Solution of the basin problem for Hénon-like attractors

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Abstract

For a large class of non-uniformly hyperbolic attractors of dissipative diffeomorphisms, we prove that there are no "holes" in the basin of attraction: stable manifolds of points in the attractor fill-in a full Lebesgue measure subset. Then, Lebesgue almost every point in the basin is generic for the SRB (Sinai-Ruelle-Bowen) measure of the attractor. This solves a problem posed by Sinai and by Ruelle, for this class of systems.

1 Introduction

For most dynamical systems, *are time averages well-defined at Lebesgue almost every orbit ?* This is always the case if the system preserves Lebesgue measure, according to the ergodic theorem. However, in general this fundamental problem, raised by Sinai and by Ruelle in the seventies, remains essentially open. For dissipative systems one usually looks at the dynamics in the basin of each attractor, and then the problem can be restated *Is almost every orbit in the basin of attraction asymptotic to some orbit contained in the attractor ? Is it generic for some SRB measure supported in the attractor ?* See [11, Sec. IV], [16, p. 148].

By *attractor* one means a compact invariant subset Λ of the phase space M, dynamically indecomposable (e.g. Λ contains dense orbits), and whose *basin* $B(\Lambda) = \{z \in M$ whose future orbit accumulates on $\Lambda\}$ is a large set (a neighbourhood of Λ, say). One wants to focus on attractors having some degree of robustness under perturbations of the dynamical system, which is often associated to some form of hyperbolicity. Let $W^s(\xi)$ denote the set of all $z \in M$ whose orbit approaches the orbit of a point ξ as time goes to $+\infty$. Then the first question may be formulated

(B1) does $B(\Lambda) = \bigcup_{\xi\in\Lambda} W^s(\xi)$ up to a zero Lebesgue measure set ?

Suppose Λ supports an invariant ergodic measure μ which is hyperbolic (all the Lyapunov exponents are nonzero) and whose conditional measures on unstable manifolds are absolutely continuous with respect to Lebesgue measure. Then, see [17, 18], μ is an *SRB measure*, in the

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sense that its *basin* $B(\mu)$ has positive Lebesgue measure in M. By definition, $B(\mu)$ is the set of points $z \in M$ such that the time average of every continuous function $\varphi : M \to \mathbb{R}$ on the orbit of z exists and coincides with the space average $\int \varphi \, d\mu$ (one also says that z is a *generic point* for μ). Then, supposing that μ is unique,

(B2) does $B(\mu) = B(\Lambda)$ up to a zero Lebesgue measure set ?

It is now classical that both versions (B1) and (B2) of the basin problem have an affirmative answer in the case of uniformly hyperbolic (Axiom A) attractors, where a main ingredient is the uniform shadowing property. See [7, 8, 19, 20, 21]. On the other hand, although these problems have been around for some time, little is known in the non-uniformly hyperbolic setting: exceptions include the geometric Lorenz-like attractors [1, 12] (for which a stable foliation exists, essentially, by definition), and systems preserving a smooth ergodic measure μ (where $B(\mu)$ has full measure as a direct consequence of the ergodic theorem).

Here we give a positive solution to the basin problem for Hénon-like attractors. This type of attractor was first constructed in [2], where it was shown that the Hénon model

$$h(x, y) = (1 - ax^2 + y, bx) \tag{1}$$

has a "strange" (non-hyperbolic) attractor for a set of values of the parameters (a, b) with positive Lebesgue measure. Based on these methods, attractors combining hyperbolic behaviour with presence of "folding" regions were shown to occur persistently in certain general bifurcation mechanisms [10, 15]. Moreover, it was proved in [5] that all these *Hénon-like attractors* support a unique invariant measure μ as above. It is for this class of systems that we state our results.

Theorem A. *Let Λ be a Hénon-like attractor of a surface diffeomorphism $f : M \to M$. Then, through Lebesgue almost every point $z \in B(\Lambda)$ passes a stable leaf $W^s(\xi)$ of some $\xi \in \Lambda$: $\operatorname{dist}(f^n(z), f^n(\xi)) \to 0$ exponentially fast as $n \to +\infty$.*

Theorem B. *Denoting by μ the SRB measure of f on Λ, then for Lebesgue almost every $z \in B(\Lambda)$ one has*

$$\lim_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(z)) = \int \varphi \, d\mu \quad \textit{for every continuous } \varphi : M \to \mathbb{R}.$$

As a by-product of the proofs we also get that the stable manifold $W^s(P)$ is dense in $B(\Lambda)$, where $P \in \Lambda$ denotes a hyperbolic saddle-point such that $\Lambda = \operatorname{closure}(W^u(P))$.

Let us mention a few other recent developments in the ergodic theory of Hénon-like attractors. It is shown in [6] that the system (f, μ) has exponential decay of correlations in the space of Hölder continuous observable functions. In [4] we prove that these systems are stochastically stable with respect to small random perturbations with absolutely continuous transitions. Moreover, an alternative approach to the construction of the SRB measure has been announced by the authors of [13].

According to [22], persistent Hénon-like attractors exist for diffeomorphisms on manifolds of arbitrary dimension. Not all our arguments carry on to higher dimensions, but we expect Theorems A and B to hold in such generality, and it would be nice to establish this.

We recall in the next section those known properties of Hénon-like attractors that are used in our arguments. In the remainder of this Introduction we comment on ideas involved in the proof of Theorem A. We begin by pointing out that a local version of the basin problem, which holds in the Axiom A case, is *false* for Hénon-like maps: the local stable sets of points $\xi \in \Lambda$, *do not* fill-in a full Lebesgue measure subset of a neighbourhood of the attractor. This means that global control of the stable lamination is needed in the present case.

Our strategy is to identify a positive Lebesgue measure set H formed by stable leaves of points in Λ, and to show that almost every $z \in B(\Lambda)$ eventually reaches this set. The points in H are characterized by a bounded recurrence property that ensures that their orbits do not return too often to the folding region and, most important, these returns are always "tangential" in the sense of [2], see Section 2 below.

The arrival time to H, a parameter of "nonlocalness", depends in a very discontinuous way on the point z. The fact that it is finite Lebesgue almost everywhere relies on a statistical argument that we present in Section 4. On its turn, this is based on a geometric pseudo-Markov construction on the basin of the attractor Λ, which we describe in Section 3. A main feature is the following bounded geometry property: the set of points sharing the same finite itinerary is always a rectangle (bounded by a pair of stable segments and a pair of unstable segments).

In Section 5 we put these ideas together to prove Theorem A. This includes some description of the topological basin of attraction, for which it is convenient to consider the orientation-preserving and orientation-reversing maps separately. In the orientation-preserving case, we add a mild technical assumption that may exclude some of the parameters in previous papers, but keeping a positive Lebesgue measure set of them. Theorem B is a corollary of Theorem A, using the fact that $B(\mu)$ consists of entire stable leaves $W^s(\xi)$.

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2 Hénon-like attractors revisited

Let us recall a number of known facts about Hénon-like attractors, from [2], [5], [15], that we use in Sections 3 through 5. First of all we fix some notations.

We deal with parametrised families of diffeomorphisms of the plane

$$f(x,y) = f_a(x,y) = (1 - ax^2, 0) + R(a,x,y), \tag{2}$$

R close to zero in the C^3 norm, which we call *Hénon-like families*. More precisely, we suppose that $\|R\|_{C^3} \le J\sqrt{b}$ on $[1,2] \times [-2,2]^2$, with

$$J^{-1}b \le |\det Df| \le Jb \quad \text{and} \quad \|D\big(\log|\det Df|\big)\| \le J, \tag{3}$$

where $J > 0$ is arbitrary and $b > 0$ is taken sufficiently small. The quadratic family $1 - ax^2$ may be replaced by any family of maps in some fixed C^3 neighbourhood of it. The Hénon model (1) is affinely conjugate to the map $f(x,y) = (1 - ax^2 + \sqrt{b}y, \sqrt{b}x)$, and so does fall into this framework if b is small.

We consider parameter values $a \in [a_1, a_2]$ with $1 \gg \delta \gg 2 - a_1 > 2 - a_2 \gg b$. The parameter interval should not be too small: $(a_2 - a_1) \geq (2 - a_2)/10$ suffices. Moreover, 2 may be replaced by any Misiurewicz parameter of the quadratic family $1 - ax^2$. In this parameter range, f has a unique fixed saddle-point P such that $\Lambda = \text{closure}(W^u(P))$ is compact, indeed Λ is contained in $(-2,2)^2$. It is well-known that the basin $B(\Lambda)$ has nonempty interior, see [2] or [16, App. III]. In all the situations concerned here it even contains a neighbourhood of Λ, see [3, 23], and Section 5 below.

2.1 Existence and properties

Besides J, let $\sqrt{e} < \sigma_1 < \sigma_2 < 2$ be fixed at the very beginning. For the next theorem, one also fixes constants $1 \gg \beta \gg \alpha > 0$, and supposes $b \ll \delta \ll \alpha$. Throughout, we use $C > 1$ to represent various large constants depending only on J, σ_1, σ_2, α, or β (not on δ or b). Analogously, $c \in (0,1)$ is a generic notation for small constants depending only on J, σ_1, σ_2, α, or β. Let $I(\delta) = \{(x,y) : |x| < \delta\}$. For $z \in W^u(P)$, let $t(z)$ be any norm 1 vector tangent to $W^u(P)$ at z (the particular choice is irrelevant). Given a non-zero vector $v = (v_1, v_2) \in \mathbb{R}^2$, $\text{slope}\, v$ will always be taken with absolute values, i.e. $\text{slope}\, v = |v_2|/|v_1|$.

Theorem 2.1. *Given any Hénon-like family, there exists a positive Lebesgue measure set E such that for every $a \in E$ the map f has a countable* critical set $\mathcal{C} \subset W^u(P) \cap I(\delta)$ *whose elements ζ satisfy*

1. *$t(\zeta)$ is almost horizontal and $t(f(\zeta))$ is almost vertical, in the sense that* $\text{slope}\, t(\zeta) \leq C\sqrt{b}$ *and* $\text{slope}\, t(f(\zeta)) \geq c/\sqrt{b}$*;*
2. *$t(f(\zeta))$ is exponentially contracted and $w_0 = (1,0)$ is exponentially expanded under positive iterates: $\|Df^n(f(\zeta))t(f(\zeta))\| \leq (C\,b)^n$ and $\|Df^n(f(\zeta))w_0\| \geq \sigma_1^n$ for all $n \geq 1$;*
3. *if $f^n(\zeta) \in I(\delta)$ then there is $\zeta_n \in \mathcal{C}$ so that* $\text{dist}(f^n(\zeta), \zeta_n) \geq e^{-\alpha n}$ *and there is a C^2 curve $L = \{(x, y(x))\}$ with $|y'(x)| \leq 1/10$ and $|y''(x)| \leq 1/10$, tangent to $t(\zeta_n)$ at ζ_n and also containing $f^n(\zeta)$.*

In addition, there exists $\zeta \in \mathcal{C}$ such that $\{f^n(\zeta) : n \geq 0\}$ is dense in Λ.

Theorem 2.1 was first proved for the Hénon model in [2]. Then the arguments were extended to the Hénon-like case in [15]. The property in part 3 plays a central role in the proof, as well as in our own arguments here, and we shall return to comment on it. From now on we always suppose $a \in E$. The remaining statements in this subsection are part of the proof of this theorem, but we also make independent use of them in Sections 3–5.

Proposition 2.2.

1. *There exists $\zeta_0 = (x_0, y_0) \in \mathcal{C}$ with $|x_0| \leq C\sqrt{b}$, so that $\mathcal{C} \cap G_0 = \{\zeta_0\}$, where G_0 denotes the segment connecting $f(\zeta_0)$ to $f^2(\zeta_0)$ in $W^u(P)$;*
2. *denoting $G_g = f^g(G_0) \setminus f^{g-1}(G_0)$, then $\mathcal{C} \cap G_g$ is finite for every $g \geq 1$, and in fact $\mathcal{C} \cap G_1$ consists of a single point ζ_1;*
3. *for every $\zeta \in \mathcal{C} \cap G_g$ and $g \geq 0$, the segment $\gamma = \gamma(\zeta)$ of radius δc^g around ζ in $W^u(P)$ may be written $\gamma = \{(x, y(x))\}$ with $|y'(x)| \leq C\sqrt{b}$ and $|y''(x)| \leq C\sqrt{b}$;*
4. *given any $\zeta \in \mathcal{C} \cap G_g$ with $g > 0$, there exist $\tilde{g} < g$ and $\tilde{\zeta} \in \mathcal{C} \cap G_{\tilde{g}}$ with* $\text{dist}(\zeta, \tilde{\zeta}) \leq b^{g/10}$.

The lower bound on the length of the segments $\gamma(\zeta)$ is important, so that we give a special name ρ to the constant c in the context of part 3 of the proposition. Moreover, we write K for the large constant C, and call a $C^2(b)$ *curve* any curve $\{(x, y(x))\}$ with $|y'(x)| \le K\sqrt{b}$ and $|y''(x)| \le K\sqrt{b}$. Note that the expanding eigenvalue of $Df(P)$ is negative and so G_0 is a neighbourhood of P and ζ_0 in $W^u(P)$. It is easy to see that G_0 and G_1 contain $C^2(b)$ curves extending from $x = -9/10$ to $x = 9/10$. For $g \ge 0$, points in G_g are said to be of *generation* g.

Since every orbit in $B(\Lambda)$ must eventually enter $[-2,2]^2$, we may always assume to be dealing with orbits which never leave $[-2,2]^2$ in positive time, and we do so. Given $\lambda > 0$, a point $z = (x, y)$ is called λ-*expanding* if

$$\|Df^j(z)w_0\| \ge \lambda^j \quad \text{for all } j \ge 1. \tag{4}$$

An important case is $z \in f(\mathcal{C})$, with $\lambda = \sigma_1$, cf. Theorem 2.1.2. We say that z is λ-expanding up to time n if the inequality in (4) holds for $1 \le j \le n$. We define the *contracting direction of order* $n \ge 1$ at z as the tangent direction $e^{(n)}(z)$ that is most contracted by $Df^n(z)$. The next proposition summarises a number of results from [2, Section 5] and [15, Section 6]. In the statement $\lambda > 0$ and $\tau > 0$ are arbitrary constants, with τ sufficiently small (e.g. $\tau \le 10^{-20}$), and one assumes that b is much smaller than either of them.

Proposition 2.3. *Let z be λ-expanding up to time $n \ge 1$, and ξ satisfy $\mathrm{dist}(f^j(\xi), f^j(z)) \le \tau^j$ for every $0 \le j \le n-1$. Then, for any point η in the τ^n-neighbourhood of ξ and for every $1 \le l \le k \le n$,*

1. *$e^{(k)}(\eta)$ is uniquely defined and nearly vertical:* $\mathrm{slope}(e^{(k)}(\eta)) \ge c/\sqrt{b}$;
2. $\mathrm{angle}(e^{(l)}(\eta), e^{(k)}(\eta)) \le (Cb)^l$ *and* $\|Df^l(\eta)e^{(k)}(\eta)\| \le (Cb)^l$;
3. $\|De^{(k)}(\eta)\| \le C\sqrt{b}$ *and* $\|D^2e^{(k)}(\eta)\| \le C\sqrt{b}$;
4. $\|D(Df^l e^{(k)})(\eta)\| \le (Cb)^l$;
5. $1/10 \le \|Df^n(\xi)w_0\| / \|Df^n(z)w_0\| \le 10$ *and* $\mathrm{angle}(Df^n(\xi)w_0, Df^n(x)w_0) \le (\sqrt{C\tau})^n$.

Parts 3 and 4 are also true for the derivatives of $e^{(k)}$ and $Df^l e^{(k)}$ with respect to the parameter a. Throughout, we write expanding to mean λ-expanding for some $\lambda \ge e^{-20}$ (cf. Remark 3.2).

Proposition 2.4. *If z is an expanding point then its stable set $W^s(z)$ contains a segment $\Gamma = \Gamma(z) = \{(x(y), y) : |y| \le 1/10\}$ with $|x'| \le C\sqrt{b}$ and $|x''| \le C\sqrt{b}$, such that $z \in \Gamma$ and*

$$\mathrm{dist}(f^n(\xi), f^n(\eta)) \le (Cb)^n \,\mathrm{dist}(\xi, \eta), \quad \textit{for all } \xi, \eta \in \Gamma \textit{ and } n \ge 1.$$

Moreover, if z_1, z_2 are expanding points then

$$\mathrm{angle}(t_\Gamma(\xi_1), t_\Gamma(\xi_2)) \le C\sqrt{b}\,\mathrm{dist}(\xi_1, \xi_2), \quad \textit{for every } \xi_1 \in \Gamma(z_1), \xi_2 \in \Gamma(z_2),$$

where $t_\Gamma(\xi_i)$ denotes any norm 1 vector tangent to $\Gamma(z_i)$ at ξ_i, $i = 1, 2$.

We call a *long stable leaf* any curve Γ as in this proposition, and a *stable leaf* any compact curve having some iterate contained in a long stable leaf. The first part of the proposition is proved in [2, Section 5.3], the arguments extending directly to Hénon-like maps [15, Section 7C]. We sketch the proof, to explain how the second part, not explicitly stated in those papers, can be deduced from the construction.

One takes $\Gamma(z) = \lim \Gamma^n(z)$, where $\Gamma^n(z)$ is the integral curve of the direction field $e^{(n)}$ (the *temporary stable leaf* of order n) through the point z. One can check directly that the first integral curve $\Gamma^1(z)$ is long, meaning that it extends from $y = -1/10$ to $y = +1/10$. To prove that the same is true for all the $\Gamma^n(z)$ one uses induction. Let τ be fixed, as in Proposition 2.3, and $b \ll \tau$. Part 2 of that proposition implies

$$\operatorname{dist}(f^j(\xi), f^j(z)) \le (Cb)^j \operatorname{dist}(\xi, z) \quad \text{for any} \quad 1 \le j \le n-1 \quad \text{and} \quad \xi \in \Gamma^{n-1}(z).$$

As a consequence, $e^{(n)}(\eta)$ is well-defined in the τ^n-neighbourhood of $\Gamma^{n-1}(z)$. Moreover, $\operatorname{angle}(e^{(n-1)}, e^{(n)}) \le (Cb)^{n-1}$ ensures that the integral curve $\Gamma^n(z)$ does not leave this τ^n-neighbourhood inside the region $|y| \le 1/10$. Thus $\Gamma^n(z)$ must be long. The previous angle estimate also implies that $\lim \Gamma^n(z)$ does exist, and this is how one gets the first claim in Proposition 2.4.

Now, given z_1, z_2, $\xi_1 \in \Gamma(z_1)$, $\xi_2 \in \Gamma(z_2)$ as in the proposition, let n be fixed such that $\tau^{n+1} \le \operatorname{dist}(\xi_1, \xi_2) \le \tau^n$. By parts 2 and 3 of Proposition 2.3, $\operatorname{angle}(t_\Gamma(\xi_i), e^{(n)}(\xi_i)) \le (Cb)^n$ for $i = 1, 2$, and $\operatorname{angle}(e^{(n)}(\xi_1), e^{(n)}(\xi_2)) \le C\sqrt{b} \operatorname{dist}(\xi_1, \xi_2)$. Then

$$\operatorname{angle}(t_\Gamma(\xi_1), t_\Gamma(\xi_2)) \le 2(Cb)^n + C\sqrt{b} \operatorname{dist}(\xi_1, \xi_2) \le 2C\sqrt{b} \operatorname{dist}(\xi_1, \xi_2).$$

This gives the last statement in Proposition 2.4.

Proposition 2.5. *Given any $k \ge 1$, any $z \in [-2, 2]^2$ satisfying $f^j(z) \notin I(\delta)$ for $0 \le j < k$, and any tangent vector v with $\|v\| = 1$ and* slope $v \le 1/5$*, then*

$$\operatorname{slope}(Df^j(z)\, v) \le (C/\delta)\sqrt{b} < 1/10 \quad \textit{and} \quad \|Df^j(z)v\| \ge c\delta\sigma_2^j$$

for $1 \le j \le k$. If either $z \in f(I(2\delta))$ or $f^k(z) \in I(2\delta)$ then $\|Df^k(z)\, v\| \ge \sigma_2^k$, and in the latter case we also have $\operatorname{slope}(Df^k(z)\, v) \le C\sqrt{b}$.

This means, in particular, that pieces of orbits outside $I(\delta)$ are (essentially) expanding. Similar statements are well-known for one-dimensional maps such as $x \mapsto 1 - ax^2$. The proposition follows using a perturbation argument, see [2, Lemmas 4.5, 4.6].

Another important notion is that of *bound period* $p(n, \zeta)$ associated to a *return* n of a critical point $\zeta \in \mathcal{C}$. These are defined through the following inductive procedure. If $n \ge 1$ does not belong to $[\nu + 1, \nu + p(\nu, \zeta)]$ for any return $1 \le \nu < n$, then n is a (*free*) return for ζ if and only if $f^n(\zeta) \in I(\delta)$. Moreover, the bound period $p = p(n, \zeta)$ is the largest integer such that

$$\operatorname{dist}(f^{n+j}(\zeta), f^j(\zeta_n)) \le e^{-\beta j} \quad \text{for all } 1 \le j \le p, \tag{5}$$

where ζ_n is the *binding point* of $f^n(\zeta)$, given by Theorem 2.1.3. If, on the contrary, n is in $[\nu + 1, \nu + p(\nu, \zeta)]$ for some previous return $1 \le \nu < n$ then, by definition, n is a (*bound*) return for ζ if and only if $n - \nu$ is a return for the binding point ζ_ν, and we let $p(n, \zeta) = p(n - \nu, \zeta_\nu)$.

Up to a slight (and otherwise irrelevant) modification of these definitions, see [2, Section 6.2] or [15, Section 8], we may suppose that bound periods are nested: whenever $n \in [\nu+1, \nu+p(\nu,\zeta)]$ then $n + p(n,\zeta) \le \nu + p(\nu,\zeta)$, that is to say, the bound period associated to n ends before the one associated to ν.

We write $d_n(\zeta) = \operatorname{dist}(f^n(\zeta), \zeta_n)$, for ζ and ζ_n as before. Moreover, $w_j(z) = Df^j(f(z))w_0$ for any point z and $j \ge 0$.

Proposition 2.6. *Let $n \ge 1$ be a free return of $\zeta \in \mathcal{C}$, and $p = p(n,\zeta)$ be the corresponding bound period. Then*

1. $(1/5)\log(1/d_n(\zeta)) \le p \le 5\log(1/d_n(\zeta))$;
2. $\|w_{n+p}(\zeta)\| \ge \sigma_1^{(p+1)/3}\|w_{n-1}(\zeta)\|$ *and* $\operatorname{slope} w_{n+p}(\zeta) \le (C/\delta)\sqrt{b}$;
3. $\|w_{n+p}(\zeta)\| d_n(\zeta) \ge c e^{-\beta(p+1)}\|w_{n-1}(\zeta)\|$;
4. $\|w_j(f^n(\zeta))\| \ge \sigma_1^j$ *for* $1 \le j \le p$, *and* $\operatorname{slope} w_p(f^n(\zeta)) \le (C/\delta)\sqrt{b}$.

A main ingredient here is the property in Theorem 2.1.3. Actually, for free returns n, a curve L as in the theorem may be taken tangent not only to $t(\zeta_n)$ at ζ_n but also to $w_{n-1}(\zeta)$ at $f^n(\zeta)$, see [2, Section 7.3] and [15, Lemma 9.5]. We shall explain below, in a more general context, how this is used in the proof.

2.2 Dynamics on the unstable manifold

The next proposition, appearing in [5], permits to extend to generic orbits in $W^u(P)$ the control given by the previous statements for orbits of critical points. This is a key step in the construction of the SRB measure of f on Λ that appeared in that paper, cf. Theorem 2.9 below.

Proposition 2.7. *Let $\tilde{z} \in W^u(P)$ be such that $f^n(\tilde{z}) \notin \mathcal{C}$ for every $n \ge 1$. Then, given any $n \ge 1$ such that $f^n(\tilde{z}) \in I(\delta)$, there exists $\zeta_n \in \mathcal{C}$ and some C^2 curve $L = \{(x, y(x))\}$ with $|y'| \le 1/10$ and $|y''| \le 1/10$, tangent to $t(\zeta_n)$ at ζ_n and also containing $f^n(\tilde{z})$.*

Let us elaborate a bit on the content and consequences of this proposition. Given a point $z \in W^u(P)$, fix $k \gg 1$ so that $\tilde{z} = f^{-k}(z)$ belongs to a small neighbourhood of P in $W^u(P)$. We can now define *returns*, *binding points*, and *bound periods* for $\tilde{z}$ in the same way as we did before for critical points. That is, corresponding to a free return n of $\tilde{z}$ we choose as binding point a critical point ζ_n as in the proposition, and define the bound period $p = p(n,\tilde{z})$ of $f^n(\tilde{z})$ with respect to this ζ_n, cf. (5). As in the case of critical points, we take the bound periods nested; see also comments following the next proposition.

We say that $z = f^k(\tilde{z})$ is a *free point* if k is outside every bound period $[\nu+1, \nu+p(\nu, z_0)]$ of $\tilde{z}$. This is an intrinsic property of the point z: the choice of k is irrelevant, as long as it is large enough. We call a segment $\gamma \subset W^u(P)$ *free* if all its points are free. While proving Proposition 2.7, it is shown in [5] that if n is a free return for $\tilde{z}$ and $\gamma \subset W^u(P)$ is a free segment containing $f^n(\tilde{z})$, then the same binding point may be assigned to all the points in $\gamma \cap I(\delta)$. More precisely, *there is a critical point ζ_γ and a curve L as in the statement, tangent to $t(\zeta_\gamma)$ at ζ_γ and containing the whole γ.* In particular, L is tangent to $t(w)$ at every $w \in \gamma$. In some cases $\zeta_\gamma \in \gamma = L$, but it is not always possible to take $L \subset W^u(P)$.

Given any maximal free segment γ intersecting $I(\delta)$, we always fix L and ζ_γ as above, and set $d_{\mathcal{C}}(w) = \operatorname{dist}(w, \zeta_\gamma)$ for each $w \in L$. We extend $t(w)$ to represent a norm 1 vector tangent to the curve L at every $w \in L$, and define the bound period $p(w)$ of every $w \in L$ with respect to this ζ_γ, cf. (5).

The following definition is a slight extension of notions with similar denominations appearing in [2, 5, 6, 15]. Given points p, q and tangent vectors u, v, we say that p is in *tangential position* relative to (q, v) if there exists a curve $\{(x, y(x))\}$ with $|y'| \le 1/5$ and $|y''| \le 1/5$, tangent to v at q and also containing p. And we say that (p, u) is in tangential position relative to (q, v) if such a curve may be chosen tangent to u at p. Thus, as we have seen, if z is a free point contained in the $W^u(P)$ then $(z, t(z))$ is in tangential position with respect to $(\zeta_\gamma, t(\zeta_\gamma))$ for some critical point ζ_γ. It is worth stressing that there can be no analog of this for points outside the unstable manifold. One key fact, that we shall prove in Section 4, is that *for points in the basin returns are almost surely eventually tangential.*

Proposition 2.8. *Given any curve L as before and $z \in L$,*

1. $(1/5)\log(1/d_{\mathcal{C}}(z)) \le p(z) \le 5\log(1/d_{\mathcal{C}}(z))$;
2. $\|Df^{p(z)+1}(z)t(z)\| \ge \sigma_1^{(p(z)+1)/3}$ *and* $\operatorname{slope}(Df^{p(z)+1}(z)t(z)) < (C/\delta)\sqrt{b}$;
3. $\|Df^{p(z)+1}(z)t(z)\| d_{\mathcal{C}}(z) \ge c e^{-\beta(p(z)+1)}$;
4. $\|w_j(z)\| \ge \sigma_1^j$ *for* $1 \le j \le p(z)$, *and* $\operatorname{slope} w_{p(z)}(z) < (C/\delta)\sqrt{b}$.

Propositions 2.6, 2.8 have similar proofs, based on the tangential position property. We outline the main steps since some features of these arguments are relevant for what follows; see also [2, Section 7.4] and [15, Section 10]. The importance of the tangential position property comes from the fact that the diffeomorphism f behaves, essentially, as a one-dimensional quadratic map over the curve L. Let us begin by explaining this.

Let L be a curve of the form $\{(x, y(x))\}$ with $|y'| \le 1/5$ and $|y''| \le 1/5$. Recall that $f(x, y) = (1 - ax^2, 0) + R(a, x, y)$, where the first and second order derivatives of R are bounded by $C\sqrt{b}$. So, the image of L may be written $f(L) = \{\xi(x), \eta(x)\}$ with $|\xi'' + 2a|$, $|\eta'|$, $|\eta''|$ all bounded above by $C\sqrt{b}$. Let $\Gamma^s = \{(x^s(y), y)\}$ be some nearly vertical curve: $|(x^s)'| \le 1/5$ and $|(x^s)''| \le 1/5$. Let ξ^s be the horizontal distance from $f(L)$ to Γ^s, that is,

$$\xi^s(x) = x^s(\eta(x)) - \xi(x) \tag{6}$$

for each x. The previous bounds on η, x^s, and their derivatives, imply that $|(\xi^s)'' + \xi''| \le C\sqrt{b}$. This gives $|(\xi^s)'' - 2a| \le C\sqrt{b}$, and so $(\xi^s)''(x) \in (3, 5)$, up to taking b small and a close to 2. Now, suppose there exists $\zeta_\gamma = (x_\gamma, y(x_\gamma)) \in L$ such that $f(L)$ is tangent to Γ^s at $f(\zeta_\gamma)$. The situation we have in mind is when ζ_γ is a critical point and Γ^s is the long stable leaf through its image, cf. Theorem 2.1 and Proposition 2.4. Then $\xi^s(x_\gamma) = 0$ and $(\xi^s)'(x_\gamma) = 0$, and so

$$3 \le \frac{(\xi^s)'(x)}{x - x_\gamma} \le 5 \quad \text{and} \quad \frac{3}{2} \le \frac{\xi^s(x)}{(x - x_\gamma)^2} \le \frac{5}{2} \tag{7}$$

for every x. Observe that $|x - x_\gamma|$ is roughly the same as the distance from $(x, y(x))$ to ζ_γ: $|x - x_\gamma| \le \operatorname{dist}((x, y(x)), \zeta_\gamma) \le (6/5)|x - x_\gamma|$, because $L = \{(x, y(x))\}$ with $|y'| \le 1/5$.

Most important, this quadratic behaviour allows one to estimate the expansion loss experienced by trajectories at tangential returns, in terms of the distance to the critical point. This goes as follows. Suppose ζ_γ is a critical point, and all the points in $f(L)$ are expanding up to some time $p \geq 1$. Let $z(s) = (s, y(s))$ be a generic point of L. By Proposition 2.3.1, the contracting direction of order p at $f(z(s))$ is well defined and almost vertical: $e(s) = e^{(p)}(f(z(s)))$ is represented by a vector $(\epsilon(s), 1)$ with $|\epsilon(s)| \leq C\sqrt{b}$. Besides,

$$|\epsilon'(s)| \leq C\sqrt{b} \qquad \text{and} \qquad \|Df^j(f(z(s)))e(s)\| \leq (Cb)^j \;\; \text{for all } 1 \leq j \leq p. \tag{8}$$

See also [2, Section 5] and [15, Section 6]. Then let us split the tangent direction to $f(L)$ into contracting and horizontal (expanding) components

$$(\xi'(s), \eta'(s)) = \alpha(s)e(s) + \beta(s)w_0\,. \tag{9}$$

Of course, $\alpha(s) = \eta'(s)$ and $\beta(s) = \xi'(s) - \epsilon(s)\eta'(s)$. As we have seen, $|\xi'' + 2a|$, $|\eta'|$, $|\eta''|$ are all bounded by $C\sqrt{b}$. Then the same is true for $|\beta' + 2a|$, $|\alpha|$, $|\alpha'|$. In particular,

$$3 \leq \frac{\beta(x_\gamma) - \beta(s)}{s - x_\gamma} \leq 5 \tag{10}$$

for every s. We also have $|\beta(x_\gamma)| \leq (Cb)^p$, as a consequence of the following two observations. By Theorem 2.1.2, the unstable manifold $W^u(P)$ is to tangent to $W^s(f(\zeta_\gamma))$ at $f(\zeta_\gamma)$. By Proposition 2.3.2, the angle between the tangent of $W^s(f(\zeta_\gamma))$ and the contracting direction e of order p is at most $(Cb)^p$.

Now we outline the proof of Proposition 2.8. Fix $z = (x, y(x))$ in L and let $p = p(z)$. As before, $z(s) = (s, y(s))$ represents a generic point of L. We write $w_j(s) = w_j(z(s))$ for each s. First, one proves a distortion estimate, see [2, Lemma 7.8] and [15, Lemma 10.5]:

$$w_j(s) = \lambda(s)\big(w_j(x_\gamma) + \epsilon_j(s)\big), \quad c \leq \lambda(s) \leq C \text{ and } \|\epsilon_j(s)\| \ll \|w_j(x_\gamma)\|\,, \tag{11}$$

for every $0 \leq j \leq p$ and $s \in [x_\gamma, x]$. The main ingredient is provided by Theorem 2.1.3: associated to every return j of ζ_γ there exists a critical point ζ_j with the tangential position property and $\operatorname{dist}(f^j(\zeta_\gamma), \zeta_j) \geq e^{-\alpha j}$. The proof of (11) combines this information with

$$\operatorname{dist}(f^j(z(s)), f^j(\zeta_\gamma)) \leq Ce^{-\beta j} \ll e^{-\alpha j} \leq \operatorname{dist}(f^j(\zeta_\gamma), \zeta_j) \tag{12}$$

for every return $0 \leq j \leq p$, which is a consequence of (5). The fact that ζ_γ is a critical point is irrelevant at this point, as long as we have (12), expansiveness, and the tangential position property. From (11) and Theorem 2.1.2 we get

$$\|w_j(s)\| \approx \|w_j(x_\gamma)\| \geq \sigma_1^j, \tag{13}$$

for all $0 \leq j \leq p$ and $s \in [x_\gamma, x]$. Unless otherwise stated, $\approx$ means that the two expressions coincide up to factors c and C. It follows that $f(z(s))$ is expanding up to time p, for every $s \in [x_\gamma, x]$. Then we may apply the arguments above leading to (10): the tangent direction to $f(L)$ at each point may be split as in (9), and the coefficient $\beta(s)$ satisfies

$$3 \leq \frac{\beta(x_\gamma) - \beta(s)}{s - x_\gamma} \leq 5 \quad \text{and} \quad |\beta(x_\gamma)| \leq (Cb)^p. \tag{14}$$

Using (8), (9), (11), and the last part of (14), we may write

$$
\begin{aligned}
f^{j+1}(\zeta_\gamma) - f^{j+1}(z) &= \int_x^{x_\gamma} \big(\alpha(s) Df^j(f(z(s)))e(s) + \beta(s) w_j(s)\big)\, ds \\
&= w_j(x_\gamma) \int_x^{x_\gamma} \lambda(s)\big(\beta(s) - \beta(x_\gamma)\big)\, ds + \delta_j
\end{aligned}
\tag{15}
$$

with $\|\delta_j\| \ll (Cb)^j + \|w_j(x_\gamma) \int_x^{x_\gamma} \lambda(s)\big(\beta(s) - \beta(x_\gamma)\big)\, ds\|$. By (10) and $c \le \lambda(s) \le C$ in (11),

$$
\int_x^{x_\gamma} \lambda(s)\big(\beta(s) - \beta(x_\gamma)\big)\, ds \approx (x - x_\gamma)^2 \approx d_{\mathcal{C}}(z)^2.
$$

Taking $j = p$ in (15),

$$
e^{-\beta p} \approx \operatorname{dist}(f^{p+1}(z), f^{p+1}(\zeta_\gamma)) \approx \|w_p(x_\gamma)\|\, d_{\mathcal{C}}(z)^2. \tag{16}
$$

Part 1 of the proposition follows from combining this with $4^p \ge \|w_p(x_\gamma)\| \ge \sigma_1^p$. From the relation $Df^{p+1}(z)\, t(z) = \beta(x) w_p(x) + \alpha(x) Df^p(f(z))\, e(x)$, using (10) and (16),

$$
\begin{aligned}
\|Df^{p+1}(z)\, t(z)\| &\ge c\, d_{\mathcal{C}}(z) \|w_p(x)\| - (Cb)^p \|w_p(x)\| - C\sqrt{b}\, (Cb)^p \\
&\ge c \left(e^{-\beta p} \|w_p(x)\|\right)^{1/2} - (4Cb)^p \quad \ge \quad \sigma_1^{(p+1)/3},
\end{aligned}
$$

which proves the first statement in part 2. The second statement uses

$$
\operatorname{slope}(Df^{p+1}(z)\, t(z)) \approx \operatorname{slope} w_p(x) \approx \operatorname{slope} w_p(x_\gamma),
$$

recall (15), (11), together with the fact that $p+1$ does not belong to any bound period of ζ_γ (because bound periods are nested). To get part 3,

$$
\|Df^{p+1}(z)\, t(z)\|\, d_{\mathcal{C}}(z) \ge c\, d_{\mathcal{C}}(z)^2 \|w_p(x)\| - (4Cb)^p \ge c e^{-\beta p}.
$$

The first half of part 4 follows from (13), and the second half is analogous to the slope statement in part 2. This ends our sketch of the proof of Proposition 2.8.

For Proposition 2.6 some extra care is needed: arguments as above assume properties of $w_j(\zeta_n)$, $1 \le j \le p(f^n(\zeta))$, from the statement of Theorem 2.1, while the proposition itself is part of the proof of the theorem. To go around this, one begins by proving that

$$
p(f^n(\zeta)) \le 5 \log\left(1/d_n(\zeta)\right) \le 5\alpha n < n,
$$

which ensures that such properties are used only in an inductive way.

It is clear from the proof that parts 2-4 of Proposition 2.8 remain true if one replaces $t(z)$ by any norm 1 tangent vector v such that (z, v) is in tangential position relative to $(\zeta_\gamma, t(\zeta_\gamma))$. We want to point out that these arguments also allow for some freedom in the very definition of bound period. For instance, let $z(s) \in L$ with $s \in [x_\gamma, x]$ and

$$
|x - s| \le c|x - x_\gamma|. \tag{17}
$$

Taking $c \in (0,1)$ small enough, then (14) and (15) give (distinguish two cases, depending on whether $\|w_j(x_\gamma)\| \, d_{\mathcal{C}}(z)^2$ is larger or smaller than $(Cb)^j$)

$$\operatorname{dist}(f^{j+1}(z(s)), f^{j+1}(\zeta_\gamma)) \approx \operatorname{dist}(f^{j+1}(z), f^{j+1}(\zeta_\gamma)) \quad \text{for any } 0 \le j \le p, \tag{18}$$

(here $\approx$ means equality up to a factor 2), except possibly if both distances are smaller than $(Cb)^j$. In any event,

$$\operatorname{dist}(f^j(z(s)), f^j(\zeta_\gamma)) \begin{cases} \le 10e^{-\beta j} & \text{for } 1 \le j \le p(z) \\ \ge \frac{1}{10}\, e^{-\beta j} & \text{for } j = p(z)+1, \end{cases} \tag{19}$$

compare (12) and (16). Then the same arguments as before apply, to prove that parts 2-4 of Proposition 2.8 remain true with $z(s)$ in the place of z, and $p(z)$ unchanged. This means that one might just as well take $p(z(s)) = p(z)$ for any such s. Accordingly, we always presume that, given any z as before there exists a segment $L(z)$ with $z \in L(z) \subset L$ such that

$$\operatorname{length}(L(z)) \ge c d_{\mathcal{C}}(z) \quad \text{and} \quad p(\cdot) \text{ is constant on } L(z). \tag{20}$$

We fix $c < 1/100$ in (20), and denote it by c_1 from now on. A similar formulation is used in [6].

We also quote the main result of [5]:

Theorem 2.9. *There exists a unique f-invariant measure μ supported in Λ, having nonzero Lyapunov exponents almost everywhere, and whose conditional measures along unstable manifolds are absolutely continuous with respect to Lebesgue measure on these manifolds. The support of μ coincides with Λ, and the system (f,μ) is ergodic (even Bernoulli).*

Given any segment $\gamma \subset W^u(P)$, almost every point in γ (with respect to the arc-length measure) is generic for μ. This can be read out from the proof as follows, see [5, Section 3]. Almost every point z in γ has infinitely many *escape times* n_i : there exists a sequence γ_i of neighbourhoods of z in γ such that $f^{n_i}(\gamma_i)$ is a long $C^2(b)$ curve (length $= \delta/10$) in $\{|x| < \delta\}$, and the maps $f^{n_i}|\gamma_i$ have uniformly bounded distortion with respect to arc-length; the images $f^{n_i}(\gamma_i)$ may be taken crossing $x = \pm\delta/2$. A positive fraction, uniformly bounded away from zero, of the points in each $f^{n_i}(\gamma_i)$ are generic for μ. So, almost every point $z \in \gamma$ is a density point for the set of generic points, and this implies the claim.

3 Symbolic dynamics in the basin of attraction

Here we construct a special sequence of partitions $\mathcal{P}_n$ in the basin of attraction, whose atoms are all *rectangles*, that is, regions bounded by two segments of $W^u(P)$ and by two stable leaves. A first step is Proposition 3.3: for each critical value $f(\zeta) \in f(\mathcal{C})$ there exists a sequence $\Gamma_r = \Gamma_r(\zeta)$ of long stable leaves accumulating $W^s(f(\zeta))$ exponentially fast. Then we introduce a notion of *itinerary* of a point z in the basin of attraction. It involves choosing a sequence of critical points $\tilde{\zeta}_j$ close to each iterate $f^{n_j}(z)$ that is near $x = 0$, and describing the position of $f^{n_j}(z)$ relative to $\tilde{\zeta}_j$ in terms of these long stable leaves. The atoms of $\mathcal{P}_n$ are the sets of points sharing the same itinerary up to time n.

3.1 Constructing long stable leaves

In all that follows $0 < c_1 < 1/100$ is the constant we fixed before in the context of (20).

Lemma 3.1. *Let $\gamma \subset W^u(P)$ be a free segment intersecting $I(\delta)$ such that* $\operatorname{length}(\gamma) \geq 2c_1 d_{\mathcal{C}}(z)$ *for $z \in \gamma$. Then there exists $z_\gamma \in \gamma$ such that $d_{\mathcal{C}}(f^n(z_\gamma)) \geq e^{-2\beta n}$ for every return $n \geq 1$ of z_γ.*

Proof. Let $L \supset \gamma$ be some nearly horizontal curve as in Proposition 2.7, ξ_0 be the midpoint of γ, and $\gamma_0 = \gamma \cap L(\xi_0)$, where $L(\xi_0) \subset L$ is a segment as in (20). Then

$$\operatorname{length}(\gamma_0) \geq c_1 d_{\mathcal{C}}(\xi_0), \tag{21}$$

and there is $p_0 \geq 1$ such that $p(z) = p_0$ for every $z \in \gamma_0$. Let $n_0 > p_0$ be minimum such that $f^{n_0}(\gamma_0)$ intersects $I(\delta)$. Note that $f^{n_0}(\gamma_0)$ is a free segment. Then, by (21) and Propositions 2.5 and 2.8.2–3

$$\begin{aligned} \operatorname{length}(f^{n_0}(\gamma_0)) &\geq \sigma_2^{n_0-p_0-1} \operatorname{length}(f^{p_0+1}(\gamma_0)) \;\geq\; \operatorname{length}(f^{p_0+1}(\gamma_0)) \\ &\geq \inf_{z\in\gamma_0} \|Df^{p_0+1}(z)\, t(z)\|\, c_1 d_{\mathcal{C}}(z) \;\geq c\, c_1 e^{-\beta(p_0+1)} \;\geq\; 20 e^{-2\beta n_0}. \end{aligned} \tag{22}$$

In the last inequality we use the remark that $n_0 > p_0$ can be supposed arbitrarily large by decreasing $\delta > 0$ (recall from the first paragraph of Section 2.1 that we fix β first, then we let $\delta \ll 1$). As a consequence, there exists a segment $\tilde{\gamma}_1 \subset f^{n_0}(\gamma_0)$ with

$$\operatorname{length}(\tilde{\gamma}_1) \geq \frac{1}{4} \operatorname{length}(f^{n_0}(\gamma_0)) \geq 5e^{-2\beta n_0} \quad \text{and} \quad d_{\mathcal{C}}(z) \geq e^{-2\beta n_0} \text{ for } z \in \tilde{\gamma}_1 \,.$$

Let $L_1 \supset \tilde{\gamma}_1$ be some nearly horizontal curve as in Proposition 2.7 and $\gamma_1 = \tilde{\gamma}_1 \cap L(\xi_1)$, where ξ_1 is the midpoint of $\tilde{\gamma}_1$ and $L(\xi_1)$ is as in (20). We consider two different cases.

If $L(\xi_1)$ is contained in $\tilde{\gamma}_1$ then $\gamma_1 = L(\xi_1)$. In particular, $\operatorname{length}(\gamma_1) \geq 2c_1 d_{\mathcal{C}}(\xi_1)$, and this ensures that (21) holds for γ_1. In this case we just repeat the previous construction with γ_1, ξ_1, in the place of γ_0, ξ_0. Letting $p_1 = p|\gamma_1$ and $n_1 > p_1$ be minimum such that $f^{n_1}(\gamma_1)$ intersects $I(\delta)$, we find a segment $\tilde{\gamma}_2 \subset f^{n_1}(\gamma_1)$ with

$$\operatorname{length}(\tilde{\gamma}_2) \geq 5e^{-2\beta n_1} \quad \text{and} \quad d_{\mathcal{C}}(z) \geq e^{-2\beta n_1} \geq e^{-2\beta(n_0+n_1)} \text{ for } z \in \tilde{\gamma}_2 \,.$$

Now suppose $L(\xi_1)$ is not contained in $\tilde{\gamma}_1$. Then $L(\xi_1)$ connects the mid-point ξ_1 to some of endpoint of $\tilde{\gamma}_1$, and so the same is true for the intersection γ_1. Consequently,

$$\operatorname{length}(\gamma_1) \geq \frac{1}{2} \operatorname{length}(\tilde{\gamma}_1) \geq e^{-2\beta n_0}$$

Now, Propositions 2.5 and 2.8.2 give

$$\begin{aligned} \operatorname{length}(f^{n_1}(\gamma_1)) &\geq \sigma_2^{n_1-p_1-1} \operatorname{length}(f^{p_1+1}(\gamma_1)) \\ &\geq \sigma_1^{n_1/3} \operatorname{length}(\gamma_1) \geq 20 \operatorname{length}(\gamma_1) \geq 20e^{-2\beta n_0}. \end{aligned} \tag{23}$$

Thus, there exists a segment $\tilde{\gamma}_2 \subset f^{n_1}(\gamma_1)$ such that

$$\operatorname{length}(\tilde{\gamma}_2) \geq \frac{1}{4} \operatorname{length}(f^{n_1}(\gamma_1)) \geq 5 \operatorname{length}(\gamma_1) \geq 5e^{-2\beta n_0}$$

and $d_{\mathcal{C}}(z) \geq e^{-2\beta n_0} \geq e^{-2\beta(n_0+n_1)}$ for every $z \in \tilde{\gamma}_2$.

Next, we take ξ_2 to be the midpoint of $\tilde{\gamma}_2$, and write $\gamma_2 = \tilde{\gamma}_2 \cap L(\xi_2)$. Then we apply the preceding steps with γ_2, ξ_2, in the place of γ_1, ξ_1: as before, we distinguish two cases according to whether $L(\xi_2)$ is contained in $\tilde{\gamma}_2$ or not. Iterating this procedure we construct a sequence n_i, $i \geq 0$, of large integers, and a sequence γ_i, $i \geq 0$, of segments in the unstable manifold $W^u(P)$, such that

$$f^{n_{i-1}}(\gamma_{i-1}) \supset \gamma_i \quad \text{and} \quad d_{\mathcal{C}}(z) \geq e^{-2\beta(n_0+\cdots+n_{i-1})} \tag{24}$$

for $z \in \gamma_i$ and every $i \geq 1$. The first property ensures that $f^{-(n_0+\cdots+n_{i-1})}(\gamma_i)$, $i \geq 1$, is a decreasing sequence of compact subsets of γ. Take z_γ a point in the intersection of all these subsets. The conclusion of the lemma for the returns of such a z_γ occurring at times $n_0 + \cdots + n_{i-1}$, $i \geq 1$, follows directly from the second part of (24). Any other return n is necessarily bound, i.e. $n = n_0 + \cdots + n_{i-1} + j$ for some $j \leq p_i$ and $i \geq 1$, and in this case the conclusion of the lemma is immediate:

$$d_{\mathcal{C}}(f^n(z_\gamma)) \geq d_{\mathcal{C}}(f^j(\tilde{\zeta}_i)) - e^{-\beta j} \geq e^{-\alpha j} - e^{-\beta j} \geq e^{-2\beta n},$$

where $\tilde{\zeta}_i$ represents the binding point of $f^{n_0+\cdots+n_{i-1}}(z_\gamma)$. □

Remark 3.1. We shall use a slightly stronger version of this lemma, where

$$\text{length}(\gamma) \geq 2c_1 \frac{d_{\mathcal{C}}(z)}{|\log d_{\mathcal{C}}(z)|^2} \quad \text{for all } z \in \gamma,$$

and the conclusion is as before. It follows from just the same proof, together with the following observation. Though we get $\text{length}(\gamma_0) \geq c_1 d_{\mathcal{C}}(\xi_0)|\log d_{\mathcal{C}}(\xi_0)|^{-2}$ instead of (21), Proposition 2.8.1 ensures that (22) is not affected:

$$\text{length}(f^{n_0}(\gamma_0)) \geq c\, c_1 \frac{e^{-\beta(p_0+1)}}{|\log d_{\mathcal{C}}(\xi_0)|^2} \geq c\, c_1 \frac{e^{-\beta(p_0+1)}}{(5p_0)^2} \geq 20e^{-2\beta n_0},$$

if $p_0 \leq n_0$ is taken large enough. As observed in the context of (22), this can be done without affecting β, because we choose $\beta > 0$ before $0 < \delta \ll 1$.

In the next lemma we do not assume the point z to be in the unstable manifold $W^u(P)$. Also, w_0 may be replaced by any norm 1 vector.

Lemma 3.2. *Let $z \in I(\delta)$ and $k \geq 1$ be such that for every $1 \leq n \leq k$ with $f^n(z) \in I(\delta)$ there exists $\zeta_n \in \mathcal{C}$ satisfying*

1. *$(f^n(z), Df^{n-1}(f(z))w_0)$ is in tangential position relative to $(\zeta_n, t(\zeta_n))$;*
2. *$\text{dist}(f^n(z), \zeta_n) \geq e^{-2\beta n}$.*

Then $f(z)$ is expanding up to time k, in fact, $\|Df^j(f(z))w_0\| \geq \sigma_1^{j/5}$ for every $1 \leq j \leq k$.

Proof. We define a pair of sequences n_i, p_i, $i \geq 1$, as follows. As a first step, we take n_1 to be the smallest integer $n \geq 1$ such that $f^n(z) \in I(\delta)$. Then, for each $i \geq 1$, we let $p_i \geq 1$ be maximum such that

$$\operatorname{dist}(f^{n_i+j}(z), f^j(\zeta_{n_i})) \leq e^{-\beta j} \quad \text{for all } 1 \leq j \leq p_i\,.$$

Finally, for each $i \geq 1$, we define n_{i+1} to be the smallest integer $n > n_i + p_i$ such that $f^n(z) \in I(\delta)$.

Since we suppose $f(z) \in f(I(\delta))$, Proposition 2.5 gives $\|Df^j(f(z))w_0\| \geq \sigma_2^j$ for every $1 \leq j \leq n_1 - 1$, which implies the conclusion of the lemma for $j < n_1$. Now we proceed by induction, in the following way. Let $i \geq 1$ and suppose we have shown that

$$\|Df^{n_i-1}(f(z))w_0\| \geq \sigma_1^{(n_i-1)/3}\,. \tag{25}$$

By assumption 1, cf. comments we made after Proposition 2.8,

$$\|Df^{n_i+p_i}(f(z))w_0\| \geq \sigma_1^{(p_i+1)/3}\|Df^{n_i-1}(f(z))w_0\| \geq \sigma_1^{(n_i+p_i)/3}.$$

As $\|Df\| \leq 4$, we conclude that, given any $0 \leq s < p_i$,

$$\|Df^{n_i+s}(f(z))w_0\| \geq 4^{s-p_i}\|Df^{n_i+p_i}(z)w_0\| \geq 4^{s-p_i}\sigma_1^{(n_i+p_i)/3}.$$

Using assumption 2 and Proposition 2.8.1, and taking $\beta > 0$ sufficiently small,

$$p_i - s \leq 5\log\frac{1}{d(f^{n_i}(z), \zeta_{n_i})} \leq 10\beta n_i \leq \frac{\log\sigma_1}{10\log 4}(n_i + s). \tag{26}$$

As a consequence, for every $0 \leq s \leq p_i$,

$$\|Df^{n_i+s}(f(z))w_0\| \geq \sigma_1^{-(n_i+s)/10}\sigma_1^{(n_i+p_i)/3} \geq \sigma_1^{(n_i+s)/5}.$$

This proves the lemma for $n_i \leq j \leq n_i + p_i$.

Next, Proposition 2.8.2 gives $\operatorname{slope}(Df^{n_i+p_i}(f(z))w_0) \leq (C/\delta)\sqrt{b} < 1/10$, and so we may use Proposition 2.5 to conclude that

$$\|Df^{n_i+p_i+s}(f(z))w_0\| \geq c\delta\sigma_2^s\|Df^{n_i+p_i}(f(z))w_0\| \geq c\delta\sigma_2^s\sigma_1^{(n_i+p_i)/3}$$

for $1 \leq s < n_{i+1} - n_i - p_i$. Now, assumption 2 implies $2\delta \geq e^{-2\beta n_i}$, and so

$$c\delta\sigma_2^s\sigma_1^{(n_i+p_i)/3} \geq \frac{c}{2}\sigma_2^s\sigma_1^{(n_i+p_i)/3}e^{-2\beta n_i} \geq \frac{c}{2}\sigma_1^{(n_i+p_i+s)/4} \geq \sigma_1^{(n_i+p_i+s)/5}.$$

In the second inequality we suppose $\beta > 0$ is small with respect to $\log\sigma_1$, in the third one we use the fact that $n_i + p_i + s$ is very large (since δ is small). We have obtained the conclusion of the lemma also for $n_i + p_i < j < n_{i+1}$. Finally, the last part of Proposition 2.5 gives

$$\begin{aligned}\|Df^{n_{i+1}-1}(f(z))w_0\| &\geq \sigma_2^{n_{i+1}-n_i-p_i-1}\|Df^{n_i+p_i}(f(z))w_0\| \\ &\geq \sigma_2^{n_{i+1}-n_i-p_i-1}\sigma_1^{(n_i+p_i)/3} \geq \sigma_1^{(n_{i+1}-1)/3}\end{aligned}$$

which restores the induction hypothesis (25). □

Remark 3.2. Keeping assumption 1 of the lemma and replacing assumption 2 by

2. $\operatorname{dist}(f^n(z), \zeta_n) \geq e^{-5n}$,

one still gets that $f(z)$ is expanding, in a weaker sense: $\|Df^j(f(z))w_0\| \geq \lambda^j$ for $1 \leq j \leq k$, with $\lambda \geq 10^{-20}$. This is proved in the same way as the lemma, just replacing (26) by

$$p_i - s \leq 5 \log \frac{1}{d(f^{n_i}(z), \zeta_{n_i})} \leq 25 n_i \leq \frac{20 \log 10}{\log 4}(n_i + s).$$

Note also that, in any case, we only need assumptions 1 and 2 at the free return times n_i.

It is convenient to take $\Delta = \log(1/\delta)$ to be a (large) integer, and we do so in what follows.

Proposition 3.3. *Given any critical value $f(\zeta) \in f(\mathcal{C})$ there exists a sequence of long stable leaves $\Gamma_r = \Gamma_r(\zeta) = \{(x_r(y), y) : |y| \leq 1/10\}$, for $r \geq \Delta$, accumulating $W^s(f(\zeta))$ exponentially fast from the left:*

$$e^{-2r} \leq x^s(y) - x_r(y) \leq 3e^{-2r} \quad \text{for every } r \geq \Delta \text{ and } |y| \leq 1/10, \tag{27}$$

where $\{(x^s(y), y) : |y| \leq 1/10\} = \Gamma^s$ is the long stable leaf through $f(\zeta)$.

Proof. First, we consider $\zeta = \zeta_0$, the critical point of generation zero in Proposition 2.2.1. Let $\gamma = \gamma(\zeta_0)$ be a segment of $W^u(P)$ extending δ to each side of ζ_0. By Proposition 2.2.3, this is a $C^2(b)$ curve. For each $r \geq \Delta$, the set $\{z \in \gamma : (9/10)\, e^{-r} \leq d(z, \zeta_0) \leq e^{-r}\}$ has two connected components. We shall use γ_r to denote either of the two. Then

$$\operatorname{length}(\gamma_r) \geq \frac{1}{10} e^{-r} \geq 4 c_1 e^{-r} \geq 2 c_1 d_{\mathcal{C}}(z) \quad \text{for every } z \in \gamma_r\,,$$

recall that $c_1 < 1/100$. So, by Lemma 3.1, there exists $z_r \in \gamma_r$ such that

$$d_{\mathcal{C}}(f^n(z_r)) = d(f^n(z_r), \zeta_n) \geq e^{-2\beta n} \tag{28}$$

for every free return $n \geq 1$ of z_r. Here $\zeta_n \in \mathcal{C}$ is the binding point of $f^n(z_r)$, recall Proposition 2.7, so that $(f^n(z_r), t(f^n(z_r)))$ is in tangential position relative to $(\zeta_n, t(\zeta_n))$. Let $\eta_r = f(z_r)$. We also need

Lemma 3.4. *The pair $(f^n(z_r), Df^{n-1}(f(z_r))w_0)$ is in tangential position relative to $(\zeta_n, t(\zeta_n))$, for every free return $n \geq 1$ of z_r.*

Proof. Take $p_0 \geq 1$ maximum such that $\operatorname{dist}(f^j(z_r), \zeta) \leq e^{-\beta j}$ for all $1 \leq j \leq p_0$, and let n_1 be the first free return of z_r, in the sense that n_1 is the smallest integer larger than p_0 so that $f^{n_1}(z_r) \in I(\delta)$. Propositions 2.5 and 2.8.4 imply

$$\|Df^{n_1-1}(\eta_r)w_0\| \geq \sigma_2^{n_1-p_0-1} \|Df^{p_0}(\eta_r)w_0\| \geq \sigma_2^{n_1-p_0-1}\, \sigma_1^{p_0} \geq 2. \tag{29}$$

On the other hand, Propositions 2.5 and 2.8.2 give

$$\|Df^{n_1}(z_r)\, t(z_r)\| \geq \sigma_2^{n_1-p_0} |Df^{p_0}(z_r)\, t(z_r)\| \geq \sigma_2^{n_1-p_0} \sigma_1^{(p_0+1)/3} \geq \sigma_1^{n_1/3}.$$

So,

$$\|Df^{n_1-1}(\eta_r)\, t(\eta_r)\| \geq \frac{1}{5}\|Df^{n_1}(z_r)\, t(z_r)\| \geq \frac{1}{5}\sigma_1^{n_1/3} \geq 2. \tag{30}$$

Since $|\det Df^{n_1-1}(\eta_r)| \leq (Cb)^{n_1-1} \ll 1$, we deduce from (29) and (30) that

$$\text{angle}\left(Df^{n_1-1}(\eta_r)w_0, Df^{n_1-1}(\eta_r)\, t(\eta_r)\right) \leq (Cb)^{n_1-1} \ll \text{dist}(f^{n_1-1}(\eta_r), \zeta_{n_r}). \tag{31}$$

By Proposition 2.7, there is a C^2 curve $L = \{(x, y(x))\}$ with $|y'|, |y''| \leq 1/10$, tangent to $t(\zeta_n)$ at ζ_n and tangent to $Df^{n_1-1}(\eta_r)\, t(\eta_r)$ at $f^{n_1-1}(\eta_r)$. In view of (31), we may easily modify L to a C^2 curve $\tilde{L} = \{(x, \tilde{y}(x)\}$ with $|\tilde{y}'|, |\tilde{y}''| \leq 1/5$ tangent to $t(\zeta_n)$ at ζ_n and to $Df^{n_1-1}(\eta_r)\, w_0$ at $f^{n_1-1}(\eta_r)$. Existence of such an $\tilde{L}$ is precisely the content of the lemma for time n_1.

A similar argument proves the claim for the subsequent free returns of z_r. For each $i \geq 1$, let $p_i \geq 1$ be maximum such that

$$\text{dist}(f^{n_i+j}(z_r), f^j(\zeta_{n_i})) \leq e^{-\beta j} \quad \text{for all } 1 \leq j \leq p_i\,,$$

and then let n_{i+1} be the smallest integer $n > n_i + p_i$ so that $f^n(z_r) \in I(\delta)$. We may assume, by induction, that $(f^{n_j-1}(\eta_r), Df^{n_j-1}(\eta_r)w_0)$ is in tangential position relative to $(\zeta_{n_j}, t(\zeta_{n_j}))$, for every free return $1 \leq j \leq i$. Then Lemma 3.2 implies, cf. the last observation in Remark 3.2,

$$\|Df^{n_{i+1}-1}(\eta_r)w_0\| \geq \sigma_1^{(n_{i+1}-1)/5} \geq 2.$$

Taking $t(\eta_r)$ in the place of w_0, we also get (the condition of tangential position, corresponding to hypothesis 1 in Lemma 3.2, results from Proposition 2.7)

$$\|Df^{n_{i+1}-1}(\eta_r)t(\eta_r)\| \geq \sigma_1^{(n_{i+1}-1)/5} \geq 2.$$

Therefore, the angle between the vectors $Df^{n_{i+1}-1}(\eta_r)w_0$ and $Df^{n_{i+1}-1}(\eta_r)t(\eta_r)$ is less than $(Cb)^{n_{i+1}-1}$, so it is much smaller than $\text{dist}(f^{n_{i+1}-1}(\eta_r), \zeta_{n_{i+1}})$. It follows, as in the previous case, that $(f^{n_{i+1}-1}(\eta_r), Df^{n_{i+1}-1}(\eta_r)w_0)$ is in tangential position to $(\zeta_{n_{i+1}}, t(\zeta_{n_{i+1}}))$, which proves our assertion for n_{i+1}. This finishes the proof of Lemma 3.4. □

Now let us go back to proving Proposition 3.3. The previous lemma and (28) mean that z_r satisfies both assumptions of Lemma 3.2 at all free return times, and so $\eta_r = f(z_r)$ is expanding. We take $\Gamma_r = \{(x_r(y), y)\}$ to be the long stable leaf through η_r granted by Proposition 2.4. As we explain next, property (27) follows from a quadratic estimate like (7). Let us write $\gamma = \{(x, y(x))\}$ and $f(\gamma) = \{(\xi(x), \eta(x))\}$. Moreover, $\xi^s(x) = x^s(\eta(x)) - \xi(x)$ is the horizontal distance from each point of $f(\gamma)$ to the long stable leaf $\Gamma^s = \{(x^s(y), y)\}$ through $f(\zeta_0)$. We write the critical point $\zeta_0 = (x_0, y(x_0))$. By (7),

$$\frac{3}{2}(x - x_0)^2 \leq \xi^s(x) \leq \frac{5}{2}(x - x_0)^2 \quad \text{for every } x.$$

By construction, the point $z_r = (x_r, y(x_r))$ has $(9/10)e^{-r} \leq |x_r - x_0| \leq e^{-r}$. Replacing this in the previous equation, we find that the horizontal distance $\xi^s(x_r)$ from $\eta_r = f(z_r)$ to Γ^s satisfies

$$\frac{6}{5}\, e^{-2r} \leq \xi^s(x_r) < \frac{5}{2}\, e^{-2r}.$$

By the Lipschitz estimate in the last part of Proposition 2.4, the horizontal distance from any other point $(x_r(y), y)$ of Γ^r differs from $\xi^s(x_r)$, at most, by a factor that is close 1 if b is small. This ensures that the previous estimate remains valid for any point of Γ^r, with slightly worse constants:

$$\frac{11}{10} e^{-2r} \leq x^s(y) - x_r(y) < \frac{11}{4} e^{-2r} \quad \text{for every } |y| \leq 1/10. \tag{32}$$

This implies (27).

Finally, we prove the proposition for a general critical point $\zeta \in \mathcal{C}$, of generation $g \geq 1$. By Proposition 2.2.3, the segment $\gamma = \gamma(\zeta)$ of radius $\delta\rho^g$ around ζ in $W^u(P)$ is a $C^2(b)$ curve. This means that precisely the same construction of $\Gamma_r = \Gamma_r(\zeta)$ as in the previous case applies here, for r large enough so that $e^{-r} \leq \delta\rho^g$. In particular, we get (32) for all such r. On the other hand, for $r < \Delta + g\log(1/\rho)$ we define $\Gamma_r(\zeta) = \Gamma_r(\tilde{\zeta})$, where $\tilde{\zeta}$ is any critical point of generation $\tilde{g} < g$ with $\operatorname{dist}(\tilde{\zeta}, \zeta) \leq b^{g/10}$, as given by Proposition 2.2.4. Proposition 2.4 implies that the horizontal distance between the long stable leaves through the points $f(\zeta)$ and $f(\tilde{\zeta})$ is bounded by

$$2b^{g/10} \leq 100^{-g}(\delta\rho^g)^2 \leq 100^{-g} e^{-2r}$$

(take $b \ll \delta \ll 1$). In view of (32) and the hierarchical form of our construction, the horizontal distance from these Γ_r to $\Gamma^s(f(\zeta))$ satisfies bounds similar to (32), with the factors $11/10$ and $11/4$ replaced by 1 and 3 (because $\sum_g 100^{-g} < 1/10$). This completes our construction. □

Remark 3.3. Γ_Δ and $\Gamma_{\Delta+1}$ as constructed above are the same for all critical values (as long as we suppose $\rho < 1/e$, which we clearly can). Every $\Gamma_r(\zeta)$, $\zeta \in \mathcal{C}$ and $r \geq \Delta$ intersects the unstable manifold $W^u(P)$, at the point η_r. Let us also record that, by (28) and Lemma 3.4, $d_{\mathcal{C}}(f^{n-1}(\eta_r)) \geq e^{-2\beta n}$ and $(f^{n-1}(\eta_r), Df^{n-1}(\eta_r)w_0)$ is in tangential position relative to $(\zeta_n, t(\zeta_n))$, for every free return $n \geq 1$.

Most of our construction can be carried out using the family of long stable leaves $\Gamma_r(\zeta)$, $\zeta \in \mathcal{C}$ and $r > \Delta$, given by Proposition 3.3. However, for Lemma 4.7 we have to define itinerary of an orbit in the basin of Λ in terms of an extended family of long stable leaves $\Gamma_{r,l} = \Gamma_{r,l}(\zeta)$, $0 \leq l \leq r^2$, with ($\approx$ means equality up to a factor 100)

(a) horiz $\operatorname{dist}(\Gamma_{r,l}, W^s(f(\zeta))) \approx e^{-2r}$ for every $0 \leq l \leq r^2$;

(b) horiz $\operatorname{dist}(\Gamma_{r,l-1}, \Gamma_{r,l}) \approx e^{-2r}/r^2$ for every $1 \leq l \leq r^2$.

(c) $\Gamma_{r,0} = \Gamma_{r-1}$, each $\Gamma_{r,l}$ is to the right of $\Gamma_{r,l-1}$, and $\Gamma_{r,r^2} = \Gamma_r$.

This last property implies that the horizontal distance from $\Gamma_{r,l}$ decreases when r increases and, for fixed r, when l increases. Such a family can be obtained by the following variation of the previous construction. For each $r \geq \Delta + 1$, decompose the segment of $W^u(P)$ bounded by z_{r-1} and z_r into $2r^2 - 1$ segments of equal length. Denote these segments $\gamma_{r,j}$, for $1 \leq j \leq 2r^2 - 1$, in such a way that the distance to z_r decreases monotonically with j. Observe that

$$\operatorname{length}(\gamma_{r,j}) \geq \frac{1}{2r^2 - 1} \operatorname{dist}(z_r, z_{r+1}) \geq \frac{1}{2r^2} \frac{1}{2} e^{-r},$$

for every j. Given any $z \in \gamma_{r,j}$, we have $d_{\mathcal{C}}(z) \le e^{-r}$, and so

$$2c_1 \frac{d_{\mathcal{C}}(z)}{|\log d_{\mathcal{C}}(z)|^2} \le 2c_1 \frac{e^{-r}}{r^2} \le \frac{e^{-r}}{4r^2} \le \text{length}(\gamma_{r,j})$$

(we took $c_1 < 1/100$). This means that every segment $\gamma_{r,j}$ satisfies the condition in Remark 3.1, and so it contains a point $z_{r,j}$ such that $f(z_{r,j})$ is expanding. We let $\Gamma_{r,l} = \Gamma_{r,l}(\zeta)$ be the long stable leaf through the point $f(z_{r,2l})$, for each $1 \le l \le r^2 - 1$. The estimates in (a) and (b) follow from the same arguments as we used to prove (27). Remark 3.3 remains valid for this extended family $\Gamma_{r,l}$.

3.2 Itineraries for orbits in the basin

To each point $z \in B(\Lambda)$ we want to associate sequences n_j, $i_j = (\tilde{\zeta}_j, r_j, l_j, \epsilon_j)$, $j \ge 0$, where n_j is an integer, $\tilde{\zeta}_j \in \mathcal{C}$, r_j and l_j are also integers. with

$$(r_j, l_j) = (0,0) \quad \text{or else} \quad r_j \ge \Delta \text{ and } 1 \le l_j \le r_j^2,$$

and $\epsilon_j \in \{+, 0, -\}$. Roughly speaking, n_j is the jth free return of z, $\tilde{\zeta}_j$ is the corresponding binding point, and r_j, l_j, ϵ_j describe the position of $f^{n_j+1}(z)$ relative to the long stable leaves $\Gamma_{r,l}(\tilde{\zeta}_j)$. The precise construction of these sequences occupies the whole of this section.

Recall that G_0, G_1 contain long $C^2(b)$ segments γ_0, γ_1, around the critical points ζ_0, ζ_1, respectively. In view of the form of our map, for each $i = 0, 1$ we may write $f(\gamma_i)$ as $\{\xi_i(x), \eta_i(x)\}$ with $\xi_i'' \approx -2a \approx 4$ and $|\eta_i|, |\eta_i'|, |\eta_i''| \le C\sqrt{b}$. In particular, $f(\gamma_i)$ intersects each $\Gamma_{r,l}(\zeta_i)$, for $0 \le l \le r^2$, in exactly two points. Let Δ_i be the region bounded by $f(\gamma_i)$ and by the long stable leaf $W^s_{loc}(P)$ passing through P, see Figure 1. Since $f(\gamma_0)$ and $f(\gamma_1)$ are disjoint, whereas Δ_0 and Δ_1 must intersect each other (e.g. extend $\{\gamma_0, \gamma_1\}$ to a foliation by nearly horizontal curves, and use that the image of each leaf intersects every vertical line in not more than two points), we have either $\Delta_1 \subset \Delta_0$ or $\Delta_0 \subset \Delta_1$.

We consider $\Delta_1 \subset \Delta_0$, as the other case is analogous. In the sequel we define $n_j(z), i_j(z)$, $j \ge 0$, for points $z \in \Delta_0$. The extension to generic points $w \in B(\Lambda)$ is, simply, by taking $n_j(w) = n + n_j(f^n(w))$ and $i_j(w) = i_j(f^n(w))$ for each $j \ge 0$, where $n \ge 0$ is the smallest integer for which $f^n(w) \in \Delta_0$. Since Lebesgue almost every point in the basin of Λ has some iterate contained in Δ_0, cf. Section 5, this leaves out only a zero Lebesgue measure subset of $B(\Lambda)$, which is negligible for our purposes.

Before proceeding, let us make a few simple conventions. In what follows (r, l) should be replaced by $(r-1, (r-1)^2 + l)$ if $l \le 0$, and by $(r+1, l-r^2)$ if $l > r^2$. We say that $(r_1, l_1) > (r_2, l_2)$ if either $r_1 > r_2$ or $r_1 = r_2$ and $l_1 > l_2$. The *region in between* two long stable leaves is open on the left and closed on the right: if $\Gamma_1 = \{(x_1(y), y) : |y| \le 1/10\}$ and $\Gamma_2 = \{(x_2(y), y) : |y| \le 1/10\}$, with $x_1 < x_2$, then the region in between Γ_1 and Γ_2 is $\{(x, y) : x_1(y) < x \le x_2(y),\ |y| \le 1/10\}$.

Let $(\hat{r}, \hat{l})$ be defined by the condition that $f(\zeta_1)$ is in the region of Δ_0 in between $\Gamma_{\hat{r},\hat{l}}(\zeta_0)$ and $\Gamma_{\hat{r},\hat{l}-1}(\zeta_0)$. For $z \in \Delta_0$ we define $n_0 = -1$ and

(a) $i_0(z) = (\zeta_0, r, l, 0)$ if z is in the region of Δ_0 in between $\Gamma_{r,l}(\zeta_0)$ and $\Gamma_{r,l-1}(\zeta_0)$, with $(r, l) > (\hat{r}, \hat{l})$;

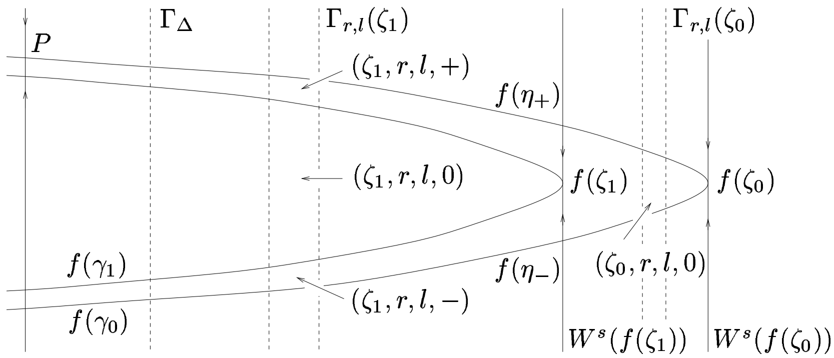

Figure 1:

(b) $i_0(z) = (\zeta_0, \hat{r}, \hat{l}, 0)$ if z is in the region of Δ_0 in between $W^s_{loc}(f(\zeta_1))$ and $\Gamma_{\hat{r},\hat{l}}(\zeta_0)$;

(c) $i_0(z) = (\zeta_1, r, l, \pm)$ if z is in either of the two regions of $\Delta_0 \setminus \Delta_1$ in between $\Gamma_{r,l}(\zeta_1)$ and $\Gamma_{r,l-1}(\zeta_1)$, the sign $+/-$ corresponding to the upper/lower region;

(d) $i_0(z) = (\zeta_1, 0, 0, \pm)$ if z is in either of the two regions of $\Delta_0 \setminus \Delta_1$ in between Γ_Δ and $W^s_{loc}(P)$, the sign $+/-$ corresponding to the upper/lower region;

(e) $i_0(z) = (\zeta_1, r, l, 0)$ if z is in the region of Δ_1 in between $\Gamma_{r,l}(\zeta_1)$ and $\Gamma_{r,l-1}(\zeta_1)$.

(f) $i_0(z) = (\zeta_1, 0, 0, 0)$ if z is in the region of Δ_1 in between Γ_Δ and $W^s_{loc}(P)$.

We also define $R(i_0) = \{z \in \Delta_0 : i_0(z) = i_0\}$ for each $i_0 = (\tilde{\zeta}_0, r_0, l_0, \epsilon_0)$ as before. This closes the first step of our definition.

The definition of these objects proceeds by recurrence. In the next paragraphs we explain how $n_1(z)$ and $i_1(z)$ are defined for z in $R(i_0)$, for each fixed i_0.

In cases (a), (b), (c), (e), define $p_1 = p_1(i_0) \geq 1$ to be the largest integer such that

$$\operatorname{dist}(f^j(z), f^j(\tilde{\zeta}_0)) \leq e^{-\beta j} \quad \text{for } 1 \leq j \leq p_1 \text{ and every } z \in f^{-1}(R(i_0)).$$

For (d), (f) just set $p_1 = 0$. In any case, let $m_1 = n_1 > p_1$ be minimum such that $f^{n_1}(R(i_0))$ intersects $I(\delta)$. Denote γ_i^u, $i = 0, 1$, and γ_j^s, $j = 0, 1$, the four segments forming the boundary of the rectangle $R(i_0)$, with the γ_i^u contained in $W^u(P)$ and the γ_j^s contained in long stable leaves. Moreover, let $z^*_{i,j} = \gamma_i^u \cap \gamma_j^s$ be the corner points of $R(i_0)$, for $i = 0, 1$ and $j = 0, 1$.

Proposition 3.5. *1.* $m_1 > p_1 \geq (4/3) r_0$*;*

2. for $i = 0, 1$*, the slope of* $f^{n_1}(\gamma_i^u)$ *is less than* $(C/\delta)\sqrt{b}$ *at every point;*

3. $\operatorname{length}(f^{n_1}(\gamma_j^s)) \leq (1/10)\, d_{\mathcal{C}}(z^*_{i,j})$ *for* $i = 0, 1$ *and* $j = 0, 1$*;*

4. $\operatorname{angle}(t(z^*_{0,j}), t(z^*_{1,j})) \leq (1/10)\, d_{\mathcal{C}}(z^*_{i,j})$ *for* $i = 0, 1$ *and* $j = 0, 1$.

This proposition will be proved in Section 3.3. As part of the proof, in Lemma 3.9, we show that p_1 is a suitable bound period for every point in the rectangle: we have (19) for $p(z) \equiv p_1$, and conclusions 2-4 of Proposition 2.8 are true at time p_1 for any point in either of the unstable boundary segments. This means that we may take the bound period constant equal to p_1 on the whole $f^{-1}(R(i_0))$. In particular, both segments $f^{n_1}(\gamma_i^u)$, $i = 0, 1$, are free. According to Proposition 2.7, each of these segments may be extended to a C^2 curve $K_i = \{(x, y_i(x)\}$ with $|y_i'|, |y_i''| \le 1/10$ and tangent to $W^u(P)$ at some critical point $\eta_i \in K_i$. By definition, $d_{\mathcal{C}}(z_{i,j}^*) = \operatorname{dist}(z_{i,j}^*, \eta_i)$ for every $j = 1, 0$. Recall that η_i may not belong to $f^{n_1}(\gamma_i^u)$. We can also not discard the possibility that $\eta_0 = \eta_1$. On the other hand, according to the next lemma, either both η_i belong to the corresponding $f^{n_1}(\gamma_i^u)$ or none does, and in the latter case we may always take the two critical points to coincide.

Lemma 3.6. *If $\eta_0 \in f^{n_1}(\gamma_0^u)$ then $\eta_1 \in f^{n_1}(\gamma_1^u)$. In the opposite case, $f^{n_1}(\gamma_1^u)$ is in tangential position relative to $(\eta_0, t(\eta_0))$: there is a C^2 curve $K_2 = \{(x, y_2(x)\}$ with $|y_2'|, |y_2''| \le 1/5$, containing $f^{n_1}(\gamma_1^u)$ and tangent to $W^u(P)$ at η_0.*

Proof. Suppose that $\eta_0 \in f^{n_1}(\gamma_0^u)$ but $\eta_1 \in K_1 \setminus f^{n_1}(\gamma_1^u)$. Fix $j = 0, 1$ so that $z_{1,j}^*$ is the boundary point of $f^{n_1}(\gamma_1^u)$ closest to η_1. In view of our definitions, $z_{0,j}^*$ is the boundary point of $f^{n_1}(\gamma_0^u)$ in the same stable leaf $f^{n_1}(\gamma_j^s)$ as $z_{1,j}^*$. For each $i = 0, 1$, write $\eta_i = (x_i, y_i(x_i))$ and $z_{i,j}^* = (x_{i,j}, y_i(x_{i,j}))$, and let $[\eta_i, z_{i,j}^*]$ be the segment of K_i connecting η_i to $z_{i,j}^*$. Let $m \ge 1$ be fixed such that

$$\tau^{m+1} < 5 \max\{d_{\mathcal{C}}(z_{i,j}^*) : i = 0, 1\} \le \tau^m,$$

where $\tau > 0$ is taken as in Proposition 2.3. According to the proposition, the contracting direction $e_i(s) = e^{(m)}(X_i(s), Y_i(s))$ of order m is well defined for any $(X_i(s), Y_i(s)) = f(s, y_i(s))$ of $f([\eta_i, z_{i,j}^*])$. This ensures that a quadratic estimate like (10) holds for each of these segments: splitting the tangent vector

$$(X_i'(s), Y_i'(s)) = \alpha_i(s) e_i(s) + \beta_i(s) w_0$$

as in (9), the coefficient β_i satisfies

$$3 \le \frac{\beta_i(x_i) - \beta_i(s)}{s - x_i} \le 5 \quad \text{and} \quad |\beta_i(x_i)| \le (Cb)^m \tag{33}$$

for every s between x_i and $x_{i,j}$. From the form of the map f and the fact that e_i and w_0 are nearly orthogonal

$$\begin{aligned} |\beta_1(x_{1,j}) - \beta_0(x_{0,j})| &\le 5 \operatorname{dist}(z_{0,j}^*, z_{1,j}^*) + C\sqrt{b} \operatorname{angle}(t(z_{0,j}^*), t(z_{1,j}^*)) \\ &\le \min\{d_{\mathcal{C}}(z_{0,j}^*), d_{\mathcal{C}}(z_{1,j}^*)\} \end{aligned} \tag{34}$$

The last inequality follows from Proposition 3.5.3 and 3.5.4. Now, suppose that $x_1 > x_{1,j}$ and $x_0 < x_{0,j}$, that is, η_1 is to the right of $z_{1,j}^*$ in K_1, and η_0 is to the left of $z_{0,j}^*$ in K_0 (the opposite case is analogous). Then (33) gives

$$\beta_1(x_{1,j}) - \beta_0(x_{0,j}) \ge 3(x_1 - x_{1,j}) + 3(x_{0,j} - x_0) - 2(Cb)^m \ge 2d_{\mathcal{C}}(z_{1,j}) + 2d_{\mathcal{C}}(z_{0,j}). \tag{35}$$

In the last inequality we use $d_{\mathcal{C}}(z_{1,j}) \le (6/5)(x_1 - x_{1,j})$ and $d_{\mathcal{C}}(z_{0,j}) \le (6/5)(x_{0,j} - x_0)$, as well as the fact that $(Cb)^m$ is much smaller than $\tau^m \approx \max\{d_{\mathcal{C}}(z_{i,j}^*) : i = 0, 1\}$. Clearly, (34) and (34) contradict each other. This proves that $\eta_1 \in f^{n_1}(\gamma_1^u)$.

To prove the second part of the lemma, let $z_{0,j}^*$ be the boundary point of $f^{n_1}(\gamma_0^u)$ closest to η_0 in K_0. Then $z_{1,j}^*$ is the boundary point of $f^{n_1}(\gamma_1^u)$ in the same stable leaf as $z_{0,j}^*$. By Proposition 3.5.3 and 3.5.4, both $\text{dist}(z_{0,j}^*, z_{1,j}^*)$ and $\text{angle}(t(z_{0,j}^*), t(z_{1,j}^*))$ are smaller than $(1/10)\,\text{dist}(z_{0,j}^*, \eta_0)$. So, we may easily modify $K_0 = \{(x, y_0(x))\}$ to get a curve K_2 as in the statement. □

We define $i_1(z)$ first when $\eta_i \in f^{n_1}(\gamma_i^u)$ for $i = 0, 1$. Up to interchanging subscripts, we may suppose that $f(\eta_0)$ is to the right of $f(\eta_1)$, meaning that its long stable leaf is to the right of the one passing through $f(\eta_1)$. Then $f(\eta_1)$ is contained in a region bounded by $f^{n_1+1}(\gamma_0^u)$ and some pair of long leaves $\Gamma_{\hat{r},\hat{l}-1}(\eta_0)$ and $\Gamma_{\hat{r},\hat{l}}(\eta_0)$. We let, see Figure 2,

(a1) $i_1(z) = (\eta_0, r, l, 0)$ if $f^{n_1+1}(z)$ is in the region of $f^{n_1+1}(R(i_0))$ in between $\Gamma_{r,l}(\eta_0)$ and $\Gamma_{r,l-1}(\eta_0)$, with $(r, l) > (\hat{r}, \hat{l})$;

(b1) $i_1(z) = (\eta_0, \hat{r}, \hat{l}, 0)$ if $f^{n_1+1}(z)$ is in the region of $f^{n_1+1}(R(i_0))$ in between $W_{loc}^s(f(\eta_1))$ and $\Gamma_{\hat{r},\hat{l}}(\zeta_0)$;

(c1) $i_1(z) = (\eta_1, r, l, \pm)$ if $f^{n_1+1}(z)$ is in either of the regions of $f^{n_1+1}(R(i_0))$ in between $\Gamma_{r,l}(\eta_1)$ and $\Gamma_{r,l-1}(\eta_1)$, the sign $+/-$ corresponding to the upper/lower region.

(d1) $i_1(z) = (\eta_1, 0, 0, \pm)$ if $f^{n_1+1}(z)$ is in either of the regions of $f^{n_1+1}(R(i_0))$ to the left of Γ_Δ, the sign $+/-$ corresponding to the upper/lower region.

It is worth keeping in mind that the stable leaves $f^{n_1+1}(\gamma_i^s)$ on the boundary of $f^{n_1+1}(R(i_0))$ can not intersect a long leaf unless they are totally contained in it.

The definition of $i_1(z)$ is slightly simpler in the case when $\eta_i \notin f^{n_1}(\gamma_i^u)$ for $i = 0, 1$. Taking advantage of the fact that both segments $f^{n_1}(\gamma_i^u)$, $i = 0, 1$, are in tangential position relative to η_0, cf. Lemma 3.6, we define

(a2) $i_1(z) = (\eta_0, r, l, +)$ if $f^{n_1+1}(z)$ is in the region of $f^{n_1+1}(R(i_0))$ in between $\Gamma_{r,l}(\eta_0)$ and $\Gamma_{r,l-1}(\eta_0)$;

(b2) $i_1(z) = (\eta_0, 0, 0, +)$ if $f^{n_1+1}(z)$ is in the region of $f^{n_1+1}(R(i_0))$ to the left of Γ_Δ.

See Figure 2. Our choice $\epsilon_j = +$ is purely conventional: the intersection of $f^{n_1+1}(R(i_0))$ with any region in between two stable leaves is connected, and so ϵ_j has no role in this case.

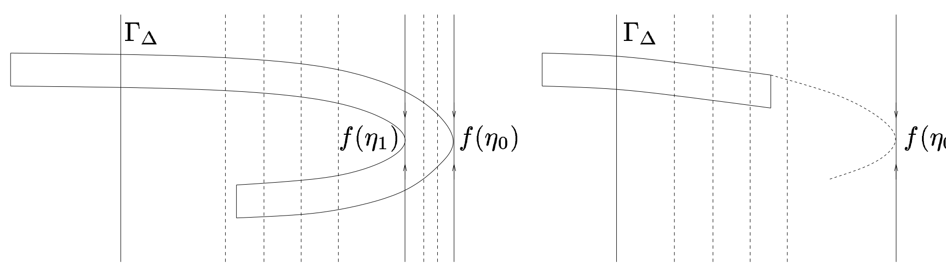

Figure 2:

This completes the definition of $i_1(z)$. We also set $R(i_0, i_1) = \{z \in R(i_0) : i_1(z) = i_1\}$, for each $i_0 = (\tilde{\zeta}_0, r_0, l_0, \epsilon_0)$ and $i_1 = (\tilde{\zeta}_1, r_1, l_1, \epsilon_1)$.

Finally, we define $i_k(z)$ for general $k \geq 1$. This is very similar to the case $k = 1$, and so we go more quickly now. Suppose $i_j(z)$, $n_j(z)$, and $R(i_0, \ldots, i_j)$ have been defined for every $j < k$. Let $i_j = (\tilde{\zeta}_j, r_j, l_j, \epsilon_j)$, $j = 0, \ldots, k-1$, be fixed, and $z \in R(i_0, \ldots, i_{k-1})$. In cases (a1), (b1), (c1), (a2), we define $p_k = p_k(i_0, \ldots, i_{k-1}) \geq 1$ to be the largest integer such that

$$\operatorname{dist}(f^j(\zeta), f^j(\tilde{\zeta}_{k-1})) \leq e^{-\beta j} \quad \text{for } 1 \leq j \leq p_k \text{ and every } \zeta \in f^{n_{k-1}}(R(i_0, \ldots, i_{k-1})).$$

For (d1), (b2) we just set $p_k = 0$. Then we let n_k be the smallest integer larger than $n_{k-1} + p_k$ such that $f^{n_k}(R(i_0, \ldots, i_{k-1}))$ intersects $I(\delta)$, and let $m_k = n_k - (n_{k-1} + 1)$. Call γ_i^u, γ_j^s the boundary segments, and $z_{i,j}^*$ the corner points of $f^{n_{k-1}+1}(R(i_0, \ldots, i_{k-1}))$, with the same conventions as before. Then,

Proposition 3.7. *1.* $m_k > p_k \geq (4/3) r_{k-1}$*;*

2. for $i = 0, 1$ *the slope of* $f^{m_k}(\gamma_i^u)$ *is less than* $(C/\delta)\sqrt{b}$ *at every point;*

3. length $f^{m_k}(\gamma_j^s) \leq (1/10)\, d_{\mathcal{C}}(z_{i,j}^*)$ *for* $i = 0, 1$ *and* $j = 0, 1$*;*

4. angle$(t(z_{0,j}^*), t(z_{1,j}^*)) \leq (1/10)\, d_{\mathcal{C}}(z_{i,j}^*)$ *for* $i = 0, 1$ *and* $j = 0, 1$.

This proposition will be proved in Section 3.3. This includes proving, in Lemma 3.9, that the bound period may be taken constant equal to p_k on the whole $f^{n_{k-1}}(R(i_0, \ldots, i_{k-1}))$, Then both $f^{m_k}(\gamma_i^u)$, $i = 0, 1$, are free segments. Thus we may use Proposition 2.7 to get the analog of Lemma 3.6 at every return:

Lemma 3.8. *Either there are two critical points* η_0, η_1 *such that* $\eta_i \in f^{m_k}(\gamma_i^u)$ *for* $i = 0$ *and* $i = 1$, *or there is a critical point* η_0 *such that both segments* $f^{m_k}(\gamma_i^u)$, $i = 0, 1$, *are in tangential position relative to* $(\eta_0, t(\eta_0))$.

In the first case we define $\hat{r}, \hat{l}$ just as before. Then we let $i_k(z)$ be given by the rules which are obtained replacing $f^{n_1+1}(z)$ by $f^{n_k+1}(z)$, and $f^{n_1+1}(R(i_0))$ by $f^{n_k+1}(R(i_0, \ldots, i_{k-1}))$ in (a1)–(d1). In the second case in the lemma we define $i_k(z)$ by the rules obtained by making the corresponding substitutions in (a2)–(b2). Finally, for each $i_0, \ldots, i_{k-1}, i_k$,

$$R(i_0, \ldots, i_{k-1}, i_k) = \{z \in R(i_0, \ldots, i_{k-1}) : i_k(z) = i_k\}.$$

Our definition of itinerary of a point z in the basin of Λ is complete. By construction, every $R(i_0, \ldots, i_k)$ is a rectangle. Note that the two segments of unstable manifold on its boundary are also contained in the boundary of $R(i_0, \ldots, i_{k-1})$. In the sequel, we call *unstable sides* of a rectangle the segments of unstable manifold on its boundary, and *unstable boundary* the union of the unstable sides. Stable sides and stable boundary are defined analogously.

3.3 Geometry of rectangles at return times

Here we prove Propositions 3.5 and 3.7. Part 1 of these propositions is trivial when $r_{k-1} = 0$ (cases (d), (f), (d1), (b2)), because $p_k = 0$. So we may suppose $r_{k-1} \geq \Delta$. Since $\|Df\| \leq 4$, the definition of r_{k-1} and p_k implies $4^{p_k+1}e^{-2r_{k-1}} \geq ce^{-\beta(p_k+1)}$. Hence,

$$m_k > p_k \geq \frac{2}{\log 4 + \beta} r_{k-1} + \log c - 1 \geq \frac{4}{3} r_{k-1}, \tag{36}$$

because $\log c - 1$ is negligible when Δ is big enough. This gives part 1 of both propositions.

Next, we are going to prove part 2. This is easy when r_{k-1} is zero: in that case $p_k = 0$ and slope $f^{m_k}(\gamma_i^u) < (C/\delta)\sqrt{b}$ is granted by Proposition 2.5. In what follows we consider $r_{k-1} \geq \Delta$. Then the main point in the proof is to show that p_k may be taken as the bound period for any point in $f^{n_{k-1}}(R(i_0, \ldots, i_{k-1}))$. The precise statement is the following

Lemma 3.9. *Suppose $p_k > 0$. For any $z \in f^{n_{k-1}}(R(i_0, \ldots, i_{k-1}))$,*

$$\operatorname{dist}(f^j(z), f^j(\tilde{\zeta}_{k-1})) \begin{cases} \leq e^{-\beta j} & \text{for } 1 \leq j \leq p_k \\ \geq \frac{1}{10} e^{-\beta(p_k+1)} & \text{for } j = p_k + 1 \end{cases} \tag{37}$$

Moreover, if z is on the unstable boundary of $f^{n_{k-1}}(R(i_0, \ldots, i_{k-1}))$ then

1. $\|Df^{p_k+1}(z)t(z)\| \geq \sigma_1^{(p_k+1)/3}$ *and* $\operatorname{slope}(Df^{p_k+1}(z)t(z)) < (C/\delta)\sqrt{b}$;
2. $\|Df^{p_k+1}(z)t(z)\| d_C(z) \geq ce^{-\beta(p_k+1)}$;
3. $\|w_j(z)\| \geq \sigma_1^j$ *for* $1 \leq j \leq p_k$, *and* $\operatorname{slope} w_{p_k}(z) < (C/\delta)\sqrt{b}$.

Proof. First we treat cases (a), (b), (e), (a1), (b1), (a2), where both unstable sides $\gamma_i = f^{-1}(\gamma_i^u)$, $i = 0, 1$ of the rectangle $f^{n_{k-1}}(R(i_0, \ldots, i_{k-1}))$ are in tangential position with respect to the binding point $\tilde{\zeta}_{k-1}$. By construction, recall the last paragraph of Section 3.1,

$$\operatorname{length}(\gamma_i) \approx \frac{e^{-r_{k-1}}}{r_{k-1}^2} \leq \frac{e^{-r_{k-1}}}{\Delta^2} \ll e^{-r_{k-1}} \approx \operatorname{dist}(\gamma_i, \tilde{\zeta}_{k-1}), \tag{38}$$

This ensures that (17) is satisfied by these γ_i, as long as $\Delta = \log(1/\delta)$ is taken large enough. Consequently, for any $0 \leq j \leq p_k$, the distances between $f^{j+1}(\tilde{\zeta}_{k-1})$ and the $(j+1)$st iterates of any two points in the same γ_i are comparable up to a factor 2, unless they are both smaller than $(Cb)^j$. Recall (18). Moreover,

$$\operatorname{length}(f^j(\gamma_l^s)) \leq C\sqrt{b}(Cb)^j \ll e^{-\beta(j+1)}, \tag{39}$$

for $0 \leq j \leq p_k$, and $l = 0, 1$, because γ_0^s and γ_1^s are contained in stable leaves. According to the definition of p_k, there exists some point $\xi \in f^{n_{k-1}}(R(i_0, \ldots, i_{k-1}))$ such that the distance from $f^{p_k+1}(\xi)$ to $f^{p_k+1}(\tilde{\zeta}_{k-1})$ exceeds $e^{-\beta(p_k+1)}$. Of course, ξ may be taken on the boundary of the rectangle. Then the distances from $f^{p_k+1}(\tilde{\zeta}_{k-1})$ to the $(p_k+1)st$ iterates of points on the boundary of $f^{n_{k-1}}(R(i_0, \ldots, i_{k-1}))$ are all much larger than $(Cb)^{p_k}$, and they are two-by-two comparable up to a factor less than 10. That is because of (39) and our previous remark that the distance varies by less than a factor 2 inside each unstable side. This shows that

$$\operatorname{dist}(f^{p_k+1}(z), f^{p_k+1}(\tilde{\zeta}_{k-1})) \geq \frac{1}{10} e^{-\beta(p_k+1)}$$

for every z on the boundary of $f^{n_{k-1}}(R(i_0,\ldots,i_{k-1}))$. It follows that the same is true for any point in the interior. This proves the upper bound in (37). The lower bound is contained in the definition of p_k, so the proof of (37) is complete. Then, as observed in Section 2.2, the arguments in the proof of Proposition 2.8 apply for any point z in $\gamma_0 \cup \gamma_1$, with $p(z) = p_k$. Claims 1, 2, 3 in the lemma follow, corresponding to parts 2, 3, 4 of Proposition 2.8.

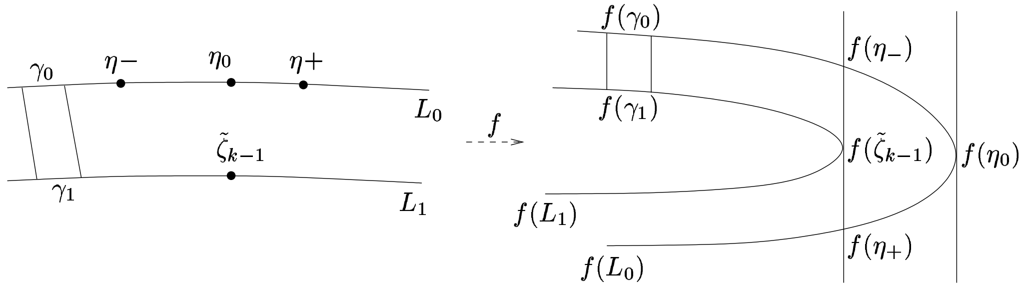

Figure 3: Binding to a non-critical point

Now we deal with cases (c) and (c1) in the definition of itineraries. The difference with respect to the previous cases is that only one of the unstable sides of $f^{n_{k-1}}(R(i_0,\ldots,i_{k-1}))$ is in tangential position with respect to the binding point $\tilde\zeta_{k-1}$. See Figures 1, 2, and 3. To fix notations, let this be γ_1 and let L_1 be a (nearly horizontal) segment of $W^u(P)$ containing γ_1 and $\tilde\zeta_{k-1}$. For the other unstable side, γ_0, there is a nearly horizontal segment L_0 of the unstable manifold connecting it to a different critical point η_0. Basically, everything we said before still applies to γ_1 but, because of the asymmetry introduced by the choice of the binding point (p_k is defined in terms of $\tilde\zeta_{k-1}$ not η_0), it is less clear why that should be true for γ_0.

In a few words, our strategy to prove that this is so is to use a point $\eta_\pm \in L_0$ such that $f(\eta_\pm)$ is contained in the long stable leaf $\Gamma(f(\tilde\zeta_{k-1}))$ through $f(\tilde\zeta_{k-1})$, as an auxiliary binding point for γ_0. On the one hand, γ_0 is in tangential position to $(\eta_0, t(\eta_0))$ and the orbits of η_0 and any $z \in L_0$ remain bound up to time p_k (an not more). On the other hand, because the image of $\eta_\pm$ is in the long stable leaf through $f(\tilde\zeta_{k-1})$, it shares the main properties required for a binding point (expansiveness, tangential returns, not too close to the critical set).

To explain this in detail, let us write $L_0 = \{(x, y_0(x))\}$ and $(X(x), Y(x)) = f(x, y_0(x))$. Since the distance from $(X(x), Y(x))$ to $\Gamma(f(\tilde\zeta_{k-1}))$ varies in a quadratic fashion, recall (7), the curve $f(L_0)$ intersects the stable leaf at exactly two points, $f(\eta_-)$ and $f(\eta_+)$. Moreover, η_0 is between η_- and η_+ inside L_0, whereas γ_0 is disjoint from the segment bounded by η_- and η_+. See Figure 3. In what follows we suppose that γ_0 is to the left of that segment (the other case is analogous) and η_- is the endpoint closest to it. Let $\eta_* = (x_*, y_0(x_*))$, for $* \in \{-, 0, +\}$, and $w_j(x) = Df^j(X(x), Y(x))w_0$ for $j \geq 1$.

Claim 1: For any $0 \leq j \leq p_k$ the distances from $f^{j+1}(\eta_-)$ to any two points in $f^{j+1}(\gamma_0)$ are either comparable, up to a factor 2, or simultaneously less than $(Cb)^j$.

Proof. Define $p \geq 1$ to be the largest integer such that $\operatorname{dist}(f^j(\xi), f^j(\eta_-)) \leq 2e^{-\beta j}$ for any $1 \leq j \leq p$ and $\xi \in \gamma_0$. According to Proposition 2.4,

$$\operatorname{dist}(f^j(\eta_-), f^j(\tilde\zeta_{k-1})) \leq (Cb)^j \quad \text{for every } j \geq 1. \tag{40}$$

Together with the definition of p_k, this shows that $\operatorname{dist}(f^j(\xi), f^j(\eta_-)) \le e^{-\beta j} + (Cb)^j < 2e^{-\beta j}$ for every $j \le p_k$. Therefore, $p \ge p_k$. We are going to prove the statements in Claim 1 for every $0 \le j \le p$. Fix $z = (x, y_0(x)) \in \gamma_0$.

The first step is a distortion bound analogous to (11): for every $1 \le j \le p$ and $s \in [x, x_-]$,

$$w_j(s) = \lambda(s)\big(w_j(x_-) + \epsilon_j(s)\big), \quad \text{with } \ c \le \lambda(s) \le C, \ \text{and } \|\epsilon_j(s)\| \ll \|w_j(x_-)\|. \tag{41}$$

This is obtained as follows. As mentioned before, all one has to know for the proof of (11) is that $f(\zeta_\gamma)$ is expanding, its free returns up to time p are tangential, and they satisfy (12). We are going to check that these facts are true for η_- in the place of ζ_γ. Then, (41) follows from the same arguments that give (11), see [2, Lemma 7.8] and [15, Lemma 10.5]. By (40) and Proposition 2.3.5 (take $\tau = Cb$),

$$\frac{1}{10} \le \frac{\|w_j(\eta_-)\|}{\|w_j(\tilde\zeta_{k-1})\|} \le 10 \quad \text{and} \quad \operatorname{angle}(w_j(\eta_-), w_j(\tilde\zeta_{k-1})) \le (Cb)^{j/2}, \tag{42}$$

for every $j \ge 1$. The first relation, combined with Theorem 2.1.2, implies that $f(\eta_-)$ is an expanding point. Let $j \ge 1$ be a free return, and ζ_j be the binding point for $f^j(\tilde\zeta_{k-1})$. Using (40) and Theorem 2.1.3,

$$\operatorname{dist}(f^j(\eta_-), \zeta_j) \approx \operatorname{dist}(f^j(\tilde\zeta_{k-1}), \zeta_j) \ge e^{-\alpha j}. \tag{43}$$

This corresponds to (12). Finally, the tangential position property may be checked as follows. By Theorem 2.1.3 and the observation near the end of Section 2.1, there exists a C^2 curve $K_j = \{(x, y_j(x))\}$ with $|y_j'|, |y_j''| \le 1/10$, such that K_j is tangent to $t(\zeta_j)$ at ζ_j and tangent to $w_{j-1}(\tilde\zeta_{k-1})$ at $f^j(\tilde\zeta_{k-1})$. Using (40), the angle estimate in (42), and (43), we may modify K_j to get another C^2 curve $\{(x, z_j(x))\}$, with $|z_j'|, |z_j''| \le 1/5$, tangent to $t(\zeta_j)$ at ζ_j and tangent to $w_{j-1}(\eta_-)$ at $f^j(\eta_-)$.

From (41) and (42) we get that $f(z(s))$ is expanding up to time p, for any $z(s) = (s, y_0(s))$ with $s \in [x, x_-]$:

$$\|w_j(s)\| \approx \|w_j(x_-)\| \ge \frac{1}{10}\,\|w_j(\tilde\zeta_{k-1})\| \ge \frac{1}{10}\,\sigma_1^j\,. \tag{44}$$

for $0 \le j \le p$. Then the contracting direction $e(s)$ of order p at $f(z(s))$ is well-defined, for any $s \in [x, x_-]$. Thus, we may split the tangent direction to $f(L_0)$ in the same way as in (9),

$$(X'(s), Y'(s)) = \alpha(s)e(s) + \beta(s)w_0\,,$$

with $|\alpha(s)|$, $|\alpha'(s)|$, $|\beta'(s) + 2a|$ bounded by $C\sqrt{b}$. Then, for $0 \le j \le p$,

$$f^{j+1}(\eta_-) - f^{j+1}(z) = \int_x^{x_-} \big(\alpha(s)Df^j(f(z(s)))e(s) + \beta(s)\lambda(s)\big(w_j(x_-) + \epsilon_j(s)\big)\big)\,ds.$$

The main difference with respect to (13) is that there is no reason why $\beta(x_-)$ should be small: $f(L_0)$ is *not* tangent to the long stable leaf through $f(\eta_-)$. But we do have

Claim 2: $\beta(x_-) \ge -(Cb)^p$.

We accept this fact for a while, and proceed with the proof of Claim 1. Since $\beta' \approx -2a$ is negative, it follows from Claim 2 that $\beta(s) \geq \beta(x_-) \geq -(Cb)^p$ and so $|\beta(s)| \leq \beta(s) + 2(Cb)^p$, for every $s \in [x, x_-]$. Let us rewrite $f^{j+1}(\eta_-) - f^{j+1}(z)$ as

$$\begin{aligned} &\int_x^{x_-} \alpha(s) Df^j(f(z(s)))e(s)\, ds + w_j(x_-) \int_x^{x_-} \big(\beta(s) + 2(Cb)^p\big)\lambda(s)\, ds \\ &-2(Cb)^p w_j(x_-) \int_x^{x_-} \beta(s)\lambda(s)\, ds + \int_x^{x_-} \beta(s)\lambda(s)\epsilon_j(s)\, ds. \end{aligned} \tag{45}$$

The principal term in (45) is the second one. Indeed, the first term is less than $(Cb)^j$, recall Proposition 2.3.4. The third term is less than $(Cb)^p 5^j \leq (Cb)^j$, because β and λ are bounded and $\|w_j\|$ is less than 5^j. Since $\|\epsilon_j(s)\| \ll \|w_j(x_-)\|$, the fourth term is much smaller than the second one:

$$\begin{aligned} \Big\| \int_x^{x_-} \beta(s)\, \lambda(s)\, \epsilon_j(s)\, ds \Big\| &\ll \int_x^{x_-} |\beta(s)|\, \lambda(s)\, \|w_j(x_-)\|\, ds \\ &\leq \|w_j(x_-)\| \int_x^{x_-} (\beta(s) + 2(Cb)^p)\, \lambda(s)\, ds. \end{aligned}$$

These observations imply that (here $\approx$ means equality up to a factor $\sqrt{2}$)

$$\operatorname{dist}(f^{j+1}(\eta_-), f^{j+1}(z)) \approx \|w_j(x_-)\| \int_x^{x_-} (\beta(s) + 2(Cb)^p)\, \lambda(s)\, ds \tag{46}$$

unless, possibly, if the right hand side is less than $(Cb)^j$, in which case $\operatorname{dist}(f^{j+1}(\eta_-), f^{j+1}(z))$ is also bounded by $(Cb)^j$. Since λ is bounded from zero and infinity, and $\beta(s) + 2(Cb)^p$ is a positive function on $[x, x_-]$, with derivative almost constant and negative,

$$\int_x^{x_-} \big(\beta(s) + 2(Cb)^p\big)\lambda(s)\, ds \approx \big(\beta(x) + 2(Cb)^p\big)(x_- - x). \tag{47}$$

Just as in (38), the length of γ_0 is much smaller than the distance from η_- to any of its points. Therefore, the right hand side of (47) is almost constant when $z = (x, y_0(x))$ varies over the whole γ_0: at most, it changes by a factor that is close to 1 if $\Delta = \log 1/\delta$ is large. It follows that the second term in (46) oscillates by, at most, a factor $\sqrt{2}$ when z varies over the whole γ_0. So, either the distance from $f^{j+1}(z)$ to $f^{j+1}(\eta_-)$ is less than $(Cb)^j$, and then the same is true for any other point in $f^{j+1}(\gamma_0)$ (with C replaced by $2C$), or else it is comparable up to a factor 2 to the distance from $f^{j+1}(\eta_-)$ to any $f^{j+1}(z') \in f^{j+1}(\gamma_0)$. This proves Claim 1. □

Before going back to the proof of Lemma 3.9, let us prove Claim 2 stated above:

Proof. Let $\tilde{p}$ be the bound period of η_- relative to the critical point η_0. As $(\eta_-, t(\eta_-))$ is in tangential position to $(\eta_0, t(\eta_0))$, we are in the precise context of Proposition 2.8. We may split $(X'(s), Y'(s)) = \tilde{\alpha}(s)\tilde{e}(s) + \tilde{\beta}(s) w_0$, where $\tilde{e}(s) = (\tilde{\epsilon}(s), 1)$ is the contracting direction of order $\tilde{p}$ at $f(z(s))$. Recall that $z(s) = (s, y_0(s))$ parametrises L_0, and $(X(s), Y(s)) = f(z(s))$. From (10) we get

$$\tilde{\beta}(x_-) - \tilde{\beta}(x_0) \approx (x_0 - x_-) \approx \operatorname{dist}(\eta_-, \eta_0).$$

Note that $\text{dist}(\eta_-, \eta_0)$ is much larger than $(Cb)^{\tilde{p}}$: using $\|Df\| \le 5$,

$$e^{-\beta(\tilde{p}+1)} \approx \text{dist}(f^{\tilde{p}+1}(\eta_-), f^{\tilde{p}+1}(\eta_0)) \le 5^{\tilde{p}+1}\,\text{dist}(\eta_-, \eta_0).$$

On the other hand, $|\tilde{\beta}(x_0)| \le (Cb)^{\tilde{p}}$ by the second part of (14). Thence,

$$\tilde{\beta}(x_-) \approx \text{dist}(\eta_-, \eta_0) \ge c(5e^{\beta})^{-\tilde{p}} > 0. \tag{48}$$

Write $e(s) = (\epsilon(s), 1)$ for the contracting direction of order p. Then $|\epsilon(s) - \tilde{\epsilon}(s)| \le (Cb)^{\min\{p,\tilde{p}\}}$, according to Proposition 2.3.2. Using this in $\alpha(s)e(s) + \beta(s)w_0 = \tilde{\alpha}(s)\tilde{e}(s) + \tilde{\beta}(s)w_0$, we get

$$|\beta(x_-) - \tilde{\beta}(x_-)| \le C\,|\epsilon(x_-) - \tilde{\epsilon}(x_-)| \le (Cb)^{\min\{p,\tilde{p}\}}.$$

If $p \le \tilde{p}$ then Claim 2 follows immediately from this last inequality and the fact that $\tilde{\beta}(x_-)$ is positive. For $\tilde{p} \le p$ we get a stronger fact: $\beta(x_-) \ge c(5e^{\beta})^{-\tilde{p}} - (Cb)^{\tilde{p}} > 0$. □

Now that we have established Claim 1, property (37) follows in the same way as in the cases (a) through (a2), that we treated before. We just review the arguments. By the definition of p_k , there exists some point on the boundary of the rectangle $f^{n_{k-1}+p_k+1}(R(i_0, \dots, i_{k-1}))$ whose distance to $f^{p_k+1}(\tilde{\zeta}_{k-1})$ exceeds $e^{-\beta(p_k+1)}$. Since the length of the stable sides of the rectangle is less than $(Cb)^{p_k}$, which is much smaller than $e^{-\beta(p_k+1)}$, we may take this point on one of the unstable sides. Using Claim 1 and its analog for γ_1 (which we knew before), together with (40) and the upper bound on the lengths of the stable sides, we conclude that the distances from $f^{p_k+1}(\tilde{\zeta}_{k-1})$ (or from $f^{p_k+1}(\eta_-)$) to any two points on the boundary of the rectangle are comparable up to a factor less than 10. This yields

$$\text{dist}(f^{p_k+1}(z), \tilde{\zeta}_{k-1}) \ge \frac{1}{10}\,e^{-\beta(p_k+1)} \quad \text{and} \quad \text{dist}(f^{p_k+1}(z), \eta_-) \ge \frac{1}{10}\,e^{-\beta(p_k+1)}$$

for every z on the boundary of $f^{n_{k-1}}(R(i_0, \dots, i_{k-1}))$, and so also for every point in the interior. This gives one of the inequalities in (37), the other one is contained in the definition of p_k.

Finally, we prove parts 1, 2, 3 of Lemma 3.9. For points $z \in \gamma_1$ this is analogous to parts 2, 3, 4 of Proposition 2.8, because γ_1 is tangential to the binding point. For $z \in \gamma_0$ it goes along similar lines, with η_- acting as the binding point. Firstly, by (46) and (47),

$$e^{-\beta(p_k+1)} \approx \text{dist}(f^{p_k+1}(\eta_-), f^{p_k+1}(z)) \approx \|w_{p_k}(x_-)\|\,(\beta(x) + 2(Cb)^{p_k})\,(x_- - x).$$

From (33) and Claim 2, we find

$$\beta(x) > x_- - x \quad \text{and} \quad c\beta(x) \le \beta(x) + 2(Cb)^{p_k} \le C\beta(x),$$

note that $x_- - x \approx \text{dist}(z, \eta_-)$ is bounded below by $c(5e^{\beta})^{-p_k} \gg (Cb)^{p_k}$. It follows that

$$e^{-\beta(p_k+1)} \approx \beta(x)\|w_{p_k}(x_-)\|(x_- - x) \approx \beta(x)\|w_{p_k}(x)\|(x_- - x) \le \beta(x)^2\|w_{p_k}(x)\|\,. \tag{49}$$

The second step uses (44). Now, $Df^{p_k}(f(z))t(z) = \alpha(x)Df^{p_k}(f(z))e(x) + \beta(x)w_{p_k}(x)$. The first term is bounded by $C\sqrt{b}\,(Cb)^{p_k}$, which is much smaller than $e^{-\beta(p_k+1)}$. Hence

$$\|Df^{p_k}(f(z))t(z)\|\,d_{\mathcal{C}}(z) \ge \|Df^{p_k}(f(z))t(z)\|(x_- - x) \approx \beta(x)\|w_{p_k}(x)\|(x_- - x) \approx e^{-\beta(p_k+1)}.$$

This gives part 2 of the lemma. The slope statements in parts 1 and 3 follow from

$$\text{slope}\left(Df^{p_k}(f(z))\,t(z)\right) \approx \text{slope}\left(w_{p_k}(x)\right) \approx \text{slope}\left(w_{p_k}(\eta_-)\right) \approx \text{slope}\left(w_{p_k}(\tilde{\zeta}_{k-1})\right),$$

recall (41), (42). The first half of part 3 is a consequence of (41), (42), and the fact that $f(\tilde{\zeta}_{k-1})$ itself is an expanding point, cf. Theorem 2.1.2. Finally, (49) gives us the first half of part 1:

$$\|Df^{p_k}(f(z))\,t(z)\|^2 \approx \beta(x)^2\|w_{p_k}(x)\|^2 \geq c\|w_{p_k}(x)\|e^{-\beta(p_k+1)} \geq \sigma_1^{2(p_k+1)/3}.$$

The last step uses $\|w_{p_k}(x)\| \geq \sigma_1^{p_k}$ and the assumption that β and δ are small (the latter forces p_k to be large). The proof of Lemma 3.9 is complete. □

Part 2 of Propositions 3.5 and 3.7 is contained in the conclusion of Lemma 3.9. Now we move on to prove parts 3 and 4. For this we need a few additional facts about the size and shape of the rectangles $f^{n_k}(R(i_0,\ldots,i_{k-1}))$, that are obtained in Lemmas 3.10 and 3.11. The proof of these lemmas is by induction on k, using (36).

Lemma 3.10. *Given $z \in R(i_0,\ldots,i_{k-1})$ and any unstable side σ^u of $R(i_0,\ldots,i_{k-1})$, there exists $\xi_k \in \sigma^u$ such that*

1. $\text{dist}(f^{n_{k-1}+1}(z), f^{n_{k-1}+1}(\xi_k)) \leq \min\left\{10(Cb)^{n_{k-1}/2}e^{r_{k-1}},\ 10(Cb)^{n_{k-1}/4}\right\}$;
2. $\text{dist}(f^{n_k}(z), f^{n_k}(\xi_k)) \leq (Cb)^{n_k/2}$.

If z is in some stable side σ^s, we may take ξ_k the common endpoint of σ^s and σ^u.

Proof. To keep track of the constant C in the statement, we denote it C_h throughout the proof. We shall take $C_h \geq 100C_1$, where C_1 is the constant C in Proposition 2.3.

We start by proving the lemma for $k=1$. Recall that $n_0+1=0$. Then the inequality in part 1 are trivial: the left hand side is bounded by 4, whereas the right hand side can be made arbitrarily large by taking b small (these comments are for completeness only, we never use this part of the lemma with $k=1$). Part 2 is proved as follows. As a consequence of Propositions 2.8.4 and 2.5, every $z \in R(i_0)$ is expanding up to time $m_1 = n_1$. So, using Proposition 2.3 in the same way as when proving Proposition 2.4 in Section 2, the temporary stable leaf of order n_1 through z is a long nearly vertical curve: $\Gamma^{n_1}(z) = \{(x(y),y) : |y| \leq 1/10\}$ with $|x'|, |x''| \leq C\sqrt{b}$, and

$$\text{dist}(f^j(\xi), f^j(z)) \leq (Cb)^j\,\text{dist}(\xi,z) \text{ for all } \xi \in \Gamma^{n_1}(z) \text{ and } 1 \leq j \leq n_1\,.$$

If $\Gamma^{n_1}(z)$ intersects the unstable segment σ^u, take ξ_1 to be the intersection point. Then $\text{dist}(f^{n_1}(z), f^{n_1}(\xi_1)) \leq (C_1 b)^{n_1}$, which is even stronger than the claim. If $\Gamma^{n_1}(z)$ leaves the rectangle $R(i_0)$ through a stable leaf σ^s, take ξ_1 to be the vertex of $R(i_0)$ where σ^s meets σ^u, then continue as before. Noting that σ^s is also contracted by positive iterates of f, one still gets $\text{dist}(f^{n_1}(z), f^{n_1}(\xi_1)) \leq (C_1 b)^{n_1}$.

Now we assume that the lemma is true at time n_{k-1}, and prove that it must be true also at time n_k. For the same reasons as before, every point $\zeta = f^{n_{k-1}+1}(z)$, $z \in R(i_0,\ldots,i_{k-1})$, is expanding up to time m_k. So, the temporary stable leaf $\Gamma^{m_k}(\zeta)$ of order m_k through ζ is a long nearly vertical curve. If $\Gamma^{m_k}(\zeta)$ crosses the unstable segment $\gamma^u = f^{n_{k-1}+1}(\sigma^u)$, let η be the intersection point. See Figure 4. Otherwise, $\Gamma^{m_k}(\zeta)$ intersects some stable side γ^s of

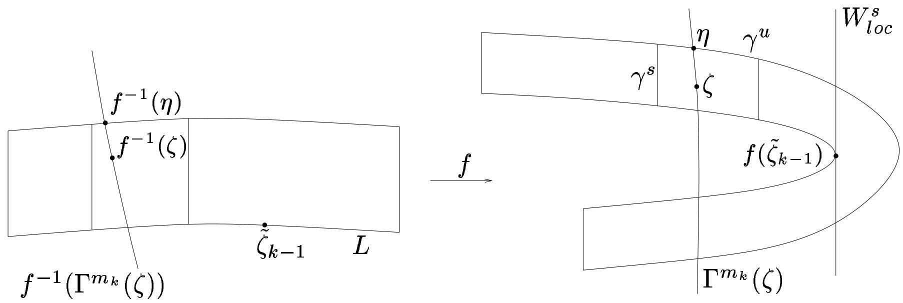

Figure 4:

$f^{n_{k-1}+1}(R(i_0,\ldots,i_{k-1}))$, and we call η the common endpoint of γ^u and γ^s. Either way, define ξ_k by $f^{n_{k-1}+1}(\xi_k) = \eta$. We are going to prove

$$\operatorname{dist}(\zeta,\eta) \le 10(C_h b)^{n_{k-1}/2} e^{r_{k-1}} \quad \text{and} \quad \operatorname{dist}(\zeta,\eta) \le 10(C_h b)^{n_k/4}. \tag{50}$$

which is just part 1 of the lemma. It immediately implies part 2. Indeed, combining the first of these inequalities with Proposition 2.3.2 we find

$$\operatorname{dist}(f^{n_k}(z), f^{n_k}(\xi_k)) = \operatorname{dist}(f^{m_k}(\zeta), f^{m_k}(\eta)) \le (C_1 b)^{m_k} 10(C_h b)^{n_{k-1}/2} e^{r_{k-1}} \le (C_h b)^{n_k/2}.$$

Note that $n_k = n_{k-1} + 1 + m_k$ and we chose $C_h \ge 100 C_1$. Moreover, $m_k > r_{k-1}$ by part 1 of Propositions 3.5 and 3.7, which was already proved. So, we have reduced Lemma 3.10 to proving (50).

For the proof of (50), it is convenient to distinguish two cases, depending on the relative size of r_{k-1} and n_{k-1}. We treat first the case when r_{k-1} is small with respect to n_{k-1}:

$$(C_h b)^{n_{k-1}} \le e^{-4 r_{k-1}}. \tag{51}$$

In this case the second inequality in (50) is a direct consequence of the first one. To prove the first inequality, consider a curve $L = \{(x, y_0(x))\}$ with $|y_0'|, |y_0''| \le 1/5$, tangent to $W^u(P)$ at $\tilde{\zeta}_{k-1} = (\bar{x}, \bar{y})$ and containing one of the unstable sides of $f^{n_{k-1}}(R(i_0,\ldots,i_{k-2}))$. We claim that a segment of $f^{-1}(\Gamma^{m_k}(\zeta))$ can be parametrised $t \mapsto z(t) = (x(t), y_0(x(t)) + t)$, with

$$|x'(t)| \le C\sqrt{b} e^{r_{k-1}} \quad \text{and} \quad |t| \le 2(C_h b)^{n_{k-1}/2}. \tag{52}$$

Let us assume this for a while. Since L is nearly horizontal, $\operatorname{dist}(z(t), L) \ge |t|/2$ for any t. Using the assumption that Lemma 3.10.1 is true for $k-1$, we conclude that $f^{-1}(\Gamma^{m_k}(\zeta))$ crosses the boundary of $f^{n_{k-1}}(R(i_0,\ldots,i_{k-1}))$ at another point $z(t_2)$ with $0 < |t_2| \le 2(C_h b)^{n_{k-1}/2}$. Moreover, $f^{-1}(\zeta) = z(t_1)$ for some t_1 between zero and t_2. Typically, $z(t_2) = f^{-1}(\eta)$ but it may also happen that $z(t_2)$ be in a stable side γ^s of $f^{n_{k-1}}(R(i_0,\ldots,i_{k-1}))$, with $f^{-1}(\eta)$ being an endpoint of γ^s. In the second case, $\operatorname{dist}(z(t_2), f^{-1}(\eta))$ is less than $(C_h b)^{n_{k-1}/2}$, by the induction hypothesis. So, using the bound for $|x'|$ given in (52), we always have,

$$\begin{aligned}\operatorname{dist}(\zeta,\eta) &\le \operatorname{dist}(z(t_1), z(t_2)) + (C_h b)^{n_{k-1}/2} \\ &\le C\sqrt{b} e^{r_{k-1}} (C_h b)^{n_{k-1}/2} + 5(C_h b)^{n_{k-1}/2} \le 10 e^{r_{k-1}} (C_h b)^{n_{k-1}/2}.\end{aligned}$$

To turn the previous paragraph into a complete proof of (50) for small r_{k-1}, we have to justify the claims in (52). Let $\tilde{\zeta}_{k-1} = (\bar{x}, \bar{y})$ be the binding critical point. Write the long leaf $W^s_{loc}(f(\tilde{\zeta}_{k-1}))$ through $f(\tilde{\zeta}_{k-1}) = (\bar{x}_1, \bar{y}_1)$ as $\{(x^s(y), y) : |y| \leq 1/10\}$, and the temporary stable leaf $\Gamma^{m_k}(\zeta)$ as $\{(x_\zeta(y), y) : |y| \leq 1/10\}$. Moreover, denote $(X_0(x), Y_0(x)) = f(x, y_0(x))$ and $(X(x,t), Y(x,t)) = f(x, y_0(x) + t)$. The condition $f(x, y_0(x) + t) \in \Gamma^{m_k}(\zeta)$ is expressed by $X(x,t) = x_\zeta(Y(x,t))$ or, equivalently,

$$x^s(Y(x,t)) - X(x,t) = x^s(Y(x,t)) - x_\zeta(Y(x,t)) \tag{53}$$

We want to show that (53) defines x as an implicit function of t, with derivative bounded by $C\sqrt{b}e^{r_{k-1}}$, on the whole interval $|t| \leq 2(C_h b)^{n_{k-1}/2}$. For this purpose, let us estimate the partial derivatives of both sides of (53). Firstly,

$$|\partial_t(x^s \circ Y - X)(x,t)| \leq C\sqrt{b} \quad \text{and} \quad |\partial_t(x^s \circ Y - x_\zeta \circ Y)(x,t)| \leq C\sqrt{b} \tag{54}$$

for every (x,t), because $(x^s)'$, $(x_\zeta)'$, $\partial_y f$ are less than $C\sqrt{b}$. By Proposition 2.3.2 and 2.3.3,

$$|(x^s - x_\zeta)'(y)| \leq C\sqrt{b}\,(x^s - x_\zeta)(y) + (Cb)^{m_k}.$$

Combining this with $m_k > r_{k-1}$ (Proposition 3.7.1) and the definition of r_{k-1}, we get

$$|(x^s - x_\zeta)(y)| \approx e^{-2r_{k-1}} \quad \text{and} \quad |(x^s - x_\zeta)'(y)| \leq C\sqrt{b}e^{-2r_{k-1}} \tag{55}$$

for every $|y| \leq 1/10$; compare Proposition 3.3. This last relation implies

$$|\partial_x(x^s \circ Y - x_\zeta \circ Y)(x,t)| \leq C\sqrt{b}e^{-2r_{k-1}} \tag{56}$$

for every (x,t). On the other hand, as in (7),

$$(x^s \circ Y_0 - X_0)(x) \approx (\bar{x} - x)^2 \quad \text{and} \quad (x^s \circ Y_0 - X_0)'(x) \approx (\bar{x} - x) \tag{57}$$

for all x. Let (x,t) be any solution of (53) with $|t| \leq 2(C_h b)^{n_{k-1}/2}$. Using (55),

$$|(x^s \circ Y - X)(x,t)| = |(x^s \circ Y - x_\zeta \circ Y)(x,t)| \approx e^{-2r_{k-1}}.$$

In view of (54), the bound on $|t|$, and assumption (51), these expressions change much less that $e^{-2r_{k-1}}$ if we replace t by zero. This means that the previous relation is not affected by taking zero in the place of t: it becomes $|(x^s \circ Y_0 - X_0)(x)| \approx e^{-2r_{k-1}}$. Using (57), we conclude that $|x - \bar{x}| \approx e^{-r_{k-1}}$ and

$$|\partial_x\big(x^s \circ Y - X\big)(x,0)| = |(x^s \circ Y_0 - X_0)'(x)| \approx e^{-r_{k-1}}.$$

Now, since the derivative of $\partial_x\big(x^s \circ Y - X\big)$ is also bounded, we may put t back in the place of zero without affecting this relation:

$$|\partial_x\big(x^s \circ Y - X\big)(x,t)| \approx e^{-r_{k-1}}. \tag{58}$$

Since $C\sqrt{b}e^{-2r_{k-1}}$ is much smaller than $e^{-r_{k-1}}$, the relations (56) and (58) give

$$|\partial_x(x^s \circ Y - X)(x,t) - \partial_x(x^s \circ Y - x_\zeta \circ Y)(x,t)| \approx e^{-r_{k-1}}$$

for any solution (x,t) with $|t| \le 2(C_h b)^{n_{k-1}/2}$. Thus, we may indeed use the implicit function theorem in (53). Moreover, by (54), the implicit function $x(t)$ has $|x'(t)| \le C\sqrt{b}e^{r_{k-1}}$. The proof of (52) is complete.

Now we prove (50) for large r_{k-1}, that is, $(C_h b)^{n_{k-1}} \ge e^{-4r_{k-1}}$. We claim that in this case

$$\operatorname{dist}(\zeta,\eta) \le C\sqrt{b}e^{-r_{k-1}} < e^{-r_{k-1}}. \tag{59}$$

Note that this implies both inequalities in the statement. To prove the claim, let $(x^s(y),y)$ and $(x_\zeta(y),y)$ be as before, and $(X(x),Y(x))$ parametrise the unstable side γ^u that contains η. Suppose first that $\Gamma^{m_k}(\zeta)$ intersects γ^u at the point η. Then the length of the segment of Γ^{m_k} connecting ζ to η is less than $C\sqrt{b}e^{-r_{k-1}}$, as a consequence of the estimates corresponding to (55) and (57), with X, Y in the place of X_0, Y_0. This proves the claim in this case. The other one corresponds to Γ^{m_k} intersecting a stable side γ^s at some point η', with η being an endpoint of γ^s. The same argument as before applies, both to the segment of Γ^{m_k} connecting ζ to η' and to the stable segment γ^s: their lengths are shorter than $C\sqrt{b}e^{-r_{k-1}}$. So we get (59) in this case too. □

Remark 3.4. The following elementary fact is used in the next lemma. Let v_1, v_2, ϵ_1, ϵ_2 be planar vectors such that $\|\epsilon_i\| < \|v_i\|/2$ for $i = 1,2$. Let $\theta = \operatorname{angle}(v_1, v_2)$ and χ be the norm of $\epsilon_1/\|v_1\| - \epsilon_2/\|v_2\|$. Then $\operatorname{angle}(v_1+\epsilon_1, v_2+\epsilon_2) \le \theta + 2(\chi + \|\epsilon_1\|\,\theta)$. A proof follows. Dividing v_i and ϵ_i by $\|v_i\|$, we may suppose $\|v_i\| = 1$ for $i=1,2$. Then $v_2 = e^{i\theta}v_1$, and so

$$\operatorname{angle}(v_1+\epsilon_1, v_2+\epsilon_2) \le \operatorname{angle}(v_1+\epsilon_1, e^{i\theta}v_1 + e^{i\theta}\epsilon_1) + \operatorname{angle}(v_2 + e^{i\theta}\epsilon_1, v_2+\epsilon_2).$$

The first term is equal to θ, and the second one is less than $2\|e^{i\theta}\epsilon_1 - \epsilon_2\| \le 2\theta\|\epsilon_1\| + 2\|\epsilon_1 - \epsilon_2\|$, because $\|\epsilon_i\| \le 1/2$ for $i = 1,2$.

Lemma 3.11. *Let ξ_k be as in Lemma 3.10. If $r_j \le 5n_j$ for every $1 \le j \le k-1$ then*

$$\operatorname{angle}(Df^{n_k}(z)w_0, t(f^{n_k}(\xi_k))) \le (Cb)^{n_k/4}.$$

Proof. During the proof we represent by C_a the constant in the statement. In a number of places we assume C_a to be large, with respect to a few other constants.

First we treat the case $k=1$. Combining Propositions 2.8.4 and 2.5 we get that ξ_1 is expanding up to time n_1. So, recall Proposition 2.3.1, the contracting direction $e_\xi = (\epsilon_\xi, 1)$ of order n_1 at ξ_1 is well defined and satisfies $|\epsilon_\xi| \le C\sqrt{b}$. Let $(1, \dot{y}_\xi)$, $|\dot{y}_\xi| \le 1/5$, represent the tangent direction to $W^u(P)$ at $f^{-1}(\xi_1)$. We split

$$Df(f^{-1}(\xi_1))(1,\dot{y}_\xi) = \alpha_\xi e_\xi + \beta_\xi w_0\,.$$

As in (10), we must have $|\alpha_\xi| \le C\sqrt{b}$ and $|\beta_\xi| \approx 2ad_{\mathcal{C}}(f^{-1}(\xi_1)) \approx 2ae^{-r_0}$. By Proposition 2.3.2, $\|Df^{n_1}(\xi_1)e_\xi\| \le (Cb)^{n_1}$. Since $t(f^{n_1}(\xi_1))$ is collinear to $\alpha_\xi Df^{n_1}(\xi_1)e_\xi + \beta_\xi Df^{n_1}(\xi_1)w_0$,

$$\operatorname{angle}(Df^{n_1}(\xi_1)w_0, t(f^{n_1}(\xi_1))) \le \frac{|\alpha_\xi|\,\|Df^{n_1}(\xi_1)e_\xi\|}{|\beta_\xi|\,\|Df^{n_1}(\xi_1)w_0\|} \le C\sqrt{b}e^{r_0}(Cb)^{n_1} \le (Cb)^{n_1/2}.$$

The last inequality uses $n_1 = m_1 > r_0$. Moreover, by construction, $\operatorname{dist}(f^j(\eta), f^j(\zeta)) \le (Cb)^j$ for $0 \le j \le n_1$. So, using Proposition 2.3.5 we get

$$\operatorname{angle}(Df^{n_1}(z)w_0, Df^{n_1}(\xi_1)w_0) \le (Cb)^{n_1/2}.$$

Adding these two angle estimates, we find that $\text{angle}(Df^{n_1}(z)w_0, t(f^{n_1}(\xi_1)))$ is bounded by $(Cb)^{n_1/2}$. The case $k=1$ of the lemma follows, taking C_a larger than this last constant C.

Now we proceed by induction. We use the same notations as in the proof of the previous lemma, in particular, $\zeta = f^{n_{k-1}+1}(z)$ and $\eta = f^{n_{k-1}+1}(\xi_k)$. As before, $\|Df^j(\eta)w_0\| \geq 1$ for $1 \leq j \leq m_k$. Recall that the distance between iterates of ζ and η is exponentially contracted during the first m_k iterates. So, using Lemma 3.10.1,

$$\text{dist}(f^j(\eta), f^j(\zeta)) \leq (Cb)^{n_{k-1}/2} e^{r_{k-1}} (Cb)^j \quad \text{for all } 0 \leq j \leq m_k . \tag{60}$$

By Lipschitz continuity of Df, for every $1 \leq j \leq m_k$ we have

$$\|Df^j(\zeta)w_0 - Df^j(\eta)w_0\| \leq (Cb)^{n_{k-1}/2} e^{r_{k-1}}. \tag{61}$$

In view of the assumption $r_{k-1} \leq 5n_{k-1}$, the last term can be made small by reducing b. So, in particular, $\|Df^j(\zeta)w_0\| \geq 1/2$ for $1 \leq j \leq m_k$. Then the contracting directions of order m_k, $e_\eta = (\epsilon_\eta, 1)$ at η and $e_\zeta = (\epsilon_\zeta, 1)$ at ζ, are well defined and satisfy $|\epsilon_\eta|, |\epsilon_\zeta| \leq C\sqrt{b}$. Let $(1, \dot{y}_\eta)$ and $(1, \dot{y}_\zeta)$, with $|\dot{y}_\eta|, |\dot{y}_\zeta| \leq 1/5$, be collinear to $t(f^{-1}(\eta))$ and to $Df^{n_{k-1}}(z)w_0$, respectively. We split

$$Df(f^{-1}(\eta))(1, \dot{y}_\eta) = \alpha_\eta e_\eta + \beta_\eta w_0 \quad \text{and} \quad Df(f^{-1}(\zeta))(1, \dot{y}_\zeta) = \alpha_\zeta e_\zeta + \beta_\zeta w_0 .$$

Then $|\alpha_\eta|$ and $|\alpha_\zeta|$ are bounded by $C\sqrt{b}$, moreover, $|\beta_\eta| \approx 2ad_{\mathcal{C}}(f^{-1}(\eta)) \approx 2ae^{-r_{k-1}}$. The induction assumption means that $|\dot{y}_\eta - \dot{y}_\zeta| \leq 2(C_a b)^{n_{k-1}/4}$. By Lemma 3.10.1 and $r_{k-1} \leq 5n_{k-1}$,

$$\text{dist}(f^{-1}(\eta), f^{-1}(\zeta)) \leq (Cb)^{n_{k-1}/2} e^{r_{k-1}} \leq (Cb)^{n_{k-1}/4}. \tag{62}$$

We take C_a larger than this last constant C. It follows that

$$|\alpha_\eta - \alpha_\zeta| \leq C\sqrt{b}(C_a b)^{n_{k-1}/4} \quad \text{and} \quad |\beta_\eta - \beta_\zeta| \leq C(C_a b)^{n_{k-1}/4}. \tag{63}$$

Using the assumption $r_{k-1} \leq 5n_{k-1}$ once more, we conclude that these expressions are much smaller than $e^{-r_{k-1}}$. In particular, $|\beta_\zeta|$ is also of order $e^{-r_{k-1}}$. Expansivity implies

$$\|Df^{m_k}(\eta)e_\eta\| \leq (Cb)^{m_k} \quad \text{and} \quad \|Df^{m_k}(\zeta)e_\zeta\| \leq (Cb)^{m_k}, \tag{64}$$

whereas Proposition 2.3.4 and Lemma 3.10.1 give

$$\|Df^{m_k}(\eta)e_\eta - Df^{m_k}(\zeta)e_\zeta\| \leq (Cb)^{m_k} \,\text{dist}(\eta, \zeta) \leq (Cb)^{m_k} e^{r_{k-1}} (Cb)^{n_{k-1}/2}. \tag{65}$$

We take $C_a \geq C$ for any of the constants appearing in (61) - (65). Then, combining these estimates through the triangle inequality,

$$\left\| \frac{\alpha_\eta \, Df^{m_k}(\eta)e_\eta}{|\beta_\eta| \, \|Df^{m_k}(\eta)w_0\|} - \frac{\alpha_\zeta \, Df^{m_k}(\zeta)e_\zeta}{|\beta_\zeta| \, \|Df^{m_k}(\zeta)w_0\|} \right\| \leq C\sqrt{b}\,(C_a b)^{n_{k-1}/4} e^{2r_{k-1}} (C_a b)^{m_k}.$$

Then, as $t(f^{n_k}(\xi_k))$ is collinear to $\alpha_\eta Df^{m_k}(\eta)e_\eta + \beta_\eta Df^{m_k}(\eta)w_0$ and $Df^{n_k}(z)w_0$ is collinear to $\alpha_\zeta Df^{m_k}(\zeta)e_\zeta + \beta_\zeta Df^{m_k}(\zeta)w_0$,

$$\begin{aligned}
&\text{angle}(Df^{n_k}(z)w_0, t(f^{n_k}(\xi_k))) \\
&\qquad \leq 2\,\text{angle}(Df^{m_k}(\eta)w_0, Df^{m_k}(\zeta)w_0) + C\sqrt{b}\,(C_a b)^{n_{k-1}/4} e^{2r_{k-1}} (C_a b)^{m_k} \\
&\qquad \leq (Cb)^{n_{k-1}/4} e^{r_{k-1}/2} (Cb)^{m_k/2} + C\sqrt{b}\,(C_a b)^{n_{k-1}/4} e^{2r_{k-1}} (C_a b)^{m_k}.
\end{aligned}$$

For the first inequality, take $v_1 = \beta_\eta Df^{m_k}(\eta)w_0$, $\epsilon_1 = \alpha_\eta Df^{m_k}(\eta)e_\eta$, $v_2 = \beta_\zeta Df^{m_k}(\zeta)w_0$, and $\epsilon_2 = \alpha_\zeta Df^{m_k}(\zeta)e_\zeta$ in Remark 3.4. The second inequality follows from Proposition 2.3.5 combined with (60). Since $n_k = n_{k-1} + 1 + m_k$ and $m_k > r_{k-1}$, this gives

$$\text{angle}(Df^{n_k}(z)w_0, t(f^{n_k}(\xi_k))) \le (C_a b)^{n_k/4}.$$

(take C_a larger than the other constants in the previous inequality). So, the inductive step is complete. □

Remark 3.5. More generally, if $0 \le t \le k-1$ is such that $r_j \le 5(n_j - n_t)$ for all $t < j \le k-1$ then

$$\text{angle}(Df^{n_k - n_t - 1}(f^{n_t+1}(z))w_0, t(f^{n_k}(\xi_k))) \le (Cb)^{(n_k - n_t - 1)/4}.$$

This is proved in the same way as the lemma, starting the induction at $k = t+1$. Observe that the assumption was used to relate r_{k-1} and n_{k-1}, in the context of (61), (62), (63). In the present situation one relates r_{k-1} and $n_{k-1} - n_t$ in much the same way.

Remark 3.6. The hypotheses in Lemma 3.11 and Remark 3.5 are unnecessarily strong: the arguments remain valid if one assumes only $r_j \le |\log Cb|\, n_j$ for $1 \le j \le k-1$, respectively, $r_j \le |\log Cb|\,(n_j - n_t)$ for $t < j \le k-1$. On the other hand, the statements given above are sufficient for all our purposes in this paper (we never use these more general hypotheses).

It is time to complete the proof of Propositions 3.5 and 3.7. Part 3 follows immediately from Lemma 3.10.2: for $z = f^{-n_k}(z_{0,j}^*)$ we may take $\xi_k = f^{-n_k}(z_{1,j}^*)$, and then, by Remark 3.3,

$$\text{dist}(z_{0,j}^*, z_{1,j}^*) \le (Cb)^{n_k/2} \le \frac{1}{10}\, e^{-2\beta n_k} \le \frac{1}{10}\, d_{\mathcal{C}}(z_{i,j}^*).$$

Finally, part 4 of Propositions 3.5 and 3.7 can be readily deduced from Lemma 3.11, in the version given in Remark 3.5. Indeed, we may take $z = f^{-n_k}(\zeta)$ for an arbitrary point $\zeta \in f^{m_k}(\gamma_i^s)$, and then $\xi_k = f^{-n_k}(z_{i,j}^*)$, for $i = 0, 1$. The way we defined itineraries, there exists $0 \le t \le k-1$ such that $f^{n_t+1}(z)$ and $f^{n_t+1}(\xi_k)$ belong to the same long stable leaf $\Gamma_{r_t, l_t}(\tilde{\zeta}_t)$. By the construction of these leaves, recall Proposition 3.3 and Remark 3.3, $r_j \le 2\beta(n_j - n_t) < 5(n_j - n_t)$ for all $j \ge t+1$. So, we may conclude that

$$\text{angle}(Df^{n_k-(n_t+1)}(f^{n_t+1}(z))w_0, t(z_{i,j}^*)) = \text{angle}(Df^{n_k-(n_t+1)}(f^{n_t+1}(z))w_0, t(f^{n_k}(\xi_k)))$$

is bounded by $(Cb)^{(n_k - n_t - 1)/4}$. Adding the inequalities corresponding to $l = 0$ and $l = 1$, we get the claims in Propositions 3.5.4 and 3.7.4.

$$\text{angle}(t(z_{0,j}^*), t(z_1^*)) \le (Cb)^{(n_k - n_t - 1)/4} \le \frac{1}{20}\, e^{-2\beta(n_k - n_t + 1)} \le \frac{1}{10}\, d_{\mathcal{C}}(z_l^*). \tag{66}$$

This finishes the proof of Propositions 3.5.4 and 3.7.4.

We conclude this section with yet another useful application of these arguments.

Corollary 3.12. *Let $0 \le t < k$ and $z \in \Delta_0$ satisfy $r_i \le 5(n_j - n_t)$ for every $t < j < k$. Then $f^{n_t+1}(z)$ is expanding up to time $n_k - n_t - 1$: there is $\lambda \ge e^{-20}$ such that*

$$\|Df^i(f^{n_t+1}(z))w_0\| \ge \lambda^i \quad \text{for every } 1 \le i \le n_k - n_t - 1.$$

Moreover, $\left(f^{n_j}(z), Df^{n_j - n_t - 1}(f^{n_t+1}(z))w_0\right)$ is in tangential position relative to $\left(\tilde{\zeta}_j, t(\tilde{\zeta}_j)\right)$ for any return n_j with $t < j < i$.

Proof. We check that Remark 3.2 can be applied to the point $\zeta = f^{n_t}(z)$. Recall that, according to Lemma 3.10.2, the height of the rectangle $f^{n_j}(R(i_0, \ldots, i_j))$ does not exceed $(Cb)^{n_j/2}$, which is much smaller than $e^{-5n_j} \le e^{-5(n_j - n_t)}$. Therefore, the assumption of the corollary implies that $\operatorname{dist}(f^{n_j - n_t}(\zeta), \tilde{\zeta}_j) \ge e^{-5(n_j - n_t)}$ and $f^{n_j - n_t}(\zeta)$ is in tangential position relative to $(\tilde{\zeta}_j, t(\tilde{\zeta}_j))$, for every $j > t$. Using Remark 3.5 we get more: $(f^{n_j - n_t}(\zeta), Df^{n_j - n_t - 1}(f(\zeta))w_0)$ is in tangential position relative to $(\tilde{\zeta}_j, t(\tilde{\zeta}_j))$, also for every $j > t$. So the assumptions of Remark 3.2 are indeed satisfied, and we may conclude that $f(\zeta)$ is expanding. □

4 Tangential positions are statistically inevitable

Now the goal is to show that *for Lebesgue almost every point in the basin of attraction, returns are eventually tangential*. The following notion is motivated by Corollary 3.12. Given an itinerary $(i_0, i_1, \ldots, i_k, \ldots)$, with return times $n_0 < n_1 < \cdots < n_k < \cdots$, we define its *close returns* $\nu_0 < \nu_1 < \cdots < \nu_s < \cdots$ as follows. Take $\nu_0 = n_0 = -1$. Then, for each $s \ge 0$, let $\nu_{s+1} = n_{k(s+1)}$ where $k(s+1)$ is maximum such that

$$r_j \le 5(n_j - \nu_s) \quad \text{for all} \quad \nu_s < n_j < n_{k(s+1)}.$$

Observe that, according to Corollary 3.12, *non-close returns are always tangential,* moreover, *points remain expanding as long as they have no close returns.* The main result in the present section is Proposition 4.10: *itineraries with many close returns are improbable.* Sections 4.1 and 4.2 contain some crucial preparatory results. Throughout, it is understood that all the constants c and C are independent of $k \ge 1$ and $i_0, i_1, \ldots, i_{k-1}$.

4.1 Unstable sides are roughly parallel

The first step is to prove that the tangent directions to the unstable sides of each rectangle $f^{n_k}(R(i_0, \ldots, i_{k-1}))$ satisfy a Lipschitz condition, expressed in the next proposition.

Proposition 4.1. *For any $\tilde{z}_0$ and $\tilde{z}_1$ in different unstable sides of $f^{n_k}(R(i_0, \ldots, i_{k-1}))$,*

$$\operatorname{angle}(t(\tilde{z}_0), t(\tilde{z}_1)) \le Cb^{-1} e^{4(n_k - \nu_s)} \operatorname{dist}(\tilde{z}_0, \tilde{z}_1)$$

where ν_s is the last close return strictly before n_k.

The proof is by induction on s: assuming the conclusion of the proposition at time ν_s, we prove that it holds for all $\nu_s < n_k \le \nu_{s+1}$. This has two main parts. We begin by obtaining, in Lemma 4.2, an estimate for the angle at time $\nu_s + 1 = n_{k(s)} + 1$ in terms of $r_{k(s)}$ only (thus, independent of the history prior to ν_s). In a second stage, we deduce the proposition for $n_k > \nu_s$ from this estimate and Lemma 4.4, which contains the statement that the tangent vectors to the unstable sides of $f^j(R(i_0, \ldots, i_{k-1}))$ are expanded under $Df^{n_k - j}$, for all $\nu_s < j < n_k$. To keep track of the induction, during the proof we use C_p to mean the constant C in the statement of Proposition 4.1. For Lemma 4.2, we assume that δ is small depending on C_p; see (75). Later, see (82) in Lemma 4.5, we take C_p much larger than a few other constants, independent of δ. Clearly, such constraints are compatible as long as δ is small enough.

Let $x \mapsto (x, y_j(x))$, $j = 0, 1$, parametrise the unstable sides of $f^{\nu_s}(R(i_0, \ldots, i_{k(s)-1}))$, and $y \mapsto (x^s(y), y)$ parametrise the long stable leaf through $f(\tilde{\zeta}_{k(s)})$. As before, $\tilde{\zeta}_{k(s)}$ is the binding critical point associated to the return $\nu_s = n_{k(s)}$. We write

$$(\tilde{\xi}_j(x), \eta_j(x)) = f(x, y_j(x)) \quad \text{and} \quad \xi_j(x) = \tilde{\xi}_j(x) - x^s(\eta_j(x)). \tag{67}$$

That is, $\xi_j(x)$ is the (signed) horizontal distance from the point $(\tilde{\xi}_j(x), \eta_j(x))$ to the stable leaf through $f(\tilde{\zeta}_{k(s)})$. Compare (6).

In the next lemma we use $\operatorname{sl} t(z)$ to represent the slope of tangent vectors *in* (ξ, η) *coordinates*: $\operatorname{sl} t(\xi_j(x), \eta_j(x)) = \eta_j'(x)/\xi_j'(x)$. Note that we do not take absolute values.

Lemma 4.2. *For any z_0 and z_1 in the unstable boundary of $f^{\nu_s+1}(R(i_0, \ldots, i_{k(s)}))$,*

$$|\operatorname{sl} t(z_0) - \operatorname{sl} t(z_1)| \leq C b^{-1} e^{3 r_{k(s)}} \operatorname{dist}(z_0, z_1).$$

Proof. For the time being we suppose that z_0 and z_1 are in different unstable sides of the rectangle $f^{\nu_s+1}(R(i_0, \ldots, i_{k(s)}))$. There are two cases to be considered, cf. Figure 5.

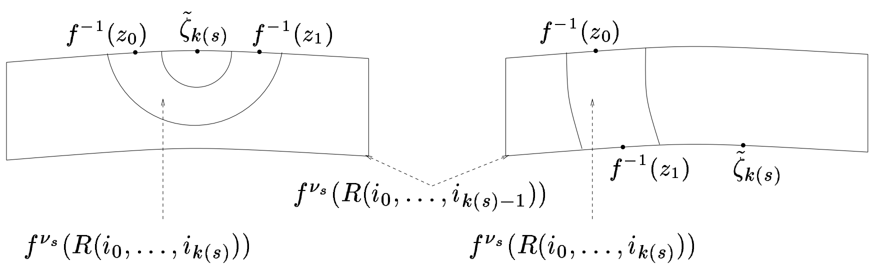

Figure 5:

First, we suppose that $f^{-1}(z_0)$ and $f^{-1}(z_1)$ belong to the same unstable side of the rectangle $f^{\nu_s}(R(i_0, \ldots, i_{k(s)-1}))$, cf. cases (a), (b), (a1), (b1) in Section 3.2. This corresponds to the left hand half of Figure 5. Up to interchanging the roles of y_0 and y_1, we may suppose that this is the unstable side parametrised by $(x, y_0(x))$. Then $\tilde{\zeta}_{k(s)} = (x_c, y_0(x_c))$ for some x_c. We also write $z_0 = f(x_0, y_0(x_0))$ and $z_1 = f(x_1, y_0(x_1))$, for some x_0 and x_1. By the definition of itineraries, $|\xi_0(x_j)| \approx e^{-2r_{k(s)}}$. Then, by the quadratic behaviour property (7),

$$(x_j - x_c)^2 \approx |\xi_0(x_j)| \approx e^{-2r_{k(s)}}, \quad \text{and so} \quad |\xi_0'(x_j)| \approx |x_j - x_c| \approx e^{-r_{k(s)}}$$

In the case we are dealing with, $f^{-1}(z_0)$ and $f^{-1}(z_0)$ are to opposite sides of the critical point $\tilde{\zeta}_{k(s)}$. So, $|x_0 - x_1| = |x_0 - x_c| + |x_1 - x_c| \geq c e^{-r_{k(s)}}$. Assumption (3) implies that $\|Df^{-1}\| \leq 4J/b$. It follows that

$$\operatorname{dist}(z_0, z_1) \geq cb(|x_0 - x_1| + |y_0(x_0) - y_1(x_1)|) \geq cb e^{-r_{k(s)}}.$$

This gives

$$|\operatorname{sl} t(z_0) - \operatorname{sl} t(z_1)| \leq \frac{|\eta_0'(x_0)|}{|\xi_0'(x_0)|} + \frac{|\eta_0'(x_1)|}{|\xi_0'(x_1)|} \leq C\sqrt{b}\, e^{r_{k(s)}} \leq C b^{-1/2}\, e^{2r_{k(s)}} \operatorname{dist}(z_0, z_1),$$

which is even stronger than the claim in the lemma.

Now we treat the situation when $f^{-1}(z_0)$ and $f^{-1}(z_1)$ are in different unstable sides of the rectangle $f^{\nu_s}(R(i_0,\ldots,i_{k(s)-1}))$, cf. (c), (d), (e), (c1) in Section 3.2. See the right hand half of Figure 5. Interchanging y_0 and y_1 if necessary, we may suppose that the binding critical point is $(x_c, y_1(x_c))$ for some x_c. Due to the form of the map f, and the fact that long stable leaves are almost vertical, we may write $\xi_j(x) = 1 - ax^2 + \psi(x, y_j(x))$, for $j = 0, 1$, and a function ψ that is $C\sqrt{b}$ close to a constant in the C^2 topology. Then,

$$\xi_0(x) = \xi_1(x) + \rho(x, (y_0 - y_1)(x)),$$

where $\rho(x,h) = \psi(x, y_1(x)+h) - \psi(x, y_1(x))$ has C^2 norm less than $C\sqrt{b}$. Let x_0, x_1 be defined by $z_0 = f(x_0, y_0(x_0))$ and $z_1 = f(x_1, y_1(x_1))$. We have

$$\begin{aligned}\xi_0'(x_0) - \xi_1'(x_1) = (\xi_1'(x_0) - \xi_1'(x_1)) &+ \partial_x \rho(x_0, (y_0 - y_1)(x_0)) \\ &+ \partial_y \rho(x_0, (y_0 - y_1)(x_0))(y_0 - y_1)'(x_0).\end{aligned}$$

The first term on the right is bounded by $4|x_0 - x_1|$. Since $\rho(x, 0) = \partial_x \rho(x, 0) = 0$ for all x, the second one is bounded by $C\sqrt{b}\,|(y_0 - y_1)(x_0)|$. As for the last term, we write it as

$$\left[\partial_y \rho(x_0, (y_0 - y_1)(x_0)) - \partial_y \rho(x_0, y_*)\right](y_0 - y_1)'(x_0) + \partial_y \rho(x_0, y_*)\,(y_0 - y_1)'(x_0) \tag{68}$$

where y_* is any point in the interval bounded by 0 and $(y_0 - y_1)(x)$, having the mean value property:

$$\partial_y \rho(x_0, y_*)(y_0 - y_1)(x_0) = \rho(x_0, (y_0 - y_1)(x_0)). \tag{69}$$

Since $|(y_0 - y_1)'| \le 1$, the first term in (68) is less than

$$C\sqrt{b}\,|(y_0 - y_1)(x_0) - y_*| \le C\sqrt{b}\,|(y_0 - y_1)(x_0)|.$$

Now we use induction: assuming that Proposition 4.1 holds for the points $(x_0, y_0(x_0))$ and $(x_0, y_1(x_0))$ at time $n_k = \nu_s$ ensures that

$$|(y_1 - y_0)'(x_0)| \le L_s |(y_1 - y_0)(x_0)|, \qquad L_s = C_p b^{-1} e^{4(\nu_s - \nu_{s-1})}.$$

Therefore, using (69),

$$|\partial_y \rho(x_0, y_*)(y_0 - y_1)'(x_0)| \le L_s |\rho(x_0, (y_0 - y_1)(x_0))| = L_s |(\xi_0 - \xi_1)(x_0)|\,.$$

At this point, putting the previous estimates together, we have shown that

$$|\xi_0'(x_0) - \xi_1'(x_1)| \le C|x_0 - x_1| + C\sqrt{b}\,|(y_0 - y_1)(x_0)| + L_s |(\xi_0 - \xi_1)(x_0)|\,.$$

Moreover, $|(y_0 - y_1)(x_0)| \le |y_0(x_0) - y_1(x_1)| + C|x_0 - x_1|$, and similarly for $(\xi_0 - \xi_1)$, since the derivatives of y_1 and ξ_1 are uniformly bounded. Hence, the previous inequality implies

$$|\xi_0'(x_0) - \xi_1'(x_1)| \le (C + L_s)|x_0 - x_1| + C\sqrt{b}\,|y_0(x_0) - y_1(x_1)| + L_s |\xi_0(x_0) - \xi_1(x_1)|\,. \tag{70}$$

Moreover, we may replace $C + L_s$ by CL_s (possibly with a larger C), since $L_s \ge 1$.

In a similar fashion, we write $\eta_0(x) = \eta_1(x) + \theta(x, (y_0 - y_1)(x))$, where

$$\theta(x, h) = R_2(x, y_1(x) + h) - R_2(x, y_1(x))$$

has C^2 norm less than $C\sqrt{b}$, and satisfies $\theta(x, 0) = \partial_x \theta(x, 0) = 0$ for every x. Then we conclude, as before, that

$$|\eta_0'(x_0) - \eta_1'(x_1)| \leq C L_s \sqrt{b}\, |x_0 - x_1| + C\sqrt{b}\, |y_0(x_0) - y_1(x_1)| + L_s |\eta_0(x_0) - \eta_1(x_1)| . \quad (71)$$

Observe that the first term comes with a better factor $C\sqrt{b}$ than the corresponding one in (70). This is because $|\eta_j'(x_0) - \eta_j'(x_1)| \leq C\sqrt{b}\, |x_0 - x_1|$ and $|\eta_j(x_0) - \eta_j(x_1)| \leq C\sqrt{b}\, |x_0 - x_1|$, which are better than the corresponding estimates for ξ_j' and ξ_j . Putting (70) and (71) together with $|\xi_j'| \leq C$ and $|\eta_j'| \leq C\sqrt{b}$, we find that $|\eta_0'(x_0)|\, |\xi_1'(x_1) - \xi_0'(x_0)| + |\xi_0'(x_0)|\, |\eta_0'(x_0) - \eta_1'(x_1)|$ is bounded by

$$C \left(\sqrt{b}\, |y_0(x_0) - y_1(x_1)| + \sqrt{b}\, L_s |x_0 - x_1| + L_s |\xi_0(x_0) - \xi_1(x_1)| + L_s |\eta_0(x_0) - \eta_1(x_1)| \right) . \quad (72)$$

It is clear that the two last terms are bounded by $C L_s \operatorname{dist}(z_0, z_1)$. We also want to bound the first two terms by some multiple of $\operatorname{dist}(z_0, z_1)$. For this we apply the mean value term to f^{-1}: (3) ensures that $\|Df^{-1}\| \leq C b^{-1}$, and so

$$|y_0(x_0) - y_1(x_1)| \leq C b^{-1} \operatorname{dist}(z_0, z_1).$$

There is a similar estimate for $|x_0 - x_1|$ but, in fact, we can do slightly better: since $|\partial_y R_1|, |\partial_y R_2|$ are less than $C\sqrt{b}$, the derivative of the first component of f^{-1} is bounded by $C b^{-1/2}$, and so the mean value theorem gives

$$|x_0 - x_1| \leq C b^{-1/2} \operatorname{dist}(z_0, z_1). \quad (73)$$

Replacing these remarks in (72) we obtain

$$|\eta_0'(x_0) \xi_1'(x_1) - \xi_0'(x_0) \eta_1'(x_1)| \leq C b^{-1/2} \operatorname{dist}(z_0, z_1) + C L_s \operatorname{dist}(z_0, z_1). \quad (74)$$

A key remark is that,

$$L_s \leq C_p b^{-1} e^{-(\nu_s - \nu_{s-1})} e^{r_{k(s)}} \leq b^{-1} e^{r_{k(s)}} . \quad (75)$$

because $r_{k(s)} > 5(\nu_s - \nu_{s-1})$ (since ν_s is a close return), and $\nu_s - \nu_{s-1} \geq p_{k(s-1)+1}$ can be made arbitrarily large by taking δ sufficiently small. Thus, (74) gives

$$|\eta_0'(x_0) \xi_1'(x_1) - \xi_0'(x_0) \eta_1'(x_1)| \leq C b^{-1} e^{r_{k(s)}} \operatorname{dist}(z_0, z_1), \quad (76)$$

where *the constant C does not depend on C_p*. Furthermore,

$$|\xi_1'(x_1)| \geq c |x_1 - x_c| \geq c e^{-r_{k(s)}} \quad \text{and} \quad |\xi_0'(x_0)| \geq c e^{-r_{k(s)}} . \quad (77)$$

The first claim follows easily from $\xi_1'(x_c) = 0$ and $\xi_1'' \approx -2a$. The second one is slightly trickier, because $\xi_0'(x_c)$ may not be zero: the critical point $\tilde{\zeta}_{k(s)} = (x_c, y_1(x_c))$ is in the unstable side parametrised by $(x, y_1(x))$ not $(x, y_0(x))$. To overcome this, we let $\bar{x}_c$ be the unique solution of $\xi_0'(\bar{x}_c) = 0$. Then $|\xi_0'(x_0)| \approx 2a|x_0 - \bar{x}_c|$, and

$$a|x_0 - \bar{x}_c|^2 \approx |\xi_0(\bar{x}_c) - \xi_0(x_0)| \geq |\xi_1(x_c) - \xi_0(x_0)| \geq e^{-2r_{k(s)}},$$

recall Proposition 3.3. This completes the proof of (77). Then, combining (76) with (77),

$$|\operatorname{sl} t(z_0) - \operatorname{sl} t(z_1)| = \left| \frac{\eta_0'(x_0)}{\xi_0'(x_0)} - \frac{\eta_1'(x_1)}{\xi_1'(x_1)} \right| \leq \frac{Cb^{-1} e^{r_{k(s)}} \operatorname{dist}(z_0, z_1)}{|\xi_0'(x_0)|\,|\xi_1'(x_1)|} \leq Cb^{-1} e^{3r_{k(s)}} \operatorname{dist}(z_0, z_1).$$

This proves the lemma in this case.

All that is left is the case when z_0, z_1 are in the same unstable side of $f^{\nu_s+1}(R(i_0, \dots, i_{k(s)}))$. Equivalently, $f^{-1}(z_0)$ and $f^{-1}(z_1)$ are in the same unstable side of $f^{\nu_s}(R(i_0, \dots, i_{k(s)}))$. Then there are x_0, x_1 so that $z_0 = f(x_0, y_j(x_0))$ and $z_1 = f(x_1, y_j(x_1))$, for either $j = 0$ or $j = 1$. The binding critical point may be written $\tilde{\zeta}_{k(s)} = (x_c, y_i(x_c))$. Here i may differ from j but, for the same reasons as in the previous paragraph, we always have $|\xi_j'(x_0)| \geq ce^{-r_{k(s)}}$ and $|\xi_j'(x_1)| \geq ce^{-r_{k(s)}}$. So,

$$|\operatorname{sl} t(z_0) - \operatorname{sl} t(z_1)| = \left| \frac{\eta_j'(x_0)}{\xi_j'(x_0)} - \frac{\eta_j'(x_1)}{\xi_j'(x_1)} \right| \leq C\sqrt{b}\, e^{2r_{k(s)}} |x_0 - x_1| \leq Ce^{2r_{k(s)}} \operatorname{dist}(z_0, z_1), \tag{78}$$

where the last inequality uses (73). This is a stronger fact than we claimed. The proof of Lemma 4.2 is now complete. □

Recall that we consider $\nu_s = n_{k(s)}$ to be the last close return before n_k. By Corollary 3.12, every $z \in f^{\nu_s+1}(R(i_0, \dots, i_k))$ is expanding up to time $\mu = n_k - \nu_s - 1$. Let $e_\mu(z)$ denote the contracting direction of order μ at z. Just as in the previous lemma, the constants C in the next corollary do not depend on C_p, and this is important for what follows.

Corollary 4.3. *There is a C^1 vector field $v_0 = (1, 0) + \phi\, e_\mu$ defined on $f^{\nu_s+1}(R(i_0, \dots, i_k))$ and tangent to the unstable sides of it, with $|\phi| \leq C\sqrt{b}e^{r_{k(s)}}$ and $\|D\phi\| \leq Cb^{-1}e^{3r_{k(s)}}$.*

Proof. First we define $\phi(z)$ for the points $z = (\xi_j(x), \eta_j(x))$ in each of the unstable sides of $f^{\nu_s+1}(R(i_0, \dots, i_k))$, by the condition that $(1, 0) + \phi(z)\, e_\mu(z)$ be collinear to the tangent direction $t(z)$. Writing

$$t(z) = \xi_j'(x)\partial_\xi + \eta_j'(x)\partial_\eta \quad \text{and} \quad e_\mu(z) = (\epsilon(z), 1) = \big[\epsilon(z) - (x^s)'(\eta_j(x))\big]\partial_\xi + \partial_\eta\,,$$

($\{\partial_\xi, \partial_\eta\}$ are the vector fields dual to the coordinates (ξ, η) in (67), that is $\xi = \tilde{\xi} - x^s(\tilde{\eta})$ and $\eta = \tilde{\eta}$, where $(\tilde{\xi}, \tilde{\eta})$ stand for the usual coordinates in the plane), this means that

$$\phi(z) = \frac{\eta_j'(x)/\xi_j'(x)}{1 - \big[\epsilon(z) - (x^s)'(\eta_j(x))\big]\big[\eta_j'(x)/\xi_j'(x)\big]}.$$

Write the image $f(\tilde{\zeta}_{k(s)})$ of the binding critical point as (ξ_c, η_c). Propositions 2.3.3 and 2.4 imply $|D(\epsilon - (x^s)')| \leq C\sqrt{b}$, and so

$$\big|\big[\epsilon(\tilde{\zeta}_{k(s)}) - (x^s)'(\eta_c)\big] - \big[\epsilon(z) - (x^s)'(\eta_j(x))\big]\big| \leq C\sqrt{b} \operatorname{dist}(z, \tilde{\zeta}_{k(s)}) \leq C\sqrt{b}\, e^{-r_{k(s)}}.$$

By Propositions 2.3.2 and 3.7.1, $|\epsilon(\tilde{\zeta}_{k(s)}) - (x^s)'(\eta_c)| \le (Cb)^\mu \ll C\sqrt{b}\, e^{-r_{k(s)}}$. So, using (77),

$$|\epsilon(z) - (x^s)'(\eta_j(x))|\, |\eta_j'(x)/\xi_j'(x)| \le C\sqrt{b}\, e^{-r_{k(s)}}\, C\sqrt{b}\, e^{r_{k(s)}} \ll 1.$$

It immediately follows that $|\phi(z)| \le 2|\operatorname{sl} t(z)| \le C\sqrt{b}\, e^{r_{k(s)}}$. It is clear that ϕ is C^1 on each unstable side. Moreover, its derivative is bounded by $Cb^{-1}e^{3r_{k(s)}}$, as a consequence of the Lipschitz estimate for the slopes $\eta_j'(x)/\xi_j'(x)$ provided by Lemma 4.2. This lemma also implies that ϕ is Lipschitz continuous on the union of the two unstable sides, with Lipschitz constant $Cb^{-1}e^{3r_{k(s)}}$. Therefore, it can be C^1 extended to the whole rectangle $f^{\nu_s+1}(R(i_0,\ldots,i_k))$, preserving the bounds on the function and the derivative. For instance, the extension may be taken affine on vertical line segments; beforehand, extend ϕ to curves slightly larger than the unstable sides, so that every relevant vertical segment intersects both curves. We still denote the extended function by ϕ, then we define $v_0 = (1,0) + \phi\, e_\mu$ on the whole rectangle. □

Next, we introduce the projectivization f_* of Df, given by $f_*(z,v) = Df(z)v/\|Df(z)v\|$, and define vector fields v_j on $f^{\nu_s+j+1}(R(i_0,\ldots,i_k))$, for $1 \le j \le \mu$, by push-forward under f_*:

$$v_j(\xi) = f_*(f^{-1}(\xi), v_{j-1}(f^{-1}(\xi))).$$

Of course, each v_j is tangent to the unstable sides of $f^{\nu_s+j+1}(R(i_0,\ldots,i_{k-1}))$.

Lemma 4.4. *Given $\zeta \in f^{n_k}(R(i_0,\ldots,i_{k-1}))$, let $\zeta_i = f^{-\mu+i}(\zeta)$ for each $0 \le i \le \mu$. Then,*

$$\|Df^i(\zeta_{\mu-i})v_{\mu-i}(\zeta_{\mu-i})\| \ge 1 \quad \textit{for all} \;\; 0 \le i \le \mu.$$

Proof. By Corollary 3.12, $(\zeta_{n_j-\nu_s-1}, Df^{n_j-\nu_s-1}(\zeta_0)w_0)$ is in tangential position relative to $(\tilde{\zeta}_j, t(\tilde{\zeta}_j))$, for every return n_j such that $\nu_s < n_j < n_k$. Moreover, $\|Df^i(\zeta_0)w_0\| \ge \lambda^i$ for every $0 \le i \le \mu$. On the other hand, $\|Df^i(\zeta_0)e_\mu\|$ is less than $(Cb)^i$, cf. Proposition 2.3.2. Since $v_0 = w_0 + \phi e_\mu$, it follows that

$$\begin{aligned}\text{angle}(v_{n_j-\nu_s-1}(\zeta_{n_j-\nu_s-1}),\ Df^{n_j-\nu_s-1}(\zeta_0)w_0) &\le (Cb/\lambda)^{n_j-\nu_s-1} \ll e^{5(n_j-\nu_s)}/2 \\ &\le e^{-r_j}/2 \le \operatorname{dist}(f^{n_j-\nu_s-1}(\zeta_0), \tilde{\zeta}_j)\end{aligned}$$

(because n_j is not a close return). Hence, $(\zeta_{n_j-\nu_s-1}, v_{n_j-\nu_s-1}(\zeta_{n_j-\nu_s-1}))$ is also in tangential position relative to $(\tilde{\zeta}_j, t(\tilde{\zeta}_j))$, for every return n_j between ν_s and n_k. In addition, as we have seen in Lemma 3.9, p_{j+1} is a suitable bound period for $f^{n_j-\nu_s-1}(\zeta_0)$. Therefore, the conclusions of Proposition 2.8 are valid in this context. In particular, part 2 of the proposition gives that

$$\|(Df^{p_{j+1}+1} \cdot v_{n_j-\nu_s-1})(\zeta_{n_j-\nu_s-1})\| \ge \sigma_1^{(p_{j+1}+1)/3}, \tag{79}$$

and the slope is less than $1/10$. Then the slope of $v_{n_j+p_{j+1}-\nu_s}(\zeta_{n_j+p_{j+1}-\nu_s})$ is also less than $1/10$ (the two vectors are collinear), and so Proposition 2.5 applies to it:

$$\|(Df^{n_{j+1}-(n_j+p_{j+1}+1)} \cdot v_{n_j+p_{j+1}-\nu_s})(\zeta_{n_j+p_{j+1}-\nu_s})\| \ge \sigma_1^{n_{j+1}-(n_j+p_{j+1}+1)}. \tag{80}$$

Now the proof of the lemma is similar to that of [2, Lemma 7.13]. The key observation is that any return (either free or bound) occurring in the time interval $[\mu - i, \mu)$ has its bound period contained in that interval, because μ is a free iterate for ζ_0. This means that whatever contraction takes place at the return is compensated for before time μ is reached, recall (79). More precisely, $[\mu - i, \mu)$ may be split as a union of subintervals during which the trajectory is outside $\{|x| \le \delta\}$, for which (80) applies, and of (complete) bound periods, where we have an analog of (79). We illustrate this with the case where $\mu - i$ is a free iterate for ζ_0, referring the reader to [2, Lemma 7.13] for the details in the general situation. Let $n_l > \mu - i$ be the first free return after time $\mu - i$. Then $n_{l-1} + p_l < \mu - i < n_l$ because we are assuming that $\mu - i$ is a free iterate. It follows that slope $v_{\mu-i}(\zeta_{\mu-i}) \approx$ slope $Df^{\mu-i}(\zeta_0)w_0$ is less than $1/10$, by Propositions 2.8.4 and 2.5. Therefore, we have an analog of (80) for $(Df^{n_l-\mu+i} \cdot v_{\mu-i})(\zeta_{\mu-i})$. Multiplying this by the product of (79) and (80) over all the n_j with $l \le j < k$, we find that $\|(Df^i \cdot v_{\mu-i})(\zeta_{\mu-i})\| \ge \sigma_1^{i/3} > 1$, as claimed in the lemma. □

Proposition 4.1 is an immediate consequence of the next lemma, with the same constants. The following elementary facts are used in the proof of the lemma. Let ξ be a generic point and v be any norm 1 tangent vector at ξ. Then $\partial_v f_*(\xi, v)\dot{v}$ coincides with the component of $Df(\xi)(\dot{v})/\|Df(\xi)v\|$ orthogonal to $Df(\xi)v$, for any vector $\dot{v}$ tangent to the v-direction at (ξ, v). In particular,

$$|\partial_v f_*(\xi, v)| \|Df(\xi)v\|^2 = |\det Df(\xi)|.$$

Similarly, $\partial_\xi f_*(\xi, v)\dot{\xi}$ coincides with the component of $D^2 f(\xi)(\dot{\xi}, v)/\|Df(\xi)v\|$ orthogonal to $Df(\xi)v$, for any tangent vector $\dot{\xi}$ tangent to the ξ-direction at (ξ, v). As a consequence, the norm of $\partial_\xi f_*(\xi, v) Df(f(\xi))^{-1}\dot{\eta}$ is bounded by

$$\|D^2 f(\xi)\left(Df(\xi)^{-1}\dot{\eta}, \frac{v}{\|Df(\xi)v\|}\right)\| \le \frac{\|D^2 f(\xi)\|}{|\det Df(\xi)|}\|\dot{\eta}\|$$

for any tangent vector $\dot{\eta}$ tangent to the ξ-direction at $(f(\xi), Df(\xi)v)$.

Lemma 4.5. $\|Dv_\mu(\zeta)\| \le Cb^{-1}e^{4(n_k - \nu_s)}$ *at every* $\zeta \in f^{n_k}(R(i_0, \dots, i_{k-1}))$.

Proof. Taking derivatives in the definition of v_μ yields

$$Dv_\mu(\zeta) = \sum_{j=0}^{\mu-1} \partial_v f_*^j(\zeta_{\mu-j}, v_{\mu-j}) \partial_\xi f_*(\zeta_{\mu-j-1}, v_{\mu-j-1}) Df^{-j-1}(\zeta) + \partial_v f_*^\mu(\zeta_0, v_0) Dv_0(\zeta_0) Df^{-\mu}(\zeta), \tag{81}$$

where $v_{\mu-j}$ means $v_{\mu-j}(\zeta_{\mu-j})$. For every $j \ge 1$,

$$\|\partial_v f_*^j(\zeta_{\mu-j}, v_{\mu-j})\| = \prod_{i=\mu-j}^{\mu-1} \frac{|\det Df(\zeta_i)|}{\|Df(\zeta_i)v_i\|^2} = \frac{|\det Df^j(\zeta_{\mu-j})|}{\|Df^j(\zeta_{\mu-j})v_{\mu-j}\|^2}.$$

On the other hand,

$$\|Df^{-j}(\zeta)\| \le \frac{\|Df^j(\zeta_{\mu-j})\|}{|\det Df^j(\zeta_{\mu-j})|}.$$

Moreover,

$$\|\partial_\xi f_*(\zeta_{\mu-j-1}, v_{\mu-j-1}) Df^{-1}(\zeta_{\mu-j})\| \le \frac{\|D^2 f(\zeta_{\mu-j-1})\|}{|\det Df(\zeta_{\mu-j-1})|} \le Cb^{-1}.$$

Replacing all this in (81),

$$\|Dv_\mu(\zeta)\| \le \sum_{j=0}^{\mu-1} Cb^{-1}\frac{\|Df^j(\zeta_{\mu-j})\|}{\|Df^j(\zeta_{\mu-j})v_{\mu-j}\|^2} + \frac{\|Df^\mu(\zeta_0)\|}{\|Df^\mu(\zeta_0)v_0\|^2}\|Dv_0(\zeta_0)\|.$$

So, in view of Corollary 4.3 and Lemma 4.4,

$$\|Dv_\mu(\zeta)\| \le \sum_{j=0}^{\mu-1} Cb^{-1}4^j + 4^\mu Cb^{-1}e^{3r_{k(s)}} \le C_p b^{-1}e^{4\mu}, \tag{82}$$

as long as we choose C_p sufficiently large with respect to the other constants. This can be done since the constants C in Lemma 4.2 and Corollary 4.3 were taken independent of C_p, recall (76). The last inequality also uses $\mu > p_{k(s)} \ge (4/3)\, r_{k(s)}$, which is contained in Proposition 3.7.1. $\square$

4.2 Area distortion bounds

Next, we obtain a uniform bound for the distortion of the Jacobian on trajectories sharing the same itinerary:

Proposition 4.6. *Given any $k \ge 1$ and $i_0, \ldots, i_{k-1}$,*

$$\frac{|\det Df^{n-l}(f^l(z))|}{|\det Df^{n-l}(f^l(w))|} \le C.$$

for every $z, w \in R(i_0, \ldots, i_{k-1})$ and every $0 \le l < n \le n_k$.

Let us observe that this statement is trivial when the map f has constant Jacobian, e.g. the Hénon model. In the general case, the proof is based on the following two lemmas.

Lemma 4.7. *Given any $k \ge 1$ and $i_0, \ldots, i_{k-1}$, we have*

$$\sum_{j=0}^{n_k-1} \text{length}(f^j(\gamma^u)) \le C$$

for either of the unstable sides γ^u of $R(i_0, \ldots, i_{k-1})$.

Proof. The statement follows from ideas from [2, Lemma 7.8]. We only outline the main points, referring the reader to [2] for details. Let $\ell_i = \text{length}(f^{n_i}(\gamma^u))$, for $0 \le i \le k$.

The first step is to show that the sum over any free period $[n_{i-1} + p_i + 1, n_i - 1]$ is bounded by $C\ell_i$.

$$\sum_{j=n_{i-1}+p_i+1}^{n_i-1} \text{length}(f^j(\gamma^u)) \le C\ell_i\,.$$

To prove this, one notes that Proposition 2.5 implies that the lengths grow exponentially fast during free periods: $\text{length}(f^j(\gamma^u)) \le C\sigma_0^{j-n_i}\,\text{length}(f^{n_i}(\gamma^u))$ with $\sigma_0 > 1$. Consequently, the sum is bounded by a multiple of length$(f^{n_i}(\gamma^u))$. Next, one shows that the sum over a bound period $[n_i+1, n_i+p_{i+1}]$ is bounded by $C\ell_i/d_i$:

$$\sum_{j=n_i+1}^{n_i+p_{i+1}} \operatorname{length}(f^j(\gamma^u)) \leq C\ell_i e^{r_i}.$$

In brief terms, the ratio between length$(f^j(\gamma^u))$ and the dist$(f^j(\gamma^u), f^j(\tilde{\zeta}_i))$ is essentially preserved during the bound period. On the other hand, this distance is bounded by a geometrically decreasing sequence $Ce^{-\beta j}$, by the definition of binding periods. So, the sum of the lengths over all $n_i < j \leq n_i + p_{i+1}$ is less than

$$C\,\frac{\operatorname{length}(f^{n_i}(\gamma^u))}{d_{\mathcal{C}}(f^{n_i}(\gamma^u))} \leq C\ell_i e^{r_i}.$$

Clearly, $\ell_i \leq \ell_i e^{r_i}$. Therefore, these two estimates imply that the sum over the whole time interval $[0, n_k - 1]$ is bounded by $C\sum_{i=0}^{k} \ell_i e^{r_i}$. Now, Propositions 2.5 and 2.8.2 imply that lengths get expanded between any consecutive free returns: $\ell_i \geq \sigma_0 \ell_{i-1}$ with $\sigma_0 > 1$. It follows that, for each fixed r, the sequence of all the $\ell_i e^{r_i}$ with $r_i \equiv r$ is geometrically increasing. Therefore, the sum of $\ell_i e^{r_i}$ over the corresponding values of i is bounded by a multiple of the last term:

$$C\sum_{i=0}^{k} \ell_i e^{r_i} = C\sum_{r>0}\sum_{i:r_i=r} \ell_i e^{r_i} \leq C\sum_{r>0} \ell_{i(r)} e^{r}$$

where, by definition, $i(r)$ is the largest value of i for which $r_i = r$. By construction, $\ell_i e^{r_i}$ is less than Cr_i^{-2} for every i: this is because we defined itineraries in terms of the extended family of long leaves introduced after Remark 3.3 (incidentally, this is the only place in the proof where that is used). So,

$$C\sum_{r>0} \ell_{i(r)} e^{r} \leq \sum_{r>0} Cr^{-2} \leq C.$$

This gives the lemma. □

Lemma 4.8. *Given any $k \geq 1$ and $i_0, \ldots, i_{k-1}$,*

$$\sum_{j=0}^{n_k-1} \operatorname{dist}(f^j(z), f^j(z^u)) \leq C$$

for every $z \in R(i_0, \ldots, i_{k-1})$ and every z^u in the unstable boundary of $R(i_0, \ldots, i_{k-1})$.

Proof. Let σ^u be any of the unstable sides of $R(i_0, \ldots, i_{k-1})$. Suppose first that there are no close returns before time n_k, other than $\nu_0 = -1$. Then, cf. Corollary 3.12, every point $z \in R(i_0, \ldots, i_{k-1})$ is expanding up to time n_k. So, the temporary stable leaf Γ^{n_k} of order n_k through z is long and nearly vertical. If Γ^{n_k} intersects σ^u, let ξ be the intersection point. Otherwise, it must intersect some stable side σ^s of $R(i_0, \ldots, i_{k-1})$, let ξ be the point where σ^s and σ^u meet. Since Γ^{n_k} and σ^s are exponentially contracted during at least n_k iterates,

$$\sum_{j=0}^{n_k-1} \operatorname{dist}(f^j(z), f^j(\xi)) \leq \sum_{j=0}^{n_k-1} (Cb)^j \leq C.$$

Finally, as a consequence of Lemma 4.7, this conclusion remains true if one replaces ξ by any other point in σ^u.

In general, let $\nu_0 < \nu_1 < \cdots < \nu_s < n_k$ be the close returns prior to n_k. We allow ourselves a slight abuse of language: take ν_{s+1} to mean n_k, and $k(s+1)$ to mean k, wherever they occur in this proof. By Corollary 3.12, $f^{\nu_l+1}(z)$ is expanding up to time $\mu_l = \nu_{l+1} - \nu_l - 1$ for any $0 \le l \le s$. Let Γ^{μ_l} be the corresponding temporary stable leaf through $f^{\nu_l+1}(z)$, and let λ^u denote the unstable side of $R(i_0, \ldots, i_{k(l+1)})$ that contains σ^u. If Γ^{μ_l} intersects $f^{\nu_l+1}(\lambda^u)$, let η_l be the point of intersection. Otherwise, Γ^{μ_l} intersects some stable side $f^{\nu_l+1}(\lambda^s)$ of $f^{\nu_l+1}(R(i_0, \ldots, i_{k(l+1)}))$, and η_l is the endpoint point shared by $f^{\nu_l+1}(\lambda^u)$ and $f^{\nu_l+1}(\lambda^s)$. In both cases, cf. Lemma 3.10.2,

$$\begin{aligned}\operatorname{dist}(f^j(z), f^{j-\nu_l-1}(\eta_l)) &\le (Cb)^{j-\nu_l-1}\left(\operatorname{dist}(f^{\nu_l+1}(z), \eta_l) + \operatorname{length}(f^{\nu_l+1}(\lambda^s))\right)\\ &\le (Cb)^{j-\nu_l-1} 2 (Cb)^{(\nu_l+1)/2}\end{aligned}$$

for all $\nu_l + 1 \le j \le \nu_{l+1}$. So,

$$\sum_{j=\nu_l+1}^{\nu_{l+1}} \operatorname{dist}(f^j(z), f^{j-\nu_l-1}(\eta_l)) \le (Cb)^{(\nu_l+1)/2}. \tag{83}$$

Now, by the same argument as in Lemma 4.7,

$$\sum_{j=\nu_l+1}^{\nu_{l+1}} \operatorname{length}(f^j(\lambda^u)) \le C \sum_{q=k(l)}^{k(l+1)} \operatorname{length}(f^{n_q}(\lambda^u))\, e^{r_q}.$$

The term corresponding to $q = k(l)$ is bounded by $Cr_{k(l)}^{-2}$. We claim that the sum over all $k(l) < q \le k(l+1)$ is bounded by

$$C \operatorname{length}(f^{n_{k(l+1)}}(\lambda^u))\, e^{r_{k(l+1)}} \le Cr_{k(l+1)}^{-2}.$$

That is a consequence of the following two observations. On the one hand, as ν_l and ν_{l+1} are consecutive close returns,

$$r_{k(l+1)} > 5(\nu_{l+1} - \nu_l) \ge 5(\nu_{l+1} - n_q) \ge 5p_{q+1} \ge 5r_q \quad \text{for } k(l) \le q < k(l+1). \tag{84}$$

The last inequality is from Proposition 3.7.1. On the other hand, by Propositions 2.5 and 2.8.2,

$$\operatorname{length}(f^{n_q}(\lambda^u)) \ge \sigma_1^{(n_q - n_{q-1})/3} \operatorname{length}(f^{n_{q-1}}(\lambda^u)) \quad \text{for } k(l) < q \le k(l+1).$$

Our claim follows by a geometric series argument. Then, given any $z^u \in \sigma^u$,

$$\sum_{j=\nu_l+1}^{\nu_{l+1}} \operatorname{dist}(f^j(z^u), f^{j-\nu_l-1}(\eta_l)) \le \sum_{j=\nu_l+1}^{\nu_{l+1}} \operatorname{length}(f^j(\lambda^u)) \le Cr_{k(l)}^{-2} + Cr_{k(l+1)}^{-2}. \tag{85}$$

Putting (83) and (85) together,

$$\sum_{j=0}^{n_k-1} \operatorname{dist}(f^j(z^u), f^{j-\nu_l-1}(\eta_l)) \le \sum_{l=0}^{s} (Cb)^{(\nu_l+1)/2} + \sum_{l=0}^{s+1} Cr_{k(l)}^{-2}.$$

It is clear that the first term on the right is bounded by some uniform constant $C > 0$. To show that the same is true about the second one, we just observe that $r_{k(l)} > 5r_{k(l)}$, by (84), so that $r_{k(l)} > 5^l$ for every $0 \le l \le s$. □

Proposition 4.6 is an easy consequence of Lemma 4.8. Indeed, from the lemma we have

$$\sum_{j=0}^{n_k-1} \operatorname{dist}(f^j(z), f^j(w)) \le C,$$

for any pair of points z, w in any rectangle $R(i_0, \dots, i_{k-1})$. Then, since J is a bound for the derivative of $\log|\det|Df||$,

$$\log \frac{|\det Df^{n-l}(f^l(z))|}{|\det Df^{n-l}(f^l(w))|} \le J \sum_{j=l}^{n-1} \operatorname{dist}(f^j(z), f^j(w)) \le JC.$$

This gives the proposition.

4.3 Close returns are exponentially improbable

Given $m \ge 1$ and $i_0, \dots, i_{m-1}$, let $S(i_0, \dots, i_{m-1})$ be the set of points $z \in R(i_0, \dots, i_{m-1})$ for which n_m is a close return. Leb denotes the two-dimensional Lebesgue measure (area). The following observation will be useful in the proof of Lemma 4.9.

Remark 4.1. Typically, the stable sides of the rectangles $f^{n_k+1}(R(i_0, \dots, i_{k-1}, i_k))$ are contained in long stable leaves $\Gamma_{r,l}(\tilde\zeta_k)$ associated to the binding point $\tilde\zeta_k$. This may fail to happen only if $R(i_0, \dots, i_{k-1}, i_k)$ is at some of the tips of $R(i_0, \dots, i_{k-1})$, in which case the two rectangles share a stable side (or both, even more exceptionally). See Figures 1 and 2. In any case, by induction on k, one may always find for each stable side γ_i^s a return $n_t \le n_k$ such that γ_i^s is contained in $f^{n_k-n_t}(\Gamma_{r,l}(\tilde\zeta_t))$ for some (r, l). Moreover, denoting $\nu_s = n_{k(s)}$ the last close return before n_k, we must have $n_t \ge \nu_s$. Indeed, suppose there was $\hat s \le s$ such that $\nu_{\hat s-1} \le n_t < \nu_{\hat s}$. According to Remark 3.3, the distance from $f^{\nu_{\hat s}-n_t}(\Gamma_{r,l}(\tilde\zeta_t))$ to the binding point $\tilde\zeta_{k(\hat s)}$ is larger than $e^{-2\beta(\nu_{\hat s}-n_t)}$, and so $r_{k(\hat s)} \le 2\beta(\nu_{\hat s} - n_t)$. This contradicts the assumption that $\nu_{\hat s}$ is a close return: $r_{k(\hat s)} \ge 5(\nu_{\hat s} - \nu_{\hat s-1}) > 2\beta(\nu_{\hat s} - n_t)$. Thus, $n_t \ge \nu_s$ as we claimed.

Lemma 4.9. *There exist $\theta = \theta(b) > 0$ such that, for any $m \ge 1$ and $i_0, \dots, i_{m-1}$,*

$$\frac{\operatorname{Leb}\big(S(i_0, \dots, i_{m-1})\big)}{\operatorname{Leb}\big(R(i_0, \dots, i_{m-1})\big)} \le \min\Big\{\frac{1}{\theta} e^{-(n_m-\nu_s)}, 1-\theta\Big\},$$

where ν_s is the last close return before n_m.

Proof. By Remark 4.1, each of the stable sides γ_0^s and γ_1^s of $f^{n_m}(R(i_0, \dots, i_{m-1}))$ is contained in some $f^{n_m-n_t-1}(\Gamma_{r,l}(\tilde\zeta_t))$, where $\tilde\zeta_t$ is the binding point corresponding to a return n_t with $\nu_s \le n_t \le n_{m-1}$. In particular, cf. Remark 3.3,

$$\operatorname{dist}(\gamma_i^s, \tilde\zeta_m) \ge e^{-2\beta(n_m-n_t-1)} \gg e^{-4(n_m-\nu_s)}. \tag{86}$$

Now there are two situations to consider, corresponding to the two possibilities in Lemma 3.8.

If the two unstable sides are in tangential position to the same critical point η_0, then γ_0^s and γ_1^s are both to the left or both to the right of η_0; see the right hand side of Figure 2. Then (86) implies that $\operatorname{dist}(\xi, \tilde{\zeta}_m)$ is much larger than $e^{-4(n_m-\nu_s)}$ for any point $\xi \in f^{n_m}(R(i_0, \dots, i_{m-1}))$. Consequently, $r_m < 4(n_m - \nu_s)$, and so n_m is *not* a close return, for any point in the rectangle. In other words, $S(i_0, \dots, i_{m-1})$ is empty, and so the lemma is trivial in this case. In what remains of the proof we treat the case when each unstable side of $f^{n_m}(R(i_0, \dots, i_{m-1}))$ contains a critical point, see the left hand side of Figure 2.

Let $\tilde{\zeta}_m = (x_0, y_0)$ and $v = v_{n_m-\nu_s-1}$ be a C^1 vector field on $f^{n_m}(R(i_0, \dots, i_{m-1}))$ as constructed in Section 4.1. The vertical line $\{x = x_0\}$ crosses the rectangle, in the sense that it intersects both unstable sides. That is because the stable sides are much shorter than their distance to η_0, by Lemma 3.10.2 and (86). Moreover, the integral curves of v are nearly horizontal, e.g., as a consequence of Proposition 3.7.2 and Lemma 4.5. So $\{x = x_0\}$ intersects every integral curve of v. We introduce coordinates $\varphi(t, y) = v^t(x_0, y)$, where $(v^t)_{t\in\mathbb{R}}$ denotes the flow of v, and we write $z(t, y) = f^{-n_m}(\varphi(t, y))$. By (86), this is well defined (at least) for every $|t| \le e^{-4(n_m-\nu_s)}$. It follows that, for each fixed y

$$\{t : r_m(z(t, y)) \ge 5(n_m - \nu_s)\} \text{ is contained in } \{t : |t| \le Ce^{-5(n_m-\nu_s)}\} \tag{87}$$

for some sufficiently large C. This implies that

$$\operatorname{Leb}\{(t, y) : n_m \text{ is a close return for } z(t, y)\} \le Ce^{-(n_m-\nu_s)} \operatorname{Leb}\{(t, y) : |t| \le e^{-4(n_m-\nu_s)}\}. \tag{88}$$

It is easy to see that $|\det D\varphi(0, y)|$ is uniformly bounded away from zero and infinity. Then Lemma 4.5 implies (Liouville's formula, see e.g. [14, Section I.3]), that there exists $C > 0$ such that

$$\exp(-Cb^{-1}) \le |\det D\varphi(t, y)| \le \exp(Cb^{-1}) \quad \text{whenever } |t| \le e^{-4(n_m-\nu_s)}.$$

So, the previous inequality implies that $\{\varphi(t, y) : n_m \text{ is a close return for } z(t, y)\}$ has Lebesgue measure bounded by

$$\begin{aligned} C \exp(2Cb^{-1})\, e^{-(n_m-\nu_s)} &\operatorname{Leb}\{\varphi(t, y) : |t| \le e^{-4(n_m-\nu_s)}\} \\ &\le C \exp(2Cb^{-1})\, e^{-(n_m-\nu_s)} \operatorname{Leb}(f^{n_m}(R(i_0, \dots, i_{m-1}))). \end{aligned}$$

Now, using the distortion bound in Proposition 4.6, for $k = m$, $n = n_m$, $l = 0$, we conclude that

$$\operatorname{Leb}\big(S(i_0, \dots, i_{m-1})\big) \le C \exp(2Cb^{-1})\, e^{-(n_m-\nu_s)} \operatorname{Leb}(R(i_0, \dots, i_{m-1}))$$

(possibly for a larger constant C). This gives the first estimate in the statement of the lemma, with $1/\theta = C \exp(2Cb^{-1})$.

The second estimate is obtained along similar lines. Firstly, the factor $Ce^{-(n_m-\nu_s)}$ in (88) can be made less than $1/2$ by reducing δ. Thus, taking complements in (88),

$$\operatorname{Leb}\{(t, y) : |t| \le e^{-4(n_m-\nu_s)} \text{ and } n_m \text{ is not a close return for } z(t, y)\}$$

is at least half of $\operatorname{Leb}\{(t, y) : |t| \le e^{-4(n_m-\nu_s)}\}$. So, arguing as before, the Lebesgue measure of $\{\varphi(t, y) : |t| \le e^{-4(n_m-\nu_s)}$ and n_m is not a close return for $z(t, y)\}$ is larger than

$$\frac{1}{2} \exp(-2Cb^{-1}) \operatorname{Leb}\{\varphi(t, y) : |t| \le e^{-4(n_m-\nu_s)}\}.$$

Now, (87) also implies that n_m is never a close return if $|t| > e^{-4(n_m - \nu_s)}$. Therefore,

$$\mathrm{Leb}\{\varphi(t,y):\ n_m \text{ is not close return for } z(t,y)\} \geq \frac{1}{2}\exp(-2Cb^{-1})\,\mathrm{Leb}(f^{n_m}(R(i_0,\dots,i_{m-1}))).$$

The set on the left hand side is precisely $f^{n_m}\big(R(i_0,\dots,i_{m-1}) \setminus S(i_0,\dots,i_{m-1})\big)$. So, using Proposition 4.6,

$$\mathrm{Leb}\,\big(R(i_0,\dots,i_{m-1}) \setminus S(i_0,\dots,i_{m-1})\big) \geq c\exp(-2Cb^{-1})\,\mathrm{Leb}(R(i_0,\dots,i_{m-1})),$$

for some $c > 0$. Equivalently,

$$\mathrm{Leb}\,\big(S(i_0,\dots,i_{m-1})\big) \leq \big(1 - c\exp(-2Cb^{-1})\big)\,\mathrm{Leb}(R(i_0,\dots,i_{m-1})).$$

This gives the second bound in the statement of the lemma, with $\theta = c\exp(-2Cb^{-1})$ (which is compatible with the expression for θ we had found before). □

Given any $k \geq 1$ and $i_0,\dots,i_{k-1}$, let $H(i_0,\dots,i_{k-1})$ be the subset of $z \in R(i_0,\dots,i_{k-1})$ for which no return n_j with $j \geq k$ is a close return. According to the next proposition, this occupies a definite fraction of the rectangle $R(i_0,\dots,i_{k-1})$, in terms of Lebesgue measure Leb.

Proposition 4.10. *There is $\theta_0 = \theta_0(b) > 0$ such that*

$$\mathrm{Leb}\,\big(H(i_0,\dots,i_{k-1})\big) \geq \theta_0\,\mathrm{Leb}\,\big(R(i_0,\dots,i_{k-1})\big)$$

for every $i_0,\dots,i_{k-1}$ and $k \geq 1$.

Proof. Let ν_s be the last close return with $\nu_s \leq n_{k-1}$. By the previous lemma, the total Lebesgue measure of the sub-rectangles $R(i_0,\dots,i_{k-1},i_k)$ for which n_k is not a close return is larger than

$$\max\big\{\theta, 1 - \theta^{-1}e^{-(n_k - \nu_s)}\big\}\,\mathrm{Leb}(R(i_0,\dots,i_{k-1})).$$

In general, given $l \geq 1$ and $i_0,\dots,i_{k+l-1}$ such that neither of $n_k,\dots,n_{k+l-1}$ is a close return, the Lebesgue measure of the union of all the sub-rectangles $R(i_0,\dots,i_{k+l-1},i_{k+l})$ for which n_{k+l} is also not a close return is at least

$$\max\big\{\theta, 1 - \theta^{-1}e^{-(n_{k+l} - \nu_s)}\big\}\,\mathrm{Leb}\,\big(R(i_0,\dots,i_{k+l-1})\big).$$

Noting that $n_{k+l} - \nu_s \geq l$, we conclude that

$$\frac{\mathrm{Leb}\,\big(H(i_0,\dots,i_{k-1})\big)}{\mathrm{Leb}\,\big(R(i_0,\dots,i_{k-1})\big)} \geq \prod_{l=0}^{\infty}\max\big\{\theta, 1 - \theta^{-1}e^{-l}\big\} \geq \theta^q \prod_{l=q}^{\infty}(1 - \theta^{-1}e^{-l}),$$

for any $q > |\log\theta|$ (this is to ensure that $1 - \theta^{-1}e^{-l}$ is positive for every $l \geq q$). We fix such a q, for instance, $q =$ integer part of $|\log\theta| + 1$. Then it suffices to take θ_0 equal to the term on the right hand side of the last inequality. □

5 Filling the holes in

Finally, we tie the previous results together to prove Theorems A and B. First we show that Lebesgue almost every point in the region Δ_0 has some positive iterate contained in a long stable leaf of some point of the attractor.

Let i_0 be fixed. By Proposition 4.10, the set $H(i_0)$ of points $z \in R(i_0)$ without close returns corresponds to a positive Lebesgue measure fraction of $R(i_0)$. By Corollary 3.12 the points in $H(i_0)$ are expanding. Clearly, the long stable leaves through these points intersect the unstable manifold $W^u(P) \subset \Lambda$. Now, by construction, the complement $R(i_0) \setminus H(i_0)$ can be written as a union of rectangles $R(i_0, \ldots, i_{l(1)})$, with variable $l(1)$, which we call *first order gaps*: $n_{l(1)}$ is the first close return. Again by Proposition 4.10, a positive fraction of each first order gap is filled-in by a set $H(i_0, \ldots, i_{l(1)})$ whose points z have no other close return. So, $f^{n_{l(1)}+1}(z)$ has a long stable leaf which, moreover, intersects the attractor Λ. The complement $R(i_0, \ldots, i_{l(1)}) \setminus H(i_0, \ldots, i_{l(1)})$ is given by a union of rectangles $R(i_0, \ldots, i_{l(1)}, \ldots, i_{l(2)})$, the *second order gaps*. Now it is clear how to proceed with the argument: Proposition 4.10 tells us that a definite fraction of each mth order gap $R(i_0, \ldots, i_{l(m)})$, $m \geq 1$, is filled-in by a subset $H(i_0, \ldots, i_{l(m)})$ whose points are in the $f^{n_{l(m)}}$-preimage of a long stable leaf through a point $\xi \in \Lambda$. And $R(i_0, \ldots i_{l(m)}) \setminus H(i_0, \ldots i_{l(m)})$ is a union of rectangles, that are the gaps of order $m+1$. In this way we conclude that

$$H = \bigcup_{m, l(1), \cdots, l(m), i_0, \cdots, i_{l(m)}} H(i_0, \ldots, i_{l(m)})$$

is a full Lebesgue measure subset of Δ_0, contained in the union $\bigcup_{\xi \in \Lambda} W^s(\xi)$ of the stable sets of points in Λ.

To complete the proof of Theorem A, we show that for almost all points $w \in B(\Lambda)$ there exists $n \geq 0$ such that $z = f^n(w)$ is in the region Δ_0, as claimed in Section 3. First of all, if $0 < b \ll \delta_* \ll 1$ then the set H^+ of points whose forward orbits remain in $[-2,2]^2$ but do not hit $R_* = [-\delta_*, \delta_*] \times [-2,2]$ has zero Lebesgue measure. This can be proved along well-known lines. One constructs invariant stable and unstable cone fields for f in $[-2,2]^2 \setminus R_*$. It follows that the set H of points whose full orbits are contained in $[-2,2]^2 \setminus R_*$ is uniformly hyperbolic for f. Since $H^+ \subset W^s(H)$ and the stable set $W^s(H)$ has zero Lebesgue measure, cf. [7, Theorem 4.11], the claim follows. Then we may restrict ourselves to points having some positive iterate in $R_* = [-\delta_*, \delta_*] \times [-2,2]$. For the sequel it is convenient to distinguish two cases, depending on whether the map f is orientation preserving (both eigenvalues of $Df(P)$ are negative) or orientation reversing (the contracting eigenvalue is positive, the expanding one is negative).

The latter case is somewhat better known, see e.g. [2, Section 4], [3], [15, Section 4]. For a compact region R as in Figure 6, bounded by a segment of $G_0 \cup G_1 \cup G_2$ and a segment in the long stable leaf of $f^3(\zeta_0)$, one can prove that Λ is contained in R, and some fundamental domain of $f \mid W^u(P)$ is contained in the interior of R. Moreover, R is contained in the basin $B(\Lambda)$, and it is forward invariant: $f(R) \subset R$. It follows that the union $W^s(P) \cup \left(\bigcup_{n \geq 0} f^{-n}(R)\right)$ contains a neighbourhood of the attractor Λ, and is equal to the basin of attraction $B(\Lambda)$. Another consequence is that the forward orbit of any point in $B(\Lambda)$ must eventually enter in R. Consider the rectangle $R_0 \subset R$ bounded by G_0, G_1, and the preimage under f of the long stable leaf $\Gamma^s(P)$ passing through P. Recall that G_i contains long nearly horizontal curves

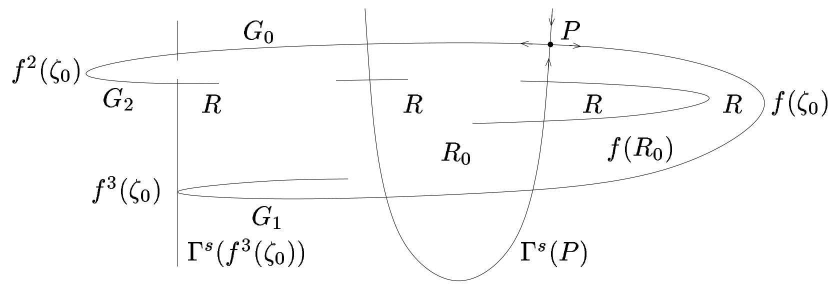

Figure 6:

around ζ_i, for $i = 0, 1$. Then, in view of the form of the map, each $f(G_i)$ must intersect $\Gamma^s(P)$ at exactly two points. Therefore $f^{-1}(\Gamma^s(P))$ intersects G_i at two points, and so R_0 is indeed well-defined. According to the previous paragraph, Lebesgue almost every point $w \in B(\Lambda)$ has some positive iterate $f^k(w)$ in R_0 (take δ_* small). Now, $f(R_0)$ is the rectangle bounded by $f(G_i)$, $i = 0, 1$, and $\Gamma^s(P)$, which is clearly contained in the domain Δ_0 defined in Section 3.2. Then, by the arguments presented so far, $z = f^{k+1}(w)$ belongs Lebesgue almost surely in the stable manifold $W^s(\xi)$ of some point $\xi \in \Lambda$. Therefore, the same is true for w, and so the proof of the theorem is complete in this case.

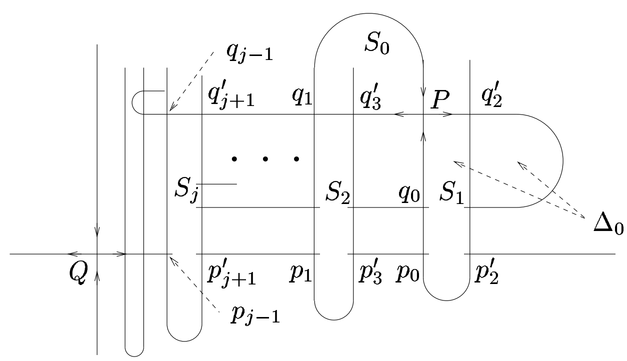

Figure 7:

Similar ideas apply when f preserves orientation. The argument is close to [23, Section 2.1], where it was shown that the basin $B(\Lambda)$ contains a neighbourhood of the attractor Λ, for certain parameter intervals. It goes as follows. Let $g = g_a$ be the one-dimensional map $g(x) = 1 - ax^2$, and P and Q be the fixed points of g, with $Q < 0 < P$ (by abuse of language, we represent by the same letters points playing similar roles in the dynamics of g and f, respectively). We consider the sequence of intervals $[p_j, p'_{j+2}]$, where $p_0 = P$, p'_2 is the point of $g^{-2}(P)$ to the right of P, and $[p_j, p'_{j+2}] = g^{-1}([p_{j-1}, p'_{j+1}]) \cap \{x < 0\}$, for every $j \geq 1$. For each large j, there is some compact interval $I_j = [a_{1,j}, a_{2,j}]$, close to $a = 2$ in parameter space, for which $g^3(0)$ is in the interior of $[p_j, p'_{j+2}]$. Each I_j may be fixed such that $(a_{2,j} - a_{1,j}) \geq (2 - a_{2,j})/10$

(we may take $(a_{2,j} - a_{1,j})/(2 - a_{2,j})$ close to $(p'_{j+2} - p_j)/(p_j - Q)$ for j large). We consider only parameters a varying inside some I_j (this is a simplifying condition, that is probably not necessary; it is possible to replace each I_j by a larger interval, at the price of rendering more involved the arguments that follow). Then we fix δ_* sufficiently small so that $g^3([-\delta_*, \delta_*])$ is contained in the interior of $[p_j, p'_{j+2}]$.

Now let $b \ll \delta_*$ and $f = f_a$ be close to the quadratic family g_a in the sense of Section 2. Since the interval I_j is not too small (cf. previous paragraph and the one preceding Section 2.1) the arguments of [2], [15] apply within I_j: after convenient parameter exclusions there remains a positive Lebesgue measure subset of parameters $a \in I_j$ for which f has the properties listed in Section 2. We want to prove that, for any $a \in I_j$, Lebesgue almost every point in $B(\Lambda)$ has some iterate in Δ_0. As already explained, we only need to consider points in $R_* = [-\delta_*, \delta_*] \times [-2, 2]$. Note that $f^3(R_*)$ is $C\sqrt{b}$-close to $g^3([-\delta_*, \delta_*]) \times \{0\}$. By the perturbation argument in [23, Section 2.1], we have that $f^3(R_*)$ is contained in a region S_j as in Figure 7: S_j is bounded by a segment in G_0 and a connected piece of the stable manifold $W^s(P)$ of P linking two nearly vertical segments of $W^s(P)$ through points p_{j-1} and p'_{j+1} in $W^s(P) \cap W^u(Q)$. Let us denote q_{j-1} and q'_{j+1}, respectively, the points where these nearly vertical segments intersect G_0 (the "vertices" of S_j). By construction, $f(p_{j-1}) = p_{j-2}$ and $f(p'_{j+1}) = p'_j$, and then $f(S_j) = \tilde{S}_{j-1} \subset S_{j-1}$, where $\tilde{S}_{j-1}$ is the region bounded by G_1 and the segment of $W^s(P)$ connecting $f(q_{j-1})$ to $f(q'_{j+1})$. So, $f^{j+2}(R_*) \subset f^{j-1}(S_j) \subset \tilde{S}_1 \subset S_1$, note that S_1 is the region bounded by the segments of $W^u(P)$ and $W^s(P)$ linking P to q'_2. Then $f(S_1) = S_0$ is bounded by the segments of $W^u(P)$ and $W^s(P)$ linking P to $q_1 = f(q'_2)$ (in particular, ζ_0 is in the boundary of S_0). Next, $f(S_0)$ is the region bounded by the segments of $W^u(P)$ and $W^s(P)$ linking P to $q_0 = f(q_1)$, that is, $f(S_0)$ coincides with Δ_0. This proves that $f^{j+4}(R_*) \subset \Delta_0$. We have shown that Lebesgue almost every point $w \in B(\Lambda)$ has some positive iterate $z = f^k(w)$ in Δ_0. By our previous arguments, z is in the stable manifold $W^s(\xi)$ of some point $\xi \in \Lambda$, Lebesgue almost surely. Then the same is true for w, and Theorem A is proved in this case too. The proof of Theorem A is complete.

Now Theorem B is a simple consequence. For each $(i_0, \ldots, i_k)$, let γ be an unstable side of $R(i_0, \ldots, i_k)$. By Corollary 3.12, every point in $f^{n_k+1}(H(i_0, \ldots, i_k))$ has a long stable leaf, that intersects $f^{n_k+1}(\gamma)$ transversely. Cf. comments at the end of Section 2, almost every point in $f^{n_k+1}(\gamma)$ belongs to the basin $B(\mu)$. Since the set of generic points of an invariant measure consists of entire stable sets, we may conclude that $f^{n_k+1}(H(i_0, \ldots, i_k)) \setminus B(\mu)$ is a union of long stable leaves intersecting $f^{n_k+1}(\gamma)$ in a set with zero arc-length measure. The second part of Proposition 2.4 implies that the lamination of $f^{n_k+1}(H(i_0, \ldots, i_k))$ by long stable leaves is Lipschitz (in the sense that the holonomy maps are Lipschitz continuous). It follows that $f^{n_k+1}(H(i_0, \ldots, i_k)) \setminus B(\mu)$ must have zero Lebesgue measure (area). Then the same is true for $H(i_0, \ldots, i_k) \setminus B(\mu)$. Taking the union over all $i_0, \ldots, i_k$, this proves that almost every point in set H we had constructed is generic for the measure μ. Since we had shown that Lebesgue almost every point in the topological basin $B(\Lambda)$ eventually reaches H, we conclude that $B(\Lambda) \setminus B(\mu)$ has zero area, as claimed. This finishes the proof of Theorem B.

Along similar lines, we can prove that the stable manifold $W^s(P)$ of the fixed point P is dense in the basin of attraction $B(\Lambda)$. Clearly, for this it is enough to show that $W^s(P)$ is dense in some full Lebesgue measure subset of $B(\Lambda)$. Thus, in view of the previous arguments in the proofs of Theorem A and B, we only have to show that, given any $i_0, \ldots, i_k$, Lebesgue almost every point in $f^{n_k+1}(H(i_0, \ldots, i_k))$ is in the closure of $W^s(P)$. Let γ be any of the unstable sides of $R(i_0, \dots, i_k)$. As we recalled at the end of Section 2, there exists a full Lebesgue measure subset $\tilde{\gamma}_k$ of $f^{n_k+1}(\gamma)$ such that any point ξ in $\tilde{\gamma}_k$ has infinitely many returns $m_i = m_i(\xi)$ for which $f^{m_i}(\xi)$ is near $x = \pm\delta/2$. Then $f^{m_i+1}(\xi)$ is between the long stable leaves Γ_Δ and $\Gamma_{\Delta+1}$. Recall Remark 3.3. By the Lipschitz property of the stable foliation, the subset $\tilde{H}_k$ of points whose stable leaves intersect $\tilde{\gamma}_k$ has full Lebesgue measure in $f^{n_k+1}(H(i_0, \dots, i_k))$. We are going to show that any point $z \in \tilde{H}_k$ is accumulated by $W^s(P)$.

We begin by noting that, since f is close to a map $(x, y) \mapsto (1 - ax^2, 0)$, with $a \approx 2$, there exists a long nearly vertical segment of $W^s(P)$ between Γ_Δ and $\Gamma_{\Delta+1}$. This follows from the fact that the negative orbit of the fixed point $p = (1/2, 0)$ under the map $x \mapsto 1 - 2x^2$ is dense in the interval $[-1, 1]$, using well-known perturbation arguments. Let ξ be the point where the long stable leaf through $z \in \tilde{H}_k$ intersects $\tilde{\gamma}_k$, and let m_i be as above. Then $f^{m_i}(z)$ is between Γ_Δ and $\Gamma_{\Delta+1}$, and so there exists a point of $W^s(P)$ at distance less than $Ce^{-\Delta} = C\delta$ from $f^{m_i}(z)$. Moreover, such a point may be chosen in the image $f^{m_i}(l_z)$ of a horizontal segment l_z through z. This is because the image of l_z is nearly horizontal near z, as a consequence of Propositions 2.5 and 2.8. The propositions also imply that l_z is exponentially expanded under f^{m_i}, so $\operatorname{dist}(z, W^s(P)) \le C\sigma_0^{-m_i}$ with $\sigma_0 > 1$. As m_i may be chosen arbitrarily large, we get that z is indeed in the closure of $W^s(P)$.

Closing this paper, let us remark that for Hénon maps $h(x, y) = (1 - ax^2 + \sqrt{b}y, \pm\sqrt{b}x)$, with b small, it is possible to give a complete characterization of the topological basin of attraction:

(TB) the basin $B(\Lambda)$ is the domain in the plane bounded by the stable manifold $W^s(Q)$ of the fixed point Q.

This was also claimed in [9], independently of the present work. Recall that Q denotes the fixed point in the region $\{x < 0\}$, whereas the attractor Λ is the closure of the other fixed point P. In what follows we give an outline of the proof of (TB).

First we suppose that h is orientation reversing, corresponding to the positive sign in the expression of h. See the left hand side of Figure 8. The stable manifold of Q contains two long nearly vertical segments located near $x \approx \pm 1$, that connect to each other in the region $\{y \ll 0\}$. Let U be the open domain bounded by this piece of $W^s(Q)$ and by a horizontal line $H = \{y = 3\}$. Then U is forward invariant, $h(U) \subset U$, and it contains the attractor Λ.

We claim that U is contained in the topological basin $B(\Lambda)$. To prove this, we begin by constructing a rectangle V bordering $W^s(Q)$ inside U, such that

- V contains the three first "tips" of $W^u(P)$, i.e., neighbourhoods of the points $f^i(\zeta_0)$, $i = 1, 2, 3$, in Figure 6
- $h^2(V) \setminus V$ is contained in the domain R introduced before in the context of Figure 6.

V may be constructed, e.g., using linearizing coordinates for f in a neighbourhood of Q. We already know that $R \subset B(\Lambda)$. Since any point in V must eventually leave V, it follows that $V \subset B(\Lambda)$. Now, let z be any point in $U \setminus V$. If z eventually reaches the folding region $[-\delta_*, \delta_*] \times \{|y| \le 3\}$, then it gets mapped to $R \cup V$ in the next iterate. Otherwise, if the orbit of z remains outside the folding region for all positive times, then z is expanding, and so it has a long stable leaf intersecting the attractor. In both cases, z is in the basin of Λ. This proves the claim.

It follows that $B(\Lambda)$ coincides with the saturation $\bigcup_{n=0}^{\infty} h^{-n}(U)$ of the domain U. In particular, the boundary of $B(\Lambda)$ is contained in $W^s(Q) \cup \alpha(H)$, where $\alpha(H)$ is the set of accumulation

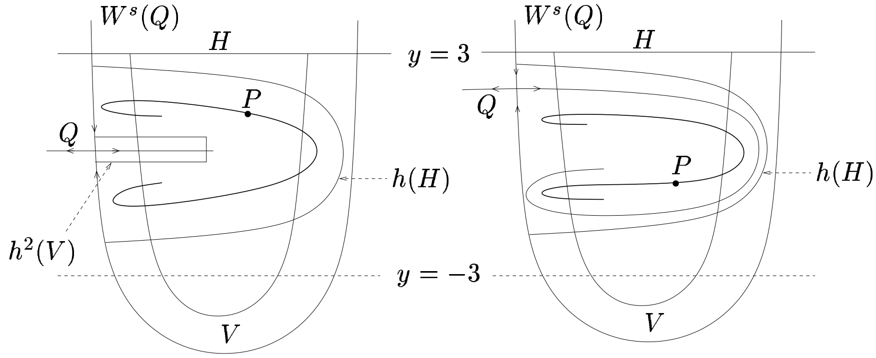

Figure 8:

points of the backward orbit of H. Now, H is contained in the region $\{|y| \geq |x| \text{ and } y \geq 2\}$. Using the form of the inverse map

$$(x_1, y_1) = h^{-1}(x, y) = (y/\sqrt{b}, (ay^2/b + x - 1)/\sqrt{b})$$

one checks easily that

$$|y| \geq |x| \text{ and } |y| \geq 2 \quad \Rightarrow \quad |y_1| \geq |x_1| \text{ and } |y_1| \geq 2|y|\,.$$

This implies that the backward orbit of any point in that region goes off to infinity. In particular, $\alpha(H)$ is empty, and so $\partial B(\Lambda) \subset W^s(Q)$. On the other hand, the forward orbits of all points in a small neighbourhood of Q outside U also go to infinity, and so they are not in $B(\Lambda)$. Combined with the previous conclusion, this gives $\partial B(\Lambda) = W^s(Q)$.

Now we explain how these arguments can be adapted to the orientation preserving case $h(x, y) = (1 - ax^2 + \sqrt{b}y, -\sqrt{b}x)$. The corresponding picture is on the right hand side of Figure 8. In this case we choose the parameter a in such a way that the folding region is mapped to some domain known to be contained in $B(\Lambda)$, e.g., the rectangle S_j introduced in the context of Figure 7. We also consider a tubular neighbourhood V of $W^s(Q)$ inside U, where h is conjugate to a linear map. For points in $U \setminus V$, either their forward orbit eventually hits the folding region, or they have long stable leaves intersecting Λ (because V is not too thin). In either case, such points are in $B(\Lambda)$. Points in V eventually move to $U \setminus V$ under forward iteration. So, the whole region U is contained in the topological basin. It remains true that the backward orbit of $H = \{y = 3\}$ has no (finite) accumulation points. Then the proof of (TB) proceeds precisely as in the orientation reversing case.

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