Dynamics Beyond Poincaré

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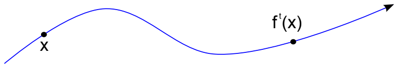
Dynamical models

discrete time: iteration of a transformation $f : M \to M$

$$x_0 \mapsto x_1 = f(x_0) \mapsto \cdots \mapsto x_{n+1} = f(x_n) \mapsto \cdots$$

continuous time: flow $f^t : M \to M$ of a differential equation $\dot{x} = F(x)$



more generally: actions of noncompact groups/semigroups; stochastic/random processes; etc

Central goals

- Describe the behavior of typical trajectories, for typical evolution laws, specially as time goes to infinity.
- Analyze the stability of such dynamical behavior under small modifications of the evolution law.

A dream theorem

Theorem

Consider $f_a(x) = x^2 + a$. For almost every $a \in \mathbb{R}$ there exists a probability measure μ_a on $\mathbb{R}$ such that for almost every $x_0 \in \mathbb{R}$

- the orbit $(x_n)_n$ goes to infinity, or else
- $\displaystyle\frac{1}{n}\sum_{j=0}^{n-1} \delta_{x_j}$ converges (weakly) to μ_a

$$\mu_a(V) = \lim_n \frac{1}{n}\#\{0 \le j < n : x_j \in V\} \quad \text{for } V \subset \mathbb{R}.$$

Sullivan, McMullen, **Lyubich**, Swiatek, Graczyk, de Melo, van Strien, Martens, Nowicki, Avila, Moreira and many others.

Poincaré

Analytic expressions for the trajectories are seldom available. Anyway, most of the times they are of little use to understand the dynamical behavior.

Instead, one should aim at a qualitative description of the global dynamics, by any means available (geometric, topological, algebraic, probabilistic, etc). Periodic motions are crucial!

Poincaré's famous mistake in *Sur le problème des trois corps et les équations de la dynamique* led him to discover homoclinic trajectories which, in turn, would lead to the theory of uniformly hyperbolic systems.

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Stability

Birkhoff (1930's): Carried out many fundamental steps along the directions set by Poincaré. In particular: *Homoclinic trajectories imply infinitely many periodic motions.*

Andronov, Pontryagin (1930's): Defined structurally stable dynamical systems: every nearby system has the same orbit structure, up to a global homeomorphism.

Peixoto (1960's): Proved that among flows on orientable surfaces, the structurally stable systems form an open and dense subset of the space of all flows.

Smale (1960's): Founded the theory of uniformly hyperbolic systems. The initial goal was to characterize structural stability, in any dimension, and to prove that most systems are stable.

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Stability Conjecture

Stability Conjecture (Palis, Smale): A dynamical system is structurally stable if and only if it is uniformly hyperbolic.

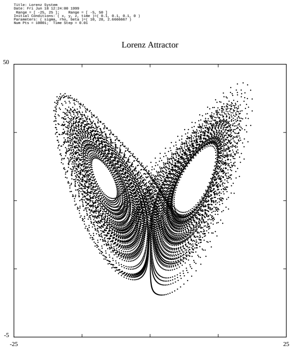
Theorem

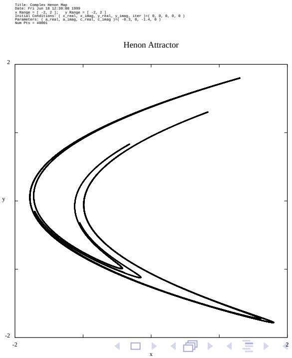
The stability conjecture is true for C^1 diffeomorphisms and flows.

Robbin, Robinson, de Melo, **Mañé**, Palis, Hayashi, Aoki among others.

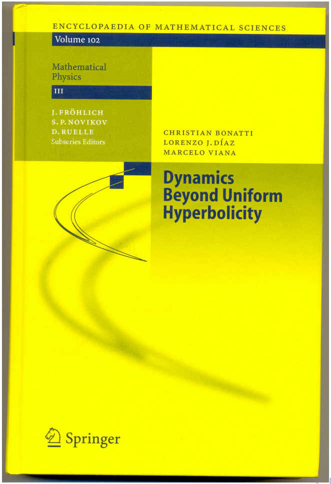
Abundance of non-stability/non-hyperbolicity

On the other hand, many (most??) interesting examples are not uniformly hyperbolic.





$1/4$ century of Dynamics beyond hyperbolicity



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International Workshop on

Global Dynamics Beyond Uniform Hyperbolicity

Bedlewo, Banach Center, Poland

May 30th to June 10th 2013

Geodesic flows

Let M be a compact Riemannian manifold.

Given $p \in M$ and $v \in T^1_p M$, let $\gamma_{p,v} : \mathbb{R} \to M$ be the unique geodesic tangent to v at p.

The geodesic flow on the unit tangent bundle $T^1 M$ is

$$f^t : (p, v) \mapsto (\gamma_{p,v}(t), \dot{\gamma}_{p,v}(t)).$$

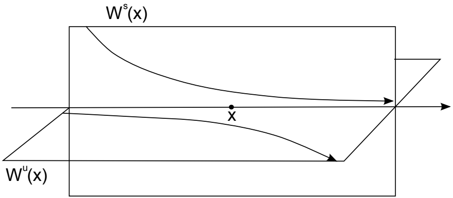
It preserves a natural volume measure on $T^1 M$ (called Liouville measure).

Ergodicity and hyperbolicity

Theorem (Anosov 1967, Hopf 1939)

If M has negative sectional curvature, then the geodesic flow is ergodic relative to the Liouville measure.

(1) The geodesic flow is uniformly hyperbolic:



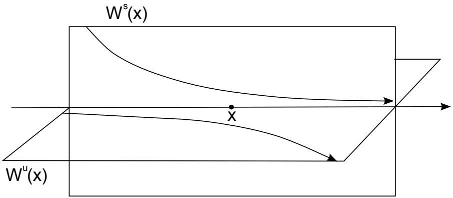
Hyperbolicity

(2) The stable foliation $\{W^s(x)\}$ and the unstable foliation $W^u(x)$ are absolutely continuous:

$$\text{a set } X \subset T^1N \text{ has volume zero in } T^1N$$
$$\Updownarrow$$
$$\text{if } X \cap W^*(x) \text{ has volume zero in } W^*(x)\text{, for almost every } x.$$
$$* = s \text{ or } * = u$$

Time-one maps

Let us consider the time-one map $f = f^1$ of the geodesic flow:



- f is not uniformly hyperbolic, just partially hyperbolic: the flow direction is “neutral” (central direction);
- the central direction is uniquely integrable: the integrable curves are the flow trajectories.

Stable ergodicity

Theorem (Grayson, Pugh, Shub 1995)

The time 1 diffeomorphism f is stably ergodic: every nearby volume preserving diffeomorphism g is ergodic.

Remarks:

- Most nearby diffeomorphisms do not embed in a flow.
- Every nearby diffeomorphism g is partially hyperbolic.
- The central foliation is usually not absolutely continuous.

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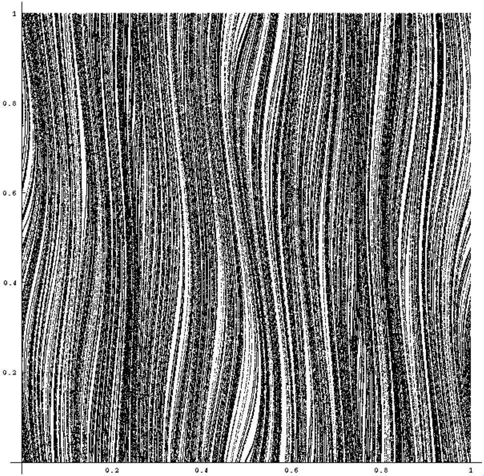
Pathological foliations



Dichotomy and Rigidity

Theorem (Avila, Viana, Wilkinson)

Let $\dim M = 2$. There exists a neighborhood $\mathcal{U}$ of the time-one map f such that for every volume preserving $g \in \mathcal{U}$,

(1) either there is $k \geq 1$ and a full volume set $Z \subset T^1M$ such that $Z \cap T^1M$ consists of exactly k orbits of g;

(2) or g is the time-one map of some smooth flow.

Lyapunov exponents

The hidden player is the central Lyapunov exponent

$$\lambda^c(g,x) = \lim_n \frac{1}{n} \log \|Dg^n \mid E^c_x\|, \qquad E^c = \text{ central direction.}$$

By ergodicity, $\lambda^c(g,x)$ does not depend on x, on a full measure subset.

Step 1: $\lambda^c(g) \neq 0$ (non-uniform hyperbolicity) implies that the center foliation is pathological.

Previous results of this kind by Shub, Wilkinson and Ruelle, Wilkinson, in a different setting.

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Invariance Principle

Step 2: The truly delicate case is $\lambda^c(g) = 0$. This is handled using a general tool, called the Invariance Principle, which was developed by Bonatti, Viana for linear systems, and by Avila, Viana in the general case.