LORENZ ATTRACTORS WITH ARBITRARY EXPANDING DIMENSION*

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Abstract. *We construct the first examples of flows with robust multidimensional Lorenz-like attractors: the singularity contained in the attractor may have any number of expanding eigenvalues, and the attractor remains transitive in a whole neighbourhood of the initial flow. These attractors support an SRB (Sinai-Ruelle-Bowen) measure and, contrary to the usual (low-dimensional) Lorenz models, they have infinite modulus of structural stability.*

ATTRACTEURS DE LORENZ DE VARIETÉ INSTABLE DE DIMENSION ARBITRAIRE

Résumé. *Nous construisons les premiers exemples de flots possédant un attracteur robuste de type Lorenz multidimensionnel: l'attracteur contient un zéro dont la variété instable est de dimension arbitraire, et l'attracteur reste transitif pour toute perturbation du flot initial. Ces attracteurs sont le support d'une mesure de Sinai-Ruelle-Bowen et, contrairement aux attracteurs de Lorenz usuels (en dimension 3), ils ont un module de déformations de dimension infinie.*

Version française abrégée: Les attracteurs de type Lorenz (ou *attracteurs singuliers*) ont été définis par [1] et [2], qui présentent un modèle géométrique pour le comportement, observé par Lorenz dans un article célèbre [3], d'une famille d'équations différentielles de $\mathbb{R}^3$ reliée à un modèle de convection des fluides. Ce sont des attracteurs transitifs de champs de vecteurs de $\mathbb{R}^3$, contenant à la fois une singularité hyperbolique de type selle et une infinité d'orbites périodiques hyperboliques régulières. Le plus important est que ces attracteurs sont robustes: tout champ de vecteurs voisin possède un attracteur du même type.

L'étude des attracteurs singuliers est un sujet important en Systèmes Dynamiques et de nombreux résultats ont été obtenus en dimension 3 . En particulier, [7] a commencé une théorie générale de ces attracteurs, et montre que leur robustesse implique une propriété d'hyperbolicité partielle.

L'étude des attracteurs singuliers sur des variétés de dimension plus grande est, par contre, un sujet presque vierge. Bien sûr, il est facile de plonger en dimension quelconque les modèles classiques, en ajoutant des directions "fortes-stables". Cependant, l'existence de champs de vecteurs possédant un attracteur singulier robuste dont la variété instable d'un zéro est de dimension strictement supérieure à 1, est resté un problème ouvert depuis l'introduction des modèles géométriques, il y a près de vingt ans. Nous annonçons ici une réponse positive:

Théorème. *Pour tout $k \geq 2$ et tout $n \geq k+3$, il existe une variété M de dimension n et*

(1) *Il existe un C^1-ouvert $\mathcal{O}_0$ de champs de vecteurs Z sur M possédant un attracteur transitif $A(Z)$ qui contient à la fois des orbites régulières et une singularité hyperbolique dont la variété instable est de dimension k.*
(2) *De plus, il existe un C^∞-ouvert $\mathcal{O}_1 \subset \mathcal{O}_0$ tel que pour tout $Z \in \mathcal{O}_1$ l'attracteur singulier $A(Z)$ est le support d'une unique mesure μ de Sinai-Ruelle-Bowen : pour toute fonction continue $\varphi: M \to \mathbb{R}$ et pour Lebesgue*

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presque tout point z dans le bassin de l'attracteur, la moyenne temporelle $\frac{1}{T}\int_0^T \varphi(Z_t(z))\,dt$ converge vers $\int \varphi\,d\mu$ quand T tend vers $+\infty$.

La partie (1) de notre théorème reste valable pour $k = 2$ et $n = 4$. Sur les variétés de dimension 3, [7] annoncent actuellement que de tels attracteurs robustes n'existent pas, par contre [5] montrent que l'ensemble des flots possédant des attracteurs singuliers de variété instable de dimension 2 contient une sous-variété de codimension 2 de l'espace des flots.

La construction de nos exemples sera présentée dans la Section 2 du texte en Anglais, et la démonstration du théorème sera ébauchée dans les Sections 3 et 4. La Section 5 présentera l'exemple construit dans le cas $k = 2$, $n = 4$, ainsi que des commentaires sur des raffinements possibles de notre résultat.

1. Introduction and statement of the main result

Lorenz-like (or *singular*) attractors were introduced by [1] and [2], as so-called geometric models for the behaviour observed by Lorenz in his famous study [3] of a three-dimensional system of differential equations related to a model of fluid convection. These are transitive attractors of smooth flows, containing both regular orbits and singularities. Most important, they are a robust phenomenon: any flow close to the initial one has an attractor with similar features.

The study of these systems is a main topic in Dynamics, and important progresses have been obtained, specially in low dimensions. Recently, [7] have been developing a theory of Lorenz-like attractors in three-dimensional manifolds, and prove that robustness implies a property of partial hyperbolicity.

The study of singular attractors for flows in higher dimensions is, however, mostly open. Of course, one may embbed the usual Lorenz models into flows in any dimension, just by "multiplying by a strong contraction" (the attractor is contained in a three-dimensional submanifold, which is invariant and normally contracting for the flow). But it has remained an open problem, ever since the introduction of the geometric models about two decades ago, whether robust attractors of flows may contain singularities with more than one expanding eigenvalue. Here we announce a positive solution to this problem:

Theorem. *Given $k \geq 2$ and any $n \geq k+3$ there exists a manifold M with dimension n, and*

(1) *There exists a C^1-open set $\mathcal{O}_0$ of vector fields Z on M exhibiting a transitive attractor $A(Z)$ that contains regular orbits together with a hyperbolic singularity whose unstable manifold has dimension k.*
(2) *Moreover, there exists a C^∞-open set $\mathcal{O}_1 \subset \mathcal{O}_0$ such that for every $Z \in \mathcal{O}_1$ the singular attractor $A(Z)$ supports a unique SRB measure μ: the time average $\frac{1}{T}\int_0^T \varphi(Z_t(z))\,dt$ converges to $\int \varphi\,d\mu$ as T goes to $+\infty$, for any continuous function $\varphi : M \to \mathbb{R}$ and Lebesgue almost every point z in the basin of the attractor.*

The first part of our theorem remains true for $k = 2$ and $n = 4$, as we shall see in the last section. In 3 dimensions, robust multidimensional Lorenz attractors do not exist, according to [7], but [5] show that transitive attractors containing a singularity with 2 expanding eigenvalues persist in certain codimension 2 submanifolds of the space of flows.

In Section 2 we describe the construction of our examples, and in Sections 3 and 4 we sketch the proofs of the properties claimed in the Theorem. In Section 5 we also comment on refinements and possible extensions of our statements.

2. Constructing multidimensional singular attractors

We begin by revisiting the classical geometric Lorenz models of [1], [2]. By construction, these systems admit a two-dimensional submanifold Σ as a partial cross-section to the flow. More precisely, there is a curve $\Gamma \subset \Sigma$ and a well-defined Poincaré first-return map $\Phi : \Sigma \setminus \Gamma \to \Sigma$. The curve Γ corresponds to the intersection of Σ with the stable manifold of a singularity O (contained in the attractor), and future trajectories of points in Γ do not intersect Σ. Most important, Φ is a hyperbolic map, in the following sense:

(1) Φ admits an invariant contracting smooth foliation $\mathcal{F}^s$ (containing Γ as a leaf): every leaf $\mathcal{F}^s_z$ is mapped completely inside some leaf $\mathcal{F}^s_{\Phi(z)}$, and $\Phi|\mathcal{F}^s_z$ is a uniform contraction;

(2) the quotient space $\Sigma/\mathcal{F}^s$ (i.e., the space of leaves of $\mathcal{F}^s$) is diffeomorphic to an interval, and the map ϕ induced by Φ on $\Sigma/\mathcal{F}^s$ is uniformly expanding (with derivative tending to infinity as one approaches Γ).

The robustness of the attractor stems from this hyperbolic character of the Poincaré map, and sensitivity on initial conditions results from the expansivity of ϕ. On the other hand, the interval map ϕ has modulus of stability 2 (the classes of topological conjugacy in a neighbourhood of it are parametrized by an open subset of $\mathbb{R}^2$) and, as a consequence, a similar statement holds for the flow.

The general strategy for defining our examples is to try and reproduce these basic ingredients, in the higher dimensional setting. Topology turns out to play a significant role in this extension, imposing certain restrictions on the kind of ambient manifolds and of cross-sections one may have. In the construction we now describe, the cross-section is the product of the k-dimensional torus T^k by the 2-dimensional disk D^2, and the flow is defined on a quotient manifold of $T^k \times D^2 \times [0,1]$, obtained by identifying points in $T^k \times D^2 \times \{0,1\}$ as explained below.

We start by considering a convenient smooth expanding map f of T^k (our conditions on f are stated along the way), and the corresponding natural extension. This last notion is usually defined as the shift map on the space of all sequences $(x_n)_n$ on T^k satisfying $f(x_n) = x_{n-1}$ for every $n \in \mathbb{Z}$. Here we deal with a concrete realization of this map, which is a generalization of the solenoid associated to an expanding map of the circle [10]. That is, for some $l \geq 1$, we consider a smooth embedding F of $N = T^k \times D^l$ into itself which preserves the vertical foliation $\{\{x\} \times D^l : x \in T^k\}$ and induces a (strong) contraction on its leaves, and such that

$$\pi_1 \circ F = f \circ \pi_1, \qquad \text{where } \pi_1 : N \to T^k \text{ is given by } \pi_1(x,y) = x.$$

Then the restriction of F to the maximal invariant set $\Lambda_F = \bigcap_{j \geq 0} F^j(N)$ is topologically conjugate to the natural extension of f. It is not difficult to see, from transversality theory, that such an embedding F does exist if l is large enough, say $l \geq k+1$. On the other hand, for certain choices of f (e.g. isotopic to a diagonal matrix with integer two-by-two prime coefficients) we are able to show that such an F exists already for $l = 2$, and we shall refer to this situation in what follows. Such a solenoid does not exist with $l = 1$, and that is why we need a different construction for 4-manifolds, see Section 5.

Now we suspend the map F to a smooth flow, in the usual way. That is, we let M be the quotient manifold of $N \times [0,1]$ by the relation $(z,1) \sim (F(z),0)$, and $h : N \times [0,1] \to M$ be the corresponding identification map. Then the suspension of F is the flow on M associated to the vector $X = h_*(\partial/\partial t)$, where (x,t) denotes the coordinate system on $N \times [0,1]$. The image of $N \times \{0\}$ under h is a global cross-section, and the corresponding Poincaré map is smoothly conjugate to F.

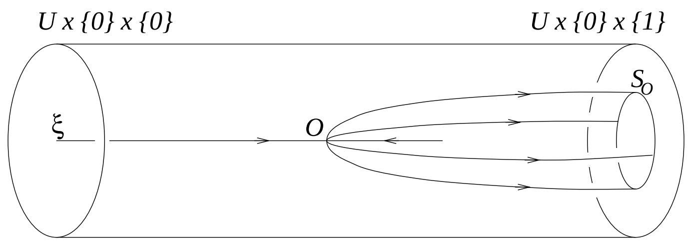

Figure 1

Next, we transform X into a vector field Y on M having a hyperbolic singularity O with k expanding eigenvalues $\sigma_1, \ldots, \sigma_k$ and 3 contracting eigenvalues $\lambda_0, \lambda_1, \lambda_2$. This surgery procedure is reminiscent of the classical construction of Cherry flows, see [8]. A 3-dimensional version was used in [4], [6] to produce new examples of Lorenz-like attractors, obtained from hyperbolic flows through a unique bifurcation. Let q be an arbitrary point in T^k, and U be a small neighbourhood of q. We fix the "tube" $V = h(U \times D^2 \times [0,1])$ around the trajectory of $\xi = h(q,0,0)$, and modify the vector field inside V in such a way as to create such a singularity O. Figure 1 describes the modified system, restricted to $h(U \times \{0\} \times [0,1])$. The stable manifold of O intersects $\Sigma = h(N \times \{0\})$ transversely along a 2-dimensional disk Γ containing ξ. The unstable manifold of O intersects $h(N \times \{1\}) \subset \Sigma$ on a submanifold diffeomorphic to S^{k-1}. There is a Poincaré map $\Phi : \Sigma \setminus \Gamma \to \Sigma$ associated to the vector field Y, which may thought of as the result of "gluing" the solenoid F with the transition map of a flow near a hyperbolic singularity such as we have been considering.

Choosing λ_0 to denote the contracting eigenvalue of O along $h(U \times \{0\} \times [0,1])$, we suppose that $|\lambda_i| \ll |\lambda_0|$ for $i = 1,2$, and $|\sigma_j|^{-1} \ll |\lambda_0|$ for $j = 1, \ldots, k$. We also take the contraction rate $\|DF \mid \{x\} \times D^2\|$ of F along every vertical leaf to be much smaller than $|\lambda_0|$. Then Φ is a hyperbolic map:

(1) Φ admits an invariant contracting smooth foliation $\mathcal{F}^s$ of Σ, whose leaves are diffeomorphic to D^2 (and include Γ);
(2) the quotient space $\Sigma/\mathcal{F}^s$ is diffeomorphic to T^k; denoting q the point in T^k corresponding to Γ, the map $\phi\colon T^k \setminus \{q\} \to T^k$ induced by Φ is expanding.

The local behaviour of ϕ is described in Figure 1. Near the point q the derivative goes to infinity and, roughly speaking, ϕ maps q to a sphere S_O of codimension 1 in T^k, the image of any small neighbourhood of q missing the "inside" of S_O.

Hyperbolicity of the first-return map Φ is a robust property, in the sense that every vector field Z which is C^1-close enough to Y has a similar hyperbolic first-return map on Σ. Moreover, if Z is C^∞-close to Y then its contracting foliation $\mathcal{F}^s_Z$ is also C^2, and it is C^2-close to $\mathcal{F}^s = \mathcal{F}^s_Y$. As a consequence, all the features of the maps Φ and ϕ we shall need in the sequel (e.g. strong expansion rate) remain valid in a neighbourhood of Y, which ensures that the arguments in Section 3, respectively Section 4, hold for any vector field Z C^1-near, respectively C^∞-near,

Y. These are our examples of multidimensional Lorenz-like flows: the attractor of Z is just the maximal invariant compact set

$$A(Z) = \bigcap_{T>0} \text{closure}\Big(\bigcup_{t>T} Z_t(\Sigma)\Big).$$

3. Topological properties of the attractor

Observe that the expansion rate σ of the map ϕ can be made arbitrarily large, by choosing f strongly expanding and taking the neighbourhood U in our construction small enough. In this section we show that the flow is transitive on the attractor, if σ is large enough: we assume that $\sigma > \max\{2, 2\frac{\Delta}{R}\}$, where Δ is the diameter of T^k and R is the radius of injectivity of the exponential map on T^k. The idea is to prove that ϕ is transitive on $T^k \sim \Sigma/\mathcal{F}^s$: then, as an easy consequence, Φ is transitive on Σ, and the flow of Z is transitive on $A(Z)$. For general C^1-vector fields Z close to Y, the contracting foliation $\mathcal{F}^s_Z$ need not be smooth, in which case ϕ may fail to be differentiable. However, this technical point is easily bypassed (e.g. dealing directly with the return map Φ), and so we need not be concerned with it in this outline of our argument.

As a matter of fact, we prove a stronger (topological mixing) property for ϕ: given any open subset W of T^k, there is $N \geq 0$ such that $\phi^{N+1}(W)$ covers the whole T^k. For any open subset $Q \subset T^k$ we define $\rho(W)$ as the radius of the largest ball contained in Q. First, we note that whenever $\rho(W) < R$, then $\rho(\phi(W)) > (\sigma/2)\rho(W)$ (dividing σ by 2 is necessary only if the discontinuity q belongs in W). Since $\sigma/2 > 1$, we get that there exists $N > 0$ such that $\rho(\phi^N(W)) > R$. Finally, using $(\sigma/2)R > \Delta$ we conclude that the image of any ball of radius R covers T^k, and so $\phi^{N+1}(W) = T^k$.

Now, let us briefly explain why these vector fields Z have infinite modulus of stability, that is, the classes of topological equivalence in a neighbourhood of each vector field cannot be parametrized by any open subset of an euclidean space $\mathbb{R}^n$, $n \geq 1$. As the singular set S_O is infinite, for $k \geq 2$, there are infinitely many degrees of freedom to change the set of combinatorial itineraries of its points, by arbitrarily small perturbations of the expanding map ϕ. Since a topological conjugacy between two such discontinuous maps must send the singular set of one into the singular set of the other, preserving itineraries, it follows that ϕ has infinite modulus of stability. The analogous statement for the flow is a direct consequence.

4. Statistical properties of the attractor

The main step to show that the flow has an SRB measure supported on the attractor is to prove that ϕ has an ergodic invariant probability measure μ_ϕ which is absolutely continuous with respect to Lebesgue measure on T^k. We follow the usual strategy of analysing spectral properties of the transfer operator $\mathcal{L} : \varphi \mapsto \mathcal{L}\varphi$, defined by

$$\mathcal{L}\varphi(y) = \sum_{\phi(x)=y} \frac{\varphi(x)}{|\operatorname{Jac}\phi(x)|}.$$

A very useful remark is that, although the map ϕ is not even continuous, its local inverse branches are rather smooth. Indeed, the inverse of the restriction of ϕ to
a small neighbourhood of q may be continuously extended to the disk H_q inside

S_O, by setting $\phi^{-1}|H_O \equiv q$. In fact, this extension is everywhere C^1, due to our assumptions on the eigenvalues at the singularity O. It follows that the operator $\mathcal{L}$ preserves the space of Lipschitz continuous functions, and even improves Lipschitz constants (above a threshold that depends only on ϕ). At this point we also need to assume that the "gluing" in Section 2 has been done in a convenient way, to ensure a fair amount of global regularity (distortion bounds) for the map ϕ. Then one deduces, along well-known lines, that the map ϕ has a unique absolutely continuous invariant probability measure μ_ϕ, and that this measure is ergodic and supported on the whole T^k.

Now, one may view μ_ϕ as a measure on the σ-algebra of subsets of Σ which are union of full leaves of $\mathcal{F}^s$. Then $\mu_\Phi = \lim \Phi_*^n(\mu_\phi)$ defines a probability measure on Σ, which is an SRB measure for Φ. Finally, the measure μ in the Theorem is obtained, simply, by suspending μ_Φ along the flow $\hat{Z}(x,t) = Z_t(x)$ of the vector field Z. More precisely, $\mu = \hat{Z}_*\big(\mu_\Phi \times dt \mid \{(x,t) \in \Sigma \times \mathbb{R} : 0 \le t < \tau(x)\}\big)$ where $\tau(x) = \inf\{t > 0 : Z_t(x) \in \Sigma\} \in (0, +\infty]$ denotes the return time to Σ of a point $x \in \Sigma$ (this is a μ_Φ-integrable function).

5. Further comments

A modification of the previous construction allows us to extend part (1) of the Theorem to the case $k = 2$ and $n = 4$. The main new step is to exhibit an expanding map $\phi : T^2 \setminus \{q\} \to T^2$, having arbitrarily large expansion rate and admitting a codimension 1 solenoid Φ. That is, Φ is an embedding of $(T^2 \setminus \{q\}) \times D^1$ into $T^2 \times D^1$, such that $\pi_1 \circ \Phi = \phi \circ \pi_1$, and which is a strong contraction along each leaf of the vertical foliation of $T^2 \times D^1$. We construct these maps in such a way that in a neighbourhood of q the singular solenoid Φ coincides with the transition map of a flow close to a hyperbolic singularity with 2 expanding eigenvalues. This is done as follows.

A construction of P. Schweitzer [9] provides an embedding ϕ_0 of $T^k \setminus \{q\}$ inside $T^k \times D^1$, transverse to the leaves of the vertical foliation. On the other hand, there is a classical construction of a dimension 2 foliation of $T^2 \times D^1$, whose leaves are injective immersions of $\mathbb{R}^2$. Using this, one may obtain an embedding of $D^2 \times D^1$ into $T^2 \times D^1$ preserving the vertical foliation and strongly contracting each leaf, and whose quotient map is a (strongly) expanding map ϕ_1 from D^2 to T^2. Then it suffices to take $\phi = \phi_1 \circ \phi_0$.

A few concluding remarks are in order on our statements. Periodic orbits are dense in the attractors we have constructed, moreover, all the periodic orbits are hyperbolic and have the same index (number of contracting eigenvalues). We mention that a different construction, to be given elsewhere by the first and the last authors, yields robust Lorenz-like attractors that contain several singularities with different indices. In these other examples, the indices of (regular) periodic orbits are also variable, and the orbits need not be hyperbolic.

Finally, it seems that solenoids may be replaced by much more general hyperbolic attractors in the construction of Section 2, although at present we only have a partial proof of this.

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