# Absolute continuity, Lyapunov exponents, rigidity

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Based on the series of papers

[AV] Artur Avila + MV

[ASV] AA + Jimmy Santamaria + MV

[AVW] AA + MV + Amie Wilkinson

[VY] MV + Jiagang Yang

and previous joint work [BGV03] with Christian Bonatti and Xavier Gomez-Mont on linear cocycles.

Further applications by

AA + MV + AW: to smooth actions of higher rank groups

F. Rodriguez-Hertz, J. Rodriguez-Hertz, R. Ures, A. Tahzibi: to entropy maximizing measures

A diffeomorphism $f : M \to M$ is partially hyperbolic if there exists a continuous decomposition of the tangent bundle

$$T_x M = E^u_x \oplus E^c_x \oplus E^s_x$$

which is invariant under the dynamics:

$$Df_x(E^*_x) = E^*_{f(x)} \quad \text{for all } * \in \{u, c, s\}.$$

and...

# Partial hyperbolicity

- $Df \mid E^s$ is uniformly contracting:

$$\|Df \mid E^s_x\| \leq \lambda < 1,$$

- $Df \mid E^u$ is uniformly expanding:

$$\|(Df \mid E^u_x)^{-1}\| \leq \lambda < 1,$$

- $Df \mid E^c$ is "in between":

$$\frac{1}{\lambda}\frac{\|Df_x(v^s)\|}{\|v^s\|} \leq \frac{\|Df_x(v^c)\|}{\|v^c\|} \leq \lambda\frac{\|Df_x(v^u)\|}{\|v^u\|}.$$

# Examples

Automorphisms: Take $f_L : \mathbb{T}^d \to \mathbb{T}^d$ induced by some $L \in SL(d,\mathbb{Z})$ whose spectrum is contained in the union of three disjoint annuli.

$E^c =$ the invariant subspace corresponding to the part of the spectrum inside the middle annulus.

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Time-1 maps: Take $f_0 = X^1$, where $X^t : M \to M$, $t \in \mathbb{R}$ is an Anosov flow.

$E^c = \partial X^t / \partial t =$ direction of the flow.

More generally: maps that embed in hyperbolic Lie group actions, $E^c =$ tangent to the orbits of the action.

# Examples

Skew-products: Take $f_0 : M \times N \to M \times N$ of the form

$$f_0(x, y) = (g(x), h(x, y))$$

where $g$ is an Anosov diffeomorphism, with hyperbolicity constant $\sigma > 1$, and

$$\sup \left\| \frac{\partial h}{\partial y} \right\| < \sigma \quad \text{and} \quad \sup \left\| \left( \frac{\partial h}{\partial y} \right)^{-1} \right\| < \sigma.$$

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Partial hyperbolicity is an open property: any diffeomorphism in a $C^1$-neighborhood is also partially hyperbolic.

Suppose $f : M \to M$ is volume preserving: $\text{vol}(f(A)) = \text{vol}(A)$ for any $A \subset M$. Then (Birkhoff ergodic theorem) the time average

$$\lim \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x))$$

exists almost everywhere, for any continuous $\varphi : M \to \mathbb{R}$.

This is not necessarily so if $f$ is dissipative, that is, not volume preserving.

# Time averages

An $f$-invariant probability $\mu$ is a Sinai-Ruelle-Bowen measure if the set $B(\mu)$ of points for which

$$\lim \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x)) = \int \varphi \, d\mu, \quad \forall \varphi \in C^0$$

has positive volume.

**Problem**

Are there SRB measures? Finitely many? Is almost every point in the basin of some SRB measure?

The center Lyapunov exponents of $f$ are the rates of growth

$$\lim \frac{1}{n} \log \|Df_x^n(v^c)\| \quad \text{of vectors } v^c \in E_x^c.$$

They are well defined almost everywhere, with respect to any invariant probability measure.

**Problem**

Can one always perturb $f$ to make the center Lyapunov exponents different from zero? What if we can not?

**Shub, Wilkinson, Baraviera, Bonatti**

For volume preserving diffeomorphisms the sum of the center exponents can be made non-zero by a $C^1$-small perturbation.

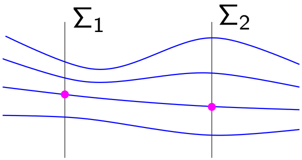
# Invariant foliations

## Anosov, Sinai, Brin, Pesin, Hirsch, Pugh, Shub

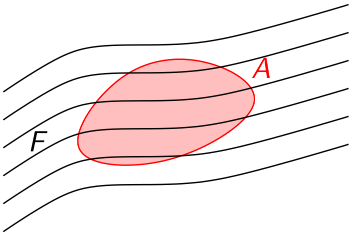
- The stable and unstable bundles are uniquely integrable: there exist (unique) foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ such that

$$T_x\mathcal{F}^s_x = E^s_x \quad \text{and} \quad T_x\mathcal{F}^u_x = E^u_x \quad \text{everywhere.}$$

- These foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ are absolutely continuous: projections along leaves send zero measure sets to zero measure sets.



# Absolute continuity




Absolute continuity implies that the volume measure $m$ admits a disintegration à la Fubini (leafwise absolute continuity):

$$m(A) = \int m_F(A \cap F)\, dF \quad \text{for any } A \subset M,$$

where $dF$ = quotient measure in the space of leaves and $m_F$ is a density (absolutely continuous measure) on each leaf $F$.

What about center ($\mathcal{F}^c$), center stable ($\mathcal{F}^{cs}$), and center unstable ($\mathcal{F}^{cu}$) unstable foliations, tangent to $E^c$, $E^c \oplus E^s$, and $E^c \oplus E^u$?

**Problem**

When do they exist? When are they unique? How frequently are these foliations absolutely continuous?

# Failure of absolute continuity

Let $f_0 = h \times \mathrm{id} : \mathbb{T}^2 \times S^1 \to \mathbb{T}^2 \times S^1$, where $h : \mathbb{T}^2 \to \mathbb{T}^2$ is the (Anosov) map induced on the torus $\mathbb{T}^2$ by

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

**Shub, Wilkinson**

There exists $f$ arbitrarily close to $f_0$, volume preserving, ergodic, and such that the Lyapunov exponent

$$\lambda^c(f) = \int \log |Df \mid E^c| \, dm \neq 0.$$

The latter implies $\mathcal{F}^c$ is not absolutely continuous.

# Atomic disintegration

**Ruelle, Wilkinson**

For any $f$ close to $f_0$, if $\lambda^c(f)$ is non-zero then there is $k \geq 1$ and a full volume set $Z$ with $\#(Z \cap F) = k$ for every center leaf $F$.

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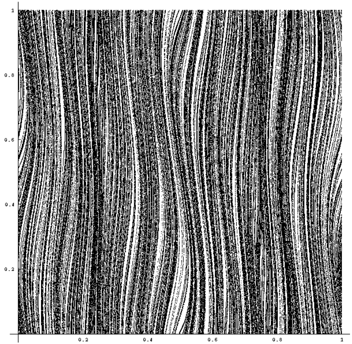

# Linear cocycles

A linear cocycle is a skew-product over a measure preserving map $f : M \to M$,

$$L : M \times \mathbb{R}^d \to M \times \mathbb{R}^d, \quad (f(x), L_x(v)),$$

where each $L_x : \mathbb{R}^d \to \mathbb{R}^d$ is a linear isomorphism.

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More generally, one considers morphisms of vector bundles

$$\begin{array}{rcll} L : \mathcal{V} & \to & \mathcal{V} & \text{acting linearly on the fibers} \\ \pi \downarrow & & \downarrow \pi & \\ f : M & \to & M & \text{preserving some probability } \mu. \end{array}$$

The extremal Lyapunov exponents are the limits

$$\lambda_+(L,x) = \lim \frac{1}{n} \log \|L_x^n\| \qquad \lambda_-(L,x) = \lim \frac{1}{n} \log \|(L_x^n)^{-1}\|^{-1}.$$

They are defined $\mu$-almost everywhere if $\log \|L_x^{\pm 1}\| \in L^1(\mu)$.

**Problem (Furstenberg)**

When is $\lambda_-(L,\cdot) < \lambda_+(L,\cdot)$?

**[BGV], [V], [ASV]**

Assuming $(f, \mu)$ is mildly hyperbolic, the set of cocycles $L$ for which $\lambda_-(L, \cdot) = \lambda_+(L, \cdot)$ has infinite codimension.

Furstenberg, Kesten, in the 1960's, for i.i.d. random matrices. Many important contributions by Ledrappier, Guivarc'h, Raugi, Gol'dsheid, Margulis and others.

**Mañé, Bochi, Viana**

Not true for cocycles that are just continuous!

# A strategy

Try to approach the problem about center Lyapunov exponents of partially hyperbolic maps as follows:

1. develop a theory of Lyapunov exponents for non-linear cocycles over mildly hyperbolic systems
2. and apply it to the fiber bundle morphism

$$\begin{array}{rcl} f : M & \longrightarrow & M \\ \pi \downarrow & & \downarrow \pi \\ g : M/\mathcal{F}^c & \longrightarrow & M/\mathcal{F}^c \end{array}$$

($g$ is a hyperbolic homeomorphism on the space of center leaves).

Let $\mathcal{E} \to M$ be a fiber bundle with smooth fibers. Consider a smooth cocycle:

$$\begin{array}{rcll} F : (\mathcal{E}, m) & \to & (\mathcal{E}, m) & \text{acting smoothly on the fibers} \\ \pi \downarrow & & \downarrow \pi & \\ f : (M, \mu) & \to & (M, \mu) & \text{mildly hyperbolic} \end{array}$$

For instance:

- $f$ is hyperbolic (Anosov) and $\mu$ has local product structure
- $f$ is partially hyperbolic, accessible, and center bunched, and $\mu =$ volume.

Assume the cocycle has invariant holonomies, that is, the stable and unstable foliations of $f$ lift to invariant foliations of $F$.

# Invariance Principle

**Invariance Principle [AV], [ASV]**

Suppose the Lyapunov exponents of $F : \mathcal{E} \to \mathcal{E}$ along the fibers vanish $m$-almost everywhere. Then $m$ admits a disintegration into conditional probabilities on the fibers

$$\{m_x : x \in M\}$$

continuous and invariant under stable and unstable holonomies.

$$m(B) = \int_M m_x(E \cap \mathcal{E}_x)\, d\mu(x) \quad \text{for any } B \subset \mathcal{E}.$$

# Symplectic perturbations of automorphisms

Let $f_L : \mathbb{T}^4 \to \mathbb{T}^4$ be the automorphism defined by a symplectic linear map $L : \mathbb{R}^4 \to \mathbb{R}^4$ with exactly two eigenvalues on the unit circle. Assume no eigenvalue is a root of unity (that is, $f_L$ is ergodic).

**Symplectic dichotomy [AV]**

There exists a neighborhood $\mathcal{U}$ of $f_L$ in the space of $C^\infty$ symplectic diffeomorphisms on $M$ such that for every $f \in \mathcal{U}$, either $f$ is volume preserving conjugate to $f_L$ or the center Lyapunov exponents of $f$ are distinct and non-zero. In either case, $f$ is Bernoulli.

F. Rodriguez-Hertz proved such automorphisms are stably ergodic (in the volume preserving category).

# A glimpse at the proof

Suppose the center Lyapunov exponents vanish.

- Then we may apply the Invariance Principle to a convenient smooth cocycle, to obtain a continuous invariant family of probabilities on the projective tangent spaces to center leaves.

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- That gives rise to a continuous invariant conformal structure on the center leaves. Each $\mathcal{F}^c_x \simeq \mathbb{C}$. We promote this to a continuous invariant translation structure on the center leaves.

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- Translations on the center leaves define an $\mathbb{R}^2$-action by torus homeomorphisms. The closure $G$ of this action is a compact Abelian subgroup of $\mathrm{Homeo}(\mathbb{T}^4)$ acting transitively on $\mathbb{T}^4$.

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- Translations on the center leaves define an $\mathbb{R}^2$-action by torus homeomorphisms. The closure $G$ of this action is a compact Abelian subgroup of Homeo($\mathbb{T}^4$) acting transitively on $\mathbb{T}^4$.
- Consequently, $G \simeq \mathbb{T}^4$. The diffeomorphism $f : \mathbb{T}^4 \to \mathbb{T}^4$ is a group automorphism. Then $f \simeq L$. Moreover, the conjugacy preserves volume.

Let $f_0 : \mathbb{T}^2 \times S^1 \to \mathbb{T}^2 \times S^1$ be defined by $f_0 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \times \mathrm{id}$. Let $f$ be any $C^\infty$ volume preserving diffeomorphism close to $f_0$.

**Rigidity Theorem [AVW]**

If the center foliation $\mathcal{F}^c$ of $f$ is absolutely continuous then $f$ is $C^\infty$ conjugate to a rotation extension

$$\mathbb{T}^2 \times S^1 \to \mathbb{T}^2 \times S^1, \quad (x, \theta) \mapsto (h(x), \theta + \omega(x))$$

of a $C^\infty$ Anosov diffeomorphism $h$, and $\mathcal{F}^c$ is actually $C^\infty$.

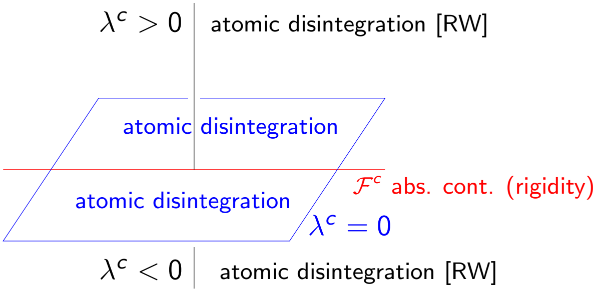
# Dichotomy theorem

A partially hyperbolic diffeomorphism is accessible if any two points in $M$ can be joined by a piecewise smooth curve whose legs are contained in $\mathcal{F}^s$ and $\mathcal{F}^u$ leaves. This is an open and dense property in the neighborhood of $f_0$.

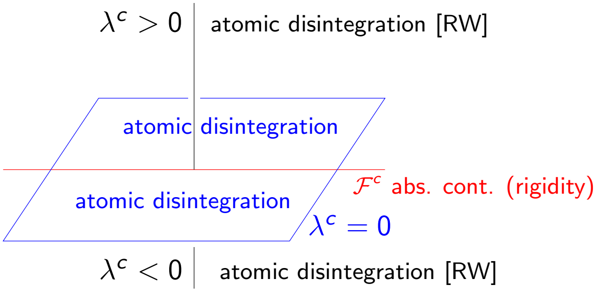
**Dichotomy Theorem [AVW]**

If $f$ is accessible then either the center foliation $\mathcal{F}^c$ is absolutely continuous or there exists $k \geq 1$ and a full measure set which intersects every center leaf on exactly $k$ points. Generically, $k = 1$.

# Summary



$\lambda^c > 0$
atomic disintegration [RW]
atomic disintegration
atomic disintegration
$\mathcal{F}^c$ abs. cont. (rigidity)
$\lambda^c = 0$
$\lambda^c < 0$
atomic disintegration [RW]




## Conjecture [VY]

1. Generically in $\{\lambda^c < 0\}$, $\mathcal{F}^{cs}$ is absolutely continuous and $\mathcal{F}^{cu}$ is not.
2. In $\{\lambda^c = 0\} \setminus \{\mathcal{F}^c$ absolutely continuous$\}$, neither $\mathcal{F}^{cs}$ nor $\mathcal{F}^{cu}$ are absolutely continuous.

Similar results (rigidity and dichotomy) hold for volume preserving perturbations of time 1 maps of Anosov flows. In particular,

**Rigidity Theorem [AVW]**

Let $f$ be a volume preserving perturbation of the time 1 map of a smooth Anosov flow in dimension 3. If the center foliation $\mathcal{F}^c$ of $f$ is absolutely continuous then $f$ is the time-1 map of a smooth Anosov flow.

# Dissipative maps

Let $f_0 : \mathbb{T}^2 \times S^1 \to \mathbb{T}^2 \times S^1$ be defined by $f_0 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \times \mathrm{id}$.

**SRB measures [VY]**

There is a $C^\infty$ neighborhood $\mathcal{U}$ of $f_0$ such that, for every $f \in \mathcal{U}$ accessible and with absolutely continuous center stable foliation, there exist finitely many physical measures and the union of their basins has full Lebesgue measure in $\mathbb{T}^2 \times S^1$.

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## [VY]

In the dissipative setting, absolute continuity of is not a rigid property: there exist open sets of maps such that all invariant foliations are absolutely continuous ($\mathcal{F}^c$, $\mathcal{F}^{cs}$, $\mathcal{F}^{cu}$, $\mathcal{F}^s$, $\mathcal{F}^u$).