|E^{c}\| \, \|(Df|E^{uu})^{-1}\| < 1$$

In other words, $Df|E^{uu}$ is uniformly expanding and *dominates* $Df|E^c$: Df expands any vector in E^c less than it expands any vector in E^{uu}. (The usual definition of partial hyperbolicity is equivalent to either f or f^{-1} satisfying this condition.) More generally, we consider diffeomorphisms with *partially hyperbolic attractors*, that is, compact subsets Λ of M such that

$$\Lambda = \bigcap_{n>0} f^n(U)$$

for some open neighbourhood U of Λ with closure $f(U) \subset U$, and there exists a splitting $T_\Lambda M = E^{uu} \oplus E^c$ of the restriction of the tangent bundle to Λ, with the same properties as before.

Partially hyperbolic systems were used by [Sh1] to give the first examples of diffeomorphisms (in the 4-torus T^4) which are robustly transitive and, yet, are not globally hyperbolic (Anosov). One calls a diffeomorphism f C^1 *robustly transitive* if any diffeomorphism g in a C^1 neighbourhood of f has orbits dense in the ambient manifold. Likewise, we say that f has a C^1 *robustly transitive attractor* Λ if for any diffeomorphism g C^1 close to f the maximal invariant set

$$\Lambda(g) = \bigcap_{n>0} g^n(U)$$

contains dense orbits. A different construction, that also produces partially hyperbolic maps, enabled [Ma1] to reduce the minimal dimension of these examples: there are C^1 robustly transitive diffeomorphisms in T^3 which are not Anosov.

All these examples have a strong form of partial hyperbolicity: there exists a continuous splitting into three nontrivial (positive dimension) subbundles

$$TM = E^{uu} \oplus E^c \oplus E^{ss}$$

where E^{ss} is uniformly contracting. On the other hand, an important restriction is that the central subbundle E^c was always 1-dimensional. This was removed by [BoDí], who constructed the first examples of C^1 robustly transitive partially hyperbolic (three nontrivial subbundles) diffeomorphisms with arbitrary central dimension.

More recently, [DPU] showed that partial hyperbolicity is, in fact, intimately related to robust transitiveness, at least in dimension three: a C^1 robustly transitive diffeomorphism of a 3-manifold must be partially hyperbolic. On the other hand, [Bon] gives examples of C^1 robustly transitive diffeomorphisms in 3-dimensional manifolds such that E^{ss} is trivial.

Here we produce further examples of this kind, and we also show that the results of [DPU] do not extend directly to higher dimensions: we obtain in T^4, cf. Theorem C, the first examples of robustly transitive diffeomorphisms that do not admit any invariant hyperbolic subbundles. On the other hand, these maps do have a weaker hyperbolicity property, namely they admit a dominated splitting.

In fact, [BDP] announce that this is always the case for a robustly transitive diffeomorphism, in any dimension.

Another result that concerns us directly, is the construction by [PeSi] of Gibbs u-states for partially hyperbolic attractors of diffeomorphisms. By *Gibbs u-states* we mean here invariant probability measures whose conditional measures [Ro] along the leaves of the strong-unstable foliation $\mathcal{F}^{uu}$ (the unique foliation tangent to the subbundle E^{uu}) are absolutely continuous with respect to the corresponding Lebesgue measure.

[Car] used their construction to exhibit SRB measures for partially hyperbolic attractors of diffeomorphisms derived from Anosov diffeomorphisms through bifurcation of a periodic orbit. The present work is partially motivated by this paper, whose results we generalize.

1.2 Statement of main results

Our first main result states that *if the central direction is mostly contracting for the diffeomorphism, then ergodic Gibbs u-states are SRB measures, and there are finitely many of them.* Let us state this in a precise form. We take $f : M \to M$ to be a C^2 diffeomorphism satisfying conditions (H1), (H2) below.

(H1) f has an attractor (not necessarily transitive), that is, a compact set $\Lambda \subset M$ which is invariant under f and is the maximal invariant set

$$\Lambda = \bigcap_{n>0} f^n(U)$$

in some open neighbourhood U of Λ with closure $f(U) \subset U$.

For instance, we may take Λ to be the whole manifold M. In general, we call *basin* of Λ the set

$$B(\Lambda) = \bigcup_{n>0} f^{-n}(U).$$

of points whose future orbits accumulate on Λ.

(H2) There is a continuous decomposition $T_\Lambda M = E^{uu} \oplus E^c$ of the tangent bundle to M over Λ and there exists $\lambda < 1$ satisfying

(i) the decomposition is invariant under Df;

(ii) $\|(Df \mid E^{uu}_x)^{-1}\| \leq \lambda$ and $\|Df \mid E^c_x\| \|(Df \mid E^{uu}_x)^{-1}\| \leq \lambda$ for all $x \in \Lambda$.

The subbundles E^{uu} and E^c in (H2) are necessarily Hölder continuous, and the *strong-unstable* subbundle E^{uu} is uniquely integrable, see [BrPe, §2]. We denote by $\mathcal{F}^{uu}$ the integral foliation, defined over the compact set Λ. Its leaves are C^2 immersed submanifolds of M, with uniformly bounded curvature, see [Sh2, p 79], and they admit the following dynamical characterization:

$$\mathcal{F}^{uu}(x) = \mathcal{F}^{uu}(y) \quad \Leftrightarrow \quad d(f^{-n}(x), f^{-n}(y)) \leq \lambda^n d(x,y) \text{ for every } n \geq 1.$$

Given any point $x \in \Lambda$, we denote

$$\lambda_+^c(x) = \limsup_{n \to +\infty} \frac{1}{n} \log \|Df^n | E_x^c\|.$$

In other words, λ_+^c is the largest Lyapunov exponent of f along the central direction, wherever this is defined. By Oseledets theorem, see [Ma2, IV.10], Lyapunov exponents are defined almost everywhere, with respect to any invariant measure.

Then we state

Theorem A. *Suppose that the diffeomorphism f satisfies (H1), (H2), and*

(H3) for every disk D^{uu} contained in a leaf of $\mathcal{F}^{uu}$ we have $\lambda_+^c(x) < 0$ for a positive Lebesgue measure subset of points $x \in D^{uu}$.

Then f has finitely many ergodic Gibbs u-states $\mu_1, \ldots, \mu_k$. They are SRB measures for f, and the union of their basins $B(\mu_i)$ is a full Lebesgue measure subset of the basin $B(\Lambda)$ of Λ.

We also prove the following statement of uniqueness of SRB measures.

Theorem B. *Suppose that the diffeomorphism f satisfies (H1), (H2), and*

(H4) all the leaves of the foliation $\mathcal{F}^{uu}$ are dense in Λ;

(H5) there exists a disk D^{uu} contained in some leaf of $\mathcal{F}^{uu}$ such that $\lambda_+^c(x) < 0$ for a positive Lebesgue measure subset of points $x \in D^{uu}$.

Then f has a unique Gibbs u-state μ, and it is ergodic. The support of μ coincides with Λ. Moreover, the basin $B(\mu)$ is a full Lebesgue measure subset of $B(\Lambda)$, in particular, μ is the unique SRB measure of f in $B(\Lambda)$.

Theorems A is proved in Sections 2 through 4. In Section 5, we explain how the arguments can be adapted to give Theorem B. (H1), (H2) are standing hypotheses throughout these sections, except if otherwise stated.

In Section 6 we describe a few examples related to these theorems. First of all, we revisit the construction of [Car]. Next, by modifying a beautiful construction of [Ma1], we obtain the examples of robustly transitive diffeomorphisms without uniformly contracting subbundle E^{ss} we mentioned before. These diffeomorphisms satisfy (H4), and the central subbundle E^c is mostly contracting in the sense of (H3) (which is stronger than (H5)), so Theorem B applies to them.

By further modifying our construction, we are able to give the first examples of robustly transitive diffeomorphisms, in four dimensions, having no invariant hyperbolic subbundle.

Theorem C. *There exists an open subset* $\mathcal{U}$ *of* $\text{Diff}^1(T^4)$ *such that any* $f \in \mathcal{U}$ *is transitive and admits a continuous invariant dominated splitting into two* 2*-dimensional subbundles*

$$TM = E^{cs} \oplus E^{cu}, \quad \|Df|E^{cs}\| \, \|(Df|E^{cu})^{-1}\| \le \lambda < 1$$

such that $Df|E^{cs}$ *is uniformly volume contracting but not uniformly contracting,* $Df|E^{cu}$ *is uniformly volume expanding but not uniformly expanding, and neither of them admits an invariant subbundle. Moreover,* $\mathcal{U}$ *contains an open subset of the space of* C^1 *volume preserving diffeomorphisms.*

A natural problem is to study the properties of SRB measures as we construct in Theorems A and B. We mention two very important recent developments. [Cas] introduces a method of 'backward inducing' and applies it to prove exponential decay of correlations and the central limit theorem (in the Banach space of Hölder functions) for a class of attractors including those in [Car]. Exponential decay and the central limit theorem are also obtained by [Do], through a different approach, for another large class of partially hyperbolic systems with mostly contracting central direction ('average contraction property').

Another question raised by our results concerns what happens when the central subbundle is mostly expanding (in this case it is natural to consider a splitting $E^{ss} \oplus E^c$ instead). This is the subject of an ongoing project, whose results will appear in [ABV]. At present, the general answer is less complete than what we obtain here for the contracting case, but SRB measures can already be constructed in fair generality, specially when E^c is 1-dimensional.

The examples of persistently transitive diffeomorphisms without uniformly hyperbolic subbundles given by our Theorem C present a new challenge. We expect ideas from [ABV] to be useful, specially when E^{cs} is mostly contracting and E^{cu} is mostly expanding.

2 Pesin theory and Gibbs u-states

The following proposition asserts that points x with $\lambda_+^c(x) < 0$ have a stable manifold, in the sense of Pesin's theory, transverse to the strong-unstable leaf passing through x.

We call *uu-disk* the image of any embedding into a strong-unstable leaf of a euclidean disk with the same dimension as the leaf. The *uu-ball* of radius r around a point x is the set of points in the strong-unstable leaf of x, and whose distance to x, with respect to the riemannian metric induced on the leaf, is at most r.

Proposition 2.1. *Let* $\lambda_+^c(x) < 0$ *for every point* x *in a positive Lebesgue measure subset* A_0 *of some uu-disk* D^{uu}*. Then*

1. *For every point $x \in A_0$ there exists a C^1 embedded disk $W^s_{loc}(x)$ tangent to E^c_x at x, and such that the diameter of $f^n(W^s_{loc}(x))$ converges exponentially fast to zero as $n \to +\infty$.*

2. *The C^1 disk $W^s_{loc}(x)$ depends in a measurable way on the point x, and the "foliation" $\{W^s_{loc}(x) : x \in A_0\}$ is absolutely continuous.*

The proposition follows from standard arguments in Pesin's theory, see [Pe1], [PuSh]. We just recall the terminology.

Given $\varepsilon > 0$ we denote $D^{uu}(\varepsilon)$ the tubular neighbourhood of radius $\varepsilon > 0$ around D^{uu}, defined as the image under the exponential map of M of all the vectors of norm less than $\varepsilon > 0$ in the orthogonal complement of E^{uu}_x, for all $x \in D^{uu}$. If $\varepsilon > 0$ is small enough then $D^{uu}(\varepsilon)$ is diffeomorphic to a cylinder, and it comes equipped with a canonical projection π onto D^{uu}, which is a C^1 map. We say that a C^1 disk γ *crosses* $D^{uu}(\varepsilon)$ if it is contained in $D^{uu}(\varepsilon)$ and π induces a diffeomorphism of γ onto D^{uu}.

Absolute continuity means that there exists a sequence $(K_n)_n$ of compact subsets of A_0 with $\text{Leb}(A_0 \setminus K_n)$ converging to zero as $n \to \infty$, and there exist maps

$$K_n \ni x \mapsto W^s_{loc}(x)$$

associating to every point x in K_n an embedded C^1 disk $W^s(x)$ and satisfying:

(a) $W^s_{loc}(x)$ depends continuously on the point x in K_n. In particular, there exists a uniform lower bound for the size of $W^s_{loc}(x)$ in K_n; in more precise terms, there exists $\delta_n > 0$ such that the preimage of $W^s_{loc}(x)$ under the exponential $\exp_x$ of M at x contains the graph of a C^1 map defined from the δ_n neighbourhood of 0 in E^c_x to E^{uu}_x.

(b) given any $0 < \varepsilon < \delta_n/2$ and any C^1 disk γ crossing the tubular neighbourhood $D^{uu}(\varepsilon)$ the holonomy map

$$p_\gamma : \bigcup_{x \in K_n} \Big(\gamma \cap W^s_{loc}(x) \Big) \to K_n$$

defined by projection along the leaves of the foliation $\{W^s_{loc}(x) : x \in K_n\}$ is absolutely continuous

$$\text{Leb}(p_\gamma(A)) = \int_A Jp_\gamma \, d(\text{Leb}) \quad \text{for every Borel subset } A$$

with jacobian Jp_γ bounded away from zero and infinity by constants that depend only on the compact set K_n and the minimum angle between γ and the local stable manifolds $W^s_{loc}(x)$.

Corollary 2.2. *Let A_0 be as in Proposition 2.1. Then there exist $\varepsilon > 0$ and $\eta > 0$ such any uu-disk γ that crosses the tubular neighbourhood $D^{uu}(\varepsilon)$ intersects the union of all $W^s_{loc}(x), x \in A_0$ in a subset whose Lebesgue measure is larger than $\eta\,\text{Leb}(\gamma)$.*

Proof. This follows easily from Proposition 2.1. Fix $n \geq 1$ such that K_n has positive Lebesgue measure, and then fix $0 < \varepsilon < \delta_n/2$. By continuity of the strong-unstable subbundle E^{uu} and of the local stable manifolds through points of K_n, the angle between any uu-disk and those local stable manifolds is uniformly bounded away from zero (up to reducing $\varepsilon > 0$, if necessary). The conclusion follows. □

Next we prove some simple facts about Gibbs u-states. By such we mean invariant probability measures whose disintegration along the leaves of the strong-unstable foliation yields measures which are absolutely continuous with respect to Lebesgue measure on the leaves. More precisely, we use the following property which is part of the definition proposed by [PeSi].

Let $\mathcal{L}$ be the strong-unstable leaf through an arbitrary point $x \in \Lambda$. Given $r > 0$ and W a C^1 (open) disk centered at x and transverse to $\mathcal{L}$, denote $\Pi(x, W, r)$ the union of all (open) uu-balls $\gamma(z, r)$ of radius r centered in the points $z \in W \cap \Lambda$. By definition, the restriction of μ to this *foliated box* $\Pi(x, W, r)$ has a disintegration $(\mu_z)_{z \in W \cap \Lambda}$ with respect to the foliation $\{\gamma(z, r) : z \in W \cap \Lambda\}$, such that every μ_z is absolutely continuous with respect to Lebesgue measure $m_{\gamma(z,r)}$ on $\gamma(z, r)$. Moreover,

$$d\mu_z(y) = \rho(y, z)\, dm_{\gamma(z,r)}$$

for some positive function ρ which is bounded away from zero and infinity, in terms only of r and W. We shall denote $\tilde{\mu}$ the quotient measure induced by μ in the space of leaves $\gamma(z, r)$. This quotient space can be canonically identified with the intersection of Λ with the disk W, and we do so.

Theorem 4 of [PeSi] implies that partially hyperbolic attractors always support Gibbs u-states:

Lemma 2.3 (PeSi). *Let σ be an arbitrary uu-disk and m_σ be the normalized restriction of Lebesgue measure in σ. Then any accumulation point of the averaged push-forwards $\lim n^{-1} \sum_{j=0}^{n-1} f_*^j(m_\sigma)$ is a Gibbs u-state.*

The following lemma will be useful in Section 5.

Lemma 2.4. *The support of any Gibbs u-state μ of f on Λ is saturated by $\mathcal{F}^{uu}$, that is, it consists of entire leaves of $\mathcal{F}^{uu}$.*

Proof. Suppose otherwise, that is, there is some strong-unstable leaf $\mathcal{L}$ that intersects $A = \operatorname{supp} \mu$ and is not entirely inside A. Take x a point in the boundary of $A \cap \mathcal{L}$ inside $\mathcal{L}$ (recall that $\mathcal{L}$ is an immersed submanifold of M, at this point we endow it with the metric induced by the immersion). Fix any r and W and consider the corresponding foliated box $\Pi(x, W, r)$. Our choice of x ensures that there exists $y_0 \in \gamma(x, r) \cap A$, and then there exists some small open neighbourhood V of y_0 in Λ, contained in $\Pi(x, W, r)$ and such that $\mu(V) = 0$. Now

$$\mu(V) = \int \mu_z(V \cap \gamma(z, r))\, d\tilde{\mu}(z) = \int \left(\int_{V \cap \gamma(z,r)} \rho(y, z)\, dm_{\gamma(z,r)}(y) \right) d\tilde{\mu}(z).$$

Recall that $\tilde{\mu}$ is the quotient measure of μ in the space of leaves $\gamma(z,r)$. Since ρ is strictly positive, the fact that $\mu(V)=0$ must come from some neighbourhood of x in $\Lambda \cap W$ having zero $\tilde{\mu}$-measure. More precisely,

$$\tilde{\mu}(W_0)=0, \quad \text{where} \quad W_0=\{z \in \Lambda \cap W : V \cap \gamma(z,r) \neq \emptyset\}.$$

As a consequence, the neighbourhood $\Pi(x, W_0, r)$ of x in Λ has zero μ-measure, which contradicts the fact that x is in the support of μ. □

The following remark explains the relation between Gibbs u-states and SRB measures *when the central direction is mostly contracting*.

Remark 2.5. Let μ be an ergodic Gibbs u-state and D^{uu} be a uu-disk contained in the support of μ. Suppose there exists a positive Lebesgue measure subset $A_0 \subset D^{uu}$ such that $\lambda_+^c(x) < 0$ for every $x \in A_0$. Then μ is an SRB measure. Indeed, cf. Corollary 2.2, the union of the local stable manifolds $W_{loc}^s(x)$ through points of $x \in A_0$ intersects any uu-disk close enough to D^{uu} in a positive Lebesgue measure subset. Since μ is an ergodic Gibbs u-state, we may take such a disk so that a full Lebesgue measure subset is contained in the basin of μ. Then, by absolute continuity, local stable manifolds $W_{loc}^s(x)$ passing through points of $B(\mu)$ form a positive Lebesgue measure subset of M which, clearly, is contained in the basin of μ.

2.1 Accessibility classes and consequences of (H3)

In this subsection we assume (H3) in addition to (H1), (H2).

Let R be the set of *regular points* of f, defined as the set of all points $x \in \Lambda$ satisfying the following pair of conditions:

1. given any continuous function $\varphi : M \to \mathbb{R}$, both limits (Birkhoff averages)

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{j=0}^{n-1} \varphi(f^j(x)) \quad \text{and} \quad \lim_{n\to-\infty} \frac{1}{n}\sum_{j=0}^{n-1} \varphi(f^j(x))$$

   exist, and coincide;

2. the largest Lyapunov exponent of f at x along the central direction is well-defined and negative:

$$\lim_{n\to+\infty} \frac{1}{n} \log \|Df^n | E_x^c\| = \lim_{n\to-\infty} \frac{1}{-n} \log \|(Df^n|E_x^c)^{-1}\| < 0.$$

A strong-unstable leaf is *regular* if Lebesgue almost every point in it is regular. We denote S the set of all regular points contained in regular leaves. By Proposition 2.1, every point $x \in S$ has a local stable manifold $W_{loc}^s(x)$ tangent to E_x^c at x.

In what follows we take μ to be a Gibbs u-state, and suppose that (H3) holds.

Lemma 2.6. *The set S has full μ-measure, for any Gibbs u-state μ.*

Proof. For any foliated box $\Pi(x, W, r)$, let $(\mu_z)_z$ be the disintegration of μ along strong-unstable plaques $\gamma(z, r)$, and $\tilde{\mu}$ be the quotient measure. According to the ergodic theorem, condition (1) holds for a full μ-measure subset of Λ. Oseledets theorem ensures that the limits in (2) exist and are equal μ-almost everywhere. So, except for the inequality in (2), all the conditions in the definition of regular point are true for μ_z-almost every point in $\gamma(z, r)$, and $\tilde{\mu}$-almost every z. Now let z be such that the largest central Lyapunov exponent is well-defined for μ_z-almost every point in $\gamma(z, r)$. Condition (H3) implies that $\lambda_+^c < 0$ on a positive μ_z-measure subset. Since the $\lim_{n\to-\infty}$ in (2) is constant over $\gamma(z, r)$ (because this is contained in an unstable manifold), it follows the largest central Lyapunov exponent has to be negative μ_z-almost everywhere in $\gamma(z, r)$. So the set of regular points has full μ-measure on the box, in fact, $\tilde{\mu}$-almost every $\gamma(z, r)$ intersects R in a full μ_z-measure subset. The lemma follows by considering a (finite or countable) covering of Λ by foliated boxes. □

Now we say that $x, z \in S$ belong in a same *accessibility class* if there are $n \geq 1$ and points $x = y_0, y_1, \ldots, y_n = z$ all in S and such that for every $i = 1, \ldots, n$ at least on of the points y_i, y_{i-1} belongs either in the local stable manifold W^s_{loc} or in the strong-unstable leaf $\mathcal{F}^{uu}$ of the other:

$$\text{either} \quad y_i \in W^s_{loc}(y_{i-1}) \cup \mathcal{F}^{uu}(y_{i-1}) \quad \text{or} \quad y_{i-1} \in W^s_{loc}(y_i) \cup \mathcal{F}^{uu}(y_i).$$

Clearly, this defines an equivalence relation. Moreover, if two points belong in a same equivalence class then they have the same Birkhoff averages, for every continuous function φ.

Lemma 2.7. *Accessibility classes are open subsets of S.*

Proof. For any given $x \in S$ and γ be a small neighbourhood of x in $\mathcal{F}^{uu}(x)$. Let $\varepsilon > 0$ be as given by Corollary 2.2. Given any point $y \in S$ close enough to x, the strong-unstable leaf of y contains a segment γ_y that crosses the tubular neighbourhood $\gamma(\varepsilon)$. Then γ_y intersects the union of local stable manifolds of points in γ in a positive Lebesgue measure subset A_y. In fact, almost every point in A_y is in the local stable manifold of a point in $S \cap \gamma$, since S has full Lebesgue measure in γ, and the stable foliation is absolutely continuous. Since S also has full Lebesgue measure in γ_y, we conclude that a full Lebesgue measure subset of A_y consists of points in S. By construction such points are in the same accessibility class as x and as y. This proves that every $y \in S$ in an open neighbourhood of x belongs in a same accessibility class as x. □

Corollary 2.8. *The ergodic components of a Gibbs u-state μ are normalized restrictions of μ to accessibility classes, and so they are also Gibbs u-states.*

Proof. Since accessibility classes are open in S there are at most countably many of them. Then, the classes which have zero measure cover only a zero measure subset of S, and so they can be discarded. Recall also that S has full μ-measure. Since Birkhoff averages are constant on accessibility classes, for any class A with $\mu(A) > 0$ the probability μ_A given by $\mu_A(B) = \mu(A \cap B)/\mu(A)$ is ergodic. So the ergodic components of μ are precisely these normalized restrictions μ_A, and so they are absolutely continuous along strong-unstable leaves. □

Lemma 2.9. *Under condition (H3), there are finitely many accessibility classes, and so f has only finitely many ergodic Gibbs u-states. Moreover, their supports are disjoint.*

Proof. Let $\mathcal{C}_n$, $n \geq 1$ be accessibility classes. Choose γ_n a ball with radius uniformly bounded from below in a regular strong-unstable leaf, such that $S \cap \gamma_n$ is nonempty and contained in $\mathcal{C}_n$. Taking a subsequence, we may suppose that γ_n converges to some uu-disk D^{uu}. By (H3) and Proposition 2.1 there exists a positive Lebesgue measure subset A_0 of D^{uu} such that each point x in A_0 has a Pesin local stable manifold. Moreover, restricting A_0 if necessary, we may suppose that $W^s_{loc}(x)$ contains a ball of uniform radius δ around x (the distance from x to the boundary of $W^s_{loc}(x)$ is larger than δ), for every x in A_0. Then these local stable manifolds intersect γ_n in a positive Lebesgue measure subset, for every large values of n. This implies that the points of $S \cap \gamma_n$ are in a same accessibility class for every large n. So there are only finitely many distinct classes $\mathcal{C}_n$. The second part of the lemma is now an easy consequence of Corollary 2.8. These arguments also prove that the supports of different ergodic Gibbs u-states are disjoint. □

Cf. Remark 2.5, under (H3) every ergodic Gibbs u-state is an SRB measure. Therefore, to prove Theorem A it is enough to show that the basins of these ergodic Gibbs u-states cover a full Lebesgue measure subset of the basin of attraction. This will be given by Proposition 4.2.

We note that in the present section, as well as in the next one, we do not need the full strength of the definition of attractor in (H1).

Remark 2.10. For the construction of Gibbs u-states by [PeSi] it is sufficient that Λ be a compact f-invariant set, and that there exist a strong-unstable foliation (uniformly contracted by negative iterates) whose leaves are contained in Λ and whose tangent bundle is Hölder continuous, cf. [PeSi, p. 421]. These assumptions, weaker than (H1)+(H2), together with (H3), are also sufficient for all our results in the present Sections 2 (and in Section 3). So, they suffice to ensure that there exist only finitely many Gibbs u-states, and they are SRB measures. That is, Λ is a measure-theoretical attractor, even if it may not be a topological attractor. This is precisely the case in the examples of [Ka].

3 Distortion bounds

In this section we prove certain bounds on the distortion of iterates of f restricted to strong-unstable leaves or, more generally, to submanifolds tangent to a strong-unstable cone field C^{uu} in a neighbourhood of the attractor. First, a few words of explanation.

We adopt the following conventions. A continuous cone field $C = (C_x)$ defined on a subset $V \subset M$ is called *centre-unstable* if it is forward invariant:

$$Df(x) \cdot C_x \subset C_{f(x)} \quad \text{for every } x \in V \cap f^{-1}(V).$$

We call it *strong-unstable* if it is strictly invariant, $Df(x) \cdot C_x$ is contained in interior$(C_{f(x)}) \cup \{0\}$, and every vector in it is uniformly expanded: there is $\sigma > 1$ so that

$$\|Df(x) \cdot v\| \geq \sigma \|v\| \quad \text{for every } v \in C_x \text{ and } x \in V \cap f^{-1}(V).$$

Finally, a continuous cone field is *centre-stable*, respectively, *strong-stable* for f if it is *centre-unstable*, respectively, *strong-unstable* for f^{-1}.

Hypothesis (H2) implies the existence of a strong-unstable cone field C^{uu} defined on a neighbourhood $V \subset U$ of Λ. For points in $x \in \Lambda$ we may take C_x^{uu} to consist of the tangent vectors whose angle to the direction of E^{uu} is less than some small constant $\xi > 0$. This defines a continuous cone field on Λ which is sent strictly inside itself by Df, and whose vectors are uniformly expanded by Df. Then it suffices to consider an arbitrary continuous extension of this cone field to a small neighbourhood V of the attractor, which we also denote C^{uu}. By (H1), V may be taken invariant under f in the sense that $f(V) \subset V$. We say that a disk $\gamma \subset V$ is *tangent to* C^{uu} if the tangent space to γ at every point x is contained in C_x^{uu}.

For a point $x \in \Lambda$ we denote $(J^{uu} f)(x)$ the absolute value of the determinant of $Df | E_x^{uu} : E_x^{uu} \to E_{f(x)}^{uu}$, and call it the *strong-unstable jacobian of f at x*.

Lemma 3.1. *Given $L > 0$ there exists $L_1 > 0$ such that, given any C^2 disk $\gamma \subset V$ tangent to the strong-unstable cone field with curvature less than L, then every positive iterate $f^j(\gamma)$ has curvature bounded by L_1.*

Proof. We start with some preliminary remarks. Clearly, the content of the claim does not depend on the choice of a smooth riemannian metric in the neighbourhood V of Λ. For convenience, we consider a metric in which the central bundle and the strong-unstable bundle be nearly orthogonal. More precisely, we choose the metric in such a way that, for some uniform constant $\lambda_1 < 1$,

(i) $\|Df \cdot v\| \geq \lambda_1^{-1} \|v\|$ for every v in a strong-unstable cone;

(ii) $(\|Df \cdot w\| / \|w\|) \leq \lambda_1 (\|Df \cdot v\| / \|v\|)$ for every v in the strong-unstable cone and w orthogonal to v.

Strictly speaking, this requires that the width of the strong-unstable cone field be small enough, but this can always be achieved by replacing V and C^{uu} by iterates $f^N(V)$ and $Df^N \cdot C^{uu}$, for fixed large N.

For the sake of clearness we treat first the case when E^{uu} has dimension 1. Let σ_j be the parametrization by arc-length of $f^j(\gamma)$, $j \geq 1$. Then the curvature of $f^j(\gamma)$ may be written

$$k(f^j(\gamma)) = \frac{|\det(\dot\sigma_j, \ddot\sigma_j)|}{\|\dot\sigma_j\|^3} = \|\ddot\sigma_j\|.$$

Given two vectors u, v in a d-dimensional euclidean space, we use $\det(u,v)$ to denote the $(d-2)$-linear form associating to each $(w_1, \ldots, w_{n-2})$ the determinant of $(u, v, w_1, \ldots, w_{n-2})$. Note that $\det(u,v)$ depends bilinearly on u and v. Now, $\theta_{j+1} = f(\sigma_j)$ is a parametrization of $f^{j+1}(\gamma)$ and

$$\dot\theta_{j+1} = Df \cdot \dot\sigma_j \quad \text{and} \quad \ddot\theta_{j+1} = Df \cdot \ddot\sigma_j + D^2 f \cdot (\dot\sigma_j, \dot\sigma_j).$$

Hence, by bilinearity,

$$k(f^{j+1}(\gamma)) \leq \frac{|\det(Df \cdot \dot\sigma_j, Df \cdot \ddot\sigma_j)|}{\|Df \cdot \dot\sigma_j\|^3} + \frac{|\det(Df \cdot \dot\sigma_j, D^2 f \cdot (\dot\sigma_j, \dot\sigma_j)|}{\|Df \cdot \dot\sigma_j\|^3}.$$

Since $\|\dot\sigma_j\| = 1$, and vectors in the strong-unstable cone are expanded by Df, the second term is bounded by

$$\frac{\|D^2 f\|}{\|Df \cdot \dot\sigma_j\|^2} \leq \|D^2 f\|.$$

Similarly, the first term is bounded by

$$\frac{\|Df \cdot \ddot\sigma_j\|}{\|Df \cdot \dot\sigma_j\|^2} \leq \lambda_1^2 \|\ddot\sigma_j\| = \lambda_1^2\, k(f^j(\gamma)).$$

In this inequality we use properties (i) and (ii) of the riemannian metric, together with the remark that $\ddot\sigma_j$ is orthogonal to $\dot\sigma_j$. Altogether, we get that

$$k(f^{j+1}(\gamma)) \leq \lambda_1^2\, k(f^j(\gamma)) + \|D^2 f\|$$

for every j. By recurrence, we find that

$$k(f^n(\gamma)) \leq \lambda_1^{2n} k(\gamma) + \frac{\|D^2 f\|}{1 - \lambda_1^2} \leq \lambda_1^{2n} L + \frac{\|D^2 f\|}{1 - \lambda_1^2},$$

for every $n \geq 1$, and this completes the proof with $L_1 = L + \|D^2 f\|/(1 - \lambda_1^2)$.

The general case $\dim E^{uu} \geq 1$ follows from the same arguments, as follows [1]. Given a point $p_1 \in f^{j+1}(\gamma)$ and a tangent vector v_1 to $f^{j+1}(\gamma)$ at p_1, let p and v

[1] We are grateful to H. Rosenberg for pointing out this argument to us.

be their preimages under f and $Df(p)$, respectively. Choose a curve $\sigma \subset f^j(\gamma)$ tangent to v at p, and whose second derivative is orthogonal at p to the disk $f^j(\gamma)$. By recurrence, we may suppose that the curvature of σ at p is bounded by some large constant L_1. Then the same calculation as before shows that the curvature of $\sigma_1 = f(\sigma)$ at p_1 is also bounded by L_1, if this has been fixed sufficiently large. The same remains all the more true for the component of the curvature normal to the $f^{j+1}(\gamma)$. This means that the second fundamental form of $f^{j+1}(\gamma)$ is uniformly bounded. As a consequence, the curvature of the $f^{j+1}(\gamma)$ is uniformly bounded over all $j \geq 0$. $\square$

Remark 3.2. It also follows that

$$k(f^n(\gamma)) \leq 1 + \frac{\|D^2 f\|}{1-\lambda_1^2},$$

for every sufficiently large $n \geq 1$.

Lemma 3.3. *Given $L > 0$ there exists $K > 0$ such that given any C^2 disk $\gamma \subset V$ tangent to the strong-unstable cone field and with curvature less than L, and given any $n \geq 1$ such that* $\operatorname{diam}(f^n(\gamma)) < 2L$, *then*

$$\frac{1}{K} \leq \frac{(J_\gamma f^n)(x)}{(J_\gamma f^n)(y)} \leq K$$

for every pair of points $x, y \in \gamma$, where $J_\gamma f(z) = |\det Df|T_z\gamma|$ is the jacobian of f along γ.

Proof. By Lemma 3.1 the curvature the iterates $f^j(\gamma)$, $j \geq 1$, of γ is uniformly bounded. So the jacobian $J_{f^j(\gamma)}$ is C-Lipschitz continuous for some uniform constant $C > 0$. On the other hand, the fact that f is uniformly expanding along any direction contained in C^{uu} implies that

$$d(f^j(x), f^j(y)) \leq \lambda^{n-j} d(f^n(x), f^n(y)) \leq \lambda^{n-j} 2L$$

for every x, y as in the statement, and every $j = 0, 1, \ldots, n$. Using the relation

$$\left|\log \frac{(J^{uu} f^n)(x)}{(J^{uu} f^n)(y)}\right| \leq \sum_{j=0}^{n-1} |\log(J^{uu} f)(f^j(x)) - \log(J^{uu} f)(f^j(y))|$$

we get that

$$\left|\log \frac{(J^{uu} f^n)(x)}{(J^{uu} f^n)(y)}\right| \leq \sum_{j=0}^{n-1} C d(f^j(x), f^j(y)) \leq \sum_{j=0}^{n-1} C(\lambda^{n-j} 2L).$$

Hence, it suffices to take $K = \exp(\sum_{i=0}^{\infty} C(\lambda^i 2L))$. $\square$

4 Proof of Theorem A

To prove Theorem A we only have to show that Lebesgue almost every point in the basin of Λ is in the basin of some ergodic Gibbs u-state.

Lemma 4.1. *Every uu-disk $\sigma \subset \Lambda$ has a positive Lebesgue measure subset of points which are in the basin of some ergodic Gibbs u-state.*

Proof. By Lemma 2.3 every accumulation point of the sequence of averaged push-forwards of Lebesgue measure supported on σ is a Gibbs u-state. Let μ_σ be any accumulation point and μ_0 be an ergodic component of μ_σ By Corollary 2.8, μ_0 is also a Gibbs u-state. Let σ_0 be a uu-disk in the support of μ_0 and such that Lebesgue almost every point of σ_0 is in the basin of μ_0. Then σ_0 is accumulated by disks contained in the iterates $f^n(\sigma)$. By (H3) and Proposition 2.1, a positive Lebesgue measure subset of points in σ_0 have a local stable manifold. Then for every large n, $f^n(\sigma)$ has a positive Lebesgue measure subset of points which are in local stable manifolds of points of σ_0 and, consequently, are in the basin of μ_0. Then the same is true with σ in the place of $f^n(\sigma)$, which proves our claim. □

Proposition 4.2. *The union of the basins of all the ergodic Gibbs u-states is a full Lebesgue measure subset of the basin of attraction.*

Proof. (assuming $\dim E^{uu} = 1$*)* Let $\mu_1, \ldots, \mu_N$ be the ergodic Gibbs u-states of f. Suppose $Z = B(\Lambda) \setminus B(\mu_1) \cup \cdots \cup B(\mu_N)$ had positive Lebesgue measure. Since the set Z is invariant, $Z \cap V$ would have positive measure for any neighbourhood V of Λ. Take V such that the strong-unstable cone field C^{uu} is defined on it. Let x_0 be a Lebesgue density point of $Z \cap V$, and fix some C^1 foliation of a neighbourhood of it, tangent to the strong-unstable bundle E^{uu} at the point x_0. The leaves of such a foliation are tangent to the cone field C^{uu}, as long as the neighbourhood is small enough. Moreover, the intersection of $Z \cap V$ with some leaf γ must have positive Lebesgue measure inside γ. Then, let x be a point of density of $\gamma \cap Z \cap V \subset \gamma \setminus \big(B(\mu_1) \cup \cdots \cup B(\mu_N)\big)$ inside γ. For each large n, let γ_n be the neighbourhood of radius L around $f^n(x)$ inside $f^n(\gamma)$. Then $f^{-n}(\gamma_n)$ form a decreasing sequence of neighbourhoods of x. Since we suppose that γ is one-dimensional, we may conclude that the relative measure of $B(\mu_1) \cup \cdots \cup B(\mu_N)$ in $f^{-n}(\gamma_n)$ goes to zero as $n \to \infty$. Using the bounded distortion Lemma 3.3, the same remains true with γ_n in the place of $f^{-n}(\gamma_n)$. By Ascoli-Arzela, there exists a subsequence γ_{n_k} converging to some uu-segment γ_∞. Lemma 4.1 tells us that γ_∞ has a positive Lebesgue measure subset S_1 of points in $B(\mu_1) \cup \cdots \cup B(\mu_N)$. Moreover, there is a positive Lebesgue measure subset $S_2 \subset S_1$ of points having local stable manifolds with size bounded from below. By Corollary 2.2, the union of these local stable manifolds cuts γ_n, large n, in a fixed proportion, and this gives a contradiction. □

The difficulty in extending the proof to higher-dimensional strong-unstable bundle lies in the construction of disks γ_n intersecting the union of the basins of the Gibbs u-states in a set with small relative measure. Note that if we take γ_n a ball of fixed radius around $f^n(x)$ as we did before, then $f^{-n}(\gamma_n)$ need not be a ball, and so we can not use the density point property. Forward iterates $f^n(\sigma_n)$ of balls σ_n around x are no good either: if we take these $f^n(\sigma_n)$ with bounded diameter, as required by the distortion lemma, they may not contain a sufficiently large ball, as needed in Corollary 2.2. This difficulty is handled in Lemma 4.3.

Lemma 4.3. *Let $L > 0$ be fixed. Given any disk $\sigma \subset V$ tangent to the strong-unstable cone field, and given any n sufficiently large, there exist open sets $V_i \subset W_i$, $i = 1, \ldots, k(n)$ such that*

(a) the V_i are two-by-two disjoint;

(b) $\mathrm{Leb}(\cup_{i=1}^{k(n)} W_i)$ converges to $\mathrm{Leb}(\sigma)$ as $n \to \infty$;

(c) each $f^n(V_i)$, $i = 1, \ldots, k(n)$, is a ball of radius L inside $f^n(\sigma)$;

(d) each $f^n(W_i)$, $i = 1, \ldots, k(n)$, is a ball of radius $2L$ inside $f^n(\sigma)$.

Proof. Given any large enough n, let $B(x_i, L)$, $i = 1, \ldots, k(n)$, be a maximal family of disjoint balls of radius L contained in $f^n(\sigma)$. This means that any other $x \in f^n(\sigma)$, the ball of radius L around x intersects either the boundary of σ or $B(x_i, L)$ for some $i = 1, \ldots, k(n)$. In particular the family $B(x_i, 2L)$ covers the set of points in $f^n(\sigma)$ whose distance to the boundary is larger than L. We take

$$V_i = f^{-n}(B(x_i, L)) \quad \text{and} \quad W_i = f^{-n}(B(x_i, 2L)).$$

We are left to prove part (b) of the statement. For this note that the union of the W_i, $i = 1, \ldots, k(n)$, contains the set of points of σ whose distance to the boundary of σ is larger than $\lambda^n L$, where λ^{-1} is the rate of expansion of Df on the strong-unstable cone field. The Lebesgue measure of the complement of this set goes to zero as n goes to infinity, and so the proof is complete. □

Now we prove the general case of Proposition 4.2.

Proof. Suppose there was a positive Lebesgue measure subset of $B(\Lambda)$ not in $B(\mu_1) \cup \cdots \cup B(\mu_N)$. Then there would be some disk γ tangent to the strong-unstable cone field and a density point x of $\gamma \setminus \big(B(\mu_1) \cup \cdots \cup B(\mu_N)\big)$ inside γ. This is proved just as in the previous case. Let σ_m be a decreasing sequence of balls around x in γ such that the relative measure of $B(\mu_1) \cup \cdots \cup B(\mu_N)$ in σ_m goes to zero as $m \to \infty$. For each m let $V_{m,i}$ and $W_{m,i}$ be the open sets obtained by taking $\sigma = \sigma_m$ in Lemma 4.3 (for each m we choose $n = n(m)$ large enough so that the lemma applies). Since the curvature of the iterates of γ is uniformly bounded. cf. Lemma 3.1, $\mathrm{Leb}(f^n(V_i)) / \mathrm{Leb}(f^n(W_i))$ is bounded away from zero (by some constant that depends only on the curvature bound, and on the dimension of γ). Properties (b), (c), (d) in the lemma, combined with the distortion Lemma 3.3, ensure that the union of $V_{m,i}$ over all i covers a fixed fraction of σ_m, for every m. Since these $V_{m,i}$ and $V_{m,j}$ are disjoint whenever $i \neq j$, and in view of the choice of the σ_m, we may choose some $V_{m,i(m)}$ so that

$$\frac{\text{Leb}\left(V_{m,i(m)} \cap (B(\mu_1) \cup \cdots \cup B(\mu_N))\right)}{\text{Leb}(V_{m,i(m)})} \to 0 \quad \text{as } m \to \infty.$$

Using the bounded distortion lemma once more, we conclude that the same is true with $\gamma_m = f^{n(m)}(V_{m,i(m)})$ in the place of $V_{m,i(m)}$. Recall from (c) that these γ_m are balls of radius L. Now the proof proceeds precisely as before. □

The proof of Theorem A is now complete.

Remark 4.4. The argument of the proof of Proposition 4.2 proves a bit more: given any disk $\gamma \subset B(\Lambda)$ tangent to the strong-unstable cone field, Lebesgue almost every point in γ is in the basin of some ergodic Gibbs u-state. So, recall Lemma 2.6, $\lambda_+^c < 0$ Lebesgue almost everywhere in γ.

5 Proof of Theorem B

Finally, we prove Theorem B. More precisely, we show that hypotheses (H4) and (H5) imply (H3), and that the set S (introduced in Section 2.1) consists of a unique accessibility class. Then Λ supports a unique ergodic Gibbs u-state μ, and $B(\mu)$ contains a full Lebesgue measure subset of the basin of Λ. We also deduce that $\operatorname{supp} \mu = \Lambda$.

We fix a uu-disk D^{uu} as in (H5), and let A_0 be a positive Lebesgue measure subset such that $\lambda_+^c(x) < 0$ for all $x \in A_0$.

Lemma 5.1. *1. Given any $\varepsilon > 0$ there exists $L_1 > 0$ such that any uu-ball γ with radius larger than L_1 is ε-dense in the attractor Λ.*

2. Given any $\varepsilon > 0$ there exists $L_2 > 0$ such that any uu-ball γ with radius larger than L_2 contains a subdisk that crosses the tubular neighbourhood $D^{uu}(\varepsilon)$ of D^{uu}.

Proof. The proof of the first part is by contradiction. Suppose that there exists a sequence γ_n of uu-balls and a sequence of points $x_n \in \Lambda$ such that radius$(\gamma_n) > n$ and $\gamma_n \cap B(x_n, \varepsilon) = \emptyset$, for every $n \geq 1$. Up to taking subsequences, we may suppose that $(x_n)_n$ converges to some point $x \in \Lambda$. Then there exists $n_0 > 1$ such that $\gamma_n \cap B(x, \varepsilon/2) = \emptyset$ for every $n > n_0$, and so

$$\text{closure}\,\Big(\bigcup_{n>n_0} \gamma_n\Big) \bigcap B\Big(x, \frac{\varepsilon}{3}\Big) = \emptyset.$$

On the other hand, closure$(\cup_{n>n_0}\gamma_n)$ must contain some leaf of $\mathcal{F}^{uu}$ (e.g. the leaf through any accumulation point of the sequence of center points of the γ_n), because radius(γ_n) $\to \infty$. Since any such leaf is dense, by (H4), we have reached a contradiction. The first claim is proved.

To prove the second one, let x_0 be in the interior of D^{uu} and choose $\delta > 0$ small enough so that any point $x \in B(x_0, \delta) \cap \Lambda$ is in a uu-disk γ_x that crosses $D^{uu}(\varepsilon)$. By the first part of the lemma, there exists $L_1 > 0$ such that any uu-ball with radius larger than L_1 intersects $B(x_0, \delta)$. Take $L_2 = L_1 + \text{diam}(D^{uu}) + 1$. Given any uu-ball γ with radius larger than L_2, let γ' be the uu-ball of radius L_1 centered at the same point. Then γ' intersects $B(x_0, \delta)$ at some point x. Our choice of L_2 ensures that γ contains γ_x, and so the proof is complete. □

Condition (H3) is an immediate consequence. Given any uu-disk γ, some iterate $f^n(\gamma)$ contains a ball of radius L_2. By Lemma 5.1 this ball intersects the union of the local stable manifolds of points in A_0 in a positive Lebesgue measure subset B_0. Then $f^{-n}(B_0) \subset \gamma$ has positive Lebesgue measure, and $\lambda_+^c(x) < 0$ for every $x \in B_0$. As a consequence we even have $\lambda_+^c < 0$ Lebesgue almost everywhere in γ, cf. Remark 4.4.

The next lemma, which is last step in the proof of Theorem B, follows directly from Lemmas 2.4 and 2.9, together with the fact that strong-unstable leaves are dense.

Lemma 5.2. *The map f has a unique Gibbs u-state μ on Λ and it is ergodic. Moreover, the support of μ is the whole attractor Λ.*

6 Examples

In this section we describe a number of examples related to our results.

The following notations are useful. Given a disk σ tangent to the strong-unstable cone field, we let $d(x, \partial\sigma)$ be the minimum length of a curve in σ connecting x to a point in the boundary of σ, and call it the *distance from x to the boundary of σ*. Then we call *internal radius* of σ to

$$\rho(\sigma) = \sup_{x \in \sigma} d(x, \partial\sigma).$$

We use similar notions for disks inside leaves of a central foliation $\mathcal{F}^c$. We also let $d_c(x, y)$ be the *central distance* between two points in a same leaf of $\mathcal{F}^c$, defined as the length of the shortest curve connecting the two points inside the central leaf. And we define the *central diameter* of a subset of a central leaf using this distance.

6.1 DA attractors

The first one, due to [Car], consists of a C^1 open set of diffeomorphisms f with transitive attractors on the torus T^3, derived from an Anosov (or globally hyperbolic) diffeomorphism f_0 through a Hopf bifurcation. More precisely,

(a) there exist a constant $\lambda < 1$ and a Df-invariant splitting of the tangent space $TM = E^{uu} \oplus E^c$ such that $\dim E^{uu} = 1$, $\dim E^c = 2$,

$$\|(Df \mid E^{uu})^{-1}\| \leq \lambda \quad \text{and} \quad \|(Df \mid E^c)\| \, \|(Df \mid E^{uu})^{-1}\| \leq \lambda,$$

and both subbundles E^{uu} and E^c are uniquely integrable;

(b) f has a hyperbolic repelling fixed point p, obtained from a hyperbolic saddle of f_0 through a Hopf bifurcation;

(c) every strong-unstable leaf of a point in $\Lambda = T^3 \setminus W^u(p)$ is dense in Λ.

(d) for any uu-segment γ there exists a full Lebesgue measure subset of points z in γ such that $\lambda^c_+(z) < 0$.

(e) f does not admit an invariant strong-stable (i.e., uniformly contracting) subbundle E^{ss}.

As a consequence of (c), Λ is nowhere dense and it is transitive for f.

For this class of systems, [Car] proves that there exists an SRB measure supported on Λ, and this was a main inspiration for our Theorem B. Since the proof of property (c) for her systems [Car, Lemma 1] seems to have a gap (there is no uniform contraction on the central bundle in the whole Λ, this was pointed out also by A. A. Castro and J. C. Martin), and this is a key assumption in Theorem B, we give here a detailed proof, based on an idea of [Ma1].

Property (d) is also crucial in Theorem B. For these examples it can be read out from [Car], but we include (in Section 6.3) a direct argument that applies also to another class of examples we introduce in the next section. Property (e) is not in [Car], and we also prove it below.

One considers an Anosov diffeomorphism $f_0 : T^3 \to T^3$, with one expanding and two contracting directions. We suppose that the norm of Df along the stable subbundle and the norm of Df^{-1} along the unstable bundle are bounded by a constant $\lambda_0 < 1/3$. Let p be a fixed point of f_0 and $\delta > 0$ be a small constant. Denote $V_2 = B(p, \delta/2)$, and $V_3 = B(p, 3\delta)$. Then we deform f_0^{-1} inside V_2 by isotopy in such a way that

(1) The continuation of the fixed point p goes through a Hopf bifurcation, and becomes a repeller (staying all the time inside V_2);

(2) In the process, there always exist a strong-unstable cone field C^{uu} and a centre-stable cone field C^{cs}, defined everywhere, such that C^{cs} contains the stable direction of the initial map f_0;

(3) Moreover, the width of the cone fields C^{uu} and C^{cs} are bounded everywhere by a small constant $\alpha > 0$.

In particular, by [HPS], the map f we obtain in this way has an invariant central foliation $\mathcal{F}^c$, tangent to the cone field C^{cs}. Moreover, this foliation is topologically conjugate to the stable foliation $\mathcal{F}_0^s$ of f_0 (because it remains normally expanding all the way during the isotopy), and so all its leaves are dense in T^3. On the other hand, there is also a unique strong-unstable foliation $\mathcal{F}^{uu}$ invariant under f and tangent to centre-stable cone field C^{uu}, whose leaves are uniformly expanded by f.

(4) There exist a constant $\sigma > 1$ and a neighbourhood V_1 of p contained in $V_2 \cap W^u(p)$, such that $J^c = |\det Df^{-1}|T\mathcal{F}^c| \geq \sigma$ outside V_1.

(5) The map f^{-1} is δ-C^0 close to f_0^{-1} everywhere and it is sufficiently C^1-close to f_0^{-1} outside V_2 so that $\|(Df^{-1}|T\mathcal{F}^c)^{-1}\| \leq \lambda \leq 1/3$ outside V_2;

These conditions hold for a whole C^1-open set $\mathcal{U}$ of diffeomorphisms of T^3.

We fix $L > 0$ large enough so that every segment of an unstable leaf of f_0 with length $L/2$ is $\delta/2$-dense in every stable leaf of f_0. Choosing α in (3) sufficiently small, we ensure that every segment with length less than $2L$ in a strong-unstable leaf of f is C^1 close to some segment in an unstable leaf of f_0, and every disk of diameter less than 5δ contained in a central leaf of f is C^1 close to some disk contained in a stable leaf of f_0. As a consequence, every segment of length L of a strong-unstable leaf of f is δ-dense in every central leaf of f (with respect to the central distance). We suppose that $\delta > 0$ is small enough so that the minimum central distance between two connected components of the intersection of V_3 with any central leaf is larger than 100δ.

Lemma 6.1. *For every $f \in \mathcal{U}$ as before, every strong-unstable leaf is dense in $T^3 \setminus W^u(p)$. As a consequence, $W^u(p)$ is dense in T^3.*

Proof. Let $W \subset T^3$ be a nonempty open set not contained in $W^u(p)$. Then there exists some central stable leaf F^c and a nonempty open disk $D \subset W \cap F^c$ which is not contained in $W^u(p)$.

Claim 1: Some negative iterate of D has central diameter larger than 100δ.

There are two possibilities. If D does not intersect $W^u(p)$ at all, then $f^{-n}(D)$ is disjoint from V_1 for every $n \geq 1$, and so its Lebesgue measure goes to infinity, by (4). As a consequence, the central diameter of $f^{-n}(D)$ also goes to infinity as $n \to \infty$. Therefore, it suffices to take any large n. In the second case, D must intersect the boundary of some connected component of $W^u(p) \cap F^c$. It follows from the local theory of Hopf bifurcations that the boundary $\partial W^u(p)_{loc}$ of the connected component C_p of $W^u(p) \cap V_2$ that contains p coincides with the local unstable manifold of the invariant circle formed at the bifurcation. In particular, it is invariant under f^{-1}. Then $f^{-n}(D)$ intersects $\partial W^u(p)_{loc}$ for every large $n \geq 1$. Since $f^{-n}(D)$ is not contained in $W^u(p)$, it must contain an open subset

D_1 outside the connected component C_p and whose boundary touches $\partial W^u(p)_{loc}$. Then the boundary of every $f^{-k}(D_1)$ touches $\partial W^u(p)_{loc}$. If $f^{-k}(D_1)$ is disjoint from V_1 for every $n \geq 1$, we may use the same argument as in the previous case, to conclude that the central diameter of $f^{-k}(D_1)$ goes to infinity as $k \to \infty$. If $f^{-k}(D_1)$ intersects V_1 for some $k \geq 1$, then the closure of $f^{-k}(D_1)$ intersects two connected components of the intersection of V_2 with a central leaf. Hence, due to our choice of δ, the central diameter of $f^{-k}(D_1)$ is larger than 100δ. We completed the proof of our claim.

This means that, up to replacing D by some iterate $f^{-m}(D)$, we may suppose right from the start that the central diameter of D is larger than 100δ.

Claim 2: There exists x in D such that $f^{-n}(x) \in T^3 \setminus V_3$ for every $n \geq 0$.

By our choice of δ, the central neighbourhoods of radius 40δ around the connected components of $V_3 \cap F^c$ are two-by-two disjoint. On the other hand, since $\Gamma_0 = D$ has central diameter larger than 100δ, it can not be contained in any of those neighbourhoods. So, by connectivity, there exists $x_0 \in \Gamma_0$ whose central ball B_0 of radius 35δ is disjoint from V_3. Since Γ_0 is too large to be contained in B_0, we may take a compact connected subset $\Gamma_0' \subset \Gamma_0$ joining x_0 to the boundary of B_0. By (5), $f^{-1}(B_0)$ contains the central ball of radius

$$\frac{1}{\lambda} 35\delta > 100\delta$$

around $f^{-1}(x_0)$. In particular, the diameter of $\Gamma_1 = f^{-1}(\Gamma_0')$ is larger than 100δ. Repeating this procedure, we construct a sequence Γ_n, $n \geq 0$, of compact connected nonempty sets such that

$$f^n(\Gamma_n) \subset f^{n-1}(\Gamma_{n-1} \setminus V_3)$$

for every $n \geq 1$. This implies that $K_n = f^n(\Gamma_n \setminus V_3)$ is a decreasing sequence of compact sets, and any point $x \in \cap_{n \geq 0} K_n$ satisfies the conclusion of Claim 2.

Now, let x be any such point, that is, $f^{-n}(x) \notin V_3$ for every $n \geq 1$. In particular, every disk of central radius 2δ around an iterate $f^{-n}(x)$ is disjoint from V_2. Now let D_ε be any small disk around x and contained in D. By (5), iterates $f^{-n}(D_\varepsilon)$ have exponentially increasing internal radius, as long as this internal radius is smaller than 2δ. Therefore, there must be some $N \geq 1$ for which the internal radius of $f^{-N}(D_\varepsilon)$ is at least 2δ. Then $f^{-N}(D_\varepsilon)$ intersects every segment of length L of any strong-unstable leaf (and so it intersects every strong-unstable leaf). Therefore, $D_\varepsilon \subset D$ intersects every strong-unstable leaf. This proves that every strong-unstable leaf is dense in the complement of $W^u(p)$. In particular, $W^u(p)$ is dense in T^3 because the strong-unstable leaf of p is dense in $T^3 \setminus W^u(p)$. □

We also observe that f can not admit an invariant strong-stable bundle E^{ss}. This is clear if f_0 has some periodic point $q \neq p$ with complex contracting eigenvalues. We may choose V_3 small enough to be disjoint from the orbit of q, and then q is also a periodic point for f. Since the contracting eigenvalues are also unchanged, there can be no invariant contracting direction.

With a bit more effort we can obtain the same conclusion when all the periodic points of the Anosov diffeomorphism have only real eigenvalues. We use the fact that the invariant circle C formed at the Hopf bifurcation is normally hyperbolic, see e.g. [RuTa, Remark 7.3]. So, such a strong-stable bundle would be tangent to the central leaf F^c containing C and transverse to C inside the TF^c. Then E^{ss} would be everywhere tangent to the central foliation and transverse to the strong-unstable manifold $W^{uu}(C)$ of C, defined as the union of the strong-unstable leaves through points of C. Let q be a periodic hyperbolic saddle point of f, of period $k \geq 1$, whose orbit is disjoint from V_3 (it exists if V_3 is small, i.e. if we fix δ small enough). Then q has stable index 2 and its stable manifold contains a central ball B_q of radius 2δ around q. In view of the way we haven chosen the constant L associated to δ, the local strong-unstable leaves of radius L around every point $\xi \in C$ intersects B_q transversely. By considering the first (i.e. the closest to ξ inside the strong-unstable leaf) intersection of these strong-unstable leaves with the stable manifold of q, we conclude that the intersection of $W^{uu}(C)$ with the stable manifold $W^s(q)$ contains some connected component $\tilde{C}$ (a circle). Then E^{ss} is transverse to every $f^{kj}(\tilde{C})$, $j \geq 1$, inside $TW^s(q)$. Since the diameter of $f^{kj}(\tilde{C})$ goes to zero as $j \to \infty$, we conclude that E^{ss} can not be continuous at the point q. This contradicts the fact that a strong-stable subbundle is necessarily Hölder continuous.

6.2 Transitive diffeomorphisms without stable bundle

Using similar arguments we are also able to exhibit new C^1 open sets of transitive diffeomorphisms in (the whole) $M = T^3$ which admit no invariant strong-stable (or strong-unstable) subbundle E^{ss}. That is, there is a Df-invariant dominated splitting $TM = E^{uu} \oplus E^c$ into a 1-dimensional strong-unstable subbundle E^{uu} and a 2-dimensional subbundle E^c. On the other hand E^c is not uniformly hyperbolic and does admit an invariant subbundle. The first examples of this kind were exhibited by [Bon]. The present construction may be thought of as a modification of an example of [Ma1].

As in the previous section, we start with an Anosov diffeomorphism f_0 and a fixed point p of f_0. We deform f_0 by isotopy in a neighbourhood $V_2 = B(p, \delta/2)$ of p, in such a way that the map f we obtain (actually, a whole C^1-open set $\mathcal{U}$ of maps)

(A) satisfies the global properties (2), (3), (5) above; in particular, f has a strong-unstable foliation $\mathcal{F}^{uu}$ and a central foliation $\mathcal{F}^c$ as before;

(B) has three hyperbolic fixed saddle points inside V_2, contained in a same

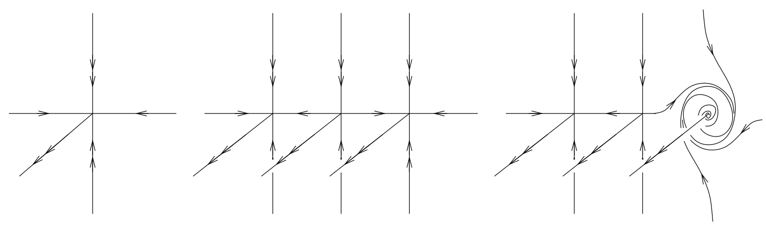

Figure 1:

central leaf F^c: one saddle with stable index 1 and two saddles with stable index 2; at least one of the index 2 saddles has complex contracting eigenvalues;

(C) there exists $\sigma > 1$ such that $J^c = |\det Df^{-1}|T\mathcal{F}^c| \geq \sigma$ at every point.

One way to obtain (B) is to have p go through a pitchfork bifurcation, as one of its contracting eigenvalues becomes 1. Then, the stable index changes from 2 to 1, and two other saddle points, of index 2, are created. Then it suffices to make the contracting eigenvalues of one of these new saddles become complex numbers. See Figure 1. As before, we suppose that $\alpha > 0$ and $\delta > 0$ are small.

Lemma 6.2. *For every $f \in \mathcal{U}$ as before, every strong-unstable leaf is dense in T^3. As a consequence, f is transitive.*

Proof. As in the first step of the proof of Lemma 6.1, we show that any disk D in a central leaf, has a negative iterate $f^{-n}(D)$ with central diameter larger than 100δ. This follows from property (C), which is a stronger version of the property (4) we had in the previous case. The second step of the proof of Lemma 6.1 translates immediately to this case, proving that some point x in $f^{-n}(D)$ has all its negative iterates outside $V_3 = B(p, 3\delta)$. The third and last step of the proof of Lemma 6.1 also applies without change here: any small disk around x has a negative iterate which has internal radius larger than 2δ, and so cuts every strong-unstable leaf. This proves the first statement in the lemma.

In particular, the unstable manifold of any periodic point is dense in T^3. Moreover, by construction, every central leaf of f is dense in T^3. Therefore, to conclude that f is transitive, it suffices to show that the stable manifolds of the periodic points with stable index 2 are dense in some central leaf. Let F^c be the central leaf in (B). The fact that f_0 is contracting on its stable leaves, together with C^0-closeness of f to f_0 (by [HPS, Theorem 7.1] this yields C^0-closeness on central leaves) imply that every point in F^c has a positive iterate in a neighbourhood of radius $50\delta \ll 1$ of the three fixed saddle points contained in F^c. Then by the local description of the dynamics near the saddles, the point is in the stable manifold of one of the saddles. That is, F^c coincides with the union of the stable manifolds of the three fixed saddles it contains, and this completes the argument. □

The presence of periodic points with complex contracting eigenvalues ensures that E^c does not admit any invariant subbundle.

Remark 6.3. We could also start with an Anosov diffeomorphism f_0 having, besides p, a periodic point q with complex contracting eigenvalues. In that case, to get the same conclusions as before one does not need the last condition in (B): it suffices to make p go through a pitchfork bifurcation, with no need to create new saddles with complex eigenvalues.

In the next subsection we show that these diffeomorphisms satisfy (H3): the central direction is mostly contracting at Lebesgue almost every point (in each strong-unstable leaf, and in the whole manifold M). Then we can also obtain the following nice consequence of our results.

Remark 6.4. We have shown that Theorem B may be applied to the maps we constructed above, and so they have a unique SRB measure μ, whose basin contains Lebesgue almost all of M. If we start with a volume preserving Anosov diffeomorphism f_0 then our construction can be carried out to give maps with the same properties as above which are also volume preserving (see, for instance, Section 6.4 where we do this in a different setting). Then, Lebesgue measure (volume) must be ergodic (Birkhoff averages are constant Lebesgue almost everywhere), in fact it coincides with μ. So, such maps are *stably ergodic* with respect to Lebesgue measure.

6.3 Control of the central Lyapunov exponents

To complete the construction of the previous examples, in Sections 6.1 and 6.2, we are left to explain why they are mostly contracting in the central direction. We begin with an abstract statement, that we apply later to the two classes of examples.

We suppose that Λ satisfies (H1), (H2), with $\dim E^{uu} = 1$. Furthermore, there exists a domain $V \subset M$ such that

(i) there exists $E > 0$, $c_0 \in (0,1)$ such that, given any uu-segment γ with $\text{length}(\gamma) \geq E$, we may partition $f(\gamma)$ into segments $\gamma(1), \ldots, \gamma(k)$ such that $E \leq \text{length}(\gamma(i)) \leq 2E$ for every $i = 1, \ldots, k$, and the total length of those $\gamma(i)$ that intersect V is less than $c_0 \,\text{length}(f(\gamma))$;

(ii) there exist $\lambda < 1$ and $\beta > 0$ such that

$$\|DF|E^c_x\| \leq (1+\beta) \text{ for } x \in V \quad \text{and} \quad \|DF|E^c_x\| \leq \lambda \text{ for } x \in M \setminus V.$$

and, for some k sufficiently large,

$$\lambda_1 = \lambda(1+\beta)^k < 1. \tag{3}$$

The precise condition k should satisfy is the following. Let E and c_0 be as in (i), let $K > 0$ be the distortion bound given by Lemma 3.3 with $L = 2E\|Df\|$, and $c = Kc_0/(1 + (K-1)c_0) < 1$. We need

$$c_1 = c\big(1 + \frac{1}{k}\big)\,(1+k)^{\frac{1}{k}} < 1 \tag{4}$$

which holds for any large k.

Proposition 6.5. *Under these assumptions (H3) holds, in fact $\lambda_+^c(x) < 0$ for Lebesgue almost every point in any uu-segment.*

Proof. The proof has two main steps. First we use (i) to show that the orbit of Lebesgue almost every point in any uu-segment γ spends a positive fraction of the time outside V. Then condition (ii) implies the conclusion.

Starting the first step, we note that it is no restriction to suppose that $\text{length}(\gamma) \geq E$. We decompose successive iterates

$$f^n(\gamma) = \bigcup_{i_1,\dots,i_j} \gamma(i_1,\dots,i_n)$$

as follows. First we write $f(\gamma) = \gamma(1) \cup \cdots \gamma(k)$ as in (i). Then, supposing $\gamma(i_1,\dots,i_{n-1})$ is defined, with length in between E and $2E$, we use (i) once more to write

$$f(\gamma(i_1,\dots,i_{n-1})) = \gamma(i_1,\dots,i_{n-1},1) \cup \cdots \cup \gamma(i_1,\dots,i_{n-1},k').$$

(k' depends on $i_1,\dots,i_{n-1}$). Given $n \geq r \geq 1$ and $1 \leq t_1 < \cdots < t_r < n$, we denote $M(t_1,\dots,t_r)$ the following subset of γ. Firstly, $M(t_1)$ consists of those points $x \in \gamma$ for which the segment $\gamma(i_1,\dots,i_{t_1})$ that contains $f^{t_1}(x)$ intersects V. Observe that then $f^t(M(t_1))$ is a union of segments $\gamma(i_1,\dots,i_t)$ for every $t \geq t_1$. Next, we proceed by recurrence: $M(t_1,\dots,t_{r-1},t_r)$, $r \geq 2$, is defined as the set of points $x \in M(t_1,\dots,t_{r-1})$ such that $f^{t_r}(x)$ is in any of the segments $\gamma(i_1,\dots,i_{t_{r-1}},\dots,i_{t_r})$ that intersects V.

Lemma 6.6. *The Lebesgue measure of $M(t_1,\dots,t_r)$ is bounded by $c^r\,\text{length}(\gamma)$.*

Proof. The way we have defined these sets, $f^t(M(t_1,\dots,t_{r-1}))$ is a union of segments $\gamma(i_1,\dots,t)$ for every $t \geq t_{r-1}$ and, in particular, for $t = t_r - 1$. We write $\iota_r = (i_1,\dots,i_{t_r-1})$, for simplicity. For each one of these segments, (i) gives

$$\text{Leb}\left(f^{t_r}(M(t_1,\dots,t_{r-1},t_r)) \cap f(\gamma(\iota_r))\right) \leq c_0\,\text{length}(f(\iota_r))).$$

Note that the length of $f(\gamma(\iota_r))$ is bounded by $2E\|Df\|$. So, using the distortion Lemma 3.3,

$$\text{Leb}\left(M(t_1,\dots,t_{r-1},t_r) \cap f^{-t_r+1}(\gamma(\iota_r))\right) \leq c\,\text{length}(f^{-t_r+1}(\gamma(\iota_r))).$$

Adding over all the $\gamma(\iota_r)$ contained in $f^{t_r-1}(M(t_1,\dots,t_{r-1}))$, we get

$$\mathrm{Leb}\left(M(t_1,\dots,t_{r-1},t_r)\right) \le c\,\mathrm{Leb}\left(M(t_1,\dots,t_{r-1})\right)$$

The lemma follows by recurrence. □

Corollary 6.7. *There exists $B > 0$, a universal constant, such that for any $n \ge 1$ the Lebesgue measure of the subset M_n of points in $x \in \gamma$ such that $f^j(x) \in V$ for at least $kn/(k+1)$ values of $j \in \{0,\dots,n-1\}$ is bounded by $Bn\,c_1^{kn/(k+1)}\,\mathrm{length}(\gamma)$.*

Proof. This set is contained in the union of all $M(t_1,\dots,t_r)$ for all choices of $n \ge r \ge kn/(k+1)$ and $t_1,\dots,t_r$. So, its Lebesgue measure is bounded by

$$\sum_{r\ge kn/(k+1)} \binom{n}{r} c^r.$$

We claim that, if $r \ge kn/(k+1)$, then

$$\binom{n}{r} \le B\left((1+\frac{1}{k})(1+k)^{\frac{1}{k}}\right)^r,$$

for some universal constant B. Summing over all such r we get the bound in the statement.

The claim is a classical consequence of Stirling's formula. Indeed, it gives

$$\binom{n}{r} = \frac{n!}{r!\,(n-r)!} \le B\,\frac{n^n}{r^r(n-r)^{n-r}}$$

for some universal constant B. The last term can be rewritten

$$\left(\frac{n}{r}\right)^r\left(\frac{n}{n-r}\right)^{n-r} = \left[(1+\frac{n-r}{r})(1+\frac{r}{n-r})^{\frac{n-r}{r}}\right]^r.$$

Then it is enough to note that $r \ge kn/(k+1)$ is just the same as $r \ge k(n-r)$. □

It follows that $\cap_{n\ge1}\cup_{j\ge n} M_j$ has zero measure. Note that the orbit of any point in the complement spends at least a fraction $1/(k+1)$ of the time outside V. So the first step in the proof of Proposition 6.5 is complete.

The second step is very short:

$$\|Df^n|E^c_x\| \le \lambda^{n/(k+1)}(1+\beta)^{kn/(k+1)} \le \lambda_1^{n/(k+1)}$$

for any point x not in M_n. Recall, from (4) that $\lambda_1 < 1$. □

Proposition 6.5 enables us to check assumption (H3) whenever f is sufficiently contracting along the central direction outside V, not too expanding in the central direction inside V, and sufficiently large expansion in the strong-unstable direction, while keeping the distortion constants not too large.

We apply it to the examples in Sections 6.1, 6.2, with V being the perturbation box V_2. To have these conditions satisfied, we just suppose that for the initial Anosov diffeomorphism f_0 any vector in the unstable subbundle is expanded by a factor 3 and any vector in the stable subbundle is contracted by a factor 1/3. Then we deform f_0 along a one-parameter family of diffeomorphisms f_μ, by isotopy inside V_2, to make it go through either a Hopf bifurcation or a pitchfork bifurcation. The expansion in the unstable/strong-unstable subbundle remains large everywhere, and the same is true for the contraction in the stable/central subbundle restricted to the outside of V_2. Then the distortion along strong-unstable leaves remains uniformly bounded in the whole family f_μ. Moreover, the deformation can be done in such a way that the f_μ be contracting along the central direction, all the way up to the bifurcation parameter μ_0.

Now we choose $E > 0$ not much smaller than the size of V_2, so that the image of any uu-segment with length bigger than E has a positive fraction c_0 (in length) outside V_2. Having fixed these constants, the distortion constant K may also be fixed, independent of the parameter μ, as we already observed. So we may fix $k > 0$ large enough to satisfy (4), and then choose β sufficiently close to zero so that (3) also holds. Finally, for parameters just slightly beyond μ_0, any expansion f_μ may display in the central direction must be smaller than $1 + \beta$. In this way we obtain the hypothesis of Proposition 6.5 (and, hence, its conclusion (H3)) for the systems in Section 6.1 and in Section 6.2.

6.4 Diffeomorphisms without hyperbolic bundles

Finally, we prove Theorem C. We start with a linear Anosov diffeomorphism f_0 induced in T^4 by a linear map of $\mathbb{R}^4$ with eigenvalues

$$0 < \lambda_1 \le \lambda_2 < 1/3 < 1 < 3 < \lambda_3 \le \lambda_4.$$

Up to replacing it by some iterate, we may suppose that f_0 has at least two fixed points p_1 and p_2. Let $\mathcal{F}_0^u$ and $\mathcal{F}_0^s$ be the unstable and the stable foliations of f_0. Let $0 < \rho \ll 1$ be fixed. For $i = 1, 2$ let $D_i^u(2\rho)$ and $D_i^s(2\rho)$ be balls of radius 2ρ around p_i in the unstable, respectively, in the stable leaf through p_i. Fix $0 < \delta \ll \rho$ small enough so that the distance along the leaves of $\mathcal{F}_0^s$ of any two distinct points in $D_1^u(2\rho) \cup D_2^u(2\rho)$ is larger than 100δ, and similarly for the distance along the leaves of $\mathcal{F}_0^u$ of any two distinct points in $D_1^s(2\rho) \cup D_2^s(2\rho)$. Then let $V_2 = B(p_1, \delta/2) \cup B(p_2, \delta/2)$ and $V_3 = B(p_1, 3\delta) \cup B(p_2, 3\delta)$. We also consider a sufficiently small constant $\alpha > 0$, the precise condition is stated below.

Now we consider the set $\mathcal{V}$ of C^1 diffeomorphisms f of T^4 satisfying the following conditions:

(1) f has a centre-unstable cone field C^{cu} and a centre-stable cone field C^{cs} both with width bounded by $\alpha > 0$ and containing, respectively, the unstable subbundle and the stable subbundle of f_0;

(2) There exists $\sigma > 1$ such that $|\det Df|TD^{cu}| \geq \sigma$ for every disk D^{cu} tangent to the cone field C^{cu} and $|\det Df|TD^{cs}| \leq \sigma^{-1}$ for every disk D^{cs} tangent to the cone field C^{cs};

(3) There exists $\lambda \leq 1/3$ we have $\|Df(x)v^{cu}\| \geq \lambda^{-1}\|v^{cu}\|$ and $\|Df^{-1}(x)v^{cs}\| \geq \lambda^{-1}\|v^{cs}\|$ for every x outside V_2 and every $v^{cu} \in C^{cu}(x)$, $v^{cs} \in C^{cs}(x)$;

(4) f has some periodic saddle point q with stable index 2, whose stable manifold intersects every disk of radius 2δ tangent to C^{cu}, and whose unstable manifold intersects every disk of radius 2δ tangent to C^{cs}.

We suppose that $\alpha > 0$ is sufficiently small so that (1), together with the way we have chosen δ, imply that every disk of radius 45δ tangent to C^{cu}, respectively C^{cs}, intersects $D_1^s(\rho) \cup D_2^s(\rho)$, respectively $D_1^u(\rho) \cup D_2^u(\rho)$, in at most one point.

Lemma 6.8. *Every diffeomorphism $f \in \mathcal{V}$ is transitive.*

Proof. In view of assumption (4), it suffices to show that, for an arbitrary open set $U \subset T^4$, some positive iterate contains a disk of radius 2δ tangent to C^{cu} and some negative iterate contains a disk of radius 2δ tangent to C^{cs}. Moreover, since our assumptions are symmetric under taking inverses, we only need to prove the first statement.

Claim: Any open set $U \subset T^4$ contains some point x such that $f^n(x)$ avoids V_3 for every sufficiently large n.

The lemma is an easy consequence. Indeed, for such a point x we may take a disk $D_0 \subset U$ around x and tangent to the centre-unstable cone field. Up to replacing U by some iterate, we may suppose that $f^n(x) \notin V_3$ for every $n \geq 0$. If D_0 does not intersect V_2 then we may use assumption (3) to conclude that $f(D_0)$ contains a disk D_1 around $f(x)$ whose radius is twice as large as the radius of D_0. Repeating this, we construct a sequence disks around the orbit of x $D_i, i = 0, 1, \ldots$, tangent to the centre-unstable cone field, with $D_i \subset f(D_{i-1})$ and whose radii increase geometrically. As long as the radius remains smaller than 2δ the disk can not intersect V_2 and the procedure can be repeated. So, eventually the radius of some D_n must be larger than 2δ, and the lemma follows.

It remains to prove the claim. The argument is similar to those in Lemmas 6.1 and 6.2, with the additional difficulty that this time there may be no invariant foliations for the map f.

It is convenient to consider a lift $\tilde{f} : \mathbb{R}^4 \to \mathbb{R}^4$ of f to the universal covering of the torus. We denote $\tilde{V}_2, \tilde{V}_3 \subset \mathbb{R}^4$ the preimages of V_2 and V_3 under the covering map (note that they have infinitely many connected components), and we use similar notations for lifts of other objects. Let π^u the projection along the stable foliation $\tilde{\mathcal{F}}_0^s$ from $\mathbb{R}^4$ to some arbitrary unstable leaf of $\tilde{f}_0$ (which we identify with $\mathbb{R}^2$). Let $\tilde{F}^u$ be an arbitrary leaf of the unstable foliation. For every $n \geq 0$, the image $\tilde{f}^n(\tilde{F}^u)$ is a graph over $\mathbb{R}^2$, in the sense that π^u induces a diffeomorphism from $\tilde{f}^n(\tilde{F}^u)$ onto $\mathbb{R}^2$. This is just because $\tilde{F}^u \mapsto \tilde{f}^n(\tilde{F}^u)$ is a proper embedding of $\mathbb{R}^2 \approx \tilde{F}^u$ into $\mathbb{R}^4$ whose tangent space, being contained in the lift of the centre-unstable cone field, avoids a cone around the vertical (stable) direction of $\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$ with width uniformly bounded from zero.

Let D be any disk contained in the intersection of $\tilde{U}$ with some unstable leaf $\tilde{F}^u$ of $\tilde{f}_0$. Assumption (2) implies that the volume of iterates $\tilde{f}^n(D)$ increases exponentially fast. Then, since these iterates are contained in graphs tangent to $\tilde{C}^{cu}$, the same is true about the diameter of $\tilde{f}^n(D)$. So, some iterate $\Gamma_0 = \tilde{f}^{n_0}(D)$ has diameter larger than 100δ. Then Γ_0 can not be contained in a ball inside $\tilde{f}^n(\tilde{F}^u)$ of radius 45δ around any point of $\tilde{D}_1^s(\rho) \cup \tilde{D}_2^s(\rho)$. Since these balls are two-by-two disjoint, due to our choice of α, and Γ_0 is connected, we conclude that there exists some $x_0 \in \Gamma_0$ such that the ball B_0 of radius 35δ around x_0 inside $\tilde{f}^n(\tilde{F}^u)$ does not intersect $\tilde{V}_3$. On the other hand, Γ_0 can not be contained in B_0 either. So, there exists a compact connected set $\gamma_0 \subset \Gamma_0$ disjoint from $\tilde{V}_3$ joining x_0 to the boundary of B_0. Then assumption (3) implies that the diameter of $\Gamma_1 = \tilde{f}(\gamma_0)$ is larger than

$$\frac{1}{\lambda}\, 35\delta > 100\delta.$$

By recurrence, we obtain a sequence Γ_n, $n \geq 0$, of compact connected nonempty sets with $\Gamma_n \subset \tilde{f}(\Gamma_{n-1} \setminus \tilde{V}_3)$. Then $\tilde{f}^{-n}(\Gamma_n \setminus \tilde{V}_3)$ is a decreasing sequence, and any point $x \in T^4$ having a lift in the intersection of these compact sets satisfies the conclusion of the claim. □

We are left to construct diffeomorphisms such any other diffeomorphism in a C^1 neighbourhood satisfies conditions (1) through (4). We do this in such a way that if the initial Anosov diffeomorphisms f_0 preserves volume then the maps obtain are also volume preserving. Roughly, our construction goes as follows. We consider two different fixed (or periodic) points p_1 and p_2. In a neighbourhood of p_1 contained in $B(p_1, \delta_2)$ we modify the map along the stable direction (keeping the unstable direction essentially unchanged) in the same way as we did for the examples in Section 6.2, see Figure 1. Then we do the same in a neighbourhood of p_2 contained in $B(p_2, \delta_2)$, exchanging the roles of the stable and the unstable direction. Let us describe this procedure in more detail.

As a first step, we consider two models of volume preserving vector fields in the unit 2-dimensional disk D^2, which are depicted in Figure 2. Both are zero in a neighbourhood of the boundary of the D^2. The first model, X, has a singularity of center type at the origin. The second model, Y, has a hyperbolic saddle at the origin. It may be obtained, for instance, as the Hamiltonian vector field of some smooth function, which is constant at the boundary and has a saddle type critical point at the origin.

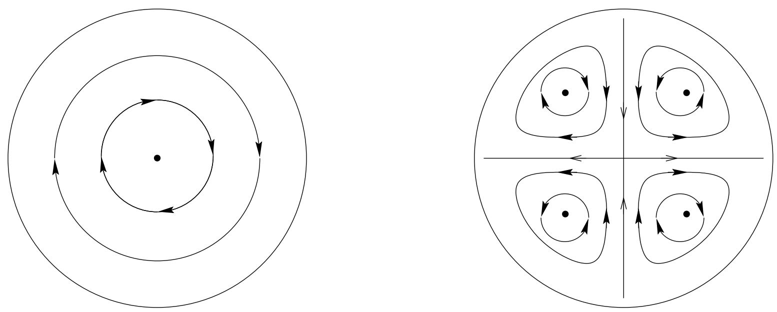

Figure 2:

Next, we construct vector fields $\tilde{X}$ and $\tilde{Y}$ in $D^2 \times D^2$, given by

$$\tilde{X}(x,y) = (\phi(y)X(x), 0) \qquad \tilde{Y}(x,y) = (\phi(y)Y(x), 0),$$

where $\phi : D^2 \to [0,1]$ is a smooth function such that $\phi(0) = 1$, and $\phi = 0$ on a neighbourhood of the boundary of D^2. Note that $\tilde{X}$, $\tilde{Y}$ are still volume preserving. Moreover, $\tilde{Y}$ has a singularity at the origin with a strong-unstable, a strong-stable, and two central directions.

As already mentioned, we obtain our examples by modifying the initial linear Anosov diffeomorphism f_0 in neighbourhoods of two different fixed (or periodic) points p_1 and p_2. We describe the modification in the neighbourhood of p_1, the construction for p_2 is just the same, with stable and unstable directions interchanged. It is useful to consider that this takes place in three stages.

First, we fix a linear chart $\varphi_1 : D^2 \times D^2 \to M$ mapping 0 to p_1, the horizontal leaves $D^2 \times \{y\}$ into the stable leaves of f_0, and the vertical leaves $\{x\} \times D^2$ into the unstable leaves of f_0. We also suppose that the local unstable manifold of the saddle point of Y is mapped parallel to the eigenspace corresponding to the weakest contracting eigenvalue λ_2 of f_0.

We consider the one-parameter family of diffeomorphisms $f_t = (\varphi_* \tilde{Y})_t \circ f_0$ obtained by composing f_0 with the flow of the push-forward of $\tilde{Y}$. The point p_1 is fixed for every f_t, on the other hand, the weakest contracting eigenvalue of $Df_t(p)$ increases as t increases from zero. Eventually, for some $t = t_0$ (depending only on λ_2 and the expanding eigenvalue of $\tilde{Y}$ at the origin) this eigenvalue becomes equal to 1, then the stable index of p_1 changes to 1. In the process new fixed saddles, with stable index 2, are created in the neighbourhood of p_1.

As a second stage, we consider $g_0 = f_{t_1}$ for some t_1 slightly larger than t_0, and let q_1 be one of the new fixed saddle points with index 2. We modify g_0 in a neighbourhood of q_1 disjoint from p_1, in the same way as we did before for f close to p_1, except that this time we use $\tilde{X}$ instead of $\tilde{Y}$. We obtain a one-parameter family of diffeomorphisms g_s such that q_1 is a fixed point of every g_s, with the contracting eigenvalues of $Dg_s(q_1)$ becoming equal, and then complex conjugate, as s becomes larger than some s_0.

We choose s_1 slightly larger than s_0, and let $h = g_{s_1}$. The reason why we are not done yet is that h may not preserve a thin centre-unstable cone field as in condition (1). We fix α as in (1). Observe that all the modifications we did took place in the direction of the stable foliation of f_0. So, on the one hand, this foliation is still invariant (but not any more contracting) for h; on the other hand, vectors in unstable subbundle of f_0 are still expanded by Dh (but the subbundle itself is no longer invariant). As a consequence, any sufficiently thin cone field around the stable foliation of f_0 is a centre-stable cone field for h.

We choose such a cone field, then its complement C is a strong-unstable cone field (but it may be very wide). Now we conjugate h in $\varphi(D^2 \times D^2)$ by some linear map $(x, y) \mapsto (rx, y)$, where (x, y) are coordinates in $D^2 \times D^2$, and $r > 0$ (the conjugated map extends correctly because $h = f_0$ is a linear map of T^4 in a neighbourhood of the image of the local chart). Taking $r > 0$ sufficiently small we have that the image C^{cu} of C under this conjugacy has width smaller than α, and so it is a thin centre-unstable cone field for the new map f. Moreover, since the stable foliation of f_0 is still invariant under f, and we have once more preserved the expansion along the unstable bundle of f_0, any cone field with width less than α centered in the stable subbundle of f_0 as centre-stable cone field of f.

Up to dual modifications, carried out independently in a neighbourhood of p_2, this f is the map we were looking for: conditions (1) through (4) stated at the beginning of this section hold in a C^1 neighbourhood of f. Indeed (1), (2), (3), follow from the construction, and for (4) it suffices to choose the chart $\varphi(D^2 \times D^2)$ sufficiently small that we keep unchanged arbitrarily large compact disks in the stable and in the unstable manifolds of the periodic point q.

We close with the following

Conjecture: The class of volume-preserving examples without uniformly hyperbolic invariant subbundles that we construct in this section contains C^1-stably ergodic diffeomorphisms (every C^1 close map that preserves Lebesgue measure is ergodic with respect to it).

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