ARE MOST DYNAMICAL SYSTEMS HYPERBOLIC ?

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Abstract. The majority of Hölder continuous, or even differentiable, linear cocycles over any hyperbolic (uniformly or not) transformation exhibit non-zero Lyapunov exponents, that is, exponential growth of the norm. Indeed, this is true on an open dense subset whose complement has ∞ codimension. These results strongly suggest that the majority of C^r dynamical systems, $r > 1$, should be hyperbolic (uniformly or not); that is known to be false for C^1 systems, by recent results of Bochi and the author.

1. Introduction

Let M be a compact manifold with dimension $d \geq 1$, and $f : M \to M$ be a C^r diffeomorphism, $r \geq 1$. Oseledets theorem [Ose68] says that, relative to any f-invariant probability μ, almost every point admits a splitting of the tangent space

$$T_x M = E_x^1 \oplus \cdots \oplus E_x^k , \quad k = k(x), \tag{1}$$

and real numbers $\lambda_1(f,x) > \cdots > \lambda_k(f,x)$ such that

$$\lim_{n \to \pm\infty} \frac{1}{n} \log \|Df^n(x) v_i\| = \lambda_i(f,x) \quad \text{for every non-zero } v_i \in E_x^i .$$

These objects are uniquely defined and they vary measurably with the point x. Moreover, the *Lyapunov exponents* $\lambda_i(f,x)$ are constant on orbits, hence they are constant μ-almost everywhere if μ is ergodic.

Assuming *hyperbolicity,* that is, that no Lyapunov exponents are zero, Pesin theory provides detailed geometric information about the system, including existence of stable and unstable sets that are smooth embedded disks at almost every point [Pes76, Rue81, FHY83, PS89]; here one takes the derivative to be Hölder continuous. Such geometric structure is at the basis of several deep results on the dynamics of hyperbolic systems, like [Pes77, Kat80, Led84, LY85, BPS99, SW00].

Which makes the following problem central to the whole theory:

Problem. Are most dynamical systems hyperbolic ?

More precisely, consider the space $\text{Diff}_\mu^r(M)$ of C^r, $r \geq 1$ diffeomorphisms that preserve a given probability μ, endowed with the corresponding C^r topology. Then the question is to be understood both in topological terms – dense, residual, or even open dense subsets – and in terms of Lebesgue measure inside generic finite-dimensional submanifolds, or parametrized families, of $\text{Diff}_\mu^r(M)$. The most interesting case is when μ is Lebesgue measure in the manifold. [1]

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[1]But the problem is just as important for general dissipative diffeomorphisms, that is, without a priori knowledge of invariant measures. E.g. [ABV00] uses hyperbolicity type properties at Lebesgue almost every point to *construct* invariant Sinai-Ruelle-Bowen measures.

As we are going to see next, systems with *zero* Lyapunov exponents are actually abundant among C^1 volume preserving diffeomorphisms. But the main results announced here, Theorems 6 and 7 below, give strong evidence that the answer to the Problem should be affirmative for C^r systems, any $r > 1$.

2. A dichotomy for C^1 conservative systems

Let μ be normalized Lebesgue measure on a compact manifold M.

Theorem 1 ([BV01, BVa]). *There exists a residual subset $\mathcal{R}$ of $\mathrm{Diff}^1_\mu(M)$ such that, for every $f \in \mathcal{R}$ and μ-almost every point x,*

(a) *either all Lyapunov exponents $\lambda_i(f, x) = 0$ for $1 \le i \le d$,*
(b) *or the Oseledets splitting of f is dominated on the orbit of x.*

The second case means there exists $m \ge 1$ such that for any y in the orbit of x

$$\frac{\|Df^m(y)v_i\|}{\|v_i\|} \ge 2\,\frac{\|Df^m(y)v_j\|}{\|v_j\|} \tag{2}$$

for any non-zero $v_i \in E^i_y$, $v_j \in E^j_y$ corresponding to Lyapunov exponents $\lambda_i > \lambda_j$. In other words, the fact that Df^n will eventually expand E^i_y more than E^j_y can be observed in finite time *uniform over the orbit.* This also implies that the angles between the Oseledets subspaces E^i_y are bounded away from zero along the orbit, in fact *the Oseledets splitting extends to a dominated splitting over the closure of the orbit.*

In some situations the conclusion gets a more global form [2] : either (a) all exponents vanish at μ-almost every point or (b) the Oseledets splitting extends to a dominated splitting on the whole ambient manifold. The latter means that $m \ge 1$ as in (2) may be chosen uniform over the whole M. It is easy to see that a dominated splitting into factors with constant dimensions is necessarily continuous. Now, existence of such a splitting is a very strong property that can often be excluded a priori. In any such case Theorem 1 is saying that generic systems must satisfy alternative (a).

A first example of this phenomenon is the 2-dimensional version of Theorem 1, proved by Bochi in 2000, partially based on a strategy proposed by Mañé in the early eighties [Mañ96].

Theorem 2 ([Boc]). *For a residual subset of C^1 area-preserving diffeomorphisms on any surface, either*

(a) *the Lyapunov exponents vanish almost everywhere or*
(b) *the diffeomorphism is uniformly hyperbolic (Anosov) on the whole M.*

Alternative (b) can only occur if M is the torus; so, C^1 *generic area preserving diffeomorphisms on any other surface have zero Lyapunov exponents almost everywhere.*

[2] It is an interesting question whether the theorem can always be formulated in this more global form. A partial positive answer is given in [BV01] for symplectic maps: generically, either the diffeomorphism is Anosov or Lebesgue almost every point has zero as Lyapunov exponent with multiplicity ≥ 2.

3. Deterministic products of matrices

Let $f : M \to M$ be a continuous transformation on a compact metric space M. A *linear cocycle* over f is a vector bundle automorphism $F : \mathcal{E} \to \mathcal{E}$ covering f, where $\pi : \mathcal{E} \to M$ is a finite-dimensional vector bundle over M. This means that

$$\pi \circ F = f \circ \pi$$

and F acts as a linear isomorphism on every fiber. The quintessential example is the derivative $F = Df$ of a diffeomorphism on a manifold (*dynamical cocycle*).

For simplicity, I focus on the case when the vector bundle is trivial $\mathcal{E} = M \times \mathbb{R}^d$, although this is not strictly necessary for what follows. Then the cocycle has the form

$$F(x, v) = (f(x), A(x)v) \quad \text{for some } A : M \to \mathrm{GL}(d, \mathbb{R}).$$

It is no restriction suppose that A takes values in $\mathrm{SL}(d, \mathbb{R})$, dividing each $A(x)$ by the determinant. Moreover, I always assume that A is at least continuous. Note that $F^n(x, v) = (f^n(x), A^n(x)v)$ for $n \in \mathbb{Z}$, with

$$A^j(x) = A(f^{j-1}(x)) \cdots A(f(x))\, A(x) \quad \text{and} \quad A^{-j}(x) = \text{ inverse of } A^j(f^{-j}(x)).$$

The theorem of Oseledets extends to linear cocycles: Given any f-invariant probability μ, then at μ-almost every point x there exists a filtration

$$\{x\} \times \mathbb{R}^d = F_x^0 > F_x^1 > \cdots > F_x^{k-1} > F_x^k = \{0\}$$

and real numbers $\lambda_1(A, x) > \cdots > \lambda_k(A, x)$ such that

$$\lim_{n \to +\infty} \frac{1}{n} \log \|A^n(x) v_i\| = \lambda_i(A, x)$$

for every $v_i \in F_x^{i-1} \setminus F_x^i$. If f is invertible there even exists an invariant splitting

$$\{x\} \times \mathbb{R}^d = E_x^1 \oplus \cdots \oplus E_x^k$$

such that

$$\lim_{n \to \pm\infty} \frac{1}{n} \log \|A^n(x) v_i\| = \lambda_i(A, x)$$

for every $v_i \in E_x^i \setminus \{0\}$. It relates to the filtration by $F_x^j = \oplus_{i>j} E_x^i$.

In either case, the largest Lyapunov exponent $\lambda(A, x) = \lambda_1(A, x)$ describes the exponential rate of growth of the norm

$$\lambda(A, x) = \lim_{n \to +\infty} \frac{1}{n} \log \|A^n(x)\|.$$

If μ is an ergodic probability, the exponents are constant μ-almost everywhere. I represent by $\lambda_j(A, \mu)$ and $\lambda(A, \mu)$ these constants.

Theorem 1 also extends to linear cocycles over any transformation. I state the ergodic invertible case:

Theorem 3 ([Boc00, BV01]). *Assume $f : (M, \mu) \to (M, \mu)$ is invertible and ergodic. There exists a residual subset $\mathcal{R}$ of maps $A \in C^0(M, \mathrm{SL}(d, \mathbb{R}))$ for which either the Lyapunov exponents $\lambda_i(A, \mu)$ are all zero at μ-almost every point, or the Oseledets splitting of A extends to a dominated splitting over the support of μ.*

The next couple of examples describe two simple mechanism that exclude a priori the dominated splitting alternative in the dichotomy:

Example 4. Let $f : M \to M$ and $A : M \to \mathrm{SL}(d,\mathbb{R})$ be such that for every $1 \le i < d$ there exists a periodic point p_i in the support of μ, with period q_i , such that the eigenvalues $\{\beta_j^i : 1 \le j \le d\}$ of $A^{q_i}(p_i)$ satisfy

$$|\beta_1^i| \ge \cdots \ge |\beta_{i-1}^i| > |\beta_{i-1}^i| = |\beta_i^i| > |\beta_{i+1}^i| \ge \cdots \ge |\beta_d^i| \tag{3}$$

and β_i^i, β_{i+1}^i are complex conjugate (not real). Such an A may be found, for instance, starting with a constant cocycle and deforming it on disjoint neighborhoods of the periodic orbits. Property (3) remains valid for every B in a C^0 neighborhood $\mathcal{U}$ of A. It implies that no B admits an invariant dominated splitting over the support of μ: if such a splitting $E \oplus F$ existed then, at every periodic point, the $\dim E$ largest eigenvalues would be strictly larger than the other eigenvalues, which is incompatible with (3). It follows, by Theorem 3, that every cocycle in a residual subset $\mathcal{U} \cap \mathcal{R}$ of the neighborhood has all the Lyapunov exponents equal to zero.

Example 5. Let $f : S^1 \to S^1$ be a homeomorphism and μ be any invariant ergodic measure with $\operatorname{supp}\mu = S^1$. Let $\mathcal{N}$ be the set of all continuous $A : S^1 \to \mathrm{SL}(2,\mathbb{R})$ non-homotopic to a constant. For a residual subset of $\mathcal{N}$, the Lyapunov exponents of the corresponding cocycle over (f,μ) are zero. That is because the cocycle has no invariant continuous subbundle if A is non-homotopic to a constant (this may be shown by the same kind of arguments as in Example 13 below).

4. Abundance of non-zero exponents

We are now going to see that the conclusions of the previous section change radically if one considers linear cocycles which are better than just continuous: assuming the base dynamics is hyperbolic, the *overwhelming majority of Hölder continuous or differentiable cocycles admit non-zero Lyapunov exponents.*

For $0 < \nu \le \infty$ denote by $C^\nu(M, \mathrm{SL}(d,\mathbb{R}))$ the space of C^ν maps from M to $\mathrm{SL}(d,\mathbb{R})$ endowed with the C^ν norm. When $\nu \ge 1$ it is implicit that M has a smooth structure. For integer ν the notation is slightly ambiguous: C^ν means either that f is ν times differentiable with continuous ν:th derivative, or that it is $\nu - 1$ times differentiable with Lipschitz continuous derivative. All the statements are meant for both interpretations.

Let $f : M \to M$ be a C^1 diffeomorphism with Hölder continuous derivative. An f-invariant probability measure μ is *hyperbolic* if every $\lambda_i(f,x)$ is different from zero at μ-almost every point. The notion of measure with local product structure is recalled at the end of this section, and I also observe that this class contains most interesting invariant measures.

Theorem 6 ([Via]). *Assume $f : (M,\mu) \to (M,\mu)$ is ergodic and hyperbolic with local product structure. Then, for every $\nu > 0$, the set of cocycles A with largest Lyapunov exponent $\lambda(A,x) > 0$ at μ-almost every point contains an open dense subset $\mathcal{A}$ of $C^\nu(M, \mathrm{SL}(d,\mathbb{R}))$. Moreover, its complement has ∞-codimension.*

The last property means that the set of cocycles with vanishing exponents is locally contained inside finite unions of closed submanifolds of $C^\nu(M, \mathrm{SL}(d,\mathbb{R}))$ with arbitrary codimension. Thus, generic parametrized families of cocycles do not intersect this exceptional set at all!

Now suppose $f : M \to M$ is uniformly hyperbolic, for instance, a two-sided shift of finite type, or an Axiom A diffeomorphism restricted to a hyperbolic basic set.

Then every invariant measure is hyperbolic. The main novelty is that the set $\mathcal{A}$ may be taken the same for all invariant measures with local product structure.

Theorem 7 ([BGMV, Via]). *Assume $f : M \to M$ is a uniformly hyperbolic homeomorphism. Then, for every $\nu > 0$, the set of cocycles A with largest Lyapunov exponent $\lambda(A, x) > 0$ at μ-almost every point and for every invariant measure with local product structure contains an open dense subset $\mathcal{A}$ of $C^\nu(M, \mathrm{SL}(d, \mathbb{R}))$. Moreover, its complement has ∞-codimension.*

Theorem 7 was first proved in [BGMV], under an additional hypothesis called domination. Under this additional hypothesis [BVb] gets a stronger conclusion: all Lyapunov exponents have multiplicity 1, in other words, the Oseledets subspaces E^i are one-dimensional. I expect this to extend to full generality:

Conjecture. Theorems 6 and 7 should remain true if one replaces $\lambda(A, x) > 0$ by all Lyapunov exponents $\lambda_i(A, x)$ having multiplicity 1.

Theorems 6 and 7 extend to cocycles over non-invertible transformations, respectively, local diffeomorphisms equipped with invariant non-uniformly expanding probabilities (all Lyapunov exponents positive), and uniformly expanding continuous maps, like one-sided shifts of finite type, or smooth expanding maps.

Finally, I recall the notion of *local product structure* for invariant measures. Let μ be a hyperbolic measure. I also assume that μ has no atoms. By Pesin's stable manifold theorem [Pes76], μ-almost every $x \in M$ has a local stable set $W^s_{loc}(x)$ and a local unstable set $W^u_{loc}(x)$ which are C^1 embedded disks. Moreover, these disks vary in a measurable fashion with the point. So, for every $\varepsilon > 0$ we may find $M_\varepsilon \subset M$ with $\mu(M_\varepsilon) > 1-\varepsilon$ such that $W^s_{loc}(x)$ and $W^u_{loc}(x)$ vary continuously with $x \in M_\varepsilon$ and, in particular, their sizes are uniformly bounded from zero. Thus for any $x \in M_\varepsilon$ we may construct sets $\mathcal{H}(x, \delta)$ with arbitrarily small diameter δ, such that (i) $\mathcal{H}(x, \delta)$ contains a neighborhood of x inside M_ε, (ii) every point of $\mathcal{H}(x, \delta)$ is in the local stable manifold and in the local unstable manifold of some pair of points in M_ε, and (iii) given y, z in $\mathcal{H}(x, \delta)$ the unique point in $W^s(y) \cap W^u(z)$ is also in $\mathcal{H}(x, \delta)$.

Lebesgue measure has local product structure if it is hyperbolic; this follows from the absolute continuity of Pesin's stable and unstable foliations [Pes76]. The same is true, more generally, for any hyperbolic probability having absolutely continuous conditional measures along unstable manifolds or along stable manifolds. Also, in the uniformly hyperbolic case, every equilibrium state of a Hölder continuous potential [Bow75] has local product structure.

5. About the proofs

I discuss some main ingredients, focussing the case when the base dynamics $f : M \to M$ is uniformly expanding, and μ is ergodic with $\operatorname{supp} \mu = M$. The general cases of Theorems 6 and 7 follow from a more local version of similar arguments.

In fact, similar methods apply to much more general *non-linear cocycles*, that is, with values in a large class of subgroups G of $\mathrm{Diff}^\nu(N)$, N a compact manifold. The linear case treated here corresponds to $G = \mathrm{SL}(d, \mathbb{R})$ seen as a subgroup of $\mathrm{Diff}^r(\mathbb{RP}^{d-1})$.

Notice that it is no restriction to consider $\nu \geq 1$: the Hölder cases $0 < \nu < 1$ are immediately reduced to the Lipschitz one $\nu = 1$ by replacing the metric $\mathrm{dist}(x,y)$ in M by $\mathrm{dist}(x,y)^\nu$.

Definition 8. $A : M \to \mathrm{SL}(d,\mathbb{R})$ is called *bundle-free* if it admits no finite-valued Lipschitz continuous invariant line bundle: in other words, given any $\eta \geq 1$, there exists no Lipschitz continuous map $\psi : x \mapsto \{v_1(x), \ldots, v_\eta(x)\}$ assigning to each $x \in M$ a subset of $\mathbb{RP}^{d-1}$ with exactly η elements, such that

$$A(x)\big(\{v_1(x), \ldots, v_\eta(x)\}\big) = \{v_1(f(x)), \ldots, v_\eta(f(x))\} \quad \text{for all } x \in M.$$

A is called *stably bundle-free* if all Lipschitz maps in a neighborhood are bundle-free.

The case $\eta = 1$ means that the cocycle has no invariant *Lipschitz* subbundles. The regularity requirement is crucial in view of the next theorem: invariant Lipschitz subbundles are exceptional, whereas Hölder invariant subbundles with poor Hölder constants are often robust! The following exercise illustrates these issues.

Exercise 9. Let $G : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}$, $G(\theta, x) = (f(\theta), g(\theta, x))$ be a smooth map with

$$\sigma_1 \geq |f'| \geq \sigma_2 > \sigma_3 > |\partial_x g| > \sigma_4 > 1.$$

Let θ_0 be a fixed point of f and x_0 be the fixed point of $g(\theta_0, \cdot)$. Then

1. The set of points whose forward orbit is bounded is the graph of a continuous function $u : S^1 \to \mathbb{R}$ with $u(\theta_0) = x_0$. This function is ν-Hölder for any $\nu < \log\sigma_4 / \log\sigma_1$. Typically it is not Lipschitz:
2. The fixed point $p_0 = (\theta_0, x_0)$ has a strong-unstable set $W^{uu}(p_0)$ invariant under G and which is locally a Lispchitz graph over S^1. If u is Lipschitz then its graph must coincide with $W^{uu}(p_0)$.
3. However, for an open dense subset of choices of g the strong-unstable set is not globally a graph: it intersects vertical lines at infinitely many points.

Theorem 10. *Suppose $A \in C^\nu(M, SL(d,\mathbb{R}))$ has $\lambda(A,x) = 0$ with positive probability, for some invariant measure μ. Then A is approximated in $C^\nu(M, SL(d,\mathbb{R}))$ by stably bundle-free maps.*

Here is a sketch of the proof. The first step is to deduce from the hypothesis

$$\lim_{n\to\infty} \frac{1}{n} \log \|A^n(x)\| = 0 \quad \text{for } \mu\text{−almost all } x$$

that Birkhoff averages of $\log\|A^i\|$ are also small: given δ there is $N \geq 1$ such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \frac{1}{N} \log \|A^N(f^{jN}(x))\| < \delta \quad \text{for } \mu\text{−almost all } x. \tag{4}$$

Using the shadowing lemma, one finds periodic points $p \in M$ satisfying (4) with δ replaced by 2δ. This implies that the eigenvalues β_j of $A^q(p)$, $q = \mathrm{per}(p)$ are all close to 1:

$$2(1-d)\delta < \frac{1}{q} \log |\beta_j| < 2\delta \quad \text{for all } j = 1, \ldots, d.$$

We may take all the norms $|\beta_j|$ to be distinct. Now the argument is very much inspired by Exercise 9. The eigenspaces of $A^q(p)$, seen as periodic points of the cocycle acting in the projective space, have strong-unstable sets that are *locally* Lipschitz graphs over M. Any Lipschitz continuous invariant line bundle ψ as in

Definition 8 has to coincide with the strong-unstable sets. But a simple transversality argument shows that *globally* the strong-unstable sets are not graphs (not even up to finite covering), if certain configurations with positive codimension are avoided.

Another key ingredient is the following result, which may be thought of as a geometric version of a classical result of Furstenberg [Fur63] about products of i.i.d. random matrices:

Theorem 11. *Suppose $A \in C^\nu(M, \mathrm{SL}(d,\mathbb{R}))$ is bundle-free and there exists some periodic point $p \in M$ of f such that the norms of the eigenvalues of A over the orbit of p are all distinct. Then $\lambda(A,\mu) > 0$ for any ergodic measure μ with local product structure and* $\operatorname{supp}\mu = M$.

The condition on the existence of some periodic point over which the cocycle is all eigenvalues with different norm is satisfied by an open and dense subset of $C^\nu(M, \mathrm{SL}(d,\mathbb{R}))$, that I denote SP. See the last section of [BVb]. I also denote by BF the subset of bundle-free maps. The proof of Theorem 11 may be sketched as follows.

Let $\hat{f} : \hat{M} \to \hat{M}$ be the natural extension of f, and $\hat{\mu}$ be the lift of μ to $\hat{M}$. Let $\hat{f}_A : \hat{M} \times \mathbb{RP}^{d-1} \to \hat{M} \times \mathbb{RP}^{d-1}$ be the projective cocycle induced by A over $\hat{f}$. Let us suppose that $\lambda(A,\mu) = 0$, and conclude that A is not bundle-free.

The first step is to prove that all points in the projective fiber of $\hat{\mu}$-almost every $\hat{x} \in \hat{M}$ have strong-stable and strong-unstable sets for $\hat{f}_A$ that are Lipschitz graphs over the stable manifold and the unstable manifold of $\hat{x}$ for $\hat{f}$. This follows from (4) and the corresponding fact for negative iterates. The strong-stable sets are locally horizontal: by definition, the cocycle is constant over local stable sets of the natural extension $\hat{f}$.

Next, one considers invariant probability measures m on $\hat{M} \times \mathbb{RP}^{d-1}$, invariant under $\hat{f}_A$ and projecting down to μ. One constructs such a measure admitting a family of conditional probabilities $\{m_{\hat{x}} : \hat{x} \in \hat{M}\}$ that is invariant under strong-unstable holonomies. Using the hypothesis $\lambda(A,\mu) = 0$ and a theorem of Ledrappier [Led86], one proves that the conditional measures are constant on local stable leaves (in other words, invariant under strong-stable holonomies), restricted to a full $\hat{\mu}$-measure subset of $\hat{M}$. Using local product structure and $\operatorname{supp}\hat{\mu} = \hat{M}$, one concludes that m admits some family of conditional measures $\{\tilde{m}_{\hat{x}} : \hat{x} \in \hat{M}\}$ that vary continuously with the point $\hat{x}$ on M and are invariant by both strong-stable and strong-unstable holonomies.

Finally, one considers a periodic point $\hat{p}$ of $\hat{f}$ such that the norms of the eigenvalues of $A^q(\hat{p})$, $q = \mathrm{per}(p)$ are all distinct. Then the probability $\tilde{m}_p$ is a convex combination of Dirac measures supported on the eigenspaces. Using the strong-stable and strong-unstable holonomies one propagates the support of $\tilde{m}_p$ over the whole M. This defines an invariant map ψ as in Definition 8, with $\eta \leq \#\operatorname{supp}\tilde{m}_p$. This map is Lipschitz, because strong-stable and strong-unstable holonomies are Lipschitz. Thus, A is not bundle-free.

Finally, I explain how to obtain Theorem 7, in the special case we are considering, from the two previous theorems. Let ZE be the subset of $A \in C^\nu(M, SL(d,\mathbb{R}))$ such that $\lambda(A,\mu) = 0$ for some ergodic measure with local product structure and $\operatorname{supp} = M$. Theorem 10 implies that any $A \in$ ZE is approximated by the interior of BF. Since SP is open and dense, A is also approximated by the interior of

BF ∩ SP. By Theorem 11, the latter is contained in the complement of ZE. This proves that the interior of $C^\nu \setminus \mathrm{ZE}$ is dense in ZE, and so it is dense in the whole $C^\nu(M, SL(d,\mathbb{R}))$, as claimed. To get the ∞ codimension statement observe that it suffices to avoid the positive codimension configuration mentioned before for *some* of infinitely many periodic points of f.

6. Further comments

The following couple of examples help understand the significance of Theorem 11.

Example 12. Let $M = S^1$, $f : M \to M$ be given by $f(x) = kx \bmod \mathbb{Z}$, for some $k \geq 2$, and μ be Lebesgue measure on M. Let

$$A : M \to \mathrm{SL}(2,\mathbb{R}), \qquad A(x) = \begin{pmatrix} \beta(x) & 0 \\ 0 & 1/\beta(x) \end{pmatrix}$$

for some smooth function β such that $\int \log \beta \, d\mu = 0$. It is easy to ensure that the set $\beta^{-1}(1)$ is finite and does not contain $x = 0$. Then $A \in \mathrm{SP}$ and indeed the matrix A "looks hyperbolic" at most points. Nevertheless, the Lyapunov exponent $\lambda(A,\mu) = \int \log \beta \, d\mu = 0$. Notice that A is not bundle-free.

Hence the following heuristic principle: assuming there is a source of hyperbolicity somewhere in M (here the fact that $A \in \mathrm{SP}$), the only way Lyapunov exponents may happen to vanish is by having expanding directions mapped *exactly* onto contracting directions, thus causing hyperbolic behavior to be "wasted way".

Putting Theorems 3 and 11 together we may give a sharp account of Lyapunov exponents for a whole C^0 open set of cocycles. This construction contains the main result of [You93]. It also shows that the present results are in some sense optimal.

Example 13. Let $f : S^1 \to S^1$ be a C^2 uniformly expanding map, and μ be the absolutely continuous invariant measure. Let $A : S^1 \to \mathrm{SL}(2,\mathbb{R})$ be of the form

$$A(x) = R_{\alpha(x)} A_0$$

where A_0 is some hyperbolic matrix, $\alpha : S^1 \to S^1$ is a continuous function with $\alpha(0) = 0$, and $R_{\alpha(x)}$ denotes the rotation of angle $\alpha(x)$. Assume that $2\deg(\alpha)$ is *not* a multiple of $\deg(f) - 1$, where $\deg(\,\cdot\,)$ represents the topological degree.

Corollary 14. *There exists a C^0 neighborhood $\mathcal{U}$ of A such that*

1. *for B in a residual subset $\mathcal{R} \cap \mathcal{U}$ we have $\lambda(B,\mu) = 0$;*
2. *for every $B \in \mathcal{U} \cap \mathcal{S}^{r,\nu}(M,2)$, $r > 0$, we have $\lambda(B,\mu) > 0$.*

Proof. Start by taking $\mathcal{U}$ to be the isotopy class of A in the space of continuous maps from M to $\mathrm{SL}(2,\mathbb{R})$. We claim that, given any $B \in \mathcal{U}$, there is no *continuous* B-invariant map

$$\psi : M \ni x \mapsto \{\psi_1(x), \ldots, \psi_\eta(x)\}$$

assigning a constant number $\eta \geq 1$ of elements of $\mathbb{RP}^1$ to each point $x \in M$. The proof is by contradiction. Suppose there exists such a map and

$$G = \{(x, \psi_i(x)) \in S^1 \times \mathbb{RP}^1 : x \in S^1 \text{ and } 1 \leq i \leq \eta\}$$

is connected. Then the graph G represents an element (η,ζ) of the fundamental group $\pi_1(S^1 \times \mathbb{RP}^1) = \mathbb{Z} \oplus \mathbb{Z}$. Because B is isotopic to A, the image of G under the cocycle must represent $(\eta \deg(f), \zeta + 2\deg(\alpha)) \in \pi_1(S^1 \times \mathbb{RP}^1)$; here the factor

2 comes from the fact that S^1 is the 2-fold covering of $\mathbb{RP}^1$. By the invariance of ψ we get

$$\zeta + 2\deg(\alpha) = \deg(f)\zeta$$

which contradicts the hypothesis that $\deg(f) - 1$ does not divide $2\deg(\alpha)$. If the graph G is not connected, consider the connected components instead. Since connected components are pairwise disjoint, they all represent elements with the same direction in the fundamental group. Then the same type of argument as before proves the claim in full generality.

Now let $\mathcal{R}$ be the residual subset in Theorem 3. The previous observation implies that no $B \in \mathcal{R} \cap \mathcal{U}$ may have an invariant dominated splitting. Then B must have all Lyapunov exponents equal to zero as claimed in (1). Similarly, that observation ensures that every $B \in \mathcal{U} \cap C^\nu$ is bundle-free. It is clear that A is in SP, and so is any map C^0 close to it. Thus, reducing $\mathcal{U}$ if necessary, we may apply Theorem 11 to conclude that $\lambda(B, \mu) > 0$. This proves (2). □

References

[ABV00] J. F. Alves, C. Bonatti, and M. Viana. SRB measures for partially hyperbolic systems whose central direction is mostly expanding. *Invent. Math.*, 140:351–398, 2000.

[BGMV] C. Bonatti, X. Gomez-Mont, and M. Viana. Généricité d'exposants de Lyapunov non-nuls pour des produits déterministes de matrices. *Annales de l'Inst. Henri Poincaré - Analyse Non-linéaire.* To appear.

[Boc] J. Bochi. Genericity of zero Lyapunov exponents. *Ergod. Th. & Dynam. Sys.* To appear.

[Boc00] J. Bochi. *Genericity of zero Lyapunov exponents.* PhD thesis, IMPA, 2000. Preprint www.preprint.impa.br.

[Bow75] R. Bowen. *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*, volume 470 of *Lect. Notes in Math.* Springer Verlag, 1975.

[BPS99] L. Barreira, Ya. Pesin, and J. Schmeling. Dimension and product structure of hyperbolic measures. *Ann. of Math.*, 149:755–783, 1999.

[BVa] J. Bochi and M. Viana. A sharp dichotomy for conservative systems: zero Lyapunov exponents or uniform domination. Preprint IMPA 2002.

[BVb] C. Bonatti and M. Viana. Lyapunov exponents with multiplicity 1 for deterministic products of matrices. Preprint www.preprint.impa.br 2001.

[BV01] J. Bochi and M. Viana. Uniform (projective) hyperbolicity or no hyperbolicity: a dichotomy for generic conservative maps. *Annales de l'Inst. Henri Poincaré - Analyse Non-linéaire*, 2001.

[FHY83] A. Fathi, M. Herman, and J.-C. Yoccoz. A proof of Pesin's stable manifold theorem. In *Geometric dynamics (Rio de Janeiro 1981)*, volume 1007 of *Lect. Notes in Math.*, pages 177–215. Springer Verlag, 1983.

[Fur63] H. Furstenberg. Non-commuting random products. *Trans. Amer. Math. Soc.*, 108:377–428, 1963.

[Kat80] A. Katok. Lyapunov exponents, entropy and periodic points of diffeomorphisms. *Publ. Math. IHES*, 51:137–173, 1980.

[Led84] F. Ledrappier. Propriétés ergodiques des mesures de Sinaï. *Publ. Math. I.H.E.S.*, 59:163–188, 1984.

[Led86] F. Ledrappier. Positivity of the exponent for stationary sequences of matrices. In *Lyapunov exponents (Bremen, 1984)*, volume 1186 of *Lect. Notes Math.*, pages 56–73. Springer, 1986.

[LY85] F. Ledrappier and L.-S. Young. The metric entropy of diffeomorphisms. I. Characterization of measures satisfying Pesin's entropy formula. *Ann. of Math.*, 122:509–539, 1985.

[Mañ96] R. Mañé. The Lyapunov exponents of generic area preserving diffeomorphisms. In *International Conference on Dynamical Systems (Montevideo, 1995)*, pages 110–119. Longman, 1996.

[Ose68] V. I. Oseledets. A multiplicative ergodic theorem: Lyapunov characteristic numbers for dynamical systems. *Trans. Moscow Math. Soc.*, 19:197–231, 1968.

[Pes76] Ya. Pesin. Families of invariant manifolds corresponding to non-zero characteristic exponents. *Math. USSR. Izv.*, 10:1261–1302, 1976.

[Pes77] Ya. B. Pesin. Characteristic Lyapunov exponents and smooth ergodic theory. *Russian Math. Surveys*, 324:55–114, 1977.

[PS89] C. Pugh and M. Shub. Ergodic attractors. *Trans. Amer. Math. Soc.*, 312:1–54, 1989.

[Rue81] D. Ruelle. Ergodic theory of diferentiable dynamical systems. *Publ. Math. IHES*, 50:27–58, 1981.

[SW00] M. Shub and A. Wilkinson. Pathological foliations and removable zero exponents. *Invent. Math.*, 139:495–508, 2000.

[Via] M. Viana. Most cocycles over any hyperbolic system have non-zero Lyapunov exponents. Preprint IMPA 2002.

[You93] L.-S. Young. Some open sets of nonuniformly hyperbolic cocycles. *Ergod. Th. & Dynam. Sys.*, 13(2):409–415, 1993.

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