HOLONOMY INVARIANCE: ROUGH REGULARITY AND APPLICATIONS TO LYAPUNOV EXPONENTS

par

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Résumé. — Un cocycle lisse est un produit gauche qui agit par des difféomorphismes dans les fibres. Si les exposants de Lyapounov extremaux du cocycle coincident alors les fibres possèdent certaines structures qui sont invariantes, à la fois, par la dynamique et par un pseudo-groupe canonique de transformations d'holonomie. Nous démontrons ce *principe d' invariance* pour les cocycles lisses au dessus des difféomorphismes conservatifs partiellement hyperboliques, et nous en donnons des applications aux cocycles linéaires et aux dynamiques partiellement hyperboliques.

Skew-products that act by diffeomorphisms on the fibers are called smooth cocycles. If the extremal Lyapunov exponents of a smooth cocycle coincide then the fibers carry quite a lot of structure that is invariant under the dynamics and under a canonical pseudo-group of holonomy maps. We state and prove this invariance principle for cocycles over partially hyperbolic volume preserving diffeomorphisms. It has several applications, e.g. to linear cocycles and to partially hyperbolic dynamics.

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1. Introduction

Lyapunov exponents measure the asymptotic rates of contraction and expansion, in different directions, of smooth dynamical systems such as diffeomorphisms, cocycles, or their continuous-time counterparts. These numbers are well defined on a full measure subset of phase-space, relative to any finite invariant measure. Systems whose Lyapunov exponents are distinct/non-vanishing exhibit a wealth of geometric and dynamical structure (invariant laminations, entropy formula, abundance of periodic orbits, dimension of invariant measures) on which one can build to describe their evolution. The main theme we are interested in is that systems for which the Lyapunov exponents are *not* distinct are also special, in that they satisfy a very strong invariance principle. Thus, a detailed theory can be achieved also in this case, if only using very different ingredients.

In the special case of linear systems, the invariance principle can be traced back to the classical results on random matrices by Furstenberg [**11**], Ledrappier [**18**], and others. Moreover, it has been refined in more recent works by Bonatti, Gomez-Mont, Viana [**6**], Bonatti, Viana [**7**], Viana [**23**] and Avila, Viana [**1, 2**]. An explicit and much more general formulation, that applies to smooth (possibly non-linear) systems, is proposed in Avila, Viana [**3**] and the present paper: while [**3**] deals with extensions of hyperbolic transformations, here we handle the case when the base dynamics is just partially hyperbolic and volume preserving. The two papers are contemporary and closely related: in particular, Theorem A of [**3**] relies on a version of the invariance principle proved in here, more precisely, Theorem B below.

As an illustration of the reach of our methods, let us state the following application in the realm of partially hyperbolic dynamics. Let $f : M \to M$ be a C^2 partially hyperbolic, dynamically coherent, volume preserving, accessible diffeomorphism whose center bundle E^c has dimension 2. If the center Lyapunov exponents vanish almost everywhere then f admits

(a) either an invariant continuous conformal structure on E^c,
(b) or an invariant continuous field of directions $r \subset E^c$,
(c) or an invariant continuous field of pairs of directions $r_1 \cup r_2 \subset E^c$.

Sometimes, one can exclude all three alternatives a priori. That is the case, for instance, if f is known to have periodic points p and q that are, respectively, elliptic and hyperbolic along the center bundle E^c (more precisely: the center eigenvalues of p are neither real nor pure imaginary, and the center eigenvalues of q are real and distinct). Then it follows that some center Lyapunov exponent is non-zero. When f is symplectic, this implies that both center Lyapunov exponents are different from zero; compare Theorem A in [**3**].

Precise statements of our results, including the definitions of the objects involved, will appear in the next section. Right now, let us observe that important applications of the methods developed in here have been obtained by several authors: a Livšic theory of partially hyperbolic diffeomorphism, by Wilkinson [**25**]; existence and properties of physical measures, by Viana, Yang [**24**]; construction of measures of maximal entropy, by Hertz, Hertz, Tahzibi, Ures [**13**].

2. Preliminaries and statements

2.1. Partially hyperbolic diffeomorphisms. — Throughout the paper, unless stated otherwise, $f : M \to M$ is a partially hyperbolic diffeomorphism on a compact manifold M and μ is a probability measure in the Lebesgue class of M. In this section we define these and other related notions. See [**8, 14, 15, 22**] for more information.

A diffeomorphism $f : M \to M$ of a compact manifold M is *partially hyperbolic* if there exists a nontrivial splitting of the tangent bundle

$$TM = E^s \oplus E^c \oplus E^u \tag{2.1}$$

invariant under the derivative Df, a Riemannian metric $\|\cdot\|$ on M, and positive continuous functions ν, $\hat\nu$, γ, $\hat\gamma$ with ν, $\hat\nu < 1$ and $\nu < \gamma < \hat\gamma^{-1} < \hat\nu^{-1}$ such that, for any unit vector $v \in T_pM$,

$$\|Df(p)v\| < \nu(p) \qquad \text{if } v \in E^s(p), \tag{2.2}$$

$$\gamma(p) < \|Df(p)v\| < \hat\gamma(p)^{-1} \quad \text{if } v \in E^c(p), \tag{2.3}$$

$$\hat\nu(p)^{-1} < \|Df(p)v\| \qquad \text{if } v \in E^u(p). \tag{2.4}$$

All three subbundles E^s, E^c, E^u are assumed to have positive dimension. However, in some cases (cf. Remarks 3.12 and 4.2) one may let either $\dim E^s = 0$ or $\dim E^u = 0$.

We take M to be endowed with the distance dist associated to such a Riemannian structure. The *Lebesgue class* is the measure class of the volume induced by this (or any other) Riemannian metric on M. These notions extend to any submanifold of M, just considering the restriction of the Riemannian metric to the submanifold. We say that f is *volume preserving* if it preserves some probability measure in the Lebesgue class of M.

Suppose that $f : M \to M$ is partially hyperbolic. The stable and unstable bundles E^s and E^u are uniquely integrable and their integral manifolds form two transverse continuous foliations $\mathcal{W}^s$ and $\mathcal{W}^u$, whose leaves are immersed submanifolds of the same class of differentiability as f. These foliations are referred to as the *strong-stable* and *strong-unstable* foliations. They are invariant under f, in the sense that

$$f(\mathcal{W}^s(x)) = \mathcal{W}^s(f(x)) \quad \text{and} \quad f(\mathcal{W}^u(x)) = \mathcal{W}^u(f(x)),$$

where $\mathcal{W}^s(x)$ and $\mathcal{W}^s(x)$ denote the leaves of $\mathcal{W}^s$ and $\mathcal{W}^u$, respectively, passing through any $x \in M$. These foliations are, usually, *not* transversely smooth: the holonomy maps between any pair of cross-sections are not even Lipschitz continuous, in general, although they are always γ-Hölder continuous for some $\gamma > 0$. Moreover, if f is C^2 then these foliations are absolutely continuous, meaning that the holonomy maps preserve the class of zero Lebesgue measure sets. Let us explain this key fact more precisely.

Let $d = \dim M$ and $\mathcal{F}$ be a continuous foliation of M with k-dimensional smooth leaves, $0 < k < d$. Let $\mathcal{F}(p)$ be the leaf through a point $p \in M$ and $\mathcal{F}(p, R) \subset \mathcal{F}(p)$ be the neighborhood of radius $R > 0$ around p, relative to the distance defined by the Riemannian metric restricted to $\mathcal{F}(p)$. A *foliation box* for $\mathcal{F}$ at p is the image of an embedding

$$\Phi : \mathcal{F}(p, R) \times \mathbb{R}^{d-k} \to M$$

such that $\Phi(\cdot, 0) = \text{id}$, every $\Phi(\cdot, y)$ is a diffeomorphism from $\mathcal{F}(p, R)$ to some subset of a leaf of $\mathcal{F}$ (we call the image a *horizontal slice*), and these diffeomorphisms vary continuously with $y \in \mathbb{R}^{d-k}$. Foliation boxes exist at every $p \in M$, by definition of continuous foliation with smooth leaves. A *cross-section* to $\mathcal{F}$ is a smooth codimension-k disk inside a foliation box that intersects each horizontal slice exactly once, transversely and with angle uniformly bounded from zero.

Then, for any pair of cross-sections Σ and Σ', there is a well defined *holonomy map* $\Sigma \to \Sigma'$, assigning to each $x \in \Sigma$ the unique point of intersection of Σ' with the horizontal slice through x. The foliation is *absolutely continuous* if all these homeomorphisms map zero Lebesgue measure sets to zero Lebesgue measure sets. That holds, in particular, for the strong-stable and strong-unstable foliations of partially hyperbolic C^2 diffeomorphisms and, in fact, the Jacobians of all holonomy maps are bounded by a uniform constant.

A measurable subset of M is *s-saturated* (or *$\mathcal{W}^s$-saturated*) if it is a union of entire strong-stable leaves, *u-saturated* (or $\mathcal{W}^u$-saturated) if it is a union of entire strong-unstable leaves, and *bi-saturated* if it is both s-saturated and u-saturated. We say that f is *accessible* if $\emptyset$ and M are the only bi-saturated sets, and *essentially accessible* if every bi-saturated set has either zero or full measure, relative to any probability measure in the Lebesgue class. A measurable set $X \subset M$ is *essentially s-saturated* if there exists an s-saturated set $X^s \subset M$ such that $X \Delta X^s$ has measure zero, for any probability measure in the Lebesgue class. *Essentially u-saturated* sets are defined analogously. Moreover, X is *bi-essentially saturated* if it is both essentially s-saturated and essentially u-saturated.

Pugh, Shub conjectured in [**19**] that essential accessibility implies ergodicity, for a C^2 partially hyperbolic, volume preserving diffeomorphism. In [**20**] they showed that this does hold under a few additional assumptions, called dynamical coherence and center bunching. To date, the best result in this direction is due to Burns, Wilkinson [**9**], who proved the Pugh-Shub conjecture assuming only the following mild form of center bunching:

Definition 2.1. — A C^2 partially hyperbolic diffeomorphism is *center bunched* if the functions ν, $\hat{\nu}$, γ, $\hat{\gamma}$ in (2.2)–(2.4) may be chosen to satisfy

$$\nu < \gamma\hat{\gamma} \quad \text{and} \quad \hat{\nu} < \gamma\hat{\gamma}. \tag{2.5}$$

When the diffeomorphism is just $C^{1+\alpha}$, for some $\alpha > 0$, the arguments of Burns, Wilkinson [**9**] can still be carried out, as long as one assumes what they call strong center bunching (see [**9**, Theorem 0.3]). All our results extend to this setting.

2.2. Fiber bundles. — In this paper we deal with a few different types of fiber bundles over the manifold M. The more general type we consider are *continuous fiber bundles* $\pi : \mathcal{E} \to M$ modeled on some topological space N. By this we mean that $\mathcal{E}$ is a topological space and there is a family of homeomorphisms (*local charts*)

$$\phi_U : U \times N \to \pi^{-1}(U), \tag{2.6}$$

indexed by the elements U of some finite open cover $\mathcal{U}$ of M, such that $\pi \circ \phi_U$ is the canonical projection $U \times N \to U$ for every $U \in \mathcal{U}$. Then each $\phi_{U,x} : \xi \mapsto \phi_U(x,\xi)$ is a homeomorphism between N and the *fiber* $\mathcal{E}_x = \pi^{-1}(x)$.

An important role will be played by the class of *fiber bundles with smooth fibers*, that is, continuous fiber bundles whose fibers are manifolds endowed with a continuous Riemannian metric. More precisely, take N to be a Riemannian manifold, not necessarily complete, and assume that all coordinate changes $\phi_V^{-1} \circ \phi_U$ have the form

$$\phi_V^{-1} \circ \phi_U : (U \cap V) \times N \to (U \cap V) \times N, \quad (x,\xi) \mapsto (x, g_x(\xi)) \tag{2.7}$$

where:

(i) $g_x : N \to N$ is a C^1 diffeomorphism and the map $x \mapsto g_x$ is continuous, relative to the uniform C^1 distance on $\mathrm{Diff}^1(N)$ (the uniform C^1 distance is defined by $\mathrm{dist}_{C^1}(g_x, g_y) = \sup\{|g_x(\xi) - g_y(\xi)|, \|Dg_x(\xi) - Dg_y(\xi)\| : \xi \in N\}$);

(ii) the derivatives $Dg_x(\xi)$ are $Dg_x^{-1}(\xi)$ are uniformly continuous and uniformly bounded in norm.

Endow each $\mathcal{E}_x$ with the manifold structure that makes $\phi_{U,x}$ a diffeomorphism. Condition (i) ensures that this does not depend on the choice of $U \in \mathcal{U}$ containing x. Moreover, consider on each $\mathcal{E}_x$ the Riemannian metric $\gamma_x = \sum_{U \in \mathcal{U}} \rho_U(x)\gamma_{U,x}$, where $\gamma_{U,x}$ is the Riemannian metric transported from N by the diffeomorphism $\phi_{U,x}$ and $\{\rho_U : U \in \mathcal{U}\}$ is a partition of unit subordinate to $\mathcal{U}$. It is clear that γ_x depends continuously on x. Condition (ii) ensures that different choices of the partition of unit give rise to Riemannian metrics γ_x that differ by a bounded factor only.

Restricting even further, we call $\pi : \mathcal{E} \to M$ a *continuous vector bundle* of dimension $d \geq 1$ if $N = \mathbb{K}^d$, with $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, and every g_x is a linear isomorphism, depending continuously on x and such that $\|g_x^{\pm 1}\|$ are uniformly bounded. Then each fiber $\mathcal{E}_x$ is isomorphic to $\mathbb{K}^d$ and is equipped with a scalar product (and, hence, a norm) which is canonical up to a bounded factor.

We also need to consider more regular vector bundles. Given $r \in \{0, 1, \dots, k, \dots\}$ and $\alpha \in [0,1]$, we say that $\pi : \mathcal{E} \to M$ is a $C^{r,\alpha}$ vector bundle if, for any U, $V \in \mathcal{U}$ with non-empty intersection, the map

$$U \cap V \to \mathrm{GL}(d, \mathbb{K}), \quad x \mapsto g_x \tag{2.8}$$

is of class $C^{r,\alpha}$, that is, it is r times differentiable and the derivative of order r is α-Hölder continuous.

2.3. Linear cocycles. — Let $\pi : \mathcal{V} \to M$ be a continuous vector bundle of dimension $d \geq 1$. A *linear cocycle* over $f : M \to M$ is a continuous transformation $F : \mathcal{V} \to \mathcal{V}$ satisfying $\pi \circ F = f \circ \pi$ and acting by linear isomorphisms $F_x : \mathcal{V}_x \to \mathcal{V}_{f(x)}$ on the fibers. By Furstenberg, Kesten [**12**], the *extremal Lyapunov exponents*

$$\lambda_+(F,x) = \lim_{n\to\infty} \frac{1}{n} \log \|F_x^n\| \quad \text{and} \quad \lambda_-(F,x) = \lim_{n\to\infty} \frac{1}{n} \log \|(F_x^n)^{-1}\|^{-1}$$

exist at μ-almost every $x \in M$, relative to any f-invariant probability measure μ. If (f,μ) is ergodic then they are constant on a full μ-measure set. It is clear that $\lambda_-(F,x) \leq \lambda_+(F,x)$ whenever they are defined. We study conditions under which these two numbers coincide.

Suppose that $\pi : \mathcal{V} \to M$ is a $C^{r,\alpha}$ vector bundle, for some fixed r and α, and f is also of class $C^{r,\alpha}$ (this is contained in our standing assumptions if $r+\alpha \le 2$). Then we call $F : \mathcal{V} \to \mathcal{V}$ a $C^{r,\alpha}$ linear cocycle if its expression in local coordinates

$$\phi_{U_1}^{-1} \circ F \circ \phi_{U_0} : (U_0 \cap f^{-1}(U_1)) \times \mathbb{K}^d \to U_1 \times \mathbb{K}^d, \quad (x,v) \mapsto (f(x), A(x)v) \tag{2.9}$$

is such that the function $x \mapsto A(x)$ is r times differentiable and the derivative of order r is bounded and α-Hölder continuous. The assumption on the vector bundle ensures that this condition does not depend on the choice of local charts.

The set $\mathcal{G}^{r,\alpha}(\mathcal{V}, f)$ of all $C^{r,\alpha}$ linear cocycles $F : \mathcal{V} \to \mathcal{V}$ over $f : M \to M$ is a $\mathbb{K}$-vector space and carries a natural $C^{r,\alpha}$ norm:

$$\|F\|_{r,\alpha} = \sup_{U,V\in\mathcal{U}} \left(\sup_{0\le i\le r} \sup_{x\in U\cap f^{-1}(V)} \|D^i A(x)\| + \sup_{x\ne y} \frac{\|D^r A(x) - D^r A(y)\|}{\operatorname{dist}(x,y)^\alpha} \right) \tag{2.10}$$

(for $\alpha = 0$ one may omit the last term). *We always assume that* $r + \alpha > 0$. Then every $F \in \mathcal{G}^{r,\alpha}(\mathcal{V}, f)$ is β-Hölder continuous, with

$$\beta = \begin{cases} \alpha & \text{if } r = 0 \\ 1 & \text{if } r \ge 1. \end{cases} \tag{2.11}$$

Definition 2.2. — We say that a cocycle $F \in \mathcal{G}^{r,\alpha}(\mathcal{V}, f)$ is *fiber bunched* if

$$\|F_x\| \, \|(F_x)^{-1}\| \, \nu(x)^\beta < 1 \quad \text{and} \quad \|F_x\| \, \|(F_x)^{-1}\| \, \hat\nu(x)^\beta < 1, \tag{2.12}$$

for every $x \in M$, where $\beta > 0$ is given by (2.11) and ν, $\hat\nu$ are functions as in (2.2)–(2.4), fixed once and for all.

Remark 2.3. — This notion appeared in [**6, 7, 23**], where it was called *domination*. The present terminology seems preferable, on more than one account. To begin with, there is the analogy with the notion of center bunching in Definition 2.1. Perhaps more important, the natural notion of domination for smooth cocycles, that we are going to introduce in Definition 3.9, corresponds to a rather different condition. The relation between the two is explained in Remark 3.13: if a linear cocycle is fiber bunched then the associated *projective* cocycle is dominated. Finally, a notion of fiber bunching can be defined for smooth cocycles as well (see [**3**]), similar to (2.12) and stronger than domination.

Theorem A. — *Let $f : M \to M$ be a C^2 partially hyperbolic, volume preserving, center bunched, accessible diffeomorphism and let μ be an invariant probability in the Lebesgue class. Assume that $F \in \mathcal{G}^{r,\alpha}(\mathcal{V}, f)$ is fiber bunched.*

Then F is approximated, in the $C^{r,\alpha}$ norm, by open sets of cocycles $G \in \mathcal{G}^{r,\alpha}(\mathcal{V}, f)$ such that $\lambda_-(G,x) < \lambda_+(G,x)$ almost everywhere. Moreover, the set of $F \in \mathcal{G}^{r,\alpha}(\mathcal{V}, f)$ for which the extremal Lyapunov exponents do coincide has infinite codimension in the fiber bunched domain: locally, it is contained in finite unions of closed submanifolds with arbitrarily high codimension.

Notice that the Lyapunov exponents are constant on a full measure subset of M, because (cf. [**9**]) the hypothesis implies that f is ergodic.

There is an analogous statement in the space of $\mathrm{SL}(d, \mathbb{K})$-cocycles, that is, such that the functions $x \mapsto g_x$ and $x \mapsto A(x)$ in (2.8) and (2.9), respectively, take values in $\mathrm{SL}(d,\mathbb{K})$. In fact, our proof of Theorem A deals with the projectivization of the cocycle, and so it treats both cases, $\mathrm{GL}(d,\mathbb{K})$ and $\mathrm{SL}(d,\mathbb{K})$, on the same footing. It would be interesting to investigate the case of G-valued cocycles for more general subgroups of $\mathrm{GL}(d,\mathbb{K})$, for instance the symplectic group.

2.4. Smooth cocycles - invariant holonomies. — Let $\pi : \mathcal{E} \to M$ be a fiber bundle with smooth fibers modeled on some Riemannian manifold N. A *smooth cocycle* over $f : M \to M$ is a continuous transformation $\mathfrak{F} : \mathcal{E} \to \mathcal{E}$ such that $\pi \circ \mathfrak{F} = f \circ \pi$, every $\mathfrak{F}_x : \mathcal{E}_x \to \mathcal{E}_{f(x)}$ is a C^1 diffeomorphism depending continuously on x, relative to the uniform C^1 distance in the space of C^1 diffeomorphisms on the fibers, and the norms of the derivative $D\mathfrak{F}_x(\xi)$ and its inverse are uniformly bounded. In particular, the functions

$$(x,\xi) \mapsto \log \|D\mathfrak{F}_x(\xi)\| \quad \text{and} \quad (x,\xi) \mapsto \log \|D\mathfrak{F}_x(\xi)^{-1}\|$$

are bounded. Then (Kingman [**17**]), given any $\mathfrak{F}$-invariant probability m on $\mathcal{E}$, the *extremal Lyapunov exponents* of $\mathfrak{F}$

$$\lambda_+(\mathfrak{F},x,\xi) = \lim_{n\to\infty} \frac{1}{n} \log \|D\mathfrak{F}_x^n(\xi)\| \quad \text{and} \quad \lambda_-(\mathfrak{F},x,\xi) = \lim_{n\to\infty} \frac{1}{n} \log \|D\mathfrak{F}_x^n(\xi)^{-1}\|^{-1}.$$

are well defined at m-almost every $(x,\xi) \in \mathcal{E}$. Clearly, $\lambda_-(\mathfrak{F},x,\xi) \le \lambda_+(\mathfrak{F},x,\xi)$. Notice that if m is $\mathfrak{F}$-invariant then its projection $\mu = \pi_* m$ is f-invariant. Most of the times we will be interested in measures m for which the projection is in the Lebesgue class of M.

Let $R > 0$ be fixed. The *local strong-stable leaf* $\mathcal{W}^s_{loc}(p)$ of a point $p \in M$ is the neighborhood of radius R around p inside $\mathcal{W}^s(p)$. The local strong-unstable leaf $\mathcal{W}^u_{loc}(p)$ is defined analogously. The choice of R is very much arbitrary, but in Section 5 we will be a bit more specific.

Definition 2.4. — We call *invariant stable holonomy* for $\mathfrak{F}$ a family H^s of homeomorphisms $H^s_{x,y} : \mathcal{E}_x \to \mathcal{E}_y$, defined for all x and y in the same strong-stable leaf of f and satisfying

(a) $H^s_{y,z} \circ H^s_{x,y} = H^s_{x,z}$ and $H^s_{x,x} = \mathrm{id}$;
(b) $\mathfrak{F}_y \circ H^s_{x,y} = H^s_{f(x),f(y)} \circ \mathfrak{F}_x$;
(c) $(x,y,\xi) \mapsto H^s_{x,y}(\xi)$ is continuous when (x,y) varies in the set of pairs of points in the same local strong-stable leaf;
(d) there are $C > 0$ and $\gamma > 0$ such that $H^s_{x,y}$ is (C,γ)-Hölder continuous for every x and y in the same local strong-stable leaf.

Invariant unstable holonomy is defined analogously, for pairs of points in the same strong-unstable leaf.

Condition (c) in Definition 2.4 means that, given any $\varepsilon > 0$ and any (x,y,ξ) with $y \in \mathcal{W}^s_{loc}(x)$, there exists $\delta > 0$ such that $\mathrm{dist}(H^s_{x,y}(\xi), H^s_{x',y'}(\xi')) < \varepsilon$ for every (x',y',ξ') with $y' \in \mathcal{W}^s_{loc}(x')$ and $\mathrm{dist}(x,x') < \delta$ and $\mathrm{dist}(y,y') < \delta$ and $\mathrm{dist}(\xi,\xi') < \delta$; for this to make sense, take the fiber bundle to be trivialized in the neighborhoods of $\mathcal{E}_x$ and $\mathcal{E}_y$. Condition (d), together with the invariance property (b), implies that

$H^s_{x,y}$ is γ-Hölder continuous for every x and y in the same strong-stable leaf (the multiplicative Hölder constant C may not be uniform over global leaves).

Remark 2.5. — Uniformity of the multiplicative Hölder constant C on local strong-stable leaves is missing in the related definition in [**3**, Section 2.4], but is assumed in [**3**, Section 4.4] when arguing that the transformation $\tilde{G}$ is a deformation of G.

Example 2.6. — The *projective bundle* associated to a vector bundle $\mathcal{V} \to M$ is the continuous fiber bundle $\mathbb{P}(\mathcal{V}) \to M$ whose fibers are the projective quotients of the fibers of $\mathcal{V}$. Clearly, this is a fiber bundle with smooth leaves modeled on $N = \mathbb{P}(\mathbb{K}^d)$. The *projective cocycle* associated to a linear cocycle $F : \mathcal{V} \to \mathcal{V}$ is the smooth cocycle $\mathfrak{F} : \mathbb{P}(\mathcal{V}) \to \mathbb{P}(\mathcal{V})$ whose action $\mathfrak{F}_x : \mathbb{P}(\mathcal{V}_x) \to \mathbb{P}(\mathcal{V}_{f(x)})$ on the fibers is given by the projectivization of $F_x : \mathcal{V}_x \to \mathcal{V}_{f(x)}$:

$$\mathfrak{F}_x(\xi) = \frac{F_x(\xi)}{\|F_x(\xi)\|} \quad \text{for each } \xi \in \mathbb{P}(\mathcal{V}_x) \text{ and } x \in M$$

(on the right hand side of the equality, think of ξ as a unit vector in $\mathbb{K}^d$). Then $\mathfrak{F}^n_x(\xi) = F^n_x(\xi)/\|F^n_x(\xi)\|$ for every ξ, x and n. It follows that,

$$D\mathfrak{F}^n_x(\xi)\dot{\xi} = \frac{\operatorname{proj}_{F^n_x(\xi)}\left(F^n_x(\dot{\xi})\right)}{\|F^n_x(\xi)\|},$$

where $\operatorname{proj}_w v = v - w(w \cdot v)/(w \cdot w)$ is the projection of a vector v to the orthogonal complement of w. This implies that

$$\|D\mathfrak{F}^n_x(\xi)\| \le \|F^n_x\|/\|F^n_x(\xi)\| \le \|F^n_x\|\|(F^n_x)^{-1}\| \tag{2.13}$$

for every ξ, x and n. Analogously, replacing each F by its inverse,

$$\|D\mathfrak{F}^n_x(\xi)^{-1}\| \le \|(F^n_x)^{-1}\|\|F^n_x\| \tag{2.14}$$

for every ξ, x and n. These two inequalities imply

$$\lambda_+(\mathfrak{F}, x, \xi) \le \lambda_+(F, x) - \lambda_-(F, x) \quad \text{and} \quad \lambda_-(\mathfrak{F}, x, \xi) \ge \lambda_-(F, x) - \lambda_+(F, x)$$

whenever these exponents are defined. We will observe in Remark 3.13 that if F is fiber bunched then both F and $\mathfrak{F}$ admit invariant stable and unstable holonomies.

Example 2.7. — Suppose that the partially hyperbolic diffeomorphism $f : M \to M$ is *dynamically coherent*, that is, there exist invariant foliations $\mathcal{W}^{cs}$ and $\mathcal{W}^{cu}$ with smooth leaves tangent to $E^c \oplus E^s$ and $E^c \oplus E^u$, respectively. Intersecting the leaves of $\mathcal{W}^{cs}$ and $\mathcal{W}^{cu}$ one obtains a center foliation $\mathcal{W}^c$ whose leaves are tangent to the center subbundle E^c at every point. Let $\mathcal{E}$ be the disjoint union of the leaves of $\mathcal{W}^c$. In many cases (see Avila, Viana, Wilkinson [**4**]), the natural projection $\pi : \mathcal{E} \to M$ given by $\pi \mid \mathcal{W}^c(x) \equiv x$ is a fiber bundle with smooth fibers. Also, the map f induces a smooth cocycle $\mathfrak{F} : \mathcal{E} \to \mathcal{E}$, mapping each $y \in \mathcal{W}^c(x)$ to $f(y) \in \mathcal{W}^c(f(x))$. Moreover, the cocycle $\mathfrak{F}$ admits invariant stable and unstable holonomies: for x close to y the image $H^s_{x,y}(\xi)$ is the point where the local strong-stable leaf through $\xi \in \mathcal{W}^c(x)$ intersects the center leaf $\mathcal{W}^c(y)$, and analogously for the unstable holonomy. This kind of construction, combined with Theorem 6.1 below, is used by Wilkinson [**25**] in her recent development of a Livšic theory for partially hyperbolic diffeomorphisms.

2.5. Lyapunov exponents and rigidity. — Theorem A will be deduced, in Section 8, from certain perturbation arguments together with an invariance principle for cocycles whose extremal Lyapunov exponents coincide. Here we state this invariance principle.

Let $\mathfrak{F} : \mathcal{E} \to \mathcal{E}$ be a smooth cocycle that admits invariant stable holonomy. Let m be a probability measure on $\mathcal{E}$, let $\mu = \pi_* m$ be its projection, and let $\{m_x : x \in M\}$ be a *disintegration* of m into *conditional probabilities* along the fibers, that is, a measurable family of probability measures $\{m_x : x \in M\}$ such that $m_x(\mathcal{E}_x) = 1$ for μ-almost every $x \in M$ and

$$m(U) = \int m_x(\mathcal{E}_x \cap U)\, d\mu(x)$$

for every measurable set $U \subset \mathcal{E}$. Such a family exists and is essentially unique, meaning that any two coincide on a full measure subset. See Rokhlin [**21**].

Definition 2.8. — A disintegration $\{m_x : x \in M\}$ is *s-invariant* if

$$(H^s_{x,y})_* m_x = m_y \quad \text{for every } x \text{ and } y \text{ in the same strong-stable leaf.} \tag{2.15}$$

One speaks of *essential s-invariance* if this holds for x and y in some full μ-measure subset of M. The definitions of *u-invariance* and *essential u-invariance* are analogous. The disintegration is *bi-invariant* if it is both s-invariant and u-invariant and we call it *bi-essentially invariant* if it is both essentially s-invariant and essentially u-invariant.

First, we state the invariance principle in the special case of linear cocycles:

Theorem B. — *Let $f : M \to M$ be a C^2 partially hyperbolic, volume preserving, center bunched diffeomorphism and μ be an invariant probability in the Lebesgue class. Let $F \in \mathcal{G}^{r,\alpha}(\mathcal{V}, f)$ be fiber bunched and suppose that $\lambda_-(F, x) = \lambda_+(F, x)$ at μ-almost every point.*

Then every $\mathbb{P}(F)$-invariant probability m on the projective fiber bundle $\mathbb{P}(\mathcal{V})$ with $\pi_* m = \mu$ admits a disintegration $\{\tilde{m}_x : x \in M\}$ along the fibers such that

(a) *the disintegration is bi-invariant over a full measure bi-saturated set $M_F \subset M$;*
(b) *if f is accessible then $M_F = M$ and the conditional probabilities $\tilde{m}_x$ depend continuously on the base point $x \in M$, relative to the weak* topology.*

Invariant probability measures m that project down to μ always exist in this setting, because $\mathbb{P}(F)$ is continuous and the domain $\mathbb{P}(\mathcal{V})$ is compact. The statement of Theorem B extends to smooth cocycles:

Theorem C. — *Let $f : M \to M$ be a C^2 partially hyperbolic, volume preserving, center bunched diffeomorphism and μ be an invariant probability in the Lebesgue class. Let $\mathfrak{F}$ be a smooth cocycle over f admitting invariant stable and unstable holonomies. Let m be an $\mathfrak{F}$-invariant probability measure on $\mathcal{E}$ with $\pi_* m = \mu$, and suppose that $\lambda_-(\mathfrak{F}, x, \xi) = 0 = \lambda_+(\mathfrak{F}, x, \xi)$ at m-almost every point.*

Then m admits a disintegration $\{\tilde{m}_x : x \in M\}$ into conditional probabilities along the fibers such that

(a) *the disintegration is bi-invariant over a full measure bi-saturated set $M_{\mathfrak{F}} \subset M$;*

(b) *if f is accessible then $M_{\mathfrak{F}} = M$ and the conditional probabilities $\tilde{m}_x$ depend continuously on the base point $x \in M$, relative to the weak* topology.*

It is clear from the observations in Example 2.6 that Theorem B is contained in Theorem C. The proof of Theorem C is given in Sections 4 through 7. There are two main stages.

The first one, that will be stated as Theorem 4.1, is to show that every disintegration of m is essentially s-invariant and essentially u-invariant. This is based on a non-linear extension of an abstract criterion of Ledrappier [**18**] for linear cocycles, proposed in Avila, Viana [**3**] and quoted here as Theorem 4.4. At this stage we only need f to be a C^1 partially hyperbolic diffeomorphism (volume preserving, center bunching and accessibility are not needed) and μ can be any invariant probability, not necessarily in the Lebesgue class.

The second stage, that we state in Theorem D below, is to prove that any disintegration essentially s-invariant and essentially u-invariant is, in fact, fully invariant under both the stable holonomy and the unstable holonomy; moreover, it is continuous if f is accessible. This is a different kind of argument, that is more suitably presented in the following framework.

2.6. Sections of continuous fiber bundles. — Let $\pi : \mathcal{X} \to M$ be a continuous fiber bundle with fibers modeled on some topological space P. The next definition refers to the strong-stable and strong-unstable foliations of the partially hyperbolic diffeomorphism $f : M \to M$.

Definition 2.9. — A *stable holonomy* on $\mathcal{X}$ is a family $h^s_{x,y} : \mathcal{X}_x \to \mathcal{X}_y$ of γ-Hölder homeomorphisms, with uniform Hölder constant $\gamma > 0$, defined for all x, y in the same strong-stable leaf and satisfying

(α) $h^s_{y,z} \circ h^s_{x,y} = h^s_{x,z}$ and $h^s_{x,x} = \mathrm{id}$

(β) the map $(x, y, \xi) \mapsto h^s_{x,y}(\xi)$ is continuous when (x, y) varies in the set of pairs of points in the same local strong-stable leaf.

Unstable holonomy is defined analogously, for pairs of points in the same strong-unstable leaf.

The special case we have in mind are the *invariant* stable and unstable holonomies of smooth cocycles on fiber bundles with smooth leaves. Clearly, conditions (α) and (β) in Definition 2.9 correspond to conditions (a) and (c) in Definition 2.4. Notice, however, that there is no analogue to the invariance condition (b); indeed, cocycles are not mentioned at all in this section. We also have no analogue to condition (d) in Definition 2.4.

In what follows μ is a probability measure in the Lebesgue class of M, not necessarily invariant under f: here we do not assume f to be volume preserving. The next definition is a straightforward extension of Definition 2.8 to the present setting:

Definition 2.10. — Let $\pi : \mathcal{X} \to P$ be a continuous fiber bundle admitting stable holonomy. A measurable section $\Psi : M \to \mathcal{X}$ is *s-invariant* if

$$h^s_{x,y}(\Psi(x)) = \Psi(y) \quad \text{for every } x,\, y \text{ in the same strong-stable leaf}$$

and *essentially s-invariant* if this relation holds restricted to some full μ-measure subset. The definitions of *u-invariant* and *essentially u-invariant* functions are analogous, assuming that $\pi : \mathcal{X} \to M$ admits unstable holonomy and considering strong-unstable leaves instead. We call Ψ *bi-invariant* if it is both s-invariant and u-invariant, and we call it *bi-essentially invariant* if it is both essentially s-invariant and essentially u-invariant.

These notions extend, immediately, to measurable sections of $\mathcal{X}$ whose domain is just a bi-saturated subset of M. A measurable section Ψ is *essentially bi-invariant* if it coincides almost everywhere with a bi-invariant section defined on some full measure bi-saturated set.

Definition 2.11. — A (Hausdorff) topological space P is *refinable* if there exists an increasing sequence of finite or countable partitions $\mathcal{Q}_1 \prec \cdots \prec \mathcal{Q}_n \prec \cdots$ into Borel subsets such that any sequence $(Q_n)_n$ with $Q_n \in \mathcal{Q}_n$ for every n and $\cap_n Q_n \neq \emptyset$ converges to some point $\eta \in P$, in the sense that every neighborhood of η contains Q_n for all large n. (Then, clearly, η is unique and $\cap_n Q_n = \{\eta\}$.)

Notice that every Hausdorff space with a countable basis $\{U_n : n \in \mathbb{N}\}$ of open sets is refinable: just take $\mathcal{Q}_n$ to be the partition generated by $\{U_1, \ldots, U_n\}$.

Theorem D. — *Let $f : M \to M$ be a C^2 partially hyperbolic, center bunched diffeomorphism and μ be any probability measure in the Lebesgue class. Let $\pi : \mathcal{X} \to M$ be a continuous fiber bundle with stable and unstable holonomies and assume that the fiber P is refinable. Then,*

(a) *every bi-essentially invariant section $\Psi : M \to \mathcal{X}$ coincides μ-almost everywhere with a bi-invariant section $\tilde{\Psi}$ defined on a full measure bi-saturated set $M_\Psi \subset M$;*
(b) *if f is accessible then $M_\Psi = M$ and $\tilde{\Psi}$ is continuous.*

The proof of part (a) is given in Section 6 (see Theorem 6.1), based on ideas of Burns, Wilkinson [**9**] that we recall in Section 5 (see Proposition 5.13). Concerning part (b), we should point out that the measure μ plays no role in it: if f is accessible then any non-empty bi-saturated set coincides with M and then one only has to check that bi-invariance implies continuity. That is done in Section 7 and uses neither center bunching nor refinability.

Actually, in Section 7 we prove a stronger fact: bi-continuity implies continuity, when f is accessible. The notion of bi-continuity is defined as follows:

Definition 2.12. — A measurable section $\Psi : M \to \mathcal{X}$ of the continuous fiber bundle $\pi : \mathcal{X} \to M$ is *s-continuous* if the map $(x, y, \Psi(x)) \mapsto \Psi(y)$ is continuous on the set of pairs of points (x, y) in the same local strong-stable leaf. The notion of *u-continuity* is analogous, considering strong-unstable leaves instead. Finally, Ψ is *bi-continuous* if it is both s-continuous and u-continuous.

More explicitly, a measurable section Ψ is s-continuous if for every $\varepsilon > 0$ and every (x, y) with $y \in \mathcal{W}^s_{loc}(x)$ there exists $\delta > 0$ such that $\operatorname{dist}(\Psi(y), \Psi(y')) < \varepsilon$ for every (x', y') with $y' \in \mathcal{W}^s_{loc}(x')$ and $\operatorname{dist}(x, x') < \delta$ and $\operatorname{dist}(y, y') < \delta$ and

$\operatorname{dist}(\Psi(x), \Psi(x')) < \delta$; it is implicit in this formulation that the fiber bundle has been trivialized in the neighborhoods of the fibers $\mathcal{X}_x$ and $\mathcal{X}_y$.

Remark 2.13. — If a section $\Psi : M \to \mathcal{X}$ is s-invariant then it is s-continuous:

$$(x, y, \Psi(x)) \mapsto \Psi(y) = h^s_{x,y}(\Psi(x))$$

is continuous on the set of pairs of points in the same local strong-stable leaf. Moreover, s-continuity ensures that the section Ψ is continuous on every strong-stable leaf: taking $x = x' = y$ in the definition, we get that $\operatorname{dist}(\Psi(y), \Psi(y')) < \varepsilon$ for every $y' \in \mathcal{W}^s_{loc}(y)$ with $\operatorname{dist}(y, y') < \delta$. Analogously, u-invariance implies u-continuity and that implies continuity on every strong-unstable leaf.

Thus, part (b) of Theorem D is a direct consequence of the following result:

Theorem E. — *Let $f : M \to M$ be a C^1 partially hyperbolic, accessible diffeomorphism. Let $\pi : \mathcal{X} \to M$ be a continuous fiber bundle. Then every bi-continuous section $\Psi : M \to \mathcal{X}$ is continuous in M.*

The proof of this theorem is given in Section 7. Notice that we make no assumptions on the continuous fiber bundle: at this stage we do not need stable and unstable holonomies, and the fibers need not be refinable either.

The logical connections between our main results can be summarized as follows:

$$\begin{array}{ccccccccc}
 & & Thm\ B(a) & \leftarrow & Thm\ C(a) & \leftarrow & Thm\ D(a) & \leftarrow & Thm\ 6.1 \\
 & \swarrow & & & \uparrow & & & & \uparrow \\
Thm\ A & \leftarrow & Prop\ 8.2 & & Thm\ 4.1 & \leftarrow & Thm\ 4.4 & & Prop\ 5.13 \\
 & \nwarrow & & & \downarrow & & & & \\
 & & Thm\ B(b) & \leftarrow & Thm\ C(b) & \leftarrow & Thm\ D(b) & \leftarrow & Thm\ E
\end{array}$$

Acknowledgements. — We are most grateful to Amie Wilkinson for explanations on the use of the methods of [**9**] in connection with the proof of Theorem D and for useful comments on earlier versions of this paper. The referee has thoroughly revised the text and provided several comments that helped improve the presentation. In particular, Theorem 4.1 is now stronger than previously. This work was partly conducted during the period A. A. served as a Clay Research Fellow. J. S. was supported by a CNPq doctoral scholarship. M. V. was partially supported by CNPq, FAPERJ, and PRONEX-Dynamical Systems.

3. Cocycles with holonomies

First, we explore the notions of domination and fiber bunching for linear cocycles. In Section 3.1 we prove that if a linear cocycle is fiber bunched then it admits invariant stable and unstable holonomies, and so does its projectivization. Moreover, in Section 3.2 we check that these invariant holonomies depend smoothly on the cocycle. Then, in Section 3.3, we discuss corresponding facts for smooth cocycles.

We will often use the following notational convention: given a continuous function $\tau : M \to \mathbb{R}^+$, we denote

$$\tau^n(p) = \tau(p)\tau(f(p)) \cdots \tau(f^{n-1}(p)) \quad \text{for any } n \geq 1.$$

3.1. Fiber bunched linear cocycles. — For simplicity of the presentation, we will focus on the case when the vector bundle $\pi : \mathcal{V} \to M$ is trivial, that is, $\mathcal{V} = M \times \mathbb{K}^d$ and $\pi : M \times \mathbb{K}^d \to M$ is the canonical projection. The general case is treated in the same way, using local charts (but the notations become rather cumbersome).

In the trivial bundle case, every linear cocycle $F : \mathcal{V} \to \mathcal{V}$ may be written in the form $F(x, v) = (f(x), A(x)v)$ for some continuous $A : M \to \mathrm{GL}(d, \mathbb{K})$. Notice that $F^n(x, v) = (f^n(x), A^n(x)v)$ for each $n \in \mathbb{Z}$, with

$$A^n(x) = A(f^{n-1}(x)) \cdots A(x) \quad \text{and} \quad A^{-n}(x) = A(f^{-1}(x))^{-1} \cdots A(f^n(x))^{-1}$$

for $n \neq 0$ and $A^0(x) = \mathrm{id}$. Notice also that $F \in \mathcal{G}^{r,\alpha}(\mathcal{V}, f)$ if, and only, if A belongs to the space $\mathcal{G}^{r,\alpha}(M, d, \mathbb{K})$ of $C^{r,\alpha}$ maps from M to $\mathrm{GL}(d, \mathbb{K})$. The $C^{r,\alpha}$ norm in $\mathcal{G}^{r,\alpha}(M, d, \mathbb{K})$ is defined by

$$\|A\|_{r,\alpha} = \sup_{0 \leq i \leq r} \sup_{x \in M} \|D^i A(x)\| + \sup_{x \neq y} \frac{\|D^r A(x) - D^r A(y)\|}{\mathrm{dist}(x, y)^\alpha}. \tag{3.1}$$

Recall that we assume that $r + \alpha > 0$ and take $\beta = \alpha$ if $r = 0$ and $\beta = 1$ if $r \geq 1$. Then every $A \in \mathcal{G}^{r,\alpha}(M, d\, K)$ is β-Hölder continuous. By the definition (2.12), the cocycle F is fiber bunched if

$$\|A(x)\| \, \|A(x)^{-1}\| \, \nu(x)^\beta < 1 \quad \text{and} \quad \|A(x)\| \, \|A(x)^{-1}\| \, \hat{\nu}(x)^\beta < 1 \tag{3.2}$$

for every x in M. In this case we also say that the function A is fiber bunched. Up to suitable adjustments, all our arguments in the sequel hold under the weaker assumption that (3.2) holds for some power A^ℓ, $\ell \geq 1$.

Notice that fiber bunching is an open condition: if A is fiber bunched then so is every B in a neighborhood, just because M is compact. Even more, still by compactness, if A is fiber bunched then there exists $m < 1$ such that

$$\|B(x)\| \, \|B(x)^{-1}\| \nu(x)^{\beta m} < 1 \quad \text{and} \quad \|B(x)\| \, \|B(x)^{-1}\| \hat{\nu}(x)^{\beta m} < 1 \tag{3.3}$$

for every $x \in M$ and every B in a C^0 neighborhood of A. It is in this form that the definition will be used in the proofs.

Lemma 3.1. — *Suppose that $A \in \mathcal{G}^{r,\alpha}(M, d, \mathbb{K})$ is fiber bunched. Then there is $C > 0$ such that*

$$\|A^n(y)\| \, \|A^n(z)^{-1}\| \leq C \nu^n(x)^{-\beta m}$$

for all y, $z \in \mathcal{W}^s_{\mathrm{loc}}(x)$, $x \in M$, and $n \geq 1$. Moreover, the constant C may be taken uniform on a neighborhood of A.

Proof. — Since $A \in \mathcal{G}^{r,\alpha}(M, d, \mathbb{K})$ is β-Hölder continuous, there exists $L_1 > 0$ such that

$$\begin{aligned}\|A(f^j(y))\| / \|A(f^j(x))\| &\leq \exp(L_1 \, \mathrm{dist}(f^j(x), f^j(y))^\beta) \\ &\leq \exp(L_1 \nu^j(x)^\beta \, \mathrm{dist}(x, y)^\beta)\end{aligned}$$

and similarly for $\|A(f^j(z))^{-1}\|/\|A(f^j(x))^{-1}\|$. By sub-multiplicativity of the norm

$$\|A^n(y)\|\,\|A^n(z)^{-1}\| \le \prod_{j=0}^{n-1} \|A(f^j(y))\|\,\|A(f^j(z))^{-1}\|.$$

In view of the previous observations, the right hand side is bounded by

$$\exp\Big[L_1 \sum_{j=0}^{n-1} \nu^j(x)^\beta (\operatorname{dist}(x,y)^\beta + \operatorname{dist}(x,z)^\beta)\Big] \prod_{j=0}^{n-1} \|A(f^j(x))\|\,\|A(f^j(x))^{-1}\|$$

Since $\nu(\cdot)$ is bounded away from 1, the first factor is bounded by some $C > 0$. By fiber bunching (3.3), the second factor is bounded by $\nu^n(x)^{-\beta m}$. It is clear from the construction that L_1 and C may be chosen uniform on a neighborhood. $\square$

Proposition 3.2. — *Suppose that $A \in \mathcal{G}^{r,\alpha}(M,d,\mathbb{K})$ is fiber bunched. Then there is $L > 0$ such that for every pair of points x, y in the same leaf of the strong-stable foliation $\mathcal{W}^s$,*

(a) $H^s_{x,y} = \lim_{n\to\infty} A^n(y)^{-1} A^n(x)$ *exists (a linear isomorphism of $\mathbb{K}^d$)*
(b) $H^s_{f^j(x),f^j(y)} = A^j(y) \circ H^s_{x,y} \circ A^j(x)^{-1}$ *for every $j \ge 1$*
(c) $H^s_{x,x} = \mathrm{id}$ *and* $H^s_{x,y} = H^s_{z,y} \circ H^s_{x,z}$
(d) $\|H^s_{x,y} - \mathrm{id}\,\| \le L \operatorname{dist}(x,y)^\beta$ *whenever $y \in \mathcal{W}^s_{\mathrm{loc}}(x)$.*
(e) *Given $a > 0$ there is $\Gamma(a) > 0$ such that $\|H^s_{x,y}\| < \Gamma(a)$ for any x, $y \in M$ with $y \in \mathcal{W}^s(x)$ and $\operatorname{dist}_{\mathcal{W}^s}(x,y) < a$.*

Moreover, L and the function $\Gamma(\cdot)$ may be taken uniform on a neighborhood of A.

Proof. — In order to prove claim (a), it is sufficient to consider the case $y \in \mathcal{W}^s_{\mathrm{loc}}(x)$ because $A^{n+j}(y)^{-1}A^{n+j}(x) = A^j(y)^{-1}A^n(f^j(y))^{-1}A^n(f^j(x))A^j(x)$. Furthermore, once this is done, claim (2) follows immediately from this same relation. Each difference $\|A^{n+1}(y)^{-1}A^{n+1}(x) - A^n(y)^{-1}A^n(x)\|$ is bounded by

$$\|A^n(y)^{-1}\|\,\|A(f^n(y))^{-1}A(f^n(x)) - \mathrm{id}\,\|\,\|A^n(x)\|.$$

Since A is β-Hölder continuous, there is $L_2 > 0$ such that the middle factor in this expression is bounded by

$$L_2 \operatorname{dist}(f^n(x), f^n(y))^\beta \le L_2\big[\nu^n(x)\operatorname{dist}(x,y)\big]^\beta.$$

Using Lemma 3.1 to bound the product of the other factors, we obtain

$$\|A^{n+1}(y)^{-1}A^{n+1}(x) - A^n(y)^{-1}A^n(x)\| \le CL_2\big[\nu^n(x)^{(1-m)}\operatorname{dist}(x,y)\big]^\beta. \tag{3.4}$$

The sequence $\nu^n(x)^{\beta(1-m)}$ is uniformly summable, since $\nu(\cdot)$ is bounded away from 1. Let $K > 0$ be an upper bound for the sum. It follows that $A^n(y)^{-1}A^n(x)$ is a Cauchy sequence, and so it does converge. This finishes the proof of claims (a) and (b). Claim (c) is a direct consequence.

Moreover, adding the last inequality over all n, we also get $\|H^s_{x,y} - \mathrm{id}\,\| \le L \operatorname{dist}(x,y)^\beta$ with $L = CL_2K$. This proves claim (d). As a consequence, we also get that there exists $\gamma > 0$ such that $\|H^s_{x,y}\| < \gamma$ for any points x, y in the same local strong-stable leaf. To deduce claim (e), notice that for any x, y in the same (global) strong-stable leaf there exist points $z_0, \ldots, z_n$, where n depends only on an upper bound for the distance between x and y along the leaf, such that $z_0 = x$, $z_n = y$, and each z_i belongs to the local strong-stable leaf of z_{i-1} for every $i = 1, \ldots, n$. Together with (c), this implies $\|H^s_{x,y}\| < \gamma^n$. It is clear from the construction that L_2 and $\Gamma(\cdot)$ may be taken uniform on a neighborhood. The proof of the proposition is complete. ☐

To show that the family of maps $H^s_{x,y}$ given by this proposition is an invariant stable holonomy for F (we also say that it is an invariant stable holonomy for A) we also need to check that these maps vary continuously with the base points. That is a consequence of the next proposition:

Proposition 3.3. — *Suppose that $A \in C^{r,\alpha}(M,d,\mathbb{K})$ is fiber bunched. Then the map*

$$(x,y) \mapsto H^s_{x,y}$$

is continuous on $W^s_N = \{(x,y) \in M \times M : f^N(y) \in \mathcal{W}^s_{\mathrm{loc}}(f^N(x))\}$, for every $N \geq 0$.

Proof. — Notice that $\operatorname{dist}(x,y) \leq 2R$ for all $(x,y) \in W^s_0$, by our definition of local strong-stable leaves. So, the Cauchy estimate in (3.4)

$$\begin{aligned} \|A^{n+1}(y)^{-1}A^{n+1}(x) - A^n(y)^{-1}A^n(x)\| &\leq CL_2\big[\nu^n(x)^{(1-m)}\operatorname{dist}(x,y)\big]^{\beta}. \\ &\leq CL_2(2R)^{\beta}\nu^n(x)^{\beta(1-m)} \end{aligned} \tag{3.5}$$

is uniform on W^s_0. This implies that the limit in part (a) of Proposition 3.2 is uniform on W^s_0. That implies case $N = 0$ of the present proposition. The general case follows immediately, using property (b) in Proposition 3.2. ☐

Remark 3.4. — Since the constants C and L_2 are uniform on some neighborhood of A, the Cauchy estimate (3.5) is also locally uniform on A. Thus, the limit in part (a) of Proposition 3.2 is locally uniform on A as well. Consequently, the stable holonomy also depends continuously on the cocycle, in the sense that

$$(A,x,y) \mapsto H^s_{A,x,y} \quad \text{is continuous on } \mathcal{G}^{r,\alpha}(M,d,\mathbb{K}) \times W^s_0.$$

Using property (b) in Proposition 3.2 we may even replace W^s_0 by any W^s_N.

Dually, one finds an *invariant unstable holonomy* $(x,y) \mapsto H^u_{x,y}$ for A (or the cocycle F), given by

$$H^u_{x,y} = \lim_{n\to-\infty} A^n(y)^{-1}A^n(x)$$

whenever x and y are on the same strong-unstable leaf, and it is continuous on $W^u_N = \{(x,y) \in M \times M : f^{-N}(y) \in \mathcal{W}^s_{\mathrm{loc}}(f^{-N}(x))\}$, for every $N \geq 0$. Even more,

$$(A,x,y) \mapsto H^u_{A,x,y} \quad \text{is continuous on every } \mathcal{G}^{r,\alpha}(M,d,\mathbb{K}) \times W^u_N.$$

3.2. Differentiability of holonomies. — Now we study the differentiability of stable holonomies $H^s_{A,x,y}$ as functions of $A \in \mathcal{G}^{r,\alpha}(M,d,\mathbb{K})$. Notice that $\mathcal{G}^{r,\alpha}(M,d,\mathbb{K})$ is an open subset of the Banach space of $C^{r,\alpha}$ maps from M to the space of all $d \times d$ matrices and so the tangent space at each point of $\mathcal{G}^{r,\alpha}(M,d,\mathbb{K})$ is naturally identified with that Banach space. The next proposition is similar to Lemma 2.9 in [**23**], but our proof is neater: the previous argument used a stronger fiber bunching condition.

Proposition 3.5. — *Suppose that $A \in \mathcal{G}^{r,\alpha}(M,d,\mathbb{K})$ is fiber bunched. Then there exists a neighborhood $\mathcal{U} \subset \mathcal{G}^{r,\alpha}(M,d,\mathbb{K})$ of A such that, for any $x \in M$ and any y, $z \in \mathcal{W}^s(x)$, the map $B \mapsto H^s_{B,y,z}$ is of class C^1 on $\mathcal{U}$, with derivative*

$$\begin{aligned}(3.6)\quad \partial_B H^s_{B,y,z} : \dot{B} \mapsto \sum_{i=0}^{\infty} B^i(z)^{-1} \Big[& H^s_{B,f^i(y),f^i(z)} B(f^i(y))^{-1} \dot{B}(f^i(y)) \\ & - B(f^i(z))^{-1} \dot{B}(f^i(z)) H^s_{B,f^i(y),f^i(z)} \Big] B^i(y).\end{aligned}$$

Proof. — There are three main steps. Recall that fiber bunching is an open condition and the constants in Lemma 3.1 and Proposition 3.2 may be taken uniform on some neighborhood $\mathcal{U}$ of A. First, we suppose that y, z are in the local strong-stable leaf of x, and prove that the expression $\partial_B H^s_{B,y,z} \dot{B}$ is well defined for every $B \in \mathcal{U}$ and every $\dot{B}$ in $T_B \mathcal{G}^{r,\alpha}(M,d,\mathbb{K})$. Next, still in the local case, we show that this expression indeed gives the derivative of our map with respect to the cocycle. Finally, we extend the conclusion to arbitrary points on the global strong-stable leaf of x.

Step 1. For each $i \geq 0$, write

$$(3.7)\qquad H^s_{B,f^i(y),f^i(z)} B(f^i(y))^{-1} \dot{B}(f^i(y)) - B(f^i(z))^{-1} \dot{B}(f^i(z)) H^s_{B,f^i(y),f^i(z)}$$

as the following sum

$$\begin{aligned}(H^s_{B,f^i(y),f^i(z)} - \mathrm{id}) B(f^i(y))^{-1} \dot{B}(f^i(y)) + B(f^i(z))^{-1} \dot{B}(f^i(z))(\mathrm{id} - H^s_{B,f^i(y),f^i(z)}) \\ + [B(f^i(y))^{-1} \dot{B}(f^i(y)) - B(f^i(z))^{-1} \dot{B}(f^i(z))].\end{aligned}$$

By property (d) in Proposition 3.2, the first term is bounded by

$$\begin{aligned}(3.8)\quad L\, \|B(f^i(y))^{-1}\|\, \|\dot{B}(f^i(y))\|\, \mathrm{dist}(f^i(y), f^i(z))^{\beta} \\ \leq L\, \|B^{-1}\|_{0,0}\, \|\dot{B}\|_{0,0} \Big[\nu^i(x)\, \mathrm{dist}(y,z)\Big]^{\beta}\end{aligned}$$

and analogously for the second one. The third term may be written as

$$B(f^i(y))^{-1}[\dot{B}(f^i(y)) - \dot{B}(f^i(z))] + [B(f^i(y))^{-1} - B(f^i(z))^{-1}]\dot{B}(f^i(z)).$$

Using the triangle inequality, we conclude that this is bounded by

$$\begin{aligned}(3.9)\quad \Big(\|B(f^i(y))^{-1}\|\, H_\beta(\dot{B}) + H_\beta(B^{-1})\, \|\dot{B}(f^i(z))\|\Big)\, \mathrm{dist}(f^i(y), f^i(z))^{\beta}. \\ \leq\ \|B^{-1}\|_{0,\beta}\, \|\dot{B}\|_{0,\beta} \Big[\nu^i(x)\, \mathrm{dist}(y,z)\Big]^{\beta},\end{aligned}$$

where $H_\beta(\phi)$ means the smallest $C \ge 0$ such that $\|\phi(z) - \phi(w)\| \le C \operatorname{dist}(z,w)^\beta$ for all z, $w \in M$. Notice, from the definition (3.1), that

$$\|\phi\|_{0,0} + H_\beta(\phi) = \|\phi\|_{0,\beta} \le \|\phi\|_{r,\alpha} \quad \text{for any function } \phi. \tag{3.10}$$

Let $C_1 = \sup\{\|B^{-1}\|_{0,\beta} : B \in \mathcal{U}\}$. Replacing (3.8) and (3.9) in the expression preceding them, we find that the norm of (3.7) is bounded by

$$(2L+1)C_1\, \nu^i(x)^\beta \operatorname{dist}(y,z)^\beta \|\dot{B}\|_{0,\beta}$$

Hence, the norm of the ith term in the expression of $\partial_B H^s_{B,y,z}\,\dot{B}$ is bounded by

$$\begin{aligned} 2(L+1)C_1\ \nu^i(x)^\beta \|B^i(z)^{-1}\|\, \|B^i(y)\| \ \operatorname{dist}(y,z)^\beta \|\dot{B}\|_{0,\beta} \\ \le C_2\, \nu^i(x)^{\beta(1-m)} \ \operatorname{dist}(y,z)^\beta \|\dot{B}\|_{0,\beta} \end{aligned} \tag{3.11}$$

where $C_2 = 2C(L+1)C_1$ and C is the constant in Lemma 3.1. In this way we find,

$$\|\partial_B H^s_{B,y,z}(\dot{B})\| \le C_2 \sum_{i=0}^{\infty} \nu^i(x)^{\beta(1-m)} \operatorname{dist}(y,z)^\beta \|\dot{B}\|_{0,\beta} \tag{3.12}$$

for any $x \in M$ and y, $z \in \mathcal{W}^s_{\mathrm{loc}}(x)$. This shows that the series defining $\partial_B H^s_{B,y,z}(\dot{B})$ does converge at such points.

Step 2. By part (a) of Proposition 3.2 together with Remark 3.4, the map $H^s_{B,y,z}$ is the uniform limit $H^n_{B,y,z} = B^n(z)^{-1}B^n(y)$ when $n \to \infty$. Clearly, every $H^n_{B,y,z}$ is a differentiable function of B, with derivative

$$\begin{aligned} \partial_B H^n_{B,y,z}(\dot{B}) = \sum_{i=0}^{n-1} B^i(z)^{-1} \big[& H^{n-i}_{B,f^i(y),f^i(z)} B(f^i(y))^{-1} \dot{B}(f^i(y)) \\ & - B(f^i(z))^{-1} \dot{B}(f^i(z)) H^{n-i}_{B,f^i(y),f^i(z)} \big] B^i(y). \end{aligned}$$

So, to prove that $\partial_B H^s_{B,y,z}$ is indeed the derivative of the holonomy with respect to B, it suffices to show that $\partial H^n_{B,y,z}$ converges uniformly to $\partial H^s_{B,y,z}$ when $n \to \infty$.

Write $1 - m = 2\tau$. From (3.4) and the fact that $\nu(\cdot)$ is strictly smaller than 1,

$$\begin{aligned} \|H^n_{B,y,z} - H^s_{B,y,z}\| &\le CL_2 \sum_{j=n}^{\infty} \nu^j(x)^{\beta(1-m)} \operatorname{dist}(y,z)^\beta \\ &\le C_3 \nu^n(x)^{2\beta\tau} \operatorname{dist}(y,z)^\beta \le C_3 \nu^n(x)^{\beta\tau} \operatorname{dist}(y,z)^\beta \end{aligned}$$

for some uniform constant C_3 (the last inequality is trivial, but it will allow us to come out with a positive exponent for $\nu^i(x)$ in (3.13) below). More generally, and for the same reasons,

$$\begin{aligned} \|H^{n-i}_{B,f^i(y),f^i(z)} - H^s_{B,f^i(y),f^i(z)}\| &\le C_3 \nu^{n-i}(f^i(x))^{\beta\tau} \operatorname{dist}(f^i(y), f^i(z))^\beta \\ &\le C_3 \nu^{n-i}(f^i(x))^{\beta\tau} \nu^i(x)^\beta \operatorname{dist}(y,z)^\beta \\ &= C_3 \nu^n(x)^{\beta\tau} \nu^i(x)^{\beta(1-\tau)} \operatorname{dist}(y,z)^\beta \end{aligned}$$

for all $0 \leq i \leq n$, and all y, z in the same local strong-stable leaf. It follows, using also Lemma 3.1, that the norm of the difference between the ith terms in the expressions of $\partial_B H^n_{B,y,z}$ and $\partial_B H^s_{B,y,z}$ is bounded by

$$C_3 \nu^n(x)^{\beta\tau} \nu^i(x)^{\beta(1-\tau)} \operatorname{dist}(y,z)^\beta \|B^i(z)^{-1}\| \, \|B^i(y)\| \leq C C_3 \nu^n(x)^{\beta\tau} \nu^i(x)^{\beta\tau} \operatorname{dist}(y,z)^\beta. \tag{3.13}$$

Combining this with (3.11), we find that $\|\partial_B H^n_{B,y,z} - \partial_B H^s_{B,y,z}\|$ is bounded by

$$C C_3 \sum_{i=0}^{n-1} \nu^i(x)^{\beta\tau} \nu^n(x)^{\beta\tau} \operatorname{dist}(y,z)^\beta + C_2 \sum_{i=n}^{\infty} \nu^i(x)^{2\beta\tau} \operatorname{dist}(y,z)^\beta.$$

Since $\nu^i(x)$ is bounded away from 1, the sum is bounded by $C_4 \nu^n(x)^{\beta\tau} \operatorname{dist}(y,z)^\beta$, for some uniform constant C_4. This latter expression tends to zero uniformly when $n \to \infty$, and so the argument is complete.

Step 3. From property (b) in Proposition 3.2, we find that if $H^s_{B,f(y),f(z)}$ is differentiable on B then so is $H^s_{B,y,z}$ and the derivative is determined by

$$\dot{B}(z)\, H^s_{B,y,z} + B(z) \cdot \partial_B H^s_{B,y,z}(\dot{B}) = H^s_{B,y,z} \cdot \dot{B}(y) + \partial_B H^s_{B,y,z}(\dot{B}) \cdot B(y). \tag{3.14}$$

Combining this observation with the previous two steps, we conclude that $H^s_{B,y,z}$ is differentiable on B for any pair of points y, z in the same (global) strong-stable leaf: just note that $f^n(y)$, $f^n(z)$ are in the same local strong-stable leaf for large n. Moreover, a straightforward calculation shows that the expression in (3.6) satisfies the relation (3.14). Therefore, (3.6) is the expression of the derivative for all points y, z in the same strong-stable leaf. The proof of the proposition is now complete. □

Corollary 3.6. — *Suppose that $A \in \mathcal{G}^{r,\alpha}(M,d,\mathbb{K})$ is fiber bunched. Then there exists $\theta < 1$ and a neighborhood $\mathcal{U}$ of A and, for each $a > 0$, there exists $C_5(a) > 0$ such that*

$$\Big\| \sum_{i=k}^{\infty} B^i(z)^{-1} \Big[H^s_{B,f^i(y),f^i(z)} B(f^i(y))^{-1} \dot{B}(f^i(y)) - B(f^i(z))^{-1} \dot{B}(f^i(z)) H^s_{B,f^i(y),f^i(z)} \Big] B^i(y) \Big\| \leq C_5(a)\, \theta^k \, \|\dot{B}\|_{0,\beta} \tag{3.15}$$

for any $B \in \mathcal{U}$, $k \geq 0$, $x \in M$, and y, $z \in \mathcal{W}^s(x)$ with $\operatorname{dist}_{\mathcal{W}^s}(y,z) < a$.

Proof. — Let $\theta < 1$ be an upper bound for $\nu(\cdot)^{\beta(1-m)}$. Begin by supposing that $\operatorname{dist}_{\mathcal{W}^s}(y,z) < R$. Then y, z are in the same local strong-stable leaf, and we may use (3.11) to get that the expression in (3.15) is bounded above by

$$C_2 \sum_{i=k}^{\infty} \nu^i(x)^{\beta(1-m)} \operatorname{dist}(y,z)^\beta \|\dot{B}\|_{0,\beta} \leq C_5' \, \theta^k \, \|\dot{B}\|_{0,\beta}$$

for some uniform constant C_5'. This settles the case $a \leq R$, with $C_5(a) = C_5'$.

In general, there is $l \geq 0$ such that $\operatorname{dist}_{\mathcal{W}^s}(y,z) < a$ implies $\operatorname{dist}_{\mathcal{W}^s}(f^l(y), f^l(z)) < R$. Suppose first that $k \geq l$. Clearly, the expression in (3.15) does not change if we replace y, z by $f^l(y)$, $f^l(z)$ and replace k by $k-l$. Then, by the previous special case, (3.15) is bounded above by

$$C_5'\,\theta^{k-l}\,\|\dot{B}\|_{0,\beta}$$

and so it suffices to choose $C_5(a) \geq C_5'\theta^{-l}$. If $k < l$ then begin by splitting (3.15) into two sums, respectively, over $k \leq i < l$ and over $i \geq l$. The first sum is bounded by $C_5''(a)\|\dot{B}\|_{0,\beta}$ for some constant $C_5''(a) > 0$ that depends only on a (and l, which is itself a function of a). The second one is bounded by $C_5'\,\|\dot{B}\|_{0,\beta}$, as we have just seen. The conclusion follows, assuming we choose $C_5(a) \geq C_5'\theta^{-l} + C_5''(a)\theta^{-l}$. □

For future reference, let us state the analogues of Proposition 3.5 and Corollary 3.6 for invariant unstable holonomies:

Proposition 3.7. — *Suppose that $A \in \mathcal{G}^{r,\alpha}(M,d,\mathbb{K})$ is fiber bunched. Then there exists a neighborhood $\mathcal{U} \subset \mathcal{G}^{r,\alpha}(M,d,\mathbb{K})$ of A such that, for any $x \in M$ and any $y,z \in \mathcal{W}^u(x)$, the map $B \mapsto H^u_{B,y,z}$ is of class C^1 on $\mathcal{U}$ with derivative*

$$\begin{aligned}(3.16)\quad \partial_B H^u_{B,y,z} : \dot{B} \mapsto -\sum_{i=1}^{\infty} B^{-i}(z)^{-1}\Big[&H^u_{B,f^{-i}(y),f^{-i}(z)}B(f^{-i}(y))^{-1}\dot{B}(f^{-i}(y))\\ &- B(f^{-i}(z))^{-1}\dot{B}(f^{-i}(z))H^u_{B,f^{-i}(y),f^{-i}(z)}\Big]B^{-i}(y).\end{aligned}$$

Corollary 3.8. — *In the same setting as Proposition 3.7,*

$$\begin{aligned}(3.17)\quad \|\sum_{i=k}^{\infty} B^{-i}(z)^{-1}\Big[&H^u_{B,f^{-i}(y),f^{-i}(z)}B(f^{-i}(y))^{-1}\dot{B}(f^{-i}(y))\\ &- B(f^{-i}(z))^{-1}\dot{B}(f^{-i}(z))H^u_{B,f^{-i}(y),f^{-i}(z)}\Big]B^{-i}(y)\| \leq C_5(a)\,\theta^k\,\|\dot{B}\|_{0,\beta}.\end{aligned}$$

for any $B \in \mathcal{U}$, $k \geq 0$, $x \in M$, and y, $z \in \mathcal{W}^u(x)$ with $\operatorname{dist}_{\mathcal{W}^u}(y,z) < a$.

3.3. Dominated smooth cocycles. — Now we introduce a concept of domination for smooth cocycles, related to the notion of fiber bunching in the linear setting. We observe that dominated smooth cocycles admit invariant stable and unstable holonomies, and these holonomies vary continuously with the cocycle. These facts are included to make the analogy to the linear case more apparent but, otherwise, they are not used in the present paper: whenever dealing with smooth cocycles we just assume that invariant stable and unstable holonomies do exist. In this section we do not consider any invariant measure.

Let $\beta > 0$ be fixed. A fiber bundle with smooth leaves $\pi : \mathcal{E} \to M$ is called β-Hölder if there exists $C > 0$ such that the coordinate changes (2.7) satisfy

$$(3.18)\qquad \operatorname{dist}_{C^1}(g_x^{\pm 1}, g_y^{\pm 1}) \leq C\operatorname{dist}(x,y)^\beta \quad \text{for every } x \text{ and } y.$$

Then we say that a smooth cocycle $\mathfrak{F} : \mathcal{E} \to \mathcal{E}$ is β-Hölder if its local expressions $\phi_{U_1}^{-1} \circ \mathfrak{F} \circ \phi_{U_0} : (U_0 \cap f^{-1}(U_1)) \times N \to U_1 \times N$, $(x,\xi) \mapsto (f(x), \mathfrak{F}_x^U(\xi))$ satisfy

$$(3.19)\qquad \operatorname{dist}_{C^1}(\mathfrak{F}_x^U, \mathfrak{F}_y^U) \leq C_U \operatorname{dist}(x,y)^\beta \quad \text{for some } C_U > 0 \text{ and every } x \text{ and } y.$$

This does not depend on the choice of the local charts. Indeed, any other local expression has the form $\mathfrak{F}_x^V = g'_{f(x)} \circ \mathfrak{F}_x^U \circ g_x^{-1}$ on the intersection of the domains of definition. Then, a straightforward use of the triangle inequality gives

$$\operatorname{dist}_{C^1}(\mathfrak{F}_x^V, \mathfrak{F}_y^V) \le C_V \operatorname{dist}(x,y)^\beta \quad \text{for every } x \text{ and } y,$$

where C_V depends on β, C, C_U and upper bounds for the norms of $D\mathfrak{F}_x^U$, Dg'_y, Dg_x^{-1} and Df.

Definition 3.9. — Denote by $\mathcal{C}^\beta(f,\mathcal{E})$ the space of cocycles $\mathfrak{F}$ that are β-Hölder continuous. A cocycle $\mathfrak{F} \in \mathcal{C}^\beta(f,\mathcal{E})$ is *s-dominated* if there is $\theta < 1$ such that

$$\|D\mathfrak{F}_x(\xi)^{-1}\| \, \nu(x)^\beta \le \theta \quad \text{for all } (x,\xi) \in \mathcal{E} \tag{3.20}$$

and it is *u-dominated* if there is $\theta < 1$ such that

$$\|D\mathfrak{F}_x(\xi)\| \, \hat\nu(x)^\beta \le \theta \quad \text{for all } (x,\xi) \in \mathcal{E}. \tag{3.21}$$

We say that F is *dominated* if it is both s-dominated and u-dominated.

In geometric terms, (3.20) means that the contractions of $\mathfrak{F}$ along the fibers are strictly weaker than the contractions of f along strong-stable leaves and (3.21) expresses a similar property for the expansions of $\mathfrak{F}$. These conditions are designed so that the usual graph transform argument yields a "strong-stable" lamination and a "strong-unstable" lamination for the map $\mathfrak{F}$, as we are going to see. Then the holonomy maps for these laminations constitute invariant stable and unstable holonomies for the cocycle.

Observe that both conditions (3.20)–(3.21) become stronger as β decreases to zero; this may be seen as a sort of compensation for the decreasing regularity (Hölder continuity) of the cocycle. The observations that follow extend, up to straightforward adjustments, to the case when these conditions hold for some iterate $\mathfrak{F}^\ell$, $\ell \ge 1$.

Proposition 3.10. — *Let $\mathfrak{F} \in \mathcal{C}^\beta(f,\mathcal{E})$ be s-dominated. Then there exists a unique partition $\mathcal{W}^s = \{\mathcal{W}^s(x,\xi) : (x,\xi) \in \mathcal{E}\}$ of $\mathcal{E}$ and there exists $C > 0$ such that*

(a) *every $\mathcal{W}^s(x,\xi)$ is a (C,β)-Hölder continuous graph over $\mathcal{W}^s(x)$;*
(b) *the partition is invariant: $\mathfrak{F}(\mathcal{W}^s(x,\xi)) \subset \mathcal{W}^s(\mathfrak{F}(x,\xi))$ for all $(x,\xi) \in \mathcal{E}$.*

Consider the family of maps $H^s_{x,y} : \mathcal{E}_x \to \mathcal{E}_y$ defined by $(y, H^s_{x,y}(\xi)) \in \mathcal{W}^s(x,\xi)$ for each $y \in \mathcal{W}^s(x)$. Then, for every x, y and z in the same strong-stable leaf,

(c) $H^s_{y,z} \circ H^s_{x,y} = H^s_{x,z}$ *and* $H^s_{x,x} = \mathrm{id}$
(d) $\mathfrak{F}_y \circ H^s_{x,y} = H^s_{f(x),f(y)} \circ \mathfrak{F}_x$
(e) *$H^s_{x,y} : \mathcal{E}_x \to \mathcal{E}_y$ is the uniform limit of $(\mathfrak{F}^n_y)^{-1} \circ \mathfrak{F}^n_x$ as $n \to \infty$;*
(f) *$H^s_{x,y} : \mathcal{E}_x \to \mathcal{E}_y$ is γ-Hölder continuous, where $\gamma > 0$ depends only on $\mathfrak{F}$, and $H^s_{x,y}$ is (C,γ)-Hölder continuous if x and y are in the same strong-stable leaf;*
(g) *$(x,y,\xi) \mapsto H^s_{x,y}(\xi)$ is continuous when (x,y) varies in the set of pairs of points in the same local strong-stable leaf.*

Moreover, there are dual statements for strong-unstable leaves, assuming that $\mathfrak{F}$ is u-dominated.

Outline of the proof. — This follows from the same normal hyperbolicity methods (Hirsch, Pugh, Shub [**15**]) that were used in the previous section for linear cocycles. Existence (a) and invariance (b) of the family $\mathcal{W}^s$ follow from a standard application of the graph transform argument (see Chapter 5 of [**22**]). The pseudo-group property (c) is a direct consequence of the definition of $H^s_{x,y}$. The invariance property (d) is a restatement of (b). To prove (e), notice that

$$H^s_{x,y} = (\mathfrak{F}^n_y)^{-1} \circ H^s_{f^n(x),f^n(y)} \circ \mathfrak{F}^n_x,$$

because the lamination $\mathcal{W}^s$ is invariant under $\mathfrak{F}$. Also, by (a), the uniform C^0 distance from $H^s_{f^n(x),f^n(y)}$ to the identity is bounded by

$$C \operatorname{dist}(f^n(x), f^n(y))^\beta \le C\big[\nu^n(x) \operatorname{dist}(x,y)\big]^\beta.$$

Putting these two observations together, we find that

$$\begin{aligned}\operatorname{dist}_{C^0}(H^s_{x,y}, (\mathfrak{F}^n_y)^{-1} \circ \mathfrak{F}^n_x) &\le \operatorname{Lip}\left((\mathfrak{F}^n_y)^{-1}\right) \operatorname{dist}_{C^0}(H^s_{f^n(x),f^n(y)}, \mathrm{id}) \\ &\le C \sup_\xi \|D\mathfrak{F}^n_y(\xi)^{-1}\| \, \nu^n(x)^\beta \operatorname{dist}(x,y)^\beta.\end{aligned}$$

So, by the domination condition (3.20),

$$\operatorname{dist}_{C^0}(H^s_{x,y}, (\mathfrak{F}^n_y)^{-1} \circ \mathfrak{F}^n_x) \le C\theta^n \operatorname{dist}(x,y)^\beta. \tag{3.22}$$

This proves (e). For pairs (x,y) in the same local strong-stable leaf, the right hand side of (3.22) is uniformly bounded by $CR^\beta\theta^n$. Since this converges to zero, we also get that the limit map $(x,y,\xi) \mapsto H^s_{x,y}(\xi)$ is continuous, as stated in (g).

The Hölder continuity property is another by-product of normal hyperbolicity theory. In this instance it can be derived as follows. In view of the invariance property (d), it suffices to consider the case when x and y are in the same local strong-stable leaf. Given nearby points $\xi, \eta \in \mathcal{E}_x$, let ξ', η' be their images under the holonomy map $H^s_{x,y}$. The domination hypothesis (3.20) ensures that there exists $n \le -c_1 \log \operatorname{dist}(\xi', \eta')$ (where $c_1 > 0$ is a uniform constant) such that the distance $\operatorname{dist}(f^n(x), f^n(y))$ between the fibers is much smaller than the distance $\operatorname{dist}(\mathfrak{F}^n_x(\xi'), \mathfrak{F}_x(\eta'))$ along the fiber, in such a way that,

$$\operatorname{dist}(\mathfrak{F}^n_x(\xi), \mathfrak{F}^n_x(\eta)) \ge \frac{1}{2} \operatorname{dist}(\mathfrak{F}^n_y(\xi'), \mathfrak{F}^n_y(\eta')).$$

Let $c_2 > 0$ be an upper bound for $\log \|D\mathfrak{F}^{\pm 1}_w\|$ over all $w \in M$. Then

$$\frac{\operatorname{dist}(\xi', \eta')}{\operatorname{dist}(\xi, \eta)} \le e^{2c_2 n} \frac{\operatorname{dist}(\mathfrak{F}^n_y(\xi'), \mathfrak{F}^n_y(\eta'))}{\operatorname{dist}(\mathfrak{F}^n_x(\xi), \mathfrak{F}^n_x(\eta))} \le 2e^{2c_2 n} \le 2d(\xi', \eta')^{-2c_1c_2}.$$

This gives $\operatorname{dist}(\xi', \eta') \le 2^\gamma \operatorname{dist}(\xi, \eta)^\gamma$ with $\gamma = 1/(1 + 2c_1c_2)$. □

Next, let $\mathcal{D}^{s,\beta}(f, \mathcal{E}) \subset \mathcal{C}^\beta(f, \mathcal{E})$ be the subset of s-dominated cocycles. It is clear from the definition that $\mathcal{D}^{s,\beta}(f, \mathcal{E})$ is an open subset, relative to the uniform C^1 distance

$$\operatorname{dist}_{C^1}(\mathfrak{F}, \mathfrak{G}) = \sup\{\operatorname{dist}_{C^1}(\mathfrak{F}_x, \mathfrak{G}_x) : x \in M\}.$$

We are going to see that invariant stable holonomies vary continuously with the cocycle inside $\mathcal{D}^{s,\beta}(f, \mathcal{E})$, relative to this distance. Analogously, invariant unstable holonomies vary continuously with the cocycle inside the subset $\mathcal{D}^{u,\beta}(f,\mathcal{E}) \subset \mathcal{C}^{\beta}(f,\mathcal{E})$ of u-dominated cocycles. We also denote by $\mathcal{D}^{\beta}(f,\mathcal{E}) \subset \mathcal{C}^{\beta}(f,\mathcal{E})$ the (open) subset of dominated cocycles.

Let $\mathcal{W}^s(\mathfrak{G}) = \{\mathcal{W}^s(\mathfrak{G},x,\xi) : (x,\xi) \in \mathcal{E}\}$ denote the strong-stable lamination of a dominated cocycle $\mathfrak{G}$, as in Proposition 3.10, and $H^s_{\mathfrak{G}} = H^s_{\mathfrak{G},x,y}$ be the corresponding stable holonomy:

$$(y, H^s_{\mathfrak{G},x,y}(\xi)) \in \mathcal{W}^s(\mathfrak{G},x,\xi). \tag{3.23}$$

Recall that $\mathcal{W}^s(\mathfrak{G},x,\xi)$ is a graph over $\mathcal{W}^s(x)$. We also denote by $\mathcal{W}^s_{\mathrm{loc}}(\mathfrak{G},x,\xi)$ the subset of points $(y, H^s_{\mathfrak{G},x,y}(\xi))$ with $y \in \mathcal{W}^s_{\mathrm{loc}}(x)$.

Proposition 3.11. — *Let $(\mathfrak{F}_k)_k$ be a sequence of cocycles converging to $\mathfrak{F}$ in $\mathcal{D}^{s,\beta}(f,\mathcal{E})$. Then, for every $x \in M$, $y \in \mathcal{W}^s(x)$, and $\xi \in \mathcal{E}_x$,*

(a) *$\mathcal{W}^s(\mathfrak{F}_k,x,\xi)$ is a β-Hölder graph; restricted to local strong-stable leaves, the multiplicative Hölder constant is uniform on (k,x,ξ);*
(b) *the sequence $(u_k)_k$ of functions defined by $\mathcal{W}^s_{\mathrm{loc}}(\mathfrak{F}_k,x,\xi) = \operatorname{graph} u_k$ converges uniformly to the function u defined by $\mathcal{W}^s_{\mathrm{loc}}(\mathfrak{F},x,\xi) = \operatorname{graph} u$; this convergence is uniform on (x,ξ);*
(c) *$H^s_{\mathfrak{F}_k,x,y}$ converges uniformly to $H^s_{\mathfrak{F},x,y}$; this convergence is uniform on (x,y), restricted to the set of pairs of points in the same local strong-stable leaf.*

Moreover, there are dual statements for invariant unstable holonomies, in the space of u-dominated cocycles.

Outline of the proof. — This is another standard consequence of the graph transform argument [**15**]. Indeed, the assumptions imply that the graph transform of $\mathfrak{F}_k$ converges to the graph transform of $\mathfrak{F}$ in an appropriate sense, so that the corresponding fixed points converge as well. This yields (a) and (b). When $y \in \mathcal{W}^s_{\mathrm{loc}}(x)$, claim (c) is a direct consequence of (b) and the definition (3.23). The general statement follows, using the invariance property in Proposition 3.10:

$$H^s_{\mathfrak{F}_k,x,y} = (\mathfrak{F}^n_{k,y})^{-1} \circ H_{\mathfrak{F}_k,f^n(x),f^n(y)} \circ \mathfrak{F}^n_{k,x}.$$

Related facts were proved in [**23**, Section 4] for linear cocycles, along these lines. □

Remark 3.12. — The previous observations do not need the full strength of partial hyperbolicity. Indeed, the definition of s-dominated cocycle still makes sense if one allows the subbundle E^u in (2.1) to have dimension zero; moreover, all the statements about invariant stable holonomies in Propositions 3.10 and 3.11 remain valid in this case. Analogously, for defining u-domination and for the statements about invariant unstable holonomies one may allow E^s to have dimension zero.

Remark 3.13. — It follows from (2.13)-(2.14) that if a linear cocycle F is fiber bunched then the associated projective cocycle $\mathfrak{F} = \mathbb{P}(F)$ is dominated. Thus, we could use Proposition 3.10 to conclude that $\mathfrak{F}$ admits invariant stable and unstable holonomies. On the other hand, it is easy to exhibit these holonomies explicitly: if $H^s_{x,y}$ and $H^u_{x,y}$ are invariant holonomies for F then $\mathbb{P}(H^s_{x,y})$ and $\mathbb{P}(H^u_{x,y})$ are invariant holonomies for $\mathfrak{F}$.

4. Invariant measures of smooth cocycles

In this section we prove the following result and we use it to obtain Theorem C:

Theorem 4.1. — *Let f be a C^1 partially hyperbolic diffeomorphism, $\mathfrak{F}$ be a smooth cocycle over f, μ be an f-invariant probability, and m be an $\mathfrak{F}$-invariant probability on $\mathcal{E}$ such that $\pi_* m = \mu$.*

(a) *If $\mathfrak{F}$ admits invariant stable holonomies and $\lambda_-(\mathfrak{F}, x, \xi) \geq 0$ at m-almost every point $(x, \xi) \in \mathcal{E}$ then, for any disintegration $\{m_x : x \in M\}$ of m into conditional probabilities along the fibers, there exists a full μ-measure subset M^s such that $m_z = (H^s_{y,z})_* m_y$ for every y, $z \in M^s$ in the same strong-stable leaf.*

(b) *If $\mathfrak{F}$ admits invariant unstable holonomies and $\lambda_+(\mathfrak{F}, x, \xi) \leq 0$ at m-almost every point $(x, \xi) \in \mathcal{E}$ then, for any disintegration $\{m_x : x \in M\}$ of m into conditional probabilities along the fibers, there exists a full μ-measure subset M^u such that $m_z = (H^u_{y,z})_* m_y$ for every y, $z \in M^u$ in the same strong-unstable leaf.*

Remark 4.2. — Theorem 4.1 does not require full partial hyperbolicity. Indeed, the proof of part (a) that we will present in the sequel remains valid when $\dim E^u = 0$. Analogously, part (b) remains true when $\dim E^s = 0$.

Theorem C can be readily deduced from Theorem 4.1 and Theorem D, as follows. Given any disintegration $\{m_x : x \in M\}$ of the probability m, define $\Psi(x) = m_x$ at every point. According to Theorem 4.1, the function Ψ is essentially s-invariant and essentially u-invariant. By Theorem D, there exists a bi-invariant function $\tilde{\Psi}$ defined on some bi-saturated full measure set $\tilde{M}$ and coinciding with Ψ almost everywhere. Then we get a new disintegration $\{\tilde{m}_x : x \in M\}$ by setting $\tilde{m}_x = \tilde{\Psi}(x)$ when $x \in \tilde{M}$ and extending the definition arbitrarily to the complement. The conclusion of Theorem D means that this new disintegration is both s-invariant and u-invariant on $\tilde{M}$. Moreover, it is continuous if f is accessible.

The proof of Theorem 4.1 is given in Sections 4.1 through 4.4. Theorem D will be proved in Sections 6 and 7.

4.1. Abstract invariance principle. — Let $(M_*, \mathcal{M}_*, \mu_*)$ be a Lebesgue space, that is, a complete separable probability space. Every Lebesgue space is isomorphic $\bmod 0$ to the union of an interval, endowed with the Lebesgue measure, and a finite or countable set of atoms. See Rokhlin [**21**, § 2]. Let $T : M_* \to M_*$ be an invertible measurable transformation. A σ-algebra $\mathcal{B} \subset \mathcal{M}_*$ is *generating* if its iterates $T^n(\mathcal{B})$, $n \in \mathbb{Z}$ generate the whole $\mathcal{M}_*$ $\bmod 0$: for every $E \in \mathcal{M}_*$ there exists E' in the smallest σ-algebra that contains all the $T^n(\mathcal{B})$ such that $\mu_*(E \Delta E') = 0$.

***Theorem 4.3* (Ledrappier [18]).** — *Let $B : M_* \to \mathrm{GL}(d, \mathbb{K})$ be a measurable map such that the functions $x \mapsto \log \|B(x)^{\pm 1}\|$ are μ_*-integrable. Let $\mathcal{B} \subset \mathcal{M}_*$ be a generating σ-algebra such that both T and B are $\mathcal{B}$-measurable* $\bmod 0$.

If $\lambda_-(B, x) = \lambda_+(B, x)$ at μ_*-almost every $x \in M_*$ then, for any $\mathbb{P}(F_B)$-invariant probability m that projects down to μ_*, any disintegration $x \mapsto m_x$ of m along the fibers is $\mathcal{B}$-measurable $\bmod 0$.

The proof of Theorem 4.1 is based on an extension of this result to smooth cocycles that was recently proved by Avila, Viana [**3**]. For the statement one needs to introduce the following notion. A *deformation of a smooth cocycle* $\mathfrak{F}$ is a measurable transformation $\tilde{\mathfrak{F}} : \mathcal{E} \to \mathcal{E}$ which is conjugated to $\mathfrak{F}$,

$$\tilde{\mathfrak{F}} = \mathcal{H} \circ \mathfrak{F} \circ \mathcal{H}^{-1},$$

by some invertible measurable map $\mathcal{H} : \mathcal{E} \to \mathcal{E}$ of the form $\mathcal{H}(x, \xi) = (x, \mathcal{H}_x(\xi))$, such that all the $\mathcal{H}_x^{-1} : \mathcal{E}_x \to \mathcal{E}_x$ are Hölder continuous, with uniform Hölder constants: there exist positive constants γ and Γ such that

$$\operatorname{dist}(\xi, \eta) \leq \Gamma \operatorname{dist}(\mathcal{H}_x(\xi), \mathcal{H}_x(\eta))^\gamma \quad \text{for every } x \in M \text{ and } \xi, \eta \in \mathcal{E}_x.$$

To each $\mathfrak{F}$-invariant probability m corresponds an $\tilde{\mathfrak{F}}$-invariant probability $\tilde{m} = \mathcal{H}_* m$.

***Theorem 4.4* (Avila, Viana [3]).** — *Let $\tilde{\mathfrak{F}}$ be a deformation of a smooth cocycle $\mathfrak{F}$. Let $\mathcal{B} \subset \mathcal{M}_*$ be a generating σ-algebra such that both T and $x \mapsto \tilde{\mathfrak{F}}_x$ are $\mathcal{B}$-measurable* $\bmod 0$. *Let $\tilde{m}$ be an $\tilde{\mathfrak{F}}$-invariant probability that projects down to μ_*.*

If $\lambda_-(\mathfrak{F}, x, \xi) \geq 0$ for m-almost every $(x, \xi) \in \mathcal{E}$ then any disintegration $x \mapsto \tilde{m}_x$ of $\tilde{m}$ along the fibers is $\mathcal{B}$-measurable $\bmod 0$.

4.2. Global essential invariance. — For proving Theorem 4.1 it suffices to consider the claim (a): then claim (b) is obtained just by reversing time. In this section we reduce the general case to a local version of the claim (Proposition 4.5 below), whose proof is postponed until Section 4.4.

For each symbol $* \in \{s, u\}$ and $r > 0$, denote by $\mathcal{W}^*(x, r)$ the neighborhood of radius r around x inside the leaf $\mathcal{W}^*(x)$. Recall that we write $\mathcal{W}^*_{\text{loc}}(x) = \mathcal{W}^*(x, R)$.

***Proposition 4.5*.** — *Consider the setting of Theorem 4.1(a). Let Σ be a cross-section to the strong-stable foliation $\mathcal{W}^s$ of f and let $\delta \in (0, R/2)$. Denote*

$$\mathcal{N}(\Sigma, \delta) = \bigcup_{z \in \Sigma} \mathcal{W}^s(z, \delta)$$

Then there exists a full μ-measure subset $\mathcal{N}^s$ of $\mathcal{N}(\Sigma, \delta)$ such that $m_y = (H^s_{x,y})_* m_x$ for every x, $y \in \mathcal{N}^s$ in the same $\mathcal{W}^s(z, \delta)$, $z \in \Sigma$.

Fix any $\delta \in (0, R/2)$. For each $p \in M$, consider a cross-section $\Sigma(p)$ such that $\mathcal{N}(\Sigma(p), \delta)$ contains p in its interior and let $\mathcal{N}^s(p) \subset \mathcal{N}(\Sigma(p), \delta)$ be a full measure subset as in Proposition 4.5. By compactness, we may find $\varepsilon \ll \delta$ and points $p_1, \ldots, p_N$ such that the ball of radius ε around every point of M is contained in some $\mathcal{N}(\Sigma(p_j), \delta)$. Since the measure m is invariant under $\mathfrak{F}$, there exists an f-invariant set $M_m \subset M$ with full μ-measure such that $m_{f(x)} = (\mathfrak{F}_x)_* m_x$ for every $x \in M_m$. Take

$$M^s = \{x \in M_m : f^n(x) \notin \mathcal{N}(\Sigma(p_j), \delta) \setminus \mathcal{N}^s(p_j) \text{ for all } n \geq 0 \text{ and } j = 1, \ldots, N.\}$$

Given any pair of points x, $y \in M^s$ in the same strong-stable leaf, take $n \geq 0$ large enough so that the distance from $f^n(x)$ to $f^n(y)$ along the corresponding strong-stable leaf is less than ε. Next, fix j such that $\mathcal{N}(\Sigma(p_j), \delta)$ contains the ball of radius

ε around $f^n(x)$. Since $x,\, y \in M^s$, both points $f^n(x)$, $f^n(y)$ belong to $\mathcal{N}^s(p_j)$. So, by Proposition 4.5,

$$m_{f^n(y)} = (H^s_{f^n(x),f^n(y)})_* m_{f^n(x)}. \tag{4.1}$$

Since $x,\, y \in M_m$, we also have that $m_{f^n(x)} = (\mathfrak{F}^n_x)_* m_x$ and analogously for y. Then, using the invariance relation $H^s_{f^n(x),f^n(y)} \circ \mathfrak{F}^n_x = \mathfrak{F}^n_y \circ H^s_{x,y}$, the equality in (4.1) becomes $m_y = (H^s_{x,y})_* m_x$.

This proves claim (a) in Theorem 4.1. Claim (b) is analogous, up to time reversion. Thus, we have reduced the proof of Theorem 4.1 to proving Proposition 4.5.

4.3. A local Markov construction. — The proof of Proposition 4.5 can be outlined as follows. The assumption that the cocycle admits stable holonomy allows us to construct a special deformation $\tilde{\mathfrak{F}}$ of the smooth cocycle $\mathfrak{F}$ which is measurable mod 0 with respect to a certain σ-algebra $\mathcal{B}$. Applying Theorem 4.4 we get that the disintegration of $\tilde{m}$ is also $\mathcal{B}$-measurable mod 0, where $\tilde{m}$ is the $\tilde{\mathfrak{F}}$-invariant measure corresponding to m. When translated back to the original setting, this $\mathcal{B}$-measurability property means that the disintegration of m is essentially invariant on the domain $\mathcal{N}(\Sigma, \delta)$, as stated in Proposition 4.5.

In this section we construct $\tilde{\mathfrak{F}}$ and $\mathcal{B}$. The next proposition is the main tool. It is essentially taken from Proposition 3.3 in [**23**], so here we just outline the construction.

Proposition 4.6. — *Let Σ be a cross-section to the strong-stable foliation $\mathcal{W}^s$ and $\delta \in (0, R/2)$. Then there exists $N \geq 1$ and a measurable family of sets $\{S(z) : z \in \Sigma\}$ such that*

(a) *$\mathcal{W}^s(z,\delta) \subset S(z) \subset \mathcal{W}^s_{\mathrm{loc}}(z)$ for all $z \in \Sigma$;*
(b) *for all $l \geq 1$ and $z, \zeta \in \Sigma$, if $f^{lN}(S(\zeta)) \cap S(z) \neq \emptyset$ then $f^{lN}(S(\zeta)) \subset S(z)$.*

Outline of the proof. — Fix N big enough so that $\nu^N(x) < 1/4$ for all $x \in M$, and denote $g = f^N$. For each $z \in \Sigma$ define $S_0 = \mathcal{W}^s(z, \delta)$ and

$$S_{n+1}(z) = S_0(z) \cup \bigcup_{(j,w)\in Z_n(z)} g^j(S_n(w)) \tag{4.2}$$

where $Z_n(z) = \big\{(j,w) \in \mathbb{N} \times \Sigma : g^j(S_n(w)) \cap S_0(z) \neq \emptyset\big\}$. Clearly, $S_0(z) \subset S_1(z)$ and $Z_0(z) \subset Z_1(z)$. Notice that if $S_{n-1}(z) \subset S_n(z)$ and $Z_{n-1}(z) \subset Z_n(z)$ for every $z \in \Sigma$, then,

$$\bigcup_{(j,w)\in Z_{n-1}(z)} g^j(S_{n-1}(w)) \subset \bigcup_{(j,w)\in Z_n(z)} g^j(S_n(w)).$$

Therefore, by induction, $S_n(z) \subset S_{n+1}(z)$ and $Z_n(z) \subset Z_{n+1}(z)$ for every $n \geq 0$. Define

$$S_\infty(z) = \bigcup_{n=0}^{\infty} S_n(z) \quad\text{and}\quad Z_\infty(z) = \bigcup_{n=0}^{\infty} Z_n(z).$$

Then $Z_\infty(z)$ is the set of $(j,w) \in \mathbb{N} \times \Sigma$ such that $g^j(S_\infty(w))$ intersects $S_0(z)$, and

$$S_\infty(z) = S_0(z) \cup \bigcup_{(j,w)\in Z_\infty(z)} g^j(S_\infty(w)).$$

The choice of N ensures that $S_\infty(z) \subset \mathcal{W}^s(z, 2\delta)$. Finally, define

$$S(z) = S_\infty(z) \setminus \bigcup_{(k,\xi)\in V(z)} g^k(S_\infty(\xi))$$

where $V(z) = \{(k,\xi) \in \mathbb{N} \times \Sigma : g^k(S_\infty(\xi)) \not\subset S_\infty(z)\}$. This family of sets satisfies the conclusion of the proposition. $\square$

Since the conclusion of Proposition 4.5 is not affected when f and $\mathfrak{F}$ are replaced by its iterates f^N and $\mathfrak{F}^N$, we may take the integer N in Proposition 4.6 to be equal to 1. Let $M_* = M$ and $T = f$. Let $\mathcal{M}_*$ be the μ-completion of the Borel σ-algebra of M and μ_* be the canonical extension of μ to $\mathcal{M}_*$. Then $(M_*, \mathcal{M}_*, \mu_*)$ is a Lebesgue space and T is an automorphism in it.

For each $z \in \Sigma$, let $r(z) \geq 0$ be the largest integer (possibly infinite) such that $f^j(S(z))$ does not intersect any of the $S(w)$, $w \in \Sigma$ for all $0 < j \leq r(z)$. Let $\mathcal{B}$ be the σ-algebra of sets $E \in \mathcal{M}_*$ such that, for every z and j, either E contains $f^j(S(z))$ or is disjoint from it. Notice that an $\mathcal{M}$-measurable function on M is $\mathcal{B}$-measurable precisely if it is constant on every $f^j(S(z))$. Define $\tilde{\mathfrak{F}} : \mathcal{E} \to \mathcal{E}$ to be $\tilde{\mathfrak{F}} = \mathcal{H} \circ \mathfrak{F} \circ \mathcal{H}^{-1}$, where

$$\mathcal{H}_x = \begin{cases} H^s_{x,f^j(z)} & \text{if } x \in f^j(S(z)) \text{ for some } z \in \Sigma \text{ and } 0 \leq j \leq r(z) \\ \mathrm{id} & \text{otherwise.} \end{cases}$$

Recall that $S(z) \subset \mathcal{W}^s_{\mathrm{loc}}(z)$ for every z, by construction. Reducing δ if necessary, we may assume that $f^j(S(z)) \subset \mathcal{W}^s_{\mathrm{loc}}(f^j(z))$ for every z and every $j \geq 0$. Then condition (d) in Definition 2.9 ensures that the family $\{\mathcal{H}_x : x \in M\}$ is uniformly Hölder continuous. The definition implies that

$$\tilde{\mathfrak{F}}_x = H^s_{f(x),f^{j+1}(z)} \circ \mathfrak{F}_x \circ H^s_{f^j(z),x} = \mathfrak{F}_{f^j(z)} \tag{4.3}$$

if $x \in f^j(S(z))$ for some $z \in \Sigma$ and $0 \leq j < r(z)$. Moreover,

$$\tilde{\mathfrak{F}}_x = H^s_{f(x),w} \circ \mathfrak{F}_x \circ H^s_{f^{r(z)}(z),x} \tag{4.4}$$

if $x \in f^{r(z)}(S(z))$ for some $z \in \Sigma$, where $w \in \Sigma$ is given by $f^{r(z)+1}(S(z)) \subset S(w)$. In all other cases, $\tilde{\mathfrak{F}}_x = \mathfrak{F}_x$.

Lemma 4.7. — *The following properties hold*

(a) $T = f$ *and* $x \mapsto \tilde{\mathfrak{F}}_x$ *are* $\mathcal{B}$*-measurable*
(b) $\mathrm{dist}_{C^0}(\mathcal{H}_x, \mathrm{id})$ *is uniformly bounded*
(c) $\{T^n(\mathcal{B}) : n \in \mathbb{N}\}$ *generates* $\mathcal{M}_* \bmod 0$.

Proof. — The relations (4.3) and (4.4) show that $\tilde{\mathfrak{F}}_x$ is constant on $f^j(S(z))$ for every $z \in \Sigma$ and $0 \leq j \leq r(z)$. Thus, $x \mapsto \tilde{\mathfrak{F}}_x$ is $\mathcal{B}$-measurable. $\mathcal{B}$-measurability of f is a simple consequence of the Markov property in Proposition 4.6. Indeed, let $E \in \mathcal{B}$ and let $z \in \Sigma$ and $0 \leq j \leq r(z)$ be such that $f^{-1}(E)$ intersects $f^j(S(z))$. Then E intersects $f^{j+1}(S(z))$. We claim that E contains $f^{j+1}(S(z))$. When $j+1 \leq r(z)$ this follows immediately from $E \in \mathcal{B}$. When $j = r(z)$, notice that $f^{j+1}(S(z)) \subset S(w)$ for some $w \in S(z)$, and $E \in \mathcal{B}$ must contain $S(w)$. So the claim holds in all cases.

It follows that $f^{-1}(E)$ contains $f^j(S(z))$. This proves that $f^{-1}(E) \in \mathcal{B}$, and so the proof of part (a) is complete. To prove part (b), observe that

$$\operatorname{diam} f^j(S(z)) \leq \operatorname{diam}_{\mathcal{W}^s} S(z) \leq R,$$

for all $z \in \Sigma$ and $j \geq 0$, and so

$$\sup_{x \in M} \operatorname{dist}_{C^0}(\mathcal{H}_x, \operatorname{id}) \leq \sup_{\operatorname{dist}(a,b) \leq R} \operatorname{dist}_{C^0}(H^s_{a,b}, \operatorname{id}).$$

The right hand side is uniformly bounded, since the stable holonomy depends continuously on the base points, and the space of $(a, b) \in M \times M$ with $\operatorname{dist}(a, b) \leq R$ is compact. This proves part (b). To prove the last claim, observe that $f^n(\mathcal{B})$ is the σ-algebra of sets $E \in \mathcal{M}_*$ such that every $f^{j+n}(S(z))$ either is contained in E or is disjoint from E. Observe that the diameter of $f^{j+n}(S(z))$ goes to zero, uniformly, when n goes to ∞. It follows that every open set can be written as a union of sets $E_n \in f^n(\mathcal{B})$ and, hence, belongs to the σ-algebra generated by $\{f^n(\mathcal{B}) : n \in \mathbb{N}\}$. This proves that the latter σ-algebra coincides mod 0 with the completion $\mathcal{M}_*$ of the Borel σ-algebra, as stated in (c). □

4.4. Local essential invariance. — Next, we deduce Proposition 4.5. By assumption, $\lambda_-(\mathfrak{F}, x, \xi) \geq 0$ at m-almost every point. Lemma 4.7 ensures that all the other assumptions of Theorem 4.4 are fulfilled as well. We conclude from the theorem that the disintegration $\{\tilde{m}_x : x \in M\}$ of the measure $\tilde{m} = \mathcal{H}_* m$ is measurable mod 0 with respect to the σ-algebra $\mathcal{B}$. Then, there exists a full μ-measure set $X^s \subset M$ such that this restriction of the disintegration to X^s is constant on every $f^j(S(z))$ with $z \in \Sigma$ and $0 \leq j \leq r(z)$. The disintegrations of m and $\tilde{m}$ are related to one another by

$$\tilde{m}_x = (\mathcal{H}_x)_* m_x = \begin{cases} (H^s_{x, f^j(z)})_* m_x & \text{if } x \in f^j(S(z)) \text{ for } z \in \Sigma \text{ and } 0 \leq j \leq r(z) \\ m_x & \text{otherwise.} \end{cases}$$

Define $\mathcal{N}^s = X^s \cap \mathcal{N}(\Sigma, \delta)$. Recall that $\mathcal{W}(z, \delta) \subset S(z)$ for all $z \in \Sigma$. Then, for every x, $y \in \mathcal{N}^s$ in the same $\mathcal{W}(z, \delta)$,

$$(H^s_{x,z})_* m_x = \tilde{m}_x = \tilde{m}_y = (H^s_{y,z})_* m_y,$$

and so $m_y = (H^s_{y,z})_*^{-1} (H^s_{x,z})_* m_x = (H_{x,y})_* m_x$. This proves Proposition 4.5. The proof of Theorem 4.1 is now complete.

5. Density points

In this section we recall some ideas of Burns, Wilkinson [**9**] that will be important in Section 6. The conclusions that interest us more directly are collected in Proposition 5.13.

Let us start with a few preparatory remarks. Recall that we take M to carry a Riemannian metric adapted to $f : M \to M$, meaning that properties (2.2)-(2.4) hold. Clearly, these properties are not affected by rescaling. At a few steps in the course of the arguments that follow we do allow for the Riemannian metric to be multiplied by some large constant.

Recall that we write $\mathcal{W}^*_{\rm loc}(x) = \mathcal{W}^*(x,R)$ for every $x \in M$ and $* \in \{s,u\}$, where is R a fixed constant. In the sequel we suppose that $R > 1$. Up to rescaling the metric, we may assume that the Riemannian ball $B(p,R)$ is contained in foliation boxes for both $\mathcal{W}^s$ and $\mathcal{W}^u$, for every $p \in M$. By further rescaling the metric, we may ensure that, given any $p \in M$ and x, $y \in B(p,R)$,

$$y \in \mathcal{W}^s_{\rm loc}(x) \quad \text{implies} \quad \operatorname{dist}(f(x), f(y)) \le \nu(p) \operatorname{dist}(x,y) \text{ and,}$$
$$y \in \mathcal{W}^u_{\rm loc}(x) \quad \text{implies} \quad \operatorname{dist}(f^{-1}(x), f^{-1}(y)) \le \hat\nu(f^{-1}(p)) \operatorname{dist}(x,y).$$

As a consequence, given any p, x, $y \in M$,

(I) $f(\mathcal{W}^s_{\rm loc}(x)) \subset \mathcal{W}^s_{\rm loc}(f(x))$ and $f^{-1}(\mathcal{W}^u_{\rm loc}(x)) \subset \mathcal{W}^u_{\rm loc}(f^{-1}(x))$.

(II) If $f^j(x) \in B(f^j(p),R)$ for $0 \le j < n$, and $y \in \mathcal{W}^s_{\rm loc}(x)$, then
$$\operatorname{dist}(f^n(x), f^n(y)) \le \nu^n(p) \operatorname{dist}(x,y);$$

(III) If $f^{-j}(x) \in B(f^{-j}(p),R)$ for $0 \le j < n$, and $y \in \mathcal{W}^u_{\rm loc}(x)$, then
$$\operatorname{dist}(f^{-n}(x), f^{-n}(y)) \le \hat\nu^{-n}(p) \operatorname{dist}(x,y).$$

These properties of the strong-stable and strong-unstable foliations of f are useful guidelines to the notion of fake foliations, that we are going to recall in Section 5.2.

5.1. Density sequences. — Let λ be the volume associated to the (adapted) Riemannian metric on M. We denote by λ_S the volume of the Riemannian metric induced on any immersed submanifold S. Given a continuous foliation $\mathcal{F}$ of M with smooth leaves, we denote by $\lambda_{\mathcal{F}}(A)$ the volume of a measurable subset A of some leaf F, relative to the Riemannian metric λ_F induced on that leaf.

By definition, λ and the invariant volume μ have the same zero measure sets. More important for our proposes, they have the same Lebesgue density points. Recall that $x \in M$ is a *Lebesgue density point* of a set $X \subset M$ if

$$\lim_{\delta \to 0} \lambda(X : B(x,\delta)) = 1$$

where $\lambda(A : B) = \lambda(A \cap B)/\lambda(B)$ is defined for general subsets A, B with $\lambda(B) > 0$. The Lebesgue Density Theorem asserts that $\lambda(X \,\Delta\, \mathrm{DP}(X)) = 0$ for any measurable set X, where $\mathrm{DP}(X)$ is the set of Lebesgue density points of X.

Balls may be replaced in the definition by other, but not arbitrary, families of neighborhoods of the point.

Definition 5.1. — A sequence of measurable sets $(Y_n)_n$ is a *Lebesgue density sequence at $x \in M$* if

(a) $(Y_n)_n$ *nests* at the point x: $Y_n \supset Y_{n+1}$ for every n and $\cap_n Y_n = \{x\}$

(b) $(Y_n)_n$ is *regular*: there is $\delta > 0$ such that $\lambda(Y_{n+1}) \ge \delta\lambda(Y_n)$ for every n

(c) x is a Lebesgue density point of an arbitrary measurable set X if and only if $\lim_{n\to\infty} \lambda(X : Y_n) = 1$.

Some of the sequences we are going to mention satisfy these conditions for special classes of sets only. In particular, we say that $(Y_n)_n$ is a *Lebesgue density sequence at x for bi-essentially saturated sets* if (c) holds for every bi-essentially saturated set X (this notion was defined in Section 2.1).

Burns, Wilkinson [**9**] propose two main techniques for defining new Lebesgue density sequences: internested sequences and Cavalieri's principle. The first one is quite simple and applies to general measurable sets. Two sequences $(Y_n)_n$ and $(Z_n)_n$ that nest at x are said to be *internested* if there is $k \geq 1$ such that

$$Y_{n+k} \subseteq Z_n \quad \text{and} \quad Z_{n+k} \subseteq Y_n \quad \text{for all } n \geq 0.$$

***Lemma 5.2* (Lemma 2.1 in [9]).** — *If $(Y_n)_n$ and $(Z_n)_n$ are internested then one sequence is regular if and only if the other one is. Moreover,*

$$\lim_{n\to\infty} \lambda(X : Y_n) = 1 \quad \Longleftrightarrow \quad \lim_{n\to\infty} \lambda(X : Z_n) = 1,$$

for any measurable set $X \subset M$.

Consequently, if two sequences are internested then one is a Lebesgue density sequence (respectively, a Lebesgue density sequence for bi-essentially saturated sets) if and only if the other is.

The second technique (Cavalieri's principle) is a lot more subtle and is specific to subsets essentially saturated by some absolutely continuous foliation $\mathcal{F}$ (with bounded Jacobians). Let U be a foliation box for $\mathcal{F}$ and Σ be a cross-section to $\mathcal{F}$ in U. The *fiber* of a set $Y \subset U$ over a point $q \in \Sigma$ is the intersection of Y with the local leaf of $\mathcal{F}$ in U containing q. The *base* of $Y \subset U$ is the set Σ_Y of points $q \in \Sigma$ whose fiber $Y(q)$ is a measurable set and has positive $\lambda_{\mathcal{F}}$-measure. The absolute continuity of $\mathcal{F}$ ensures that the base is a measurable set. We say that Y *fibers* over some set $Z \subset \Sigma$ if the basis Σ_Y equals Z. Given $c \geq 1$, a sequence of sets Y_n contained in U has *c-uniform fibers* if

$$c^{-1} \leq \frac{\lambda_{\mathcal{F}}(Y_n(q_1))}{\lambda_{\mathcal{F}}(Y_n(q_2))} \leq c \quad \text{for all } q_1,\, q_2 \in \Sigma_{Y_n} \text{ and every } n \geq 0. \tag{5.1}$$

***Proposition 5.3* (Proposition 2.7 in [9]).** — *Let $(Y_n)_n$ be a sequence of measurable sets in U with c-uniform fibers, for some c. Then, for any locally $\mathcal{F}$-saturated measurable set $X \subset U$,*

$$\lim_{n\to\infty} \lambda(X : Y_n) = 1 \quad \Longleftrightarrow \quad \lim_{n\to\infty} \lambda_\Sigma(\Sigma_X : \Sigma_{Y_n}) = 1.$$

By *locally $\mathcal{F}$-saturated* we mean that the set is a union of local leaves of $\mathcal{F}$ in the foliation box U. Sets that differ from a locally $\mathcal{F}$-saturated one by zero Lebesgue measure subsets are called *essentially locally $\mathcal{F}$-saturated*.

***Proposition 5.4* (Proposition 2.5 in [9]).** — *Let $(Y_n)_n$ and $(Z_n)_n$ be two sequences of measurable subsets of U with c-uniform fibers, for some c, and $\Sigma_{Y_n} = \Sigma_{Z_n}$ for all n. Then, for any essentially locally $\mathcal{F}$-saturated set $X \subset U$,*

$$\lim_{n\to\infty} \lambda(X : Y_n) = 1 \quad \Longleftrightarrow \quad \lim_{n\to\infty} \lambda(X : Z_n) = 1.$$

5.2. Fake foliations and juliennes. — Juliennes were proposed by Pugh, Shub [**19**] as density sequences particularly suited for partially hyperbolic dynamical systems. These are sets constructed by means of invariant foliations that are assumed to exist (dynamical coherence) tangent to the invariant subbundles E^s, E^u, $E^{cs} = E^c \oplus E^s$, $E^{cu} = E^c \oplus E^u$, and E^c, and they have nice properties of invariance under iteration and under the holonomy maps of the strong-stable and strong-unstable foliations. As mentioned before, strong-stable and strong-unstable foliations (tangent to the subbundles E^s and E^u, respectively) always exist in the partially hyperbolic setting. However, that is not always true about the center, center-stable, center-unstable subbundles E^c, E^{cs}, E^{cu}.

One main novelty in Burns, Wilkinson [**9**] was that, for the first time, the authors avoided the dynamical coherence assumption. A version of the julienne construction is still important in their approach, but now the definition is in terms of certain "approximations" to the (possibly nonexistent) invariant foliations, that they call fake foliations. We will not need to use fake foliations nor fake juliennes directly in this paper but, for the reader's convenience, we briefly describe their main features.

5.2.1. Fake foliations. — The central result about fake foliations is Proposition 3.1 in [**9**]: for any $\varepsilon > 0$ there exist constants $0 < \rho < r < R$ such that the ball of radius r around every point admits foliations

$$\widehat{\mathcal{W}}^u_p, \quad \widehat{\mathcal{W}}^s_p, \quad \widehat{\mathcal{W}}^c_p, \quad \widehat{\mathcal{W}}^{cu}_p, \quad \widehat{\mathcal{W}}^{cs}_p.$$

with the following properties, for any $* \in \{u, s, c, cs, cu\}$:

(i) For every $x \in B(p, \rho)$, the leaf $\widehat{\mathcal{W}}^*_p(x)$ is C^1 and the tangent space $T_x\widehat{\mathcal{W}}^*_p(x)$ is contained in the cone of radius ε around E^*_x.

(ii) For every $x \in B(p, \rho)$,
$$f(\widehat{\mathcal{W}}^*_p(x, \rho)) \subset \widehat{\mathcal{W}}^*_{f(p)}(f(x)) \quad \text{and} \quad f^{-1}(\widehat{\mathcal{W}}^*_p(x, \rho)) \subset \widehat{\mathcal{W}}^*_{f^{-1}(p)}(f^{-1}(x)).$$

(iii) Given $x \in B(p, \rho)$ and $n \geq 1$ such that $f^j(x) \in B(f^j(p), r)$ for $0 \leq j < n$, if $y \in \widehat{\mathcal{W}}^s_p(x, \rho)$ then $f^n(y) \in \widehat{\mathcal{W}}^s_{f^n(p)}(f^n(x), \rho)$ and
$$\operatorname{dist}(f^n(x), f^n(y)) \leq \nu^n(p) \operatorname{dist}(x, y).$$
Similarly for $\widehat{\mathcal{W}}^u$, with f replaced by its inverse.

(iv) Given $x \in B(p, \rho)$ and $n \geq 1$ such that $f^j(x) \in B(f^j(p), r)$ for $0 \leq j < n$, if $f^j(y) \in \widehat{\mathcal{W}}^{cs}_p(f^j(q), \rho)$ for $0 \leq j < n$ then $f^n(y) \in \widehat{\mathcal{W}}^{cs}_{f^n(p)}(f^n(x))$ and
$$\operatorname{dist}(f^n(x), f^n(y)) \leq \hat{\gamma}^n(p)^{-1} \operatorname{dist}(x, y).$$
Similarly for $\widehat{\mathcal{W}}^{cu}$, with f replaced by its inverse.

(v) $\widehat{\mathcal{W}}^u_p$ and $\widehat{\mathcal{W}}^c_p$ sub-foliate $\widehat{\mathcal{W}}^{cu}_p$, and $\widehat{\mathcal{W}}^s_p$ and $\widehat{\mathcal{W}}^c_p$ sub-foliate $\widehat{\mathcal{W}}^{cs}_p$.

(vi) $\widehat{\mathcal{W}}^s_p(p) = \mathcal{W}^s(p, r)$ and $\widehat{\mathcal{W}}^u_p(p) = \mathcal{W}^u(p, r)$.

(vii) All the fake foliations $\widehat{\mathcal{W}}^*$, $* \in \{u, s, c, cs, cu\}$ are Hölder continuous, and so are their tangent distributions.

(viii) Assuming f is center bunched, every leaf of $\widehat{\mathcal{W}}^{cs}_p$ is C^1 foliated by leaves of $\widehat{\mathcal{W}}^s_p$ and every leaf of $\widehat{\mathcal{W}}^{cu}_p$ is C^1 foliated by leaves of $\widehat{\mathcal{W}}^u_p$.

Properties (i) and (vi) are what we mean by "approximations". Concerning the latter, let us emphasize that the fake strong-stable and strong-unstable foliations need not coincide with the genuine ones, $\mathcal{W}^s$ and $\mathcal{W}^u$, at points other than p. The local invariance property (ii) and the exponential bounds (iii) and (iv) should be compared to the corresponding properties (I), (II), (III) of, stated at the beginning of Section 5. The regularity properties (vi) and (vii) hold uniformly in $p \in M$.

5.2.2. Juliennes. — Another direct use of the center bunching condition, besides the smoothness property (viii) above, is in the definition of juliennes. In view of the first center bunching condition, $\nu < \gamma\hat{\gamma}$ (there is a dual construction starting from $\hat{\nu} < \gamma\hat{\gamma}$ instead), we may find continuous functions τ and σ such that

$$\nu < \tau < \sigma\gamma \quad \text{and} \quad \sigma < \min\{\hat{\gamma}, 1\}.$$

Let $p \in M$ be fixed. For any $x \in \mathcal{W}^s(p,1)$ and $n \geq 0$, define

$$\widehat{B}^c_n(x) = \widehat{\mathcal{W}}^c_p(x, \sigma^n(p)) \quad \text{and} \quad S_n(p) = \bigcup_{x \in \mathcal{W}^s(p,1)} \widehat{B}^c_n(x).$$

The (fake) *center-unstable julienne* of order $n \geq 0$ centered at $x \in \mathcal{W}^s(p,1)$ is defined by

$$\widehat{J}^{cu}_n(x) = \bigcup_{y \in \widehat{B}^c_n(x)} \widehat{J}^u_n(y), \quad \text{where} \quad \widehat{J}^u_n(y) = f^{-n}(\widehat{\mathcal{W}}^u_{f^n(p)}(f^n(y), \tau^n(p))).$$

The latter is the (fake) *unstable julienne* of order $n \geq 0$ centered at y, and is defined for every $y \in S_n(p)$. See Figure 1.

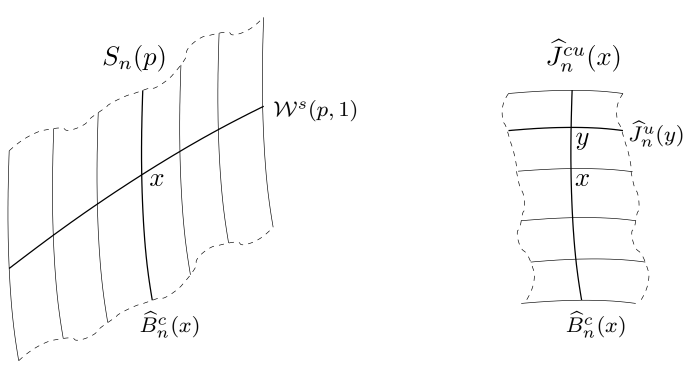

FIGURE 1.

Observe that $\widehat{J}^{cu}_n(x)$ is contained in the smooth submanifold $\widehat{\mathcal{W}}^{cu}_p(x)$, by the coherence property (v) of fake foliations. Moreover, $\widehat{J}^{cu}_n(x)$ has positive measure relative to the Riemannian volume $\lambda_{\widehat{cu}}$ defined by the restriction of the Riemannian metric to $\widehat{\mathcal{W}}^{cu}_p(x)$. Notice also that fake center-unstable leaves are transverse to the strong-stable foliation, as a consequence of property (i) of fake foliations. One key feature of center-unstable juliennes is that, unlike balls for instance, they are approximately preserved by the holonomy maps of the strong-stable foliation:

***Proposition 5.5* (Proposition 5.3 in [9]).** — *For any x, $x' \in \mathcal{W}^s(p,1)$, the sequences $h^s(\widehat{J}_n^{cu}(x))$ and $\widehat{J}_n^{cu}(x')$ are internested, where $h^s : \widehat{\mathcal{W}}_p^{cu}(x) \to \widehat{\mathcal{W}}_p^{cu}(x')$ is the holonomy map induced by the strong-stable foliation $\mathcal{W}^s$.*

5.3. Lebesgue and julienne density points. — Let S be a locally s-saturated set in a neighborhood of p. For notational simplicity, we write

$$\lambda_{\widehat{cu}}(S : \widehat{J}_n^{cu}(x)) = \lambda_{\widehat{cu}}(S \cap \widehat{\mathcal{W}}_p^{cu}(x) : \widehat{J}_n^{cu}(x)).$$

Notice that $S \cap \widehat{\mathcal{W}}_p^{cu}(x)$ coincides with the base of S over $\widehat{\mathcal{W}}_p^{cu}(x)$.

***Definition 5.6*.** — We call $x \in \mathcal{W}^s(p,1)$ a *cu-julienne density point* of S if

$$\lim_{n\to\infty} \lambda_{\widehat{cu}}(S : \widehat{J}_n^{cu}(x)) = 1.$$

Another crucial property of center-unstable juliennes is

***Proposition 5.7* (Proposition 5.5 in [9]).** — *Let X be a measurable set that is both s-saturated and essentially u-saturated. Then $x \in \mathcal{W}^s(p)$ is a Lebesgue density point of X if and only if x is a cu-julienne density point of X.*

We can not use this proposition directly, because the saturation hypotheses are not fully satisfied by the sets we deal with. However, we can rearrange the arguments in the proof of the proposition to obtain a statement that does suit our purposes. For this, let us recall the main steps in the proof of Proposition 5.7. They involve several nesting sequences $B_n(x)$, $C_n(x)$, $D_n(x)$, $G_n(x)$, that we introduce along the way.

By definition, $B_n(x)$ is just the Riemannian ball of radius $\sigma^n(p)$ centered at x:

$$B_n(x) = B(x, \sigma^n(p)).$$

***Lemma 5.8*.** — *Let $S \subset M$ be any measurable set. Then, x is a Lebesgue density point of S if and only if $\lim_{n\to\infty} \lambda(S : B_n(x)) = 1$.*

Proof. — This follows from the fact that the ratio $\sigma^{n+1}(p)/\sigma^n(p) = \sigma(f^n(p))$ of successive radii is less than 1, and is uniformly bounded away from both 0 and 1. $\square$

Next, for $x \in \mathcal{W}^s(p,1)$, let

$$C_n(x) = \bigcup_{q \in D_n^{cs}(x)} \mathcal{W}^u(q, \sigma^n(p)) \quad \text{and} \quad D_n(x) = \bigcup_{q \in D_n^{cs}(x)} f^{-n}(\mathcal{W}^u(f^n(q), \tau^n(p))).$$

Notice that these two nesting sequences fiber over the same sequence of bases

$$D_n^{cs}(x) = \bigcup_{y \in \widehat{\mathcal{W}}_p^s(x, \sigma^n(p))} \widehat{B}_n^c(y) = \bigcup_{y \in \widehat{\mathcal{W}}_p^s(x, \sigma^n(p))} \widehat{\mathcal{W}}_p^c(y, \sigma^n(p)).$$

Also, by the coherence property (v) of fake foliations, each set $D_n^{cs}(x)$ is contained in the submanifold $\widehat{\mathcal{W}}^{cs}(x)$.

***Lemma 5.9*.** — *Let $S \subset M$ be any measurable set. Then,*

$$\lim_{n\to\infty} \lambda(S : B_n(x)) = 1 \Longleftrightarrow \lim_{n\to\infty} \lambda(S : C_n(x)) = 1.$$

Proof. — Continuity and transversality of the fake foliations $\widehat{\mathcal{W}}_p^c$ and $\widehat{\mathcal{W}}_p^s$ imply that the sequences $D_n^{cs}(x)$ and $\widehat{\mathcal{W}}^{cs}(x, \sigma^n(p))$ are internested. Then, similarly, continuity and transversality of the foliations $\mathcal{W}^u$ and $\widehat{\mathcal{W}}_p^{cs}$ imply that the sequences $C_n(x)$ and $B_n(x)$ are internested. So, the claim follows from Lemma 5.2. □

Lemma 5.10. — *Let $S \subset M$ be locally essentially u-saturated. Then,*

$$\lim_{n\to\infty} \lambda(S : C_n(x)) = 1 \Longleftrightarrow \lim_{n\to\infty} \lambda(S : D_n(x)) = 1.$$

Proof. — By definition, $C_n(x)$ and $D_n(x)$ both fiber over $D_n^{cs}(x)$, with fibers contained in strong-unstable leaves. The fibers of $C_n(x)$ are uniform, in the sense of (5.1), because they are all comparable to balls of fixed radius $\sigma^n(p)$ inside strong-unstable leaves. Proposition 5.4 in [**9**] gives that the fibers of $D_n(x)$ are uniform as well. Thus, the claim follows from Proposition 5.4 above. □

Finally, define

$$G_n(x) = \bigcup_{q \in \widehat{J}_n^{cu}(x)} \mathcal{W}^s(q, \sigma^n(p)).$$

Lemma 5.11. — *Let $S \subset M$ any measurable set. Then,*

$$\lim_{n\to\infty} \lambda(S : D_n(x)) = 1 \Longleftrightarrow \lim_{n\to\infty} \lambda(S : G_n(x)) = 1.$$

Proof. — The sequences $D_n(x)$ and $G_n(x)$ are internested, according to Lemma 8.1 and Lemma 8.2 in [**9**]. So, the claim follows from Lemma 5.2. □

Lemma 5.12. — *Let $S \subset M$ be locally s-saturated. Then,*

$$\lim_{n\to\infty} \lambda(S : G_n(x)) = 1 \Longleftrightarrow \lim_{n\to\infty} \lambda_{\widehat{cu}}(S : \widehat{J}_n^{cu}(x)) = 1.$$

Proof. — By definition, $G_n(x)$ fibers over $\widehat{J}_n^{cu}(x)$. The fibers are uniform, in the sense of (5.1), because they are all comparable to balls of fixed radius $\sigma^n(p)$ inside strong-stable leaves. Then the claim follows from Proposition 5.3 above. □

Proposition 5.7 was obtained in [**9**] by concatenating Lemmas 5.8 through 5.12. A variation of these arguments yields:

Proposition 5.13. — *Let $x \in \mathcal{W}^s(p, 1)$ and $\delta > 0$.*

(a) *Let $X \subset M$ be a locally essentially u-saturated set in $B(x, \delta)$ and let Y be its local s-saturation inside $B(x, \delta)$. If x is a Lebesgue density point of X then x is a cu-julienne density point of Y.*
(b) *Let $X \subset M$ be a locally essentially s-saturated set in $B(x, \delta)$ and let Y be its local u-saturation inside $B(x, \delta)$. If x is a cu-julienne density point of X then x is a Lebesgue density point of Y.*
(c) *Let $S \subset M$ be any measurable set. If x is a cu-julienne density point of S then so is every $x' \in \mathcal{W}^s(p, 1)$.*

Proof. — Applying Lemmas 5.8 through 5.11 to $S = X$, we get that

$$\lim_{n\to\infty} \lambda(X : G_n(x)) = 1$$

(Lemma 5.10 uses the assumption that X is essentially u-saturated). It follows that

$$\lim_{n\to\infty} \lambda(Y : G_n(x)) = 1,$$

because $Y \supset X$. Thus, applying Lemma 5.12 to $S = Y$, we get that x is a cu-julienne density point of Y, as claimed in part (a) of the proposition.

Next, we prove part (b). Given an essentially s-saturated set X in $B(x, \delta)$, we may use Lemmas 5.12 and 5.11 with $S = X$ to conclude that

$$\lim_{n\to\infty} \lambda(X : D_n(x)) = 1$$

(Lemma 5.12 uses the assumption that X is essentially s-saturated). It follows that

$$\lim_{n\to\infty} \lambda(Y : D_n(x)) = 1,$$

because $Y \supset X$. Then Lemmas 5.10 through 5.8, with $S = Y$, to conclude that x is a Lebesgue density point of Y, as claimed.

Finally, absolute continuity (with bounded Jacobians) of the strong-stable foliation gives that

$$\lim_{n\to\infty} \lambda_{\widehat{cu}}(S : \widehat{J}^{cu}_n(x)) = 1 \quad \Rightarrow \quad \lim_{n\to\infty} \lambda_{\widehat{cu}}(S : h^s(\widehat{J}^{cu}_n(x))) = 1.$$

By Proposition 5.5, the sequences $h^s(\widehat{J}^{cu}_n(x))$ and $\widehat{J}^{cu}_n(x')$ are internested. Hence, by Lemma 5.2,

$$\lim_{n\to\infty} \lambda_{\widehat{cu}}(S : h^s(\widehat{J}^{cu}_n(x))) = 1 \quad \Rightarrow \quad \lim_{n\to\infty} \lambda_{\widehat{cu}}(S : \widehat{J}^{cu}_n(x')) = 1.$$

This proves part (c) of the theorem. □

6. Bi-essential invariance implies essential bi-invariance

We call a continuous fiber bundle $\mathcal{X}$ *refinable* if the fibers $\mathcal{X}_x$, $x \in M$ are refinable.

Theorem 6.1. — *Let $f : M \to M$ be a C^2 partially hyperbolic center bunched diffeomorphism and $\mathcal{X}$ be a refinable fiber bundle with stable and unstable holonomies. Then, given any bi-essentially invariant section $\Psi : M \to \mathcal{X}$, there exists a bi-saturated set M_Ψ with full measure, and a bi-invariant section $\tilde{\Psi} : M_\Psi \to \mathcal{X}$ that coincides with Ψ at almost every point.*

Theorem D(a) is a particular case of this result, as we are going to explain. Indeed, let P be the space of probability measures on N, endowed with the weak* topology, that is, the smallest topology for which the integration operator

$$P \to \mathbb{R}, \quad \eta \mapsto \int \varphi \, d\eta$$

is continuous, for every bounded continuous function $\varphi : N \to \mathbb{R}$. It is well known (see [**5**, Section 6]) that this topology is separable and metrizable, because N is a separable metric space (if we were to assume that N is complete then the weak* topology would also be complete). In particular, P admits a countable basis of open sets and so it is refinable.

Associated to $\pi : \mathcal{E} \to M$, we have a new fiber bundle $\Pi : \mathcal{X} \to M$, whose fiber over a point $x \in M$ is the space of probability measures on the corresponding $\mathcal{E}_x$. It is easy to see that this is a continuous fiber bundle with leaves modeled on the space P we have just introduced: if $\pi^{-1}(U) \to U \times N$, $v \mapsto (\pi(v), \psi_{\pi(v)}(v))$ is a continuous local chart for $\mathcal{E}$ then

$$\Pi^{-1}(U) \to U \times P, \quad \eta \mapsto (\Pi(\eta), (\psi_{\Pi(\eta)})_*(\eta))$$

is a continuous local chart for $\mathcal{X}$. The cocycle $\mathfrak{F} : \mathcal{E} \to \mathcal{E}$ induces a cocycle on $\mathcal{X}$, by push-forward, but this will not be needed here.

More important for our purposes, the stable and unstable holonomies of $\mathfrak{F}$ induce homeomorphisms

$$h^s_{x,y} = (H^s_{x,y})_* : \mathcal{X}_x \to \mathcal{X}_y \quad \text{and} \quad h^u_{x,y} = (H^u_{x,y})_* : \mathcal{X}_x \to \mathcal{X}_y$$

for points x, y in the same strong-stable leaf or the same strong-unstable leaf, respectively. These homeomorphisms form stable and unstable holonomies on $\mathcal{X}$. Indeed, the group property (α) in Definition 2.9 is an immediate consequence of property (a) in Definition 2.4, and the continuity property (β) can be verified as follows. Since the statement is local, we may pretend that the fiber bundle is trivial and the holonomies $H^s_{x,y}$ are homeomorphisms of N. Consider any sequence (x_k, y_k, ν_k) in $\mathcal{X}$ converging to $(x, y, \nu) \in \mathcal{X}$, with $y_k \in \mathcal{W}^s_{\mathrm{loc}}(x_k)$ and $y \in \mathcal{W}^s_{\mathrm{loc}}(x)$. Property (c) in Definition 2.4 implies that $H^s_{x_k,y_k}$ converges to $H^s_{x,y}$ uniformly on compact subsets. On its turn, this implies that $(H^s_{x_k,y_k})_*\nu_k$ converges to $(H^s_{x,y})_*\nu$ in the weak* topology.

Now it is clear that Theorem D(a) corresponds to the statement of Theorem 6.1 in the special case of the section $\Psi(x) = m_x$ of the fiber bundle $\mathcal{X}$ we have defined. In the remainder of this section we prove Theorem 6.1.

6.1. Lebesgue densities. — Let $\Psi : M \to P$ be a measurable function with values in a refinable space.

Definition 6.2. — We say that $x \in M$ is a *point of measurable continuity* of Ψ if there is $\upsilon \in P$ such that x is a Lebesgue density point of $\Psi^{-1}(V)$ for every neighborhood $V \subset P$ of υ. Then υ is called the *density value* of Ψ at x.

Clearly, the density value at x is unique, when it exists. Let $\mathrm{MC}(\Psi)$ denote the set of measurable continuity points of Ψ. The function $\tilde{\Psi} : \mathrm{MC}(\Psi) \to P$ assigning to each point x of measurable continuity its density value $\tilde{\Psi}(x)$ is called *Lebesgue density* of Ψ. Recall that $\mathrm{DP}(X)$ denotes the set of density points of a set X. The hypothesis that P is refinable is used in the next lemma:

Lemma 6.3. — *For any measurable function $\Psi : M \to P$, the set $\mathrm{MC}(\Psi)$ has full Lebesgue measure and $\Psi = \tilde{\Psi}$ almost everywhere.*

Proof. — Let $\mathcal{Q}_1 \prec \cdots \prec \mathcal{Q}_n \prec \cdots$ be a sequence of partitions of the space P as in Definition 2.11. Let

$$\tilde{M} = \bigcap_{n\geq 1} \bigcup_{Q\in\mathcal{Q}_n} \Psi^{-1}(Q) \cap \mathrm{DP}(\Psi^{-1}(Q)).$$

Since $\Psi^{-1}(Q) \cap \mathrm{DP}(\Psi^{-1}(Q))$ has full measure in $\Psi^{-1}(Q)$, and $\{\Psi^{-1}(Q) : Q \in \mathcal{Q}_n\}$ is a partition of M for every n, the set on the right hand side has full measure in M for every n. This proves that $\tilde{M}$ is a full measure subset of M. Next, we check that $\tilde{M}$ is contained in the set of points of measurable continuity of Ψ. Indeed, given any point $x \in \tilde{M}$, let $Q_n \in \mathcal{Q}_n$ be the sequence of atoms such that $x \in \Psi^{-1}(Q_n)$. Then x is a density point of $\Psi^{-1}(Q_n)$ for every $n \geq 1$, in view of the definition of $\tilde{M}$. Notice that $\cap_n Q_n$ is non-empty, since it contains $\Psi(x)$. Then, according to Definition 2.11, there exists $v \in \mathcal{X}$ such that every neighborhood V contains some Q_n. It follows that x is a density point of $\Psi^{-1}(V)$ for any neighborhood $V \subset \mathcal{X}$ of v, that is, v is the density value for Ψ at x. This shows that $x \in \mathrm{MC}(\Psi)$ with $\tilde{\Psi}(x) = v$. Moreover, v must coincide with $\Psi(x)$, since the intersection of all Q_n contains exactly one point. In other words, $\tilde{\Psi}(x) = \Psi(x)$ for every $x \in \tilde{M}$. □

More generally, let $\Psi : M \to \mathcal{X}$ be a measurable section of a refinable fiber bundle $\mathcal{X}$. Let $x \in M$ be fixed and U be a small neighborhood. Using a local chart, one may view $\Psi \mid U$ as a function with values in $\mathcal{X}_x$. Two such local expressions $\Psi_1 : U \to \mathcal{X}_x$ and $\Psi_2 : U \to \mathcal{X}_x$ of the section Ψ are related by

$$\Psi_1(z) = h_z(\Psi_2(z)),$$

where $(z, \xi) \mapsto (z, h_z(\xi))$ is a homeomorphism from $U \times \mathcal{X}_x$ to itself, with $h_x = \mathrm{id}$. So, a point $v \in \mathcal{X}_x$ is the density value of Ψ_1 at x if and only if it is the density value of Ψ_2 at x. More generally, given any point $y \in U$, the corresponding local expression $\Psi_3 : U \to \mathcal{X}_y$ of the section Ψ is related to $\Psi_1 : U \to \mathcal{X}_x$ by

$$\Psi_1(z) = g_z(\Psi_3(z)),$$

where $(z, \xi) \mapsto (z, g_z(\xi))$ is a homeomorphism from $U \times \mathcal{X}_y$ to $U \times \mathcal{X}_x$. So, a point z is a point of measurable continuity for Ψ_3 if and only if it is a point of measurable continuity for Ψ_1.

These observations allow us to extend Definition 6.2 to sections of refinable fiber bundles, as follows. We call $v \in \mathcal{X}_x$ a *density value* of the section $\Psi : M \to \mathcal{X}$ at the point x if it is the density value for some (and, hence, any) local expression $U \mapsto \mathcal{X}_x$ as before. We call x a *point of measurable continuity* of the section Ψ if it admits a density value or, equivalently, if it is a point of measurable continuity for some (and, hence, any) local expression of Ψ. The subset $\mathrm{MC}(\Psi)$ of points of measurable continuity has full Lebesgue measure in M, since it intersects every domain U of local chart on a full Lebesgue measure subset. Recall Lemma 6.3. Finally, the *Lebesgue density* of Ψ is the section $\mathrm{MC}(\Psi) \to \mathcal{X}$ assigning to each point x of measurable continuity its density value.

6.2. Proof of bi-invariance. — Now Theorem 6.1 is a direct consequence of the next proposition: it suffices to take $M_\Psi = \mathrm{MC}(\Psi)$ and $\tilde{\Psi}$ = the Lebesgue density of Ψ, and apply the following proposition together with Lemma 6.3.

Proposition 6.4. — *Let $f : M \to M$ be a C^2 partially hyperbolic center bunched diffeomorphism and $\mathcal{X}$ be a refinable fiber bundle with stable and unstable holonomies. For any bi-essentially invariant section $\Psi : M \to \mathcal{X}$, the set $\mathrm{MC}(\Psi)$ is bi-saturated and the Lebesgue density $\tilde{\Psi} : \mathrm{MC}(\Psi) \to \mathcal{X}$ is bi-invariant on $\mathrm{MC}(\Psi)$.*

Proof. — For any $x \in \mathrm{MC}(\Psi)$ and $y \in \mathcal{W}^s(x,1)$, we are going to prove $h^s_{x,y}(\tilde{\Psi}(x))$ is the density value of Ψ at y. It will follow that $y \in \mathrm{MC}(\Psi)$ and $\tilde{\Psi}(y) = h^s_{x,y}(\tilde{\Psi}(x))$. Analogously, one gets that if $x \in \mathrm{MC}(\Psi)$ and $y \in \mathcal{W}^u(x,1)$ then $y \in \mathrm{MC}(\Psi)$ and $\tilde{\Psi}(y) = h^u_{x,y}(\tilde{\Psi}(x))$. The proposition is an immediate consequence of these facts.

It is convenient to think of $\pi : \mathcal{X} \to M$ as a trivial bundle on neighborhoods U_x of x and U_y of y, identifying $\pi^{-1}(U_x) \approx U_x \times P$ and $\pi^{-1}(U_y) \approx U_y \times P$ via local coordinates, and we do so in what follows. Let $V \subset P$ be a neighborhood of $h^s_{x,y}(\tilde{\Psi}(x))$. We are going to show that y is a density point of $\Psi^{-1}(V)$.

By the continuity property (β) in Definition 2.9, we can find $\varepsilon > 0$ and a neighborhood $W \subset V$ of $h^s_{x,y}(\tilde{\Psi}(x))$ such that

$$h^u_{w_1,w_2}(W) \subset V \quad \text{for all } w_1,\, w_2 \in B(y,\varepsilon) \text{ with } w_1 \in \mathcal{W}^u_{\mathrm{loc}}(w_2). \tag{6.1}$$

Similarly, up to reducing $\varepsilon > 0$, there exists a neighborhood $U \subset P$ of $\tilde{\Psi}(x)$ such that

$$h^s_{z,w}(U) \subset W \quad \text{for every } z \in B(x,\varepsilon) \text{ and } w \in B(y,\varepsilon) \text{ with } z \in \mathcal{W}^s_{\mathrm{loc}}(w). \tag{6.2}$$

The assumption that Ψ is bi-essentially invariant (Definition 2.10) implies that there exists a full measure set S^{su} such that

$$\begin{aligned} &h^s_{\xi,\eta}(\Psi(\xi)) = \Psi(\eta) \quad \text{for any } \xi,\, \eta \in S^{su} \text{ in the same strong-stable leaf} \\ &h^u_{\xi,\eta}(\Psi(\xi)) = \Psi(\eta) \quad \text{for any } \xi,\, \eta \in S^{su} \text{ in the same strong-unstable leaf.} \end{aligned} \tag{6.3}$$

Lemma 6.5. — *Let $x \in \mathcal{W}^s(p,1)$ be a point of measurable continuity of Ψ. Then for any open neighborhood U of the point $\tilde{\Psi}(x) \in P$ there exist $\delta > 0$ and $L \subset B(x,\delta)$ such that*

(a) $\Psi(L \cap S^{su}) \subset U$.
(b) *L is a union of local strong-stable leaves inside $B(x,\delta)$.*
(c) *Each of these local leaves contains some point of S^{su}.*
(d) *x is a cu-julienne density point of L: $\lim_{n\to\infty} \lambda_{\widehat{cu}}(L : \widehat{J}^{cu}_n(x)) = 1$.*

Proof. — By the continuity property (β) in Definition 2.9, there exists $\delta_2 > 0$ and a neighborhood $U_2 \subset U$ of $\tilde{\Psi}(x)$ such that

$$(h^s_{z_1,z_2})(U_2) \subset U \quad \text{if } z_1,\, z_2 \in B(x,\delta_2) \text{ are in the same local strong-stable leaf.}$$

and there exists $\delta_1 > 0$ and a neighborhood $U_1 \subset U_2$ of $\tilde{\Psi}(x)$ such that

$$(h^u_{z_1,z_2})(U_1) \subset U_2 \quad \text{if } z_1,\, z_2 \in B(x,\delta_1) \text{ are in the same local strong-unstable leaf.}$$

Let $\delta = \min\{1, \delta_1, \delta_2\}$. Since x is a point of measurable continuity of Ψ, it is a Lebesgue density point of $\Psi^{-1}(U_1)$. Then, x is also a density point of $L_1 = \Psi^{-1}(U_1) \cap$

S^{su}, because S^{su} has full Lebesgue measure. Let L_1^u be the local u-saturate of L_1 inside $B(x,\delta)$ and let $L_2 = L_1^u \cap S^{su}$. Then x is a Lebesgue density point of L_1^u, because $L_1^u \supset L_1$, and so it is also a density point of L_2, because S^{su} has full measure. Take L to be the local s-saturate of L_2 inside $B(x,\delta)$.

Consider any point $z \in L \cap S^{su}$. By definition, there exist $z_1 \in \Psi^{-1}(U_1) \cap S^{su}$ and $z_2 \in L_1^u \cap S^{su}$ such that z_1 is in the local strong-unstable leaf of z_2, and z_2 in the local strong-stable leaf of z. Consequently, in view of our choices of U_1 and U_2,

$$\Psi(z_2) = h^u_{z_1,z_2}(\Psi(z_1)) \in U_2 \quad \text{and then} \quad \Psi(z) = h^s_{z_2,z}(\Psi(z_2)) \in U.$$

This proves claim (a) in the lemma. Claims (b) and (c) are clear from the construction: L is a local s-saturate of a subset of S^{su}. Finally, applying Proposition 5.13(a) to $X = L_2$ we get that x is a cu-julienne density point of $Y = L$. This gives claim (d), and completes the proof of the lemma. $\square$

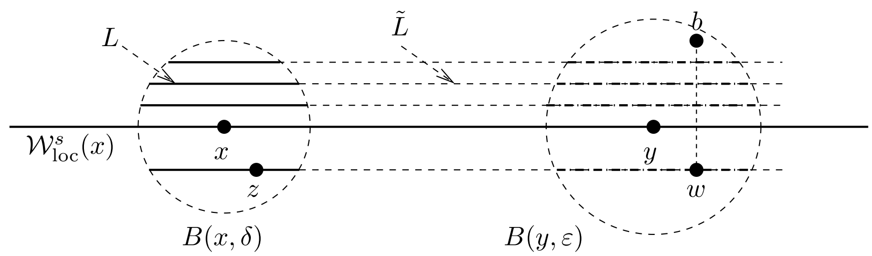

Figure 2.

Let L and δ be as in Lemma 6.5. Of course, we may suppose $\delta < \varepsilon$. We extend the local leaves in L along $\mathcal{W}^s_{\mathrm{loc}}(x)$, long enough so as to cross $B(y,\varepsilon)$. Let $\tilde{L}$ denote this extended set. See Figure 2. As we have seen in Proposition 5.13(c), cu-julienne density points of locally s-saturated sets are preserved by stable holonomy. Hence, Lemma 6.5(d) ensures that y is a cu-julienne density point of $\tilde{L}$. Then, clearly, y is also a cu-julienne density point of $X = \tilde{L} \cap S^{su} \cap B(y,\varepsilon)$. Let Y be the local u-saturation of X inside $B(y,\varepsilon)$. Since X is locally essentially s-saturated, we may use Proposition 5.13(b) to conclude that y is a Lebesgue density point of Y and, hence, also of $B = S^{su} \cap Y$. Thus, to prove that y is a Lebesgue density point of $\Psi^{-1}(V)$, as we claimed, it suffices to show that $\Psi(B) \subset V$.

Consider any point $b \in Y$. By definition, $b \in S^{su} \cap B(y,\varepsilon)$ and there exists some $w \in X$ such that b and w are in the same local strong-unstable leaf. By part (c) of Lemma 6.5, there exists $z \in L \cap S^{su}$ in the same local strong-stable leaf as w. By part (a) of Lemma 6.5, we have that $\Psi(z) \in U$. So, (6.3) and (6.2) imply that $\Psi(w) = h^s_{z,w}(\Psi(z)) \in W$. Then (6.3) and (6.1) imply that $\Psi(b) = h^u_{w,b}(\Psi(w)) \in V$, as we wanted to prove. This proves Proposition 6.4. $\square$

Now the proof of Theorem 6.1 is complete.

Remark 6.6. — Let us say that a section $\Psi : M \to \mathcal{X}$ is *essentially s-continuous* if the s-continuity property (Definition 2.12) holds on some full measure subset M^s, uniformly on the neighborhood of every point. In formal terms: given any $p,\, q \in M$ and $\eta \in P$, there exists $\rho > 0$ such that for any $\varepsilon > 0$ there exists $\delta > 0$ such that (trivialize the fiber bundle near p and q), given any $x,\, x' \in B(p,\rho) \cap M^s$ and y, $y' \in B(q,\rho) \cap M^s$ with $\Psi(x),\, \Psi(x') \in B(\eta,\rho)$ and $y \in \mathcal{W}^s_{\mathrm{loc}}(x)$ and $y' \in \mathcal{W}^s_{\mathrm{loc}}(x')$,

$$\operatorname{dist}(x,x') < \delta, \quad \operatorname{dist}(y,y') < \delta, \quad \operatorname{dist}(\Psi(x),\Psi(x')) < \delta \Rightarrow \operatorname{dist}(\Psi(y),\Psi(y')) < \varepsilon.$$

Essential u-continuity is defined analogously. Moreover, Ψ is *bi-essentially continuous* if it is both essentially s-continuous and essentially u-continuous. A variation of the previous arguments yields the following statement (compare Proposition 6.4): If $f : M \to M$ is a C^2 partially hyperbolic center bunched diffeomorphism and $\mathcal{X}$ be a refinable fiber bundle then, for any bi-essentially continuous section $\Psi : M \to \mathcal{X}$, the set of points of measurable continuity is bi-saturated and the Lebesgue density $\tilde{\Psi} : \mathrm{MC}(\Psi) \to \mathcal{X}$ is bi-continuous.

7. Accessibility and continuity

Now we prove Theorem E. The main step is to show that small open sets can be reached by "nearby" su-paths starting from a fixed point in M. For the precise statement, to be given in Proposition 7.2, we need the following notion:

Definition 7.1. — Let $z,\, w \in M$. An *access sequence connecting z to w* is a finite sequence of points $[y_0, y_1, \dots, y_n]$ such that $y_0 = z$ and $y_j \in \mathcal{W}^*(y_{j-1})$ for $1 \le j \le n$, where each $* \in \{s,u\}$, and $y_n = w$.

Proposition 7.2. — *Given $x_0 \in M$, there is $w \in M$ and there is an access sequence $[y_0(w), \dots, y_N(w)]$ connecting x_0 to w and satisfying the following property: for any $\varepsilon > 0$ there exist $\delta > 0$ and $L > 0$ such that for every $z \in B(w,\delta)$ there exists an access sequence $[y_0(z), y_1(z), \dots, y_N(z)]$ connecting x_0 to z and such that*

$$\operatorname{dist}(y_j(z), y_j(w)) < \varepsilon \quad \textit{and} \quad \operatorname{dist}_{\mathcal{W}^*}(y_{j-1}(z), y_j(z)) < L \quad \textit{for } j = 1, \dots, N$$

where $\operatorname{dist}_{\mathcal{W}^*}$ denotes the distance along the strong (either stable or unstable) leaf common to the two points.

Let us deduce Theorem E from this proposition. Since the section Ψ is assumed to be bi-continuous, it suffices to prove it is continuous at *some* point in order to conclude that it is continuous everywhere. Fix $x_0 \in M$ and then let $w \in M$ and $[y_0(w), y_1(w), \dots, y_N(w)]$ be an access sequence connecting x_0 to w as in Proposition 7.2. We are going to prove that Ψ is continuous at w. Take the fiber bundle $\pi : \mathcal{X} \to M$ to be trivialized on the neighborhood of every node $y_j(w)$, via local coordinates. Let $V \subset P$ be any neighborhood of $\Psi(w) = \Psi(y_N(w))$. Since Ψ is bi-continuous, we may find numbers $\varepsilon_j > 0$ and neighborhoods V_j of $\Psi(y_j(w))$ such that $V_N = V$ and

$$\begin{aligned} &x \in B(y_{j-1}(w), \varepsilon_j), \quad y \in B(y_j(w), \varepsilon_j), \quad y \in \mathcal{W}^{*_j}(x), \\ &\qquad\qquad \text{and } \Psi(x) \in V_{j-1} \quad \Rightarrow \quad \Psi(y) \in V_j \end{aligned} \tag{7.1}$$

for every $j = 1, \dots, N$. Let $\varepsilon = \min\{\varepsilon_j : 1 \le j \le N\}$.

Using Proposition 7.2 we find $\delta > 0$ and, for each $z \in B(w, \delta)$, an access sequence $[y_0(z), y_1(z), \dots, y_N(z)]$ connecting x_0 to z, with

$$y_j(z) \in B(y_j(w), \varepsilon) \subset B(y_j(w), \varepsilon_j) \quad \text{for } j = 1, \dots, N. \tag{7.2}$$

It is no restriction to suppose that $\delta < \varepsilon$. Consider any $z \in B(w, \delta)$. Clearly, $\Psi(x) = \Psi(y_0(z)) \in V_0$. Hence, we may use (7.1)-(7.2) inductively to conclude that $\Psi(y_j(z)) \in V_j$ for every $j = 1, \dots, N$. The last case, $j = N$, gives $\Psi(z) \in V$. We have shown that $\Psi(B(w, \delta)) \subset V$. This proves that Ψ is continuous at w, as claimed.

In this way, we reduced the proof of Theorem E to proving Proposition 7.2.

7.1. Non-injective parametrizations. — In this section we prepare the proof of Proposition 7.2, that will be given in the next section. Roughly speaking, here we construct a kind of continuous parametrization of the space of su-paths with any given number of legs.

7.1.1. Exhaustion of accessibility classes. — Fix any point $x_0 \in M$. For each $r \in \mathbb{N}$, we consider the following sequence of sets $K_{r,n}$, $n \in \mathbb{N}$:

$$K_{r,1} = \{y \in \mathcal{W}^s(x_0) : \operatorname{dist}_{\mathcal{W}^s}(x_0, y) \le r\} \quad \text{and}$$
$$K_{r,n} = \bigcup_{x \in K_{r,n-1}} \{y \in \mathcal{W}^*(x) : \operatorname{dist}_{\mathcal{W}^*}(x, y) \le r\}, \quad \text{for } n \ge 2,$$

where $* = s$ when n is odd, and $* = u$ when n is even. That is, $K_{r,n}$ is the set of points that can be reached from x_0 using an access sequence with n legs whose lengths do not exceed r.

Lemma 7.3. — *Every $K_{r,n}$ is closed in M and, hence, compact.*

Proof. — It is clear from the definition that $K_{r,1}$ is closed. The general case follows by induction. Suppose $K_{r,n-1}$ is closed, and let z belong to the complement of $K_{r,n}$. Then, by definition,

$$Z = \{y \in \mathcal{W}^*(z) : \operatorname{dist}_{\mathcal{W}^*}(x, y) \le r\}$$

does not intersect the closed set $K_{r,n-1}$. It follows that $U \cap K_{r,n} = \emptyset$ for some neighborhood U of the set Z. By continuity of the strong-stable and strong-unstable foliations, and their induced Riemannian metrics, for every point w in a neighborhood of z,

$$\{y \in \mathcal{W}^*(z) : \operatorname{dist}_{\mathcal{W}^*}(x, y) \le r\} \subset U$$

and hence, the set on the left hand side is disjoint from $K_{r,n-1}$. This proves that points w in that neighborhood of z do not belong to $K_{r,n}$ either. Thus, $K_{r,n}$ is indeed closed. □

By definition, the union of $K_{r,n}$ over all (r, n) is the accessibility class of x_0. Since we are assuming that f is accessible, this union is the whole manifold:

$$M = \bigcup_{r,n \in \mathbb{N}} K_{r,n}.$$

Since M is a Baire space, it follows that $K_{r,n}$ has non-empty interior for some r and n, that we consider fixed from now on. Our immediate goal is to define a (non-injective) continuous "parametrization"

$$\Psi_n : \mathfrak{K}_{r,n} \to K_{r,n} \tag{7.3}$$

of the set $K_{r,n}$ by a convenient compact subspace $\mathfrak{K}_{r,n}$ of a Euclidean space, that we are going to introduce in the sequel. Let d_s and d_u denote the dimensions of the strong-stable leaves and the strong-unstable leaves, respectively. This Euclidean space will be the alternating product of $\mathbb{R}^{d_s}$ and $\mathbb{R}^{d_u}$, with n factors, each of which parametrizing one leg of the access sequence. The case $n = 2$ is described in Figure 3.

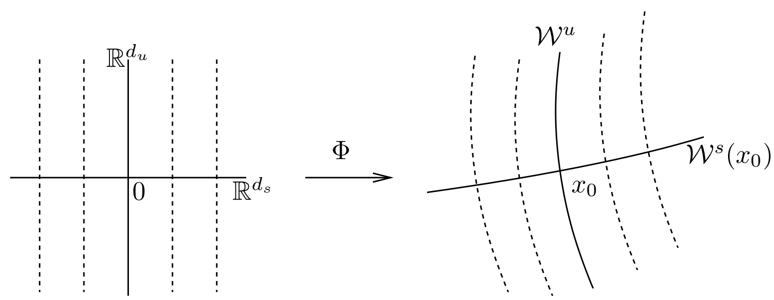

FIGURE 3.

7.1.2. Fiber bundles induced by local strong leaves. — The following lemma will be useful in the construction of (7.3). The whole point with the statement is that U does not need to be small. The diffeomorphisms in the statement are as regular as the partially hyperbolic diffeomorphism f itself.

Lemma 7.4. — *For any contractible space A, any continuous function $\Psi : A \to M$, and any symbol $* \in \{s, u\}$, there exists a homeomorphism*

$$\Theta : A \times \mathbb{R}^{d_*} \to \{(a, y) : a \in A \text{ and } y \in \mathcal{W}^*_{\text{loc}}(\Psi(a))\}$$

mapping each $\{a\} \times \mathbb{R}^{d_*}$ diffeomorphically to $\{a\} \times \mathcal{W}^*_{\text{loc}}(\Psi(a))$ with $\Theta(a, 0) = (a, \Psi(a))$ for all $a \in A$.

Proof. — We consider the case $* = s$. Since $\mathcal{W}^s$ is a continuous foliation with smooth leaves, for each $p \in M$ we may find a neighborhood U_p and a continuous map

$$\Phi_p : U_p \times \mathbb{R}^{d_s} \to M$$

such that $\Phi_p(x, 0) = x$ and $\Phi_p(x, \cdot)$ maps $\mathbb{R}^{d_s}$ diffeomorphically to $\mathcal{W}^s_{\text{loc}}(x)$, for every $x \in U_p$. Using these maps we may endow the set

$$F_s = \{(x, y) : x \in M \text{ and } y \in \mathcal{W}^s_{\text{loc}}(x)\}$$

with the structure of a fiber bundle with smooth fibers, with local charts

$$U_p \times \mathbb{R}^{ds} \to \{(x, y) : x \in U_p \text{ and } y \in \mathcal{W}^s_{\text{loc}}(x)\} \quad (x, v) \mapsto (x, \Phi_p(x, v)).$$

Then $F_\Psi^s = \{(a,y) : a \in A \text{ and } y \in \mathcal{W}^s_{\mathrm{loc}}(\Psi(a))\}$ also has a fiber bundle structure, with local coordinates

$$\Theta_p : \Psi^{-1}(U_p) \times \mathbb{R}^{d_s} \to \{(a,y) : \Psi(a) \in U_p \text{ and } y \in \mathcal{W}^s_{\mathrm{loc}}(\Psi(a))\}$$

given by $\Theta_p(a,v) = (a, \Phi_p(\Psi(a), v))$. This fiber bundle admits the space of diffeomorphisms of $\mathbb{R}^{d_s}$ that fix the origin as a structural group: all coordinate changes along the fibers belong to this group.

The core of the proof is the general fact (see [**16**, Chapter 4, Theorem 9.9]) that, for any topological group G, any fiber bundle over a contractible paracompact space that has G as a structural group is G-trivial. When applied to F_Ψ^s this result means that there exists a global chart

$$\Theta : A \times \mathbb{R}^{d_s} \to \{(a,y) : a \in A \text{ and } y \in \mathcal{W}^s_{\mathrm{loc}}(\Psi(a))\}, \quad \Theta(a,v) = (a, \Phi(a,v))$$

such that every $\Phi(a,\cdot)$ maps $\mathbb{R}^{d_s}$ to the strong-stable leaf through $\Psi(a)$, and every $\Phi(a,\cdot)^{-1} \circ \Phi_p(\Psi(a),\cdot)$ is a diffeomorphism that fixes the origin of $\mathbb{R}^{d_s}$. The latter gives that $\Phi(a,0) = \Phi_p(\Psi(a),0) = \Psi(a)$ for all $a \in A$. $\square$

7.1.3. Construction of non-injective parametrizations. — We are ready to construct $\mathfrak{K}_{r,n}$ and Ψ as in (7.3). Let $l \geq 1$ be fixed such that, for any $x \in M$,

$$\begin{aligned} &\{y \in \mathcal{W}^s(x) : \operatorname{dist}_{\mathcal{W}^s}(x,y) \leq 2r\} \subset f^{-l}\big(\mathcal{W}^s_{\mathrm{loc}}(f^l(x))\big) \\ &\{y \in \mathcal{W}^u(x) : \operatorname{dist}_{\mathcal{W}^u}(x,y) \leq 2r\} \subset f^{l}\big(\mathcal{W}^s_{\mathrm{loc}}(f^{-l}(x))\big). \end{aligned} \tag{7.4}$$

Our argument is somewhat more transparent when $l = 0$, and so the reader should find it convenient to keep that case in mind throughout the construction.

Define $E_1 = \{y \in M : f^l(y) \in \mathcal{W}^s_{\mathrm{loc}}(f^l(x_0))\}$ and $\Phi_1 : E_1 \to M$ to be the inclusion. Notice that E_1 is contractible and $\Phi_1(E_1)$ contains $K_{r,1}$. Since E_1 is a smooth disc, there exists an diffeomorphism $\Theta_1 : \mathbb{R}^{d_s} \to E_1$ with $\Theta_1(0) = x_0$. Then

$$\Psi_1 = \Phi_1 \circ \Theta_1 : \mathbb{R}^{d_s} \to M$$

is a continuous function whose image contains $K_{r,1}$. Notice that the pre-image $\mathfrak{K}_{r,1} = \Psi_1^{-1}(K_{r,1})$ is compact: $K_{r,1} = \{y \in \mathcal{W}^s(x_0) : \operatorname{dist}_{\mathcal{W}^s}(x_0,y) \leq r\}$ and we have a factor 2 in (7.4). Next, define

$$E_2 = \big\{(a,y) : a \in \mathbb{R}^{d_s} \text{ and } f^{-l}(y) \in \mathcal{W}^u_{\mathrm{loc}}(f^{-l}(\Psi_1(a)))\big\}$$

and $\Phi_2 : E_2 \to M$, $\Phi_2(a,y) = y$. Notice that $\Phi_2(E_2)$ contains $K_{r,2}$. Using Lemma 7.4 with $A = \mathbb{R}^{d_s}$, $\Psi = f^{-l} \circ \Psi_1$, and $* = u$, we find a homeomorphism

$$\Theta_2 : \mathbb{R}^{d_s} \times \mathbb{R}^{d_u} \to \{(a,y) : a \in \mathbb{R}^{d_s} \text{ and } y \in \mathcal{W}^u_{\mathrm{loc}}(f^{-l}(\Psi_1(a)))\}$$

that maps each $\{a\} \times \mathbb{R}^{d_u}$ diffeomorphically to $\{a\} \times \mathcal{W}^u_{\mathrm{loc}}(f^{-l}(\Psi_1(a)))$ and satisfies $\Theta_2(a,0) = (a, f^{-l}(\Psi_1(a)))$. Clearly, the map

$$\Gamma_2 : \{(a,y) : a \in \mathbb{R}^{d_s} \text{ and } y \in \mathcal{W}^u_{\mathrm{loc}}(f^{-l}(\Psi_1(a)))\} \to E_2, \quad \Gamma_2(a,y) = (a, f^l(y))$$

is a homeomorphism, and $\Gamma_2(\Theta_2(a,0)) = (a, \Psi_1(a))$. Then

$$\Psi_2 = \Phi_2 \circ \Gamma_2 \circ \Theta_2 : \mathbb{R}^{d_s} \times \mathbb{R}^{d_u} \to M$$

is a continuous map whose image contains $K_{r,2}$. Moreover, Ψ_2 may be viewed as a continuous extension of Ψ_1, because

$$\Psi_2(a,0)=\Phi_2(\Gamma_2(\Theta_2(a,0)))=\Phi_2(a,\Psi_1(a))=\Psi_1(a)$$

for all $a\in\mathbb{R}^{d_s}$. In general, $\Psi_2^{-1}(K_{r,2})$ needs not be compact. However,

$$\mathfrak{K}_{r,2}=\big\{(a,b)\in\mathbb{R}^{d_s}\times\mathbb{R}^{d_u}: a\in\mathfrak{K}_{r,1} \text{ and } \operatorname{dist}_{\mathcal{W}^u}(\Psi_2(a,0),\Psi_2(a,b))\le r\big\}$$

is compact and satisfies $\Psi_2(\mathfrak{K}_{r,2})=K_{r,2}$. Repeating this procedure, we construct continuous maps

$$\Psi_j:\mathbb{R}^{d_s}\times\mathbb{R}^{d_u}\times\cdots\times\mathbb{R}^{d_*}\to M$$

(there are j factors, and so $*=u$ if j is even and $*=s$ if j is odd), contractible sets E_j, and compact sets $\mathfrak{K}_{r,j}$ such that each Ψ_j is a continuous extension of Ψ_{j-1}, in the previous sense, and $\Psi_j(\mathfrak{K}_{r,j})=K_{r,j}$. We stop this procedure for $j=n$. The corresponding map Ψ_n is the parametrization announced in (7.3).

7.2. Selection of nearby access sequences. — Now we prove Proposition 7.2. We need the following general fact about regular values of continuous functions.

Definition 7.5. — Let $\Phi:\mathcal{A}\to\mathcal{B}$ be a map between topological spaces $\mathcal{A}$ and $\mathcal{B}$. A point $x\in\mathcal{A}$ is *regular* for Φ, if for every neighborhood $\mathcal{V}$ of x we have $\Phi(x)\in\Phi(\mathcal{V})^\circ$. A point $y\in B$ is a *regular value* of Φ if every point of $\Phi^{-1}(y)$ is regular.

Proposition 7.6. — *Let $\mathcal{A}$ be a compact metrizable space and $\mathcal{B}$ a locally compact Hausdorff space. If $\Phi:\mathcal{A}\to\mathcal{B}$ is continuous then the set of regular values of Φ is residual.*

Proof. — We are going to prove that the image of the set of non-regular points is meager. The assumptions imply that $\mathcal{A}$ admits a countable base $\mathcal{T}$ of open sets, and the map Φ is closed. If x is a non-regular point of Φ, then there exists $\mathcal{V}\in\mathcal{T}$ such that $\Phi(x)$ does not belong to the interior of $\Phi(\overline{\mathcal{V}})$. Therefore, $\Phi(x)$ belongs to the closed set $\partial\Phi(\overline{\mathcal{V}})$, which has empty interior because $\Phi(\overline{\mathcal{V}})$ is closed. Then, the image of non-regular points is a subset of the meager set $\bigcup\big\{\partial\Phi(\overline{\mathcal{V}}):\mathcal{V}\in\mathcal{T}\big\}$. $\square$

We apply this proposition to the continuous map $\Psi_n:\mathfrak{K}_{r,n}\to K_{r,n}$. Recall that, by construction, the image $K_{r,n}$ has non empty interior. Then, in particular, Ψ_n has some regular value $w\in K_{r,n}$. Let $(a_1,\ldots,a_n)\in\mathfrak{K}_{r,n}$ be any point in $\mathfrak{K}_{r,n}$ such that $\Psi_n(a_1,\ldots,a_n)=w$. Let $\varepsilon>0$ be as in the statement of the proposition. Since the functions Ψ_1, Ψ_2, …, Ψ_n are continuous, there exists $\rho>0$ such that if $|a_j-b_j|<\rho$, for $j=1,\ldots,n$, then

$$\operatorname{dist}(\Psi_j(a_1,\ldots,a_j),\Psi_j(b_1,\ldots,b_j))<\varepsilon \tag{7.5}$$

for all $j=1,\ldots,n$. Using that the point $(a_1,\ldots,a_n)$ is regular (Definition 7.5), we get that the image $\Psi_n(V)$ of the neighborhood

$$V=\mathfrak{K}_{r,n}\cap\{(b_1,\ldots,b_n):|a_j-b_j|<\rho,\text{ for } j=1,\ldots,n\}$$

has w in its interior. In other words, there exists $\delta > 0$ such that $B(w,\delta) \subset \Psi_n(V)$. Consider any point $z \in B(w,\delta)$. Then there exists $(b_1(z),\dots,b_n(z)) \in V$ such that $z = \Psi_n(b_1(z),\dots,b_n(z))$. Define

$$y_j(z) = \Psi_j(b_1(z)),\dots,y_j(z))$$

for $j = 1,\dots,n$, and $y_0(z) = w$. Then $[y_1(z),\dots,y_n(z)]$ is an access sequence connecting x_0 to z. The inequalities (7.5) mean that

$$\operatorname{dist}(y_j(z), y_j(w)) < \varepsilon \quad \text{for } j = 1,\dots,n.$$

Moreover, since $\Psi_n(b_1(z),\dots,b_n(z)) \in K_{r,n}$, the distance between every $y_{j-1}(z)$ and $y_j(z)$ along their common strong (stable or unstable) leaf does not exceed r. Proposition 7.2 follows taking $L = r$ and $N = n$.

8. Generic linear cocycles over partially hyperbolic maps

In this section we prove Theorem A. We will take the vector bundle $\pi : \mathcal{V} \to M$ to be trivial, that is, such that $\mathcal{V} = M \times \mathbb{K}^d$ and $\pi : M \times \mathbb{K}^d \to M$ is the canonical projection. This simplifies the presentation substantially, but is not really necessary for our arguments, which are local in nature: for obtaining the conclusion we consider modifications of the cocycle supported in a neighborhood of certain special points (the pivots, see Proposition 8.8), where triviality holds anyway, by definition.

Let us begin by giving an outline of the proof. Let $\mathbb{K}_x = \{x\} \times \mathbb{K}^d$ be the fiber of $\mathcal{V}$ and $\mathbb{P}(\mathbb{K}_x) = \{x\} \times \mathbb{P}(\mathbb{K})$ be the fiber of the projective bundle $\mathbb{P}(\mathcal{V})$ over the point x. We call *loop* of $f : M \to M$ at $x \in M$ any access sequence $\gamma = [y_0,\dots,y_n]$ connecting a point $x \in M$ to itself, that is, such that $y_0 = y_n = x$. Then we denote

$$H_\gamma = H^{*_n}_{y_{n-1},y_n} \circ \cdots \circ H^{*_j}_{y_{j-1},y_j} \circ H^{*_1}_{y_0,y_1} : \mathbb{P}(\mathbb{K}_x) \to \mathbb{P}(\mathbb{K}_x)$$

where $*_j \in \{s,u\}$ is the symbol of the strong leaf common to the nodes y_{j-1} and y_j. Theorem B implies that if $\lambda_+(F) = \lambda_-(F)$ then any F-invariant probability measure m that projects down to μ admits a disintegration $\{m_z : z \in M\}$ such that

$$(H_\gamma)_* m_x = m_x \quad \text{for any loop } \gamma. \tag{8.1}$$

We consider *loops with slow recurrence*, for which some node y_r, that we call *pivot*, is slowly accumulated by the orbits of all the nodes including its own. Using perturbations of the cocycle supported on a small neighborhood of the pivot, we prove that the map $F \mapsto H_\gamma$ assigning to each cocycle the corresponding holonomy over the loop is a submersion. In fact, we are able to consider several independent loops with slow recurrence, $\gamma_1, \dots, \gamma_m$, and prove that the map

$$F \mapsto (H_{\gamma_1},\dots,H_{\gamma_m})$$

is a submersion. Consequently, for typical cocycles, the matrices H_{γ_i} are in general position, and so they have no common invariant probability in the projective space. This shows that for typical cocycles the condition (8.1) fails and, hence, the extremal Lyapunov exponents are distinct.

8.1. Accessibility with slow recurrence. — An important step is to prove that loops with slow recurrence do exist. Beforehand, let us give the precise definition.

Definition 8.1. — A family $\{\gamma_1, \dots, \gamma_m\}$ of loops $\gamma_i = [y_0^i, \dots, y_{n(i)}^i]$ has *slow recurrence* if there exists $c > 0$ and for each $1 \leq i \leq m$ there exists $0 < r(i) < n(i)$ such that, for all $i, l = 1, \dots, m$, all $0 \leq j \leq n(i)$, and all $k \in \mathbb{Z}$,

$$\operatorname{dist}\left(f^k(y_j^i), y_{r(l)}^l\right) \geq c/(1+k^2)$$

with the exception of $k = 0$ when $(i, j) = (l, r(l))$.

It is convenient to distinguish access sequences $[y_0, y_1, \dots, y_n]$ according to the nature of the last leg: we speak of accessibility *s-sequence* if y_{n-1} and y_n belong to the same strong-stable leaf, and we speak of accessibility *u-sequence* if y_{n-1} and y_n belong to the same strong-unstable leaf. Let d_s and d_u be the dimensions of the strong-stable leaves and strong-unstable leaves, respectively.

Proposition 8.2. — *For any $m \geq 1$ and any $(x_1, \dots, x_m) \in M^m$, there exists a family γ_i of loops with slow recurrence, where each γ_i is a loop at x_i.*

The proof of this proposition requires a number of preparatory results.

Lemma 8.3. — *Given any finite set $\{w_1, \dots, w_n\} \subset M$, any $y \in M$, and any symbol $* \in \{s, u\}$, there exists a full Lebesgue measure subset of points $w \in \mathcal{W}^*_{\mathrm{loc}}(y)$ such that*

$$\operatorname{dist}(f^k(w_j), w) \geq c/(1+k^2) \tag{8.2}$$

for some $c > 0$ and for all $1 \leq j \leq n$ and all $k \in \mathbb{Z}$.

Proof. — Consider $* = s$: the case $* = u$ is analogous. Since local strong-stable leaves are a continuous family of C^2 embedded disks, there exists a constant $D_1 > 0$ such that

$$\lambda_{\mathcal{W}^s_{\mathrm{loc}}(y)}\big(\mathcal{W}^s_{\mathrm{loc}}(y) \cap B(z, c/(1+k^2))\big) \leq D_1 (c/(1+k^2))^{d_s}$$

for any $z \in M$. Thus, the Lebesgue measure of the subset of points $w \in \mathcal{W}^s_{\mathrm{loc}}(y)$ not satisfying inequality (8.2) for some fixed $c > 0$ is bounded by

$$\sum_{j=1}^{n} \sum_{k \in \mathbb{Z}} D_1 c^{d_s} (1+k^2)^{-d_s} \leq D_2\, c^{d_s} \quad \text{with } D_2 = n D_1 \sum_{k \in \mathbb{Z}} (1+k^2)^{-d_s} < \infty.$$

Making $c \to 0$, we conclude that the inequality (8.2) is indeed satisfied by Lebesgue almost every point in $\mathcal{W}^s_{\mathrm{loc}}(y)$. ◻

Corollary 8.4. — *Given any $m \geq 1$, any $(x_1, \dots, x_m) \in M^m$, and any $* \in \{s, u\}$, then for every $(z_1, \dots, z_m)$ in a full Lebesgue measure subset of M^m there exist $c > 0$ and accessibility $*$-sequences $[y_0^i, \dots, y_{n(i)}^i]$ connecting x_i to z_i such that*

$$\operatorname{dist}(f^k(y_j^i), z_l) \geq c/(1+k^2)$$

for all $i, l = 1, \dots, m$, all $0 \leq j < n(i)$, and all $k \in \mathbb{Z}$.

Proof. — Consider $* = s$: the case $* = u$ is analogous. Since the strong-stable foliation is absolutely continuous, it suffices to prove that, given any points $y_i \in M$, $1 \leq i \leq m$, the conclusion holds on a full Lebesgue measure subset of points $z_i \in \mathcal{W}^s_{\mathrm{loc}}(y_i)$, $1 \leq i \leq m$. Now, by the accessibility assumption, there exist accessibility sequences $[y^i_0, \ldots, y^i_{r(i)}]$ connecting x_i to y_i. Consider each z_i in the full Lebesgue measure subset of $\mathcal{W}^s(y_i)$ given by Lemma 8.3, applied to the finite set

$$\left\{y^i_j : 1 \leq i \leq m \text{ and } 0 \leq j \leq r(i)\right\}.$$

and the point $y = y_i$. Then the accessibility s-sequences $[y^i_0, \ldots, y^i_{k(i)}, z_i]$ satisfy the conditions in the conclusion. In view of the observation at the beginning, this proves the corollary. □

Lemma 8.5. — *For any $m \geq 1$ and any $(y_1, \ldots, y_m) \in M^m$, there exists a full Lebesgue measure subset of $(z_1, \ldots, z_m) \in \mathcal{W}^s_{\mathrm{loc}}(y_1) \times \cdots \times \mathcal{W}^s_{\mathrm{loc}}(y_m)$ such that*

$$\operatorname{dist}(f^k(z_i), z_l) \geq c/(1+k^2)$$

for some $c > 0$ and for all $i, l = 1, \ldots, m$ and all $k \geq 0$, except $k = 0$ when $i = l$. The statement remains true if one replaces $\mathcal{W}^s_{\mathrm{loc}}$ by $\mathcal{W}^u_{\mathrm{loc}}$ and $k \geq 0$ by $k \leq 0$.

Proof. — It is clear that each strong-stable leaf contains at most one periodic point. As an easy consequence we get that, that given any $\kappa \geq 1$, there exists a full Lebesgue measure subset of $(z_1, \ldots, z_m) \in \mathcal{W}^s_{\mathrm{loc}}(y_1) \times \cdots \times \mathcal{W}^s_{\mathrm{loc}}(y_m)$ such that $f^k(z_i) \neq z_l$ for all $i, l = 1, \ldots, m$ and all $0 \leq k < \kappa$, except $k = 0$ when $i = l$. Then the condition in the statement holds, for some $c > 0$, restricted to iterates $0 \leq k < \kappa$. Let us focus on $k \geq \kappa$. For each $i, l = 1, \ldots, m$, define

$$E^k_{i,l} = \left\{z_l \in \mathcal{W}^s_{\mathrm{loc}}(y_l) : \operatorname{dist}(f^k(z_i), z_l) < 1/(1+k^2) \text{ for some } z_i \in \mathcal{W}^s_{\mathrm{loc}}(y_i)\right\}.$$

The diameter of $f^k(\mathcal{W}^s_{\mathrm{loc}}(y_i))$ is bounded by $C_1\theta^k$, where $C_1 > 0$ is some uniform constant and $\theta < 1$ is an upper bound for the contraction function $\nu(x)$ in (2.2). Consequently,

$$\operatorname{diam}(E^k_{i,l}) \leq C_1\theta^k + 2/(1+k^2) \leq C_2/(1+k^2)$$

for another uniform constant $C_2 > 0$. It follows that

$$\lambda_{\mathcal{W}^s_{\mathrm{loc}}(y_l)}\Big(\bigcup_{i=1}^{m}\bigcup_{k=\kappa}^{\infty} E^k_{i,l}\Big) \leq m \sum_{k=\kappa}^{\infty} C_2(1+k^2)^{-d_s}.$$

On the one hand, the right hand side of this expression goes to 0 when κ goes to infinity. On the other hand, in view of our previous observations, for any $\kappa \geq 1$, Lebesgue almost every $(z_1, \ldots, z_m) \in \mathcal{W}^s_{\mathrm{loc}}(y_1) \times \cdots \times \mathcal{W}^s_{\mathrm{loc}}(y_m)$ with

$$z_l \notin \bigcup_{i=1}^{m}\bigcup_{k=\kappa}^{\infty} E^k_{i,l}$$

satisfies the conclusion of the lemma for some $c \in (0,1)$. This proves that the subset of $(z_1, \ldots, z_m)$ for which the conclusion of the lemma does not hold has zero Lebesgue measure, as claimed. □

Corollary 8.6. — *For any $m \geq 1$, and every $(z_1, \dots, z_m)$ in a full Lebesgue measure subset of M^m, there exists $c > 0$ such that*

$$\operatorname{dist}(f^k(z_i), z_l) \geq c/(1 + k^2)$$

for all $i, l = 1, \dots, m$ and all $k \in \mathbb{Z}$, except $k = 0$ when $i = l$.

Proof. — It suffices to prove that the conditions obtained replacing $k \in \mathbb{Z}$ by either $k \geq 0$ or $k \leq 0$ are satisfied on full Lebesgue measure subsets of M^m, and then take the intersection of these two subsets. We consider the case $k \geq 0$, as the other one is analogous. Suppose there is a positive Lebesgue measure subset of $(z_1, \dots, z_m) \in M^m$ for which the condition is *not* satisfied: the forward orbit of some z_i accumulates some z_l faster than $c/(1 + k^2)$ for any $c > 0$. Then, since M is covered by the foliation boxes of the strong-stable foliation, there exist foliation boxes U_i, $1 \leq i \leq m$ such that this exceptional subset intersects $U = U_1 \times \cdots \times U_m$ on a positive Lebesgue measure subset. The domain U is foliated by the products $\mathcal{W}^s_{\text{loc}}(y_1) \times \cdots \times \mathcal{W}^s(y_m)$ of local strong-stable leaves. We denote this foliation as $\mathcal{W}^{s,m}$. Given any holonomy maps $h_i : \Sigma^1_i \to \Sigma^2_i$ between cross-sections to the strong-stable foliation $\mathcal{W}^s$ inside U_i, the products $\Sigma^j = \Sigma^j_1 \times \cdots \times \Sigma^j_m$ are cross-sections to $\mathcal{W}^{s,m}$, and the holonomy map of $\mathcal{W}^{s,m}$ is

$$h : \Sigma^1 \to \Sigma^2, \quad h(z_1, \dots, z_m) = (h_1(z_1), \dots, h_m(z_m)).$$

Since all the h_i are absolutely continuous, so is h: the Jacobians are related by $Jh(z_1, \dots, z_m) = Jh_1(z_1) \cdots Jh_m(z_m)$. This absolute continuity property implies that every positive Lebesgue measure subset of U intersects $\mathcal{W}^s_{\text{loc}}(y_1) \times \cdots \mathcal{W}^s_{\text{loc}}(y_m)$ on a positive Lebesgue measure subset, for a subset of $(y_1, \dots, y_m)$ with positive Lebesgue measure. In particular, the exceptional set intersects some leaf of $\mathcal{W}^{s,m}$ on a positive Lebesgue measure subset. This contradicts Lemma 8.5, and this contradiction proves the corollary. □

Corollary 8.7. — *For any $m \geq 1$, any $(x_1, \dots, x_m) \in M^m$, and any $* \in \{s, u\}$, and a full Lebesgue measure set D_* of $(z_1, \dots, z_m) \in M^m$, there exists $c > 0$ such that*

$$\operatorname{dist}(f^k(z_i), z_l) \geq c/(1 + k^2) \tag{8.3}$$

for all $i, l = 1, \dots, m$ and all $k \in \mathbb{Z}$, except $k = 0$ when $i = l$, and there exist accessibility $*$-sequences $[y^i_0, \dots, y^i_{n(i)}]$ connecting x_i to z_i, for $1 \leq i \leq m$ such that

$$\operatorname{dist}(f^k(y^i_j), z_l) \geq c/(1 + k^2) \tag{8.4}$$

for all $i, l = 1, \dots, m$, all $0 \leq j < n(i)$, and all $k \in \mathbb{Z}$.

Proof. — Just take the intersections of the full Lebesgue measure subsets given in Corollary 8.4, for $* \in \{s, u\}$, and in Corollary 8.6. □

Proof of Proposition 8.2. — Given $m \geq 1$ and $(x_1, \dots, x_m) \in M^m$, let D_s and D_u be the full Lebesgue measure sets given by Corollary 8.7, and then consider

$$(z_1, \dots, z_m) \in D_s \cap D_u\,.$$

The corollary yields, for each $1 \le i \le m$, an accessibility s-sequence $[y_0^i, \ldots, y_{r(i)}^i]$ and an accessibility u-sequence $[w_0^i, \ldots, w_{t(i)}^i]$ connecting x_i to z_i. Then

$$\gamma_i = [y_0^i, \ldots, y_{r(i)}^i = w_{t(i)}^i, \ldots, w_0^i]$$

is a loop at x_i, and properties (8.3)-(8.4) mean that the family $\{\gamma_1, \ldots, \gamma_m\}$ of loops has slow recurrence. □

8.2. Holonomies on loops with slow recurrence. — As we pointed out before, the tangent space at each point $B \in \mathcal{G}^{r,\alpha}(M, d, \mathbb{K})$ is naturally identified with the Banach space of $C^{r,\alpha}$ maps from M to the space of linear maps in $\mathbb{K}^d$. This means that we may view the tangent vectors $\dot{B}$ as $C^{r,\alpha}$ functions assigning to each $z \in M$ a linear map $\dot{B}(z) : \mathbb{K}_z \to \mathbb{K}_{f(z)}$.

Let $A \in \mathcal{G}^{r,\alpha}(M, d, \mathbb{K})$ be fiber bunched. As we have seen in Section 3.2, there exists a neighborhood $\mathcal{U} \subset \mathcal{G}^{r,\alpha}(M, d, \mathbb{K})$ of A such that every $B \in \mathcal{U}$ is fiber bunched. Then, for any loop $\gamma = [y_0, \ldots, y_n]$ at a point $x \in M$, and any $0 \le k < l \le n$, we have linear holonomy maps

$$H_{B,\gamma,k,l} = H^{*_l}_{B,y_{l-1},y_l} \circ \cdots \circ H^{*_{k+1}}_{B,y_k,y_{k+1}} : \mathbb{K}_{y_k} \to \mathbb{K}_{y_l}.$$

Furthermore, all the maps $B \mapsto H_{B,\gamma,k,l}$ are C^1 on $\mathcal{U}$. In particular, the derivative of $B \mapsto H_{B,\gamma} = H_{B,\gamma,0,n}$ is given by

$$\partial_B H_{B,\gamma} : \dot{B} \mapsto \sum_{l=1}^{n} H_{B,\gamma,l,n} \big[\partial_B H_{B,\gamma,l-1,l}(\dot{B})\big] H_{B,\gamma,0,l-1}. \tag{8.5}$$

The main result in this section is

Proposition 8.8. — *Let $A \in \mathcal{G}^{r,\alpha}(M, d, \mathbb{K})$ be fiber bunched and $\mathcal{U}$ be a neighborhood as above. For each $x \in M$ and $m \ge 1$, let $\gamma_i = [y_0^i, y_1^i, \ldots, y_{n(i)}^i]$, $1 \le i \le m$ be a family of loops at x with slow recurrence. Then*

$$\mathcal{U} \ni B \mapsto (H_{B,\gamma_1}, \ldots, H_{B,\gamma_m}) \in \mathrm{GL}(d, \mathbb{K}_x)^m$$

is a submersion: the derivative is surjective at every point, even restricted to the subspace of tangent vectors $\dot{B}$ supported on a small neighborhood of the pivots.

In the proof we use (8.5) together with the expressions for the $\partial_B H_{B,\gamma,l-1,l}(\dot{B})$ given in Propositions 3.5 and 3.7. The idea is quite simple. Perturbations in the neighborhood of the pivots affect the holonomies over all the loop legs, of course. However, Corollaries 3.6 and 3.8 show that the effect decreases exponentially fast with time, and slow recurrence means that the first iterates need not be considered. Combining these two ideas one shows (Corollary 8.12) that the derivative is a small perturbation of its term of order zero. The latter is easily seen to be surjective (Lemma 8.13), and then the same is true for any small perturbation.

Remark 8.9. — Essentially the same arguments yield an $\mathrm{SL}(d, \mathbb{K})$-version of this proposition: the map $\mathcal{U} \cap \mathcal{S}^{r,\alpha}(M, d, \mathbb{K}) \ni B \mapsto (H_{B,\gamma_1}, \ldots, H_{B,\gamma_m}) \in \mathrm{SL}(d, \mathbb{K}_x)^m$ is a submersion. Clearly, it remains true that the derivative is a small perturbation of its term of order zero. Then the main point is to observe that the restriction of the operator S in Lemma 8.13 maps $T_B\mathcal{S}^{r,\alpha}(M,d,\mathbb{K})$ surjectively to $T_{H_{B,\gamma}}\operatorname{SL}(d,\mathbb{K}_x)$.

Before getting into the details, let us make an easy observation that allows for some simplification of our notations. If $\gamma=[y_0,\dots,y_n]$ is a loop with slow recurrence then so is $\bar\gamma=[y_n,\dots,y_0]$, and $H_{B,\bar\gamma}$ is the inverse of $H_{B,\gamma}$. Hence, the statement of the proposition is not affected if one reverses the orientation of any γ_i as described. So, it is no restriction to suppose that every loop γ has the orientation for which the pivot y_r satisfies

$$y_r\in\mathcal{W}^s(y_{r-1})\cap\mathcal{W}^u(y_{r+1}),\tag{8.6}$$

and we do so in all that follows.

Lemma 8.10. — *Let $\gamma=[y_0,\dots,y_n]$ be a loop with slow recurrence and y_r be the corresponding pivot. Then, there is $\tau>0$ such that for any small $\varepsilon>0$ and any tangent vector $\dot B$ supported on $B(y_r,\varepsilon)$,*

$$\|\partial_B H_{B,\gamma,l-1,l}(\dot B)\|\le\theta^{\sqrt{\tau/\varepsilon}}\,\|\dot B\|_{0,\beta}\quad\textit{for any } l\neq r,\ \textit{and}$$
$$\|\partial_B H_{B,\gamma,r-1,r}(\dot B)+B(y_r)^{-1}\dot B(y_r)H^s_{B,y_{r-1},y_r}\|\le\theta^{\sqrt{\tau/\varepsilon}}\,\|\dot B\|_{0,\beta}.$$

Proof. — By Definition 8.1, there exists $c>0$ such that

$$\operatorname{dist}(f^k(y_l),y_r)\ge c/(1+k^2)\quad\text{for all } (l,k)\in\{0,\dots,n\}\times\mathbb{Z},\ (l,k)\neq(r,0).$$

Consider $\varepsilon<c/2$. Then $B(y_r,\varepsilon)$ contains no other node of the loop. Moreover, for any $0\le l\le n$ and any $k\ge1$,

$$f^k(y_l)\in B(y_r,\varepsilon)\Longrightarrow|k|\ge t(\varepsilon),\quad\text{where } t(\varepsilon)=\sqrt{c/\varepsilon-1}.$$

Let us denote by $\partial_B H_{B,\gamma,l-1,l,t(\varepsilon)}(\dot B)$ the t-tail of the derivative, that is, the sum over $i\ge t$ in Proposition 3.5 (case $*_l=s$) or Proposition 3.7 (case $*_l=u$). Then, for any $\dot B\in T_B\mathcal{G}^{r,\alpha}(M,d,\mathbb{K})$ supported in $B(y_r,\varepsilon)$, the expression in Proposition 3.5 becomes

$$\partial_B H_{B,\gamma,l-1,l}(\dot B)=\partial_B H_{B,\gamma,l-1,l,t(\varepsilon)}(\dot B)\tag{8.7}$$

for all $l\neq r$, and

$$\partial_B H_{B,\gamma,r-1,r}(\dot B)=-B(y_r)^{-1}\dot B(y_r)H^s_{B,y_{r-1},y_r}+\partial_B H_{B,\gamma,l-1,l,t(\varepsilon)}(\dot B)\tag{8.8}$$

for $l=r$. This applies to the loop legs with symbol $*_l=s$. Observing that the sum in Proposition 3.7 does not include the term $i=0$, we conclude that (8.7) extends to all loop legs with symbol $*_l=u$. Next, by Corollaries 3.6 and 3.8,

$$\|\partial_B H_{B,\gamma,l-1,l,t}(\dot B)\|\le C_5(a)\,\theta^t\,\|\dot B\|_{0,\beta},\tag{8.9}$$

for every $1\le l\le n$ and any $t\ge0$, where a is an upper bound for the distances between consecutive loop nodes. Choose any $\tau<c/2$. The lemma follows directly from (8.7), (8.8), (8.9) with $t=t(\varepsilon)$, because $\theta<1$ and the choices of ε and τ ensure $t(\varepsilon)>\sqrt{\tau/\varepsilon}$. □

Corollary 8.11. — *Let $\gamma_i = [y_0^i, y_1^i, \ldots, y_{n(i)}^i]$, $1 \le i \le m$ be a family of loops at x with slow recurrence and $y_{r(i)}$, $1 \le i \le m$ be the corresponding pivots. Then there exists $\tau > 0$ such that, for any small $\varepsilon > 0$, any $1 \le j \le m$, and any tangent vector $\dot{B}$ supported on $B(y_r^j, \varepsilon)$, $r = r(j)$*

$$\|\partial_B H_{B,\gamma_i,l-1,l}(\dot{B})\| \le \theta^{\sqrt{\tau/\varepsilon}} \|\dot{B}\|_{0,\beta} \quad \text{for all } (i,l) \neq (j,r), \text{ and}$$

$$\|\partial_B H_{B,\gamma_j,r-1,r}(\dot{B}) + B(y_r^j)^{-1}\dot{B}(y_r^j) H^s_{B,y_{r-1}^j,y_r^j}\| \le \theta^{\sqrt{\tau/\varepsilon}} \|\dot{B}\|_{0,\beta}.$$

Proof. — The case $i = j$ is contained in Lemma 8.10. The cases $i \neq j$ follow from the same arguments, observing that

$$\operatorname{dist}(f^k(y_l^i), y_r^j) \ge c/(1+k^2) \quad \text{for every } k \in \mathbb{Z}$$

and so $f^k(y_l^i) \in B(y_r^j, \varepsilon)$ implies $|k| \ge t(\varepsilon)$, for every $0 \le l \le n(i)$. □

Corollary 8.12. — *Let $\gamma_i = [y_0^i, y_1^i, \ldots, y_{n(i)}^i]$, $1 \le i \le m$ be a family of loops at x with slow recurrence, and $y_{r(i)}$, $1 \le i \le m$ be the corresponding pivots. Then, there exists $K_1 > 0$ such that, for any small $\varepsilon > 0$, any $1 \le j \le m$, and any tangent vector $\dot{B}$ supported on $B(y_r^j, \varepsilon)$, $r = r(j)$*

$$\|\partial_B H_{B,\gamma_i}(\dot{B})\| \le K_1 \theta^{\sqrt{\tau/\varepsilon}} \|\dot{B}\|_{0,\beta} \quad \text{for all } i \neq j, \text{ and}$$

$$\|\partial_B H_{B,\gamma_j}(\dot{B}) + H_{B,\gamma_j,r,n(j)} B(y_r^j)^{-1}\dot{B}(y_r^j) H_{B,\gamma_j,0,r}\| \le K_1 \theta^{\sqrt{\tau/\varepsilon}} \|\dot{B}\|_{0,\beta}$$

Proof. — This follows from replacing in (8.5) the estimates in Corollary 8.11. By part (e) of Proposition 3.2, the factors $H_{B,\gamma_i,0,l-1}$ and $H_{B,\gamma_i,l,n(i)}$ are bounded by some uniform constant K_2 that depends only on the loops. Then, for every $i \neq j$, Corollary 8.11 and the relation (8.5) gives

$$\|\partial_B H_{B,\gamma_i}(\dot{B})\| \le \sum_{l=1}^{n(i)} K_2^2 \|\partial_B H_{B,\gamma,l-1,l}(\dot{B})\| \le K_1 \theta^{\sqrt{\tau/\varepsilon}} \|\dot{B}\|_{0,\beta},$$

as long as we choose $K_1 \ge K_2^2 \max_i n(i)$. This gives the first part of the corollary. Now we consider $i = j$. For the same reasons as before, all but one term in the expression (8.5) are bounded by $K_2^2 \theta^{\sqrt{\tau/\varepsilon}} \|\dot{B}\|_{0,\beta}$. The possible exception is

$$H_{B,\gamma_j,r,n(j)} \left[\partial_B H_{B,\gamma_j,r-1,r}(\dot{B})\right] H_{B,\gamma_j,0,r-1},$$

corresponding to $l = r$. By Corollary 8.11, this last expression differs from

$$- H_{B,\gamma_j,r,n(j)} B(y_r^j)^{-1}\dot{B}(y_r^j) H^s_{B,y_{r-1}^j,y_r^j} H_{B,\gamma_j,0,r-1} = \\ - H_{B,\gamma_j,r,n(j)} B(y_r^j)^{-1}\dot{B}(y_r^j) H_{B,\gamma_j,0,r}$$

by a term bounded by $K_2^2 \theta^{\sqrt{\tau/\varepsilon}} \|\dot{B}\|_{0,\beta}$. This completes the proof. □

Lemma 8.13. — *Let $\gamma = [y_0, \ldots, y_n]$ be a loop at $x \in M$ and $0 < r < n$ be fixed. Then the linear map*

$$\begin{array}{rccl} S: & T_B\mathcal{G}^{r,\alpha}(M,d,\mathbb{K}) & \to & T_{H_{B,\gamma}} \operatorname{GL}(d,\mathbb{K}_x) \simeq \mathcal{L}(\mathbb{K}_x^d, \mathbb{K}_x^d) \\ & \dot{B} & \mapsto & -H_{B,\gamma,r,n} B(y_r)^{-1} \dot{B}(y_r) H_{B,\gamma,0,r} \end{array}$$

is surjective, even restricted to the subspace of tangent vectors $\dot B$ vanishing outside some neighborhood of y_r. More precisely, there exists $K_3 > 0$ such that for $0 < \varepsilon < 1$ and $\Theta \in \mathcal{L}(\mathbb{K}^d, \mathbb{K}^d)$ there exists $\dot B_\Theta \in T_B\mathcal{G}^{r,\alpha}(M, d, \mathbb{K})$ vanishing outside $B(y_r, \varepsilon)$ and such that $S(\dot B_\Theta) = \Theta$ and $\|\dot B_\Theta\|_{0,\beta} \le K_3\,\varepsilon^{-\beta}\,\|\Theta\|$.

Proof. — Let $\tau : M \to [0, 1]$ be a $C^{r,\alpha}$ function vanishing outside $B(y_r, \varepsilon)$ and such that $\tau(y_r) = 1$ and the Hölder constant $H_\beta(\tau) \le 2\varepsilon^{-\beta}$. For $\Theta \in \mathcal{L}(\mathbb{K}^d, \mathbb{K}^d)$, define $\dot B_\Theta \in T_B\mathcal{G}^{r,\alpha}(M, d, \mathbb{K})$ by

$$\dot B_\Theta(w) = B(y_r)\,H^{-1}_{B,\gamma,r,n}\,\Theta\,B(y_r)^{-1}\,\tau(w)\,B(w)\,H^{-1}_{B,\gamma,0,r}.$$

Notice that $\dot B_\Theta(y_r) = B(y_r)\,H^{-1}_{B,\gamma,r,n}\,\Theta\,H^{-1}_{B,\gamma,0,r}$ and so $S(\dot B_\Theta) = \Theta$. Moreover,

$$\|\dot B_\Theta\|_{0,0} \le \|H^{-1}_{B,\gamma,r,n}\|\,\|H^{-1}_{B,\gamma,0,r}\|\,\|B(y_r)\|\,\|B(y_r)^{-1}\|\,\|B\|_{0,0}\,\|\Theta\|. \tag{8.10}$$

For any $w_1, w_2 \in M$ the norm of $\dot B_\Theta(w_1) - \dot B_\Theta(w_2)$ is bounded by

$$\|H^{-1}_{B,\gamma,r,n}\|\,\|H^{-1}_{B,\gamma,0,r}\|\,\|B(y_r)\|\,\|B(y_r)^{-1}\| \\ \Big(\|\tau(w_1) - \tau(w_2)\|\|B(w_1)\| + |\tau(w_2)|\|B(w_1) - B(w_2)\|\Big)\,\|\Theta\|.$$

Consequently, the Hölder constant $H_\beta(\dot B_\Theta)$ of $\dot B_\Theta$ is bounded above by

$$\|H^{-1}_{B,\gamma,r,n}\|\,\|H^{-1}_{B,\gamma,0,r}\|\,\|B(y_r)\|\,\|B(y_r)^{-1}\|\,\big(2\varepsilon^{-\beta}\|B\|_{0,0} + H_\beta(B)\big)\,\|\Theta\|. \tag{8.11}$$

Adding the inequalities (8.10) and (8.11), and taking

$$K_3 = \|H^{-1}_{B,\gamma,r,n}\|\,\|H^{-1}_{B,\gamma,0,r}\|\,\|B(y_r)\|\,\|B(y_r)^{-1}\|\,\|B\|_{0,\beta},$$

one obtains $\|\dot B_\Theta\|_{0,\beta} \le K_3\varepsilon^{-\beta}\|\Theta\|$. □

Proof of Proposition 8.8. — For each $1 \le j \le m$, let S_j be the operator associated to $\gamma = \gamma_j$ as in Lemma 8.13. Let Θ_j be any element of the unit sphere in $\mathcal{L}(\mathbb{K}_x, \mathbb{K}_x)$. By Lemma 8.13, for any small $\varepsilon > 0$ there exists a tangent vector $\dot B(j, \Theta_j)$ supported in $B(y^j_{r(j)}, \varepsilon)$ such that

$$S_j\big(\dot B(j, \Theta_j)\big) = \Theta_j \quad \text{and} \quad \|\dot B(j, \Theta_j)\| \le K_3\varepsilon^{-\beta}.$$

By Corollary 8.12, the norm of

$$(\partial_B H_{B,\gamma_1}, \dots, \partial_B H_{B,\gamma_j}, \dots, \partial_B H_{B,\gamma_m})(\dot B) - (0, \dots, 0, S_j(\dot B), 0, \dots, 0)$$

is bounded above by $K_3\theta^{\sqrt{\tau/\varepsilon}}\|\dot B\|$, for any tangent vector supported in $B(y^j_{r(j)}, \varepsilon)$. For $\dot B = \dot B(j, \Theta_j)$ this gives that

$$\|(\partial_B H_{B,\gamma_1}, \dots, \partial_B H_{B,\gamma_j}, \dots, \partial_B H_{B,\gamma_m})(\dot B(j, \Theta_j)) - (0, \dots, 0, \Theta_j, 0, \dots, 0)\|$$

is bounded by $K_1K_3\theta^{\sqrt{\tau/\varepsilon}}\varepsilon^{-\beta}$. Assume $\varepsilon > 0$ is small enough so that

$$K_1K_3\theta^{\sqrt{\tau/\varepsilon}}\varepsilon^{-\beta} < 1/(2m).$$

Then for any $\Theta = (\Theta_1, \dots, \Theta_m)$ with Θ_j in the unit sphere of $\mathcal{L}(\mathbb{K}_x, \mathbb{K}_x)$ we find a tangent vector $\dot{B}(\Theta) = \sum_{j=1}^m \dot{B}(j, \Theta_j)$ supported on the ε-neighborhood of the pivots and such that

$$\|\big(\partial H_{B,\gamma_1}, \dots, \partial H_{B,\gamma_m}\big)\big(\dot{B}(\Theta)\big) - \Theta\| < 1/2.$$

This implies that the image of the derivative $(\partial H_{B,\gamma_1}, \dots, \partial H_{B,\gamma_m})$ is the whole target space $\mathcal{L}(\mathbb{K}_x^d, \mathbb{K}_x^d)^m$, as claimed. □

8.3. Invariant measures of generic matrices. — Finally, we prove Theorem A. The only missing ingredient is

Proposition 8.14. — *Given $\ell \geq 1$, let $G_{2\ell}$ be the set of $(\mathtt{A}_1, \dots, \mathtt{A}_{2\ell}) \in \mathrm{GL}(d, \mathbb{K})^{2\ell}$ such that there exists some probability η in $\mathbb{P}(\mathbb{C})$ invariant under the action of $\mathtt{A}_i$ for every $1 \leq i \leq 2\ell$. Then $G_{2\ell}$ is closed and nowhere dense, and it is contained in a finite union of closed submanifolds of codimension $\geq \ell$.*

Remark 8.15. — The arguments that we are going to present remain valid if one replaces $\mathrm{GL}(d, \mathbb{K})$ by the subgroup $\mathrm{SL}(d, \mathbb{K})$ of matrices with determinant 1: just note that the curves $\mathtt{B}(t)$ defined in (8.13) and (8.17) lie in $\mathrm{SL}(d, \mathbb{K})$ if the initial matrix $\mathtt{A}$ does. Thus, the proposition holds for $\mathrm{SL}(d, \mathbb{K})$ as well.

Let us assume this proposition for a while, and use it to conclude the proof of the theorem in the complex case. Let $A \in \mathcal{G}^{r,\alpha}(M, d, \mathbb{K})$ be fiber bunched. Fix any $\ell \geq 1$ and $x \in M$. By Proposition 8.2 there is a family γ_i, $1 \leq i \leq 2\ell$, of loops at x with slow recurrence. By Proposition 8.8, the map

$$\mathcal{U} \ni B \mapsto (H_{B,\gamma_1}, \dots, H_{B,\gamma_{2\ell}}) \in \mathrm{GL}(d, \mathbb{K}_x)^{2\ell}$$

is a submersion, where $\mathcal{U}$ is a neighborhood of A independent of ℓ. Let $\mathcal{Z}$ be the pre-image of $G_{2\ell}$ under this map. Then $\mathcal{Z}$ is closed and nowhere dense, and it is contained in a finite union of closed submanifolds of codimension $\geq \ell$.

We claim that $\lambda_-(B, \mu) < \lambda_+(B, \mu)$ for all $B \in \mathcal{U} \setminus \mathcal{Z}$. Indeed, suppose the equality holds, and let m be any $\mathbb{P}(F_B)$-invariant probability that projects down to μ. By Theorem B, the measure m admits a disintegration $\{m_z : z \in M\}$ which is invariant under strong-stable holonomies $h^s = \mathbb{P}(H^s)$ and strong-unstable holonomies $h^u = \mathbb{P}(H^u)$, on the whole manifold M. In particular,

$$\mathbb{P}(H_{B,\gamma_i})_* m_x = m_x \quad \text{for every } 1 \leq i \leq 2\ell. \tag{8.12}$$

This contradicts the definition of $G_{2\ell}$, and this contradiction proves our claim. Let $\mathcal{Z}_0$ be the set of fiber bunched $B \in \mathcal{G}^{r,\alpha}(M, d, \mathbb{K})$ for which $\lambda_-(B, \mu) = \lambda_+(B, \mu)$. We have shown that any fiber bunched $A \in \mathcal{G}^{r,\alpha}(M, d, \mathbb{K})$ admits a neighborhood $\mathcal{U}$ such that, for any $\ell \geq 1$, there exists a nowhere dense subset $\mathcal{Z}$ of $\mathcal{U}$ contained in a finite union of closed submanifolds of codimension $\geq \ell$ and such that $\mathcal{Z}_0 \cap \mathcal{U} \subset \mathcal{Z}$. Thus, the closure of $\mathcal{Z}_0$ has infinite codimension and, in particular, is nowhere dense.

The proof of Theorem A has been reduced to proving Proposition 8.14. The proof of the proposition is presented in the next two sections.

8.3.1. Complex case. — Let S be the subset of matrices $\mathtt{A} \in \mathrm{GL}(d,\mathbb{C})$ whose eigenvalues are all distinct in norm. Then, S is an open and dense subset of $\mathrm{GL}(d,\mathbb{C})$ whose complement is contained in a finite union of closed manifolds of positive codimension. We use the following fact about variation of eigenvectors inside S:

Lemma 8.16. — *Let $\mathtt{A} \in S$. Then there exist C^∞ functions $\lambda_i : S_{\mathtt{A}} \to \mathbb{C}$ and $v_i : S_{\mathtt{A}} \to \mathbb{P}(\mathbb{C}^d)$ defined on an open neighborhood $S_{\mathtt{A}}$ of $\mathtt{A}$, for each $1 \le i \le d$, such that $v_i(\mathtt{B})$ is the direction of an eigenvector of $\mathtt{B}$ associated to the eigenvalue $\lambda_i(\mathtt{B})$, for any $B \in S_{\mathtt{A}}$. Furthermore, the map $S_{\mathtt{A}} \to \mathbb{P}(\mathbb{C}^d)^d$, $\mathtt{B} \mapsto (v_1(\mathtt{B}), \dots, v_d(\mathtt{B}))$ is a submersion.*

Proof. — Since each eigenvalue $\lambda_i(\mathtt{A})$ is a simple root of the polynomial $\det(\mathtt{A} - \lambda\,\mathrm{id})$, it has a C^∞ continuation $\lambda_i(\mathtt{B})$ for all nearby matrices, given by the implicit function theorem. Denote $L_i(B) = \mathtt{B} - \lambda_i(\mathtt{B})\,\mathrm{id}$. It depends smoothly on $\mathtt{B} \in S_{\mathtt{A}}$ and, since $\lambda_i(B)$ remains a simple eigenvalue of $\mathtt{B}$, it has rank $d-1$. Since the entries of $\mathrm{adj}(L_i(\mathtt{B}))$ are cofactors of $L_i(\mathtt{B})$, the adjoint is a non-zero matrix that also varies in a C^∞ fashion with $\mathtt{B}$. Moreover,

$$L_i(\mathtt{B}) \cdot \mathrm{adj}(L_i(\mathtt{B})) = \det(L_i(\mathtt{B}))\,\mathrm{id} = 0.$$

This means that any nonzero column of $\mathrm{adj}(L_i(B))$ is an eigenvector for $L_i(\mathtt{B})$, depending in a C^∞ fashion on the matrix, and so we may use it to define a function $v_i(\mathtt{B})$ as in the statement. To check that the derivative of v at $\mathtt{A}$ is onto just consider any differentiable curve $(-\varepsilon, \varepsilon) \ni t \mapsto (\beta_1(t), \dots, \beta_d(t))$ such that $\beta_i(0) = v_i(\mathtt{A})$ for all $i = 1, \dots, d$. Define $P(t) = [\beta_1(t), \dots, \beta_d(t)]$, that is, $P(t)$ is the matrix whose column vectors are the $\beta_i(t)$. Then define

$$\mathtt{B}(t) = P(t)\,\mathrm{diag}[\lambda_1(\mathtt{A}), \dots, \lambda_d(\mathtt{A})]P(t)^{-1}. \tag{8.13}$$

Then, $\mathtt{B}(0) = \mathtt{A}$ and $v(\mathtt{B}(t)) = (\beta_1(t), \dots, \beta_d(t))$ for all t. In particular, the derivative $Dv(A)$ maps $\mathtt{B}'(0)$ to $(\beta_1'(0), \dots, \beta_d'(0))$. So, the derivative is indeed surjective. □

Let $\mathcal{Z}_1$ be the subset of $\underline{\mathtt{A}} = (\mathtt{A}_1, \dots, \mathtt{A}_{2\ell})$ such that $\mathtt{A}_i \notin S$ for at least ℓ values of i. Then $\mathcal{Z}_1$ is closed and it is contained in a finite union of closed submanifolds of codimension $\ge \ell$. For every $\underline{\mathtt{A}} \notin \mathcal{Z}_1$ there are at least $\ell + 1$ matrices $\mathtt{A}_i$ whose eigenvalues all have distinct norms. Restricting to some open subset $\mathcal{V}$ of the complement of $\mathcal{Z}_1$, and renumbering if necessary, we may suppose that these matrices are $\mathtt{A}_1, \dots, \mathtt{A}_{\ell+1}$. By Lemma 8.16, reducing $\mathcal{V}$ if necessary, the map

$$\mathcal{V} \setminus \mathcal{Z}_1 \ni \underline{\mathtt{A}} \mapsto \big(v_j(\mathtt{A}_i)\big)_{1 \le j \le d,\ 1 \le i \le \ell+1} \in \mathbb{P}(\mathbb{C}^d)^{d(\ell+1)}$$

is a submersion. Consequently, there exists a closed subset $\mathcal{Z}_2$ of $\mathcal{V} \setminus \mathcal{Z}_1$ contained in a finite union of closed submanifolds of codimension $\ge \ell$ such that for every $\underline{\mathtt{A}} \in \mathcal{V} \setminus (\mathcal{Z}_1 \cup \mathcal{Z}_2)$ there exists some $1 \le i \le \ell$ such that

$$v_a(\mathtt{A}_i) \ne v_b(\mathtt{A}_{\ell+1}) \quad \text{for every } a, b \in \{1, \dots, d\}. \tag{8.14}$$

Now it suffices to prove that $G_{2\ell} \cap \mathcal{V}$ is contained in $\mathcal{Z}_1 \cup \mathcal{Z}_2$. Indeed, suppose there is $\underline{\mathtt{A}} \in G_{2\ell} \cap \mathcal{V} \setminus (\mathcal{Z}_1 \cup \mathcal{Z}_2)$. By the definition of $G_{2\ell}$, there exists some probability measure η on $\mathbb{P}(\mathbb{C}^d)$ such that

$$(\mathtt{A}_l)_*\eta = \eta \quad \text{for every } 1 \le l \le 2\ell. \tag{8.15}$$

Consider $l = i$, as in (8.14), and also $l = \ell + 1$. Since all the eigenvalues of $\mathtt{A}_i$ have distinct norms, η must be a convex combination of Dirac masses supported on the eigenspaces of $\mathtt{A}_i$. For the same reason, η must be supported on the set of eigenspaces of $\mathtt{A}_{\ell+1}$. However, (8.14) means that these two sets are disjoint, and so we reached a contradiction. This contradiction proves Proposition 8.14 in the complex case.

8.3.2. Real case. — The proof for real matrices is a bit more complicated due to the possibility of complex conjugate eigenvalues. In particular, the set of matrices whose eigenvalues are all distinct in norm is not dense. This difficulty has been met before by Bonatti, Gomez-Mont, Viana [**6**], and we use a similar approach in dimensions $d \geq 3$. For $d = 2$ we use a different argument, based on the conformal barycenter construction of Douady, Earle [**10**].

For each $r, s \geq 0$ with $r + 2s = d$, let $S(r,s)$ be the subset of matrices $\mathtt{A} \in \mathrm{GL}(d,\mathbb{R})$ having r real eigenvalues, and s pairs of (strictly) complex conjugate eigenvalues, such that all the eigenvalues that do not belong to the same complex conjugate pair have distinct norms. Every $S(r,s)$ is open and their union $S = \cup_{r,s} S(r,s)$ is an open and dense subset of $\mathrm{GL}(d,\mathbb{R})$ whose complement is contained in a finite union of closed submanifolds with positive codimension. Let $\mathrm{Grass}(k,d)$ denote the k-dimensional Grassmannian of $\mathbb{R}^d$, for $1 \leq k \leq d$. In what follows we often think of elements of $\mathrm{Grass}(2,d)$ as subsets of $\mathrm{Grass}(1,d) = \mathbb{P}(\mathbb{R}^d)$.

Lemma 8.17. — *Let* $\mathcal{F} = \{[(r_1,\ldots,r_d)e^{i\theta}] \in \mathbb{P}(\mathbb{C}^d) : \theta \in [0,2\pi], (r_1,\ldots,r_d) \in \mathbb{R}^d\}$. *Then $\mathcal{F}$ is closed in $\mathbb{P}(\mathbb{C}^d)$ and the map $\Psi : \mathbb{P}(\mathbb{C}^d)\backslash\mathcal{F} \to \mathrm{Grass}(2,d)$ defined by $\Psi(v) = \mathrm{Span}\,\{\mathrm{Re}(v), \mathrm{Im}(v)\}$ is a submersion.*

Proof. — First, we recall the usual local charts in $\mathrm{Grass}(2,d)$. Let $e_1,\ldots,e_d$ the canonical base of $\mathbb{R}^d$ and $1 \leq i < j \leq d$ be fixed. For any $d \times 2$ matrix $\mathtt{A}$ we denote by $\varphi(\mathtt{A})$ the 2×2 matrix formed by the ith and jth rows of $\mathtt{A}$ and by $\varphi^*(\mathtt{A})$ the $(d-2) \times 2$ matrix formed by the other rows of $\mathtt{A}$. Let $U_{i,j}$ be the open set of planes $L \in \mathrm{Grass}(2,d)$ such that the orthogonal projection of L to $\mathrm{Span}\,\{e_i, e_j\}$ is an isomorphism. This means that if $L \in U_{i,j}$ with $L = \mathrm{Span}\,\{v_1, v_2\}$ then $\varphi(\mathtt{A}_L)$ is invertible, where $\mathtt{A}_L = [v_1, v_2]$ is the matrix whose columns are the vectors v_1, v_2. Then the map $\phi : U_{i,j} \to \mathbb{R}^{2(d-2)}$ defined by $\phi(L) = \varphi^*(\mathtt{A}_L)\varphi(\mathtt{A}_L)^{-1}$, where we identify $(d-2) \times 2$ matrices with points in $\mathbb{R}^{2(d-2)}$, is a local chart in the Grassmannian.

Now, note that v, $\overline{v} \in \mathbb{C}^d$ are linearly independent if and only if $v \in \mathbb{P}(\mathbb{C}^d)\backslash\mathcal{F}$. Moreover, in that case $\mathrm{Re}(v), \mathrm{Im}(v)$ are $\mathbb{C}$-linearly independent and, in particular, $\Psi(v)$ is well defined. It is clear from its expression in local charts that Ψ is differentiable. Moreover, still in local charts, its derivative is given by

$$D\Psi(v)\dot{v} = \varphi^*(\dot{\mathtt{A}})\varphi(\mathtt{A})^{-1} - \varphi^*(\mathtt{A})\varphi(\mathtt{A})^{-1}\varphi(\dot{\mathtt{A}})\varphi(\mathtt{A})^{-1},$$

where $\dot{v} \in T_v\mathbb{P}(\mathbb{C}^d)$, $\mathtt{A} = [\mathrm{Re}(v), \mathrm{Im}(v)]$ and $\dot{\mathtt{A}} = [\mathrm{Re}(\dot{v}), \mathrm{Im}(\dot{v})]$. Let $\dot{\mathtt{B}}$ be in the tangent space $T_{\Psi(v)}\,\mathrm{Grass}(2,d)$. Then $\dot{\mathtt{B}}$ is a $(d-2) \times 2$ matrix with real entries. Let $\dot{\mathtt{A}}_{\dot{\mathtt{B}}}$ be the $d \times 2$ matrix defined by $\varphi^*(\dot{\mathtt{A}}_{\dot{\mathtt{B}}}) = \dot{\mathtt{B}}\varphi(\mathtt{A})$ and $\varphi(\dot{\mathtt{A}}_{\dot{\mathtt{B}}}) = 0$. Since, $\dot{\mathtt{A}}_{\dot{\mathtt{B}}} = [\dot{v}_1, \dot{v}_2]$, we have that $D\Psi(v)(\dot{v}_1 + i\dot{v}_2) = \dot{\mathtt{B}}$. This finishes the proof of the lemma. $\square$

Lemma 8.18. — *Let* $\mathtt{A} \in S(r,s)$*. Then there exists an open neighborhood* $S_{\mathtt{A}}$ *of* $\mathtt{A}$ *and there exist* C^∞ *functions*

$$\begin{aligned} &\lambda_j : S_{\mathtt{A}} \to \mathbb{R}, \quad \xi_j : S_{\mathtt{A}} \to \mathrm{Grass}(1,d), \quad \text{for } 1 \le j \le r, \text{ and} \\ &\mu_k : S_{\mathtt{A}} \to \mathbb{C} \setminus \mathbb{R}, \quad \eta_k : S_{\mathtt{A}} \to \mathrm{Grass}(2,d), \quad \text{for } 1 \le k \le s, \end{aligned}$$

such that $\xi_j(\mathtt{B})$ *is the eigenspace of* $\mathtt{B}$ *associated to the eigenvalue* $\lambda_j(\mathtt{B})$*, and* $\eta_k(\mathtt{B})$ *is the characteristic space associated to the conjugate pair of eigenvalues* $\mu_k(\mathtt{B})$ *and* $\bar\mu_k(\mathtt{B})$*. Furthermore, the map*

$$S_{\mathtt{A}} \to \mathrm{Grass}(1,d)^r \times \mathrm{Grass}(2,d)^s, \quad \mathtt{B} \mapsto (\xi_j(\mathtt{B})_{1\le j\le r}, \eta_k(\mathtt{B})_{1\le k\le s})$$

is a submersion.

Proof. — Existence and regularity of the eigenvalues λ_j and μ_k follow from the implicit function theorem. Moreover, the arguments in Lemma 8.16 imply that if $v_j(\mathtt{B})$ is an eigenvector associated to the eigenvalue $\lambda_j(\mathtt{B})$, for $j = 1, \dots, r$, and $v_{r+2k-1}(\mathtt{B}), v_{r+2k}(\mathtt{B})$ are eigenvectors associated to $\mu_k(\mathtt{B})$, $\bar\mu_k(\mathtt{B})$, respectively, for $k = 1, \dots, s$, then the map Φ defined by

$$\Phi(\mathtt{B}) = (v_1(\mathtt{B}), \dots, v_r(\mathtt{B}), v_{r+1}(\mathtt{B}), \dots, v_{r+2s}(\mathtt{B})) \in \mathbb{P}(\mathbb{R}^d)^r \times \mathbb{P}(\mathbb{C}^d)^s \tag{8.16}$$

is C^∞. We are going to show that this map is a submersion on some open neighborhood $S_{\mathtt{A}}$ of $\mathtt{A}$. For this, it is sufficient to show that the derivative $D\Phi(A)$ is onto. Consider any differentiable curve $(-\varepsilon,\varepsilon) \ni t \mapsto (\beta_1(t), \dots, \beta_{r+s}(t))$ such that $\beta_j(0) = v_j(\mathtt{A})$ for $j = 1, \dots, r$ and $\beta_{r+k}(0) = v_{r+2k-1}(\mathtt{A})$ for $k = 1, \dots, s$. Define

$$\begin{aligned} &P(t) = [\beta_1(t), \dots, \beta_r(t), \beta_{r+1}, \bar\beta_{r+1}, \dots, \beta_{r+s}, \bar\beta_{r+s}], \text{and} \\ &\mathtt{B}(t) = P(t)\, \mathrm{diag}[\lambda_1(\mathtt{A}), \dots, \lambda_r(\mathtt{A}), \mu_1(\mathtt{A}), \bar\mu_1(\mathtt{A}), \dots, \mu_s(\mathtt{A}), \bar\mu_s(\mathtt{A})]\, P(t)^{-1}. \end{aligned} \tag{8.17}$$

Observe that $t \mapsto \mathtt{B}(t)$ is a curve in $\mathrm{GL}(d,\mathbb{R})$, with $\mathtt{B}(0) = \mathtt{A}$. Observe also that $\Phi(\mathtt{B}(t)) = (\beta_1(t), \dots, \beta_{r+s}(t)$ for all $t \in (-\varepsilon,\varepsilon)$, and so $D\Phi(A)$ maps $\mathtt{B}'(0)$ to the vector $(\beta_1'(0), \dots, \beta_{r+s}'(0))$. So, the derivative is indeed surjective. Finally, define

$$\begin{aligned} &\xi_j(\mathtt{B}) = v_j(\mathtt{B}) \quad \text{for } j = 1, \dots, r \text{ and} \\ &\eta_k(\mathtt{B}) = \mathrm{Span}\,\{\mathrm{Re}(v_{r+2k-1}), \mathrm{Im}(v_{r+2k-1})\} \ \text{ for } k = 1, \dots, s. \end{aligned}$$

Clearly these maps are C^∞. Moreover, since (8.16) is a submersion, Lemma 8.17 implies that $\mathtt{B} \mapsto (\xi_j(\mathtt{B})_{1\le j\le r}, \eta_k(\mathtt{B})_{1\le k\le s})$ is a submersion. □

Let $\mathcal{Z}_1$ be the subset of $\underline{\mathtt{A}} = (\mathtt{A}_1, \dots, \mathtt{A}_{2\ell})$ such that $\mathtt{A}_i \notin S$ for at least ℓ values of i. Then $\mathcal{Z}_1$ is closed and it is contained in a finite union of closed submanifolds of codimension $\ge \ell$. For every $\underline{\mathtt{A}} \notin \mathcal{Z}_1$ there are at least $\ell + 1$ values of i such that $\mathtt{A}_i \in S$, that is, $\mathtt{A}_i \in S(r_i, s_i)$ for r_i and s_i. Restricting to some open subset $\mathcal{V}$ of the complement of $\mathcal{Z}_1$, and renumbering if necessary, we may suppose that these matrices are $\mathtt{A}_1, \dots, \mathtt{A}_{\ell+1}$. By Lemma 8.18, reducing $\mathcal{V}$ if necessary, the map

$$\mathcal{V} \setminus \mathcal{Z}_1 \ni \underline{\mathtt{A}} \mapsto \Big(\xi_j(\mathtt{A}_i)_{1\le j\le r_i},\ \eta_k(\mathtt{A}_i)_{1\le k\le s_i}\Big)_{1\le i\le \ell+1} \tag{8.18}$$

is a submersion.

Assume first that $d \ge 4$, and so $\dim \mathbb{P}(\mathbb{R}^d) \ge 3$. Since the $\xi_j(\mathtt{A})$ are points and the $\eta_k(\mathtt{A})$ are lines in the projective space, it follows that there exists a closed subset $\mathcal{Z}_2$

of $\mathcal{V} \setminus \mathcal{Z}_1$ contained in a finite union of closed submanifolds of codimension $\geq \ell$ such that for every $\mathtt{A} \in \mathcal{V} \setminus (\mathcal{Z}_1 \cup \mathcal{Z}_2)$ there exists some $1 \leq i \leq \ell$ such that

$$\xi_a(\mathtt{A}_i) \neq \xi_b(\mathtt{A}_{\ell+1}) \tag{8.19}$$

$$\xi_a(\mathtt{A}_i) \notin \eta_c(\mathtt{A}_{\ell+1}) \quad \text{and} \quad \xi_b(\mathtt{A}_i) \notin \eta_d(\mathtt{A}_{\ell+1}) \tag{8.20}$$

$$\eta_c(\mathtt{A}_i) \cap \eta_d(\mathtt{A}_{\ell+1}) = \emptyset \tag{8.21}$$

for every $1 \leq a \leq r(\mathtt{A}_i)$, $1 \leq b \leq r(\mathtt{A}_{\ell+1})$, $1 \leq c \leq s(\mathtt{A}_i)$, and $1 \leq d \leq s(\mathtt{A}_{\ell+1})$. Now it suffices to prove that $G_{2\ell} \cap \mathcal{V}$ is contained in $\mathcal{Z}_1 \cup \mathcal{Z}_2$. Indeed, suppose there is $\underline{\mathtt{A}} \in G_{2\ell} \cap \mathcal{V} \setminus (\mathcal{Z}_1 \cup \mathcal{Z}_2)$. By the definition of $G_{2\ell}$, there exists some probability measure η on $\mathbb{P}(\mathbb{C}^d)$ such that

$$(\mathtt{A}_l)_*\eta = \eta \quad \text{for every } 1 \leq l \leq 2\ell. \tag{8.22}$$

Consider both $l = i$, as in (8.19)–(8.21), and $l = \ell + 1$. Since all the eigenvalues of $\mathtt{A}_i$ have distinct norms, apart from the complex conjugate pairs, the measure η must be supported on

$$\Sigma(\mathtt{A}_i) = \bigcup_{j=1}^{r} \{\xi_j(\mathtt{A}_i)\} \cup \bigcup_{k=1}^{s} \eta_k(\mathtt{A}_i).$$

Analogously, η must be supported on $\Sigma(\mathtt{A}_{\ell+1})$. However, conditions (8.19)–(8.21) mean that the two sets $\Sigma(\mathtt{A}_i)$ and $\Sigma(\mathtt{A}_{\ell+1})$ are disjoint. This contradiction proves the proposition in any dimension $d \geq 4$.

For $d = 3$ the projective space $\mathbb{P}(\mathbb{R}^3)$ is only 2-dimensional, and so one can not force a pair of 1-dimensional submanifolds $\eta_k(\mathtt{A})$ to be disjoint, as required in (8.21). However, the argument can easily be adapted to cover the 3-dimensional case as well. Firstly, one replaces (8.21) by

$$\eta_c(\mathtt{A}_i) \neq \eta_d(\mathtt{A}_{\ell+1}) \tag{8.23}$$

for every $1 \leq c \leq s(\mathtt{A}_i)$ and $1 \leq d \leq s(\mathtt{A}_{\ell+1})$. (Both (8.21) and (8.23) are void if either $s(\mathtt{A}_i) = 0$ or $s(\mathtt{A}_{\ell+1}) = 0$; the only other possibility is $s(\mathtt{A}_i) = s(\mathtt{A}_{\ell+1}) = 1$, with $c = d = 1$.) Then the argument proceeds as before, except that we may no longer have disjointness: when $s = 1$,

$$\Sigma(\mathtt{A}_i) \cap \Sigma(\mathtt{A}_{\ell+1}) = \eta_1(\mathtt{A}_i) \cap \eta_1(\mathtt{A}_{\ell+1})$$

consists of exactly one point in projective space. Then η must be a Dirac measure supported on this point. However, in view of (8.22), this would have to be a fixed point of $\mathtt{A}_i$ contained in $\eta_1(\mathtt{A}_i)$, which is impossible because the eigenspace $\eta_i(\mathtt{A}_i)$ contains no invariant line. Thus, we reach a contradiction also in this case.

Now we deal with the case $d = 2$. Let $\mathcal{Z}_1$ be as in the previous cases: for every $\underline{\mathtt{A}} \notin \mathcal{Z}_1$ there are at least $\ell + 1$ values of i such that $\mathtt{A}_i \in S = S(2,0) \cup S(0,1)$. As before, it is no restriction to assume that these matrices are $\mathtt{A}_1, \ldots, \mathtt{A}_{\ell+1}$. There are three cases to consider:

First, suppose there exist $1 \leq i, j \leq \ell + 1$ such that $\mathtt{A}_i \in S(2,0)$, that is, it has two real (distinct) eigenvalues, and $\mathtt{A}_j \in S(0,1)$, that is, it has a pair of complex eigenvalues. We claim that in this case $\underline{\mathtt{A}}$ can not belong to $G_{2\ell}$. Indeed, on the one hand, any probability measure η on $\mathbb{P}(\mathbb{R}^2)$ which is invariant under $\mathtt{A}_i \in S(2,0)$ must be a convex combination of Dirac masses at the two eigenspaces. On the other hand, the action of $\mathtt{A}_j \in S(0,1)$ on the projective space is a rotation whose angle is *not* a multiple of π, and so it admits no such invariant measure.

Next, suppose all the matrices are hyperbolic: $\mathtt{A}_i \in S(2,0)$ for all $1 \le i \le \ell$. In this case one can use precisely the same argument as we did before in higher dimensions (conditions (8.20) and (8.21)-(8.23) become void). One finds a closed subset $\mathcal{Z}_2$ contained in a finite union of submanifolds with codimension $\ge \ell$ such that $G_{2\ell} \cap \mathcal{V}$ is contained in $\mathcal{Z}_1 \cup \mathcal{Z}_2$.

Finally, suppose all the matrices are elliptic: $\mathtt{A}_i \in S(0,1)$ for all $1 \le i \le \ell$. Recall that every matrix $\mathtt{A} \in \mathrm{GL}(2,\mathbb{R})$ with positive determinant induces an automorphism $h_{\mathtt{A}}$ of the Poincaré half plane $\mathbb{H}$:

$$\mathtt{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \longrightarrow \quad h_{\mathtt{A}}(z) = \frac{az+b}{cz+d}. \tag{8.24}$$

The action of $\mathtt{A}$ on the projective plane may be identified with the action of $h_{\mathtt{A}}$ on the boundary of $\mathbb{H}$, via

$$\partial\mathbb{H} \to \mathbb{P}(\mathbb{R}^2), \qquad x \mapsto [(x,1)]$$

(including $x = \infty$) so that $\mathbb{P}(\mathtt{A})$-invariant measures on the projective plane may be seen as $h_{\mathtt{A}}$-invariant measures sitting on the real axis. It is also easy to check that $h_{\mathtt{A}}$ has a fixed point in the open disc $\mathbb{H}$ if and only if $\mathtt{A} \in S(0,1)$. Define $\phi(\mathtt{A})$ to be this (unique) fixed point. It is easy to see that the $\mathtt{A} \mapsto \phi(\mathtt{A})$ is a C^∞ submersion: just use the explicit expression for the fixed point extracted from (8.24). The key feature is the following consequence of a classical construction of Douady, Earle [**10**]:

Lemma 8.19. — *If* $\mathtt{A}, \mathtt{B} \in S(0,1)$ *have some common invariant probability measure* μ *on* $\partial\mathbb{H}$ *then* $\phi(\mathtt{A}) = \phi(\mathtt{B})$.

Proof. — It is clear that elliptic matrices have no invariant measures with atoms of mass larger than $1/3$: such atoms would correspond to periodic points of $\mathtt{A}$ in the projective plane with period 1 or 2, which would contradict the definition of $S(0,1)$. In Proposition 1 of [**10**] a map $\mu \mapsto B(\mu)$ is constructed that assigns to each probability measure μ with no atoms of mass $\ge 1/2$ (see Remark 2 in [**10**, page26]) a point $B(\mu)$ in the half plane $\mathbb{H}$, in such a way that

$$B(h_*\mu) = h(B(\mu)) \quad \text{for every automorphism } h : \mathbb{H} \to \mathbb{H}.$$

When μ is $\mathtt{A}$-invariant this implies $h_{\mathtt{A}}(B(\mu)) = B((h_{\mathtt{A}})_*\mu) = B(\mu)$, and so the *conformal barycenter* $B(\mu)$ must coincide with the fixed point $\phi(\mathtt{A})$ of the automorphism $h_{\mathtt{A}}$. Thus, if μ is a common invariant measure then $\phi(\mathtt{A}) = B(\mu) = \phi(\mathtt{B})$. □

It follows from the previous observations that the map

$$\mathcal{V} \setminus \mathcal{Z}_1 \ni \underline{\mathtt{A}} \mapsto \big(\phi(\mathtt{A}_i)\big)_{1 \le i \le \ell+1} \in \mathbb{H}^{\ell+1}.$$

is a submersion. Hence, there exists a closed subset $\mathcal{Z}_2$ of $\mathcal{V} \setminus \mathcal{Z}_1$ contained in a finite union of closed submanifolds of codimension $\ge \ell$ such that for every $\underline{\mathtt{A}} \in \mathcal{V} \setminus (\mathcal{Z}_1 \cup \mathcal{Z}_2)$ there exists some $1 \le i \le \ell$ such that $\phi(\mathtt{A}_i) \ne \phi(\mathtt{A}_{\ell+1})$. Thus, we may apply Lemma 8.19 to conclude that if $\underline{\mathtt{A}} \in \mathcal{V} \setminus (\mathcal{Z}_1 \cup \mathcal{Z}_2)$. In other words, $G_{2\ell} \cap \mathcal{V}$ is contained in $\mathcal{Z}_1 \cup \mathcal{Z}_2$.

The proofs of Proposition 8.14 and Theorem A are now complete.

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November 2, 2012

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