MAXIMAL TRANSVERSE MEASURES OF EXPANDING FOLIATIONS

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Abstract. For an expanding (unstable) foliation of a diffeomorphism, we use a natural dynamical averaging to construct transverse measures, which we call *maximal*, describing the statistics of how the iterates of a given leaf intersect the cross-sections to the foliation. For a suitable class of diffeomorphisms, we prove that this averaging converges, even exponentially fast, and the limit measures have finite ergodic decompositions. These results are obtained through relating the maximal transverse measures to the maximal u-entropy measures of the diffeomorphism (see [UVYY]).

1. Introduction

Let $\mathcal{F}$ be a foliation on a manifold M and F be a leaf with non-exponential growth. A classical result of Plante [Pla75], Goodman, Plante [GP79] asserts that any accumulation point of the normalized intersections of large discs in F with cross-sections to $\mathcal{F}$ is an invariant transverse measure, that is, a family of measures defined on cross-sections to the foliation which is invariant under the corresponding holonomy maps. Invariant transverse measures describe the asymptotic behavior of leaves at a statistical level, and have a major role in foliation theory. See for instance [FLP12], [PPS15], [Alv18], and the references herein.

The overall purpose of this paper is to develop new methods for analysing the asymptotic behavior of foliations arising from dynamical systems such as diffeomorphisms $f : M \to M$. In this context, we has at our disposal an alternative averaging scheme, of a dynamical nature. Namely, we can fix a disk ξ inside any leaf, and look at the normalized intersections of its iterates $f^n(\xi)$ with cross-sections to the foliation. It is assumed that the foliation is expanding for f, in which case the leaves are homeomorphic to an Euclidean space and the Plante non-exponential growth condition is satisfied (compare Gromov [Gro81]).

On the other hand, the iterates $f^n(\xi)$ themselves grow exponentially fast with n. The accumulation points of their normalized intersections with cross-sections to the foliation describe the statistical behavior of the leaf's *orbit*. So it is most natural to ask whether those normalized intersections converge as $n \to \infty$ and, in any case, whether their accumulation points are invariant transverse measures? In the event of convergence, one is also interested in understanding how fast it is, an issue which is perhaps not so relevant in the classical topological setup.

Another novel point in our investigation is that we can try to relate the foliation's invariant transverse measures to the invariant measures of the diffeomorphism f itself, a class of objects for which a rich theory exists already. Indeed, we are going to see that the invariant transverse measures constructed in this paper are directly connected to the so-called measures of maximal u-entropy of f (see [UVYY])

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via projections along the foliation leaves. Thus we call them *maximal transverse measures.*

In a nutshell, we study the connections between these three different kinds of transverse objects:

- invariant transverse measures, arising from foliation theory;
- hitting measures, given by the intersections of leaf iterates with cross-sections, a topological/dynamical type of information;
- and quotients of certain invariant measures under the foliation's holonomy maps, originating from ergodic theory.

In the following we consider the class of partially hyperbolic diffeomorphism that factor over Anosov introduced in [UVYY]. Namely, we take $f : M \to M$ to be a partially hyperbolic, dynamically coherent diffeomorphism on a compact manifold M, with partially hyperbolic splitting $TM = E^{cs} \oplus E^{uu}$. We denote by $\mathcal{F}^{uu}$ the *strong-unstable* foliation, that is, the unique foliation tangent to E^{uu} at every point. In this setting, dynamical coherence means that there exists an invariant *center-stable* foliation $\mathcal{F}^{cs}$ tangent to E^{cs} at every point.

We say that f *factors over Anosov* if there exist a hyperbolic linear automorphism $A : \mathbb{T}^d \to \mathbb{T}^d$ on some torus, and a continuous surjective map $\pi : M \to \mathbb{T}^d$ such that

(H1) $\pi \circ f = A \circ \pi$;
(H2) π maps each strong-unstable leaf of f homeomorphically to an unstable leaf of A;
(H3) π maps each center-stable leaf of f to a stable leaf of A.

Several examples of diffeomorphisms that factor over Anosov are described in [UVYY, Sections 6 to 9].

Let $\mathcal{R} = \{\mathcal{R}_1, \dots, \mathcal{R}_k\}$ be a Markov partition for the linear automorphism, and $\mathcal{M} = \{\mathcal{M}_1, \dots, \mathcal{M}_k\}$ be defined by $\mathcal{M}_i = \pi^{-1}(\mathcal{R}_i)$. We denote by $\mathcal{W}^s$ and $\mathcal{W}^u$, respectively, the stable and unstable foliations of A. The connected components of the intersection of their leaves with each element of $\mathcal{R}$ will be called *stable/unstable plaques* of A. Center-stable plaques and strong-unstable plaques of f are defined analogously, considering the leaves of $\mathcal{F}^{cs}$ and $\mathcal{F}^{uu}$ instead.

Let us state our three main results. Some of the technical notions involved in the statements will be defined along the way.

Theorem A (Existence of transverse measures)**.** *Let $f : M \to M$ be a partially hyperbolic C^1 diffeomorphism that factors over Anosov. Then there exist positive constants $c_1, \dots, c_k$ such that, for any f-invariant measure μ of maximal u-entropy, $\tau_\mu = \{c_i \hat{\mu}_S : S \subset \mathcal{M}_i\}$ is an invariant transverse measure for the strong-unstable foliation $\mathcal{F}^{uu}$, where S denotes a cross-section to the strong-unstable foliation, and $\hat{\mu}_S$ is the projection of $\mu \mid \mathcal{M}_i$ to S along the strong-unstable plaques.*

A partially hyperbolic diffeomorphism $f : M \to M$ is said to have if the center Lyapunov exponents of every ergodic measure of maximal u-entropy are negative. This is a variation of the notion of diffeomorphism with mostly contracting center, which was introduced in [BV00] and has been developed by several authors, for instance in [Cas02, Dol00, DVY16]. Examples and more information on systems with c-mostly contracting center, including alternative equivalent definitions, can be found in [UVYY] and Section 4 below.

Theorem B (Exponential convergence)**.** *Let $f : M \to M$ be a partially hyperbolic diffeomorphism that factors over Anosov and has c-mostly contracting center. Let μ be an ergodic measure of maximal u-entropy whose support is connected. Then for any strong-unstable plaque $\xi^u(x)$ with $x \in \operatorname{supp} \mu$, and any cross-section S*

contained in some $\mathcal{M}_j$ with $\mu(\mathcal{M}_j) > 0$, and any Hölder real function $\hat{\varphi}$ supported inside S,

$$\frac{1}{\#(f^n(\xi^u(x)) \cap S)} \sum_{q \in f^n(\xi^u(x)) \cap S} \hat{\varphi}(q) \to \frac{1}{\|\hat{\mu}_S\|} \int \hat{\varphi}\, d\hat{\mu}_S$$

exponentially fast as $n \to \infty$.

Example 1.1. Let $f : M \to M$ be any partially hyperbolic diffeomorphism on a 3-dimensional nilmanifold $M \neq \mathbb{T}^3$. Then f factors over Anosov. Moreover, f is C^{1+} and admits some hyperbolic periodic point then any diffeomorphism in a C^1-neighborhood has c-mostly contracting center. Thus Theorems A and B apply to it. Indeed, in this case there is exactly one measure of maximal u-entropy, and its support is connected. See [UVYY, Section 7].

Remark 1.2. A version of Theorem B remains true when the support of μ is not connected. That is because in general the support has finitely many connected components, and the normalized restrictions of μ to each connected component are ergodic for a suitable iterate f^l. See Lemma 4.9 and equation (28) in [UVYY]. Then the previous statement can be applied to the restriction of f^l to each connected component. Corresponding observations apply to Theorem C, Theorem 6.1 and Theorem 8.1.

For the next theorem we assume that the center-stable bundle admits a dominated splitting $E^{cs} = E^{ss} \oplus E^c$, where E^{ss} is uniformly contracting. We denote by $\mathcal{F}^{ss}$ the *strong-stable* foliation, that is, the unique foliation tangent to E^{ss} at every point. The assumption that f is dynamically coherent implies that there exists a *center foliation* $\mathcal{F}^c$ tangent to E^c at every point.

Theorem C. *(Ergodicity) Let $f : M \to M$ be a C^1 partially hyperbolic diffeomorphism that factors over Anosov and has c-mostly contracting center. Assume furthermore that the map $\pi : M \to \mathbb{T}^d$ is a fiber bundle whose fibers are the center leaves of f, and those fibers are compact. Then, for any ergodic measure of maximal u-entropy μ, the invariant transverse measure τ_μ has a finite ergodic decomposition.*

An interesting question is whether this theory can be extended to other classes of partially hyperbolic diffeomorphisms, for example, perturbations of the time-1 map of an Anosov flow or, more generally, center-fixing diffeomorphisms (in the sense of [AVW15]).

2. Preliminaries

Throughout this paper, we take $f : M \to M$ to be a partially hyperbolic C^1 diffeomorphism which factors over an Anosov automorphism $A : \mathbb{T}^d \to \mathbb{T}^d$. We start by completing the definition of these concepts.

2.1. Partial entropy.

A C^1 diffeomorphism $f : M \to M$ on a compact manifold M is *partially hyperbolic* if there exists a Df-invariant splitting

$$TM = E^{cs} \oplus E^{uu}$$

of the tangent bundle such that $Df \mid_{E^{uu}}$ is uniformly expanding and *dominates* $Df \mid_{E^{cs}}$. By this we mean that there exist a Riemmanian metric on M and a constant $\omega < 1$ such that

$$\|Df(x)v^{uu}\| \geq \frac{1}{\omega} \text{ and } \frac{\|Df(x)v^{cs}\|}{\|Df(x)v^{uu}\|} \leq \omega \tag{1}$$

for any unit vectors $v^{cs} \in E^{cs}_x$ and $v^{uu} \in E^{uu}_x$, and any $x \in M$.

The *strong-unstable* sub-bundle E^{uu} is uniquely integrable, meaning that there exists a unique foliation $\mathcal{F}^{uu}$ which is invariant under f and tangent to E^{uu} at every point. The corresponding holonomy maps are Hölder continuous: see [BP74, Corollary 2.1] and [HP70, Theorem 6.4], which use a stronger (absolute) partial hyperbolicity condition; a proof for the (pointwise) notion of partial hyperbolicity we assume here can be found in [Via]. See also [PSW97, Theorem A'].

We assume that f is *dynamically coherent*, meaning that the sub-bundle E^{cs} is integrable, and we denote by $\mathcal{F}^{cs}$ some f-invariant integral foliation. See [HPS77]. A compact f-invariant set $\Lambda \subset M$ is *u-saturated* if it consists of entire leaves of $\mathcal{F}^{uu}$. Then it is called *u-minimal* if every strong-unstable leaf contained in Λ is dense in Λ.

The *topological u-entropy* of f, denoted by $h(f, \mathcal{F}^{uu})$, is the maximal rate of volume growth for any disk contained in an strong-unstable leaf. See Saghin, Xia [SX09]. The *u-entropy* of an f-invariant measure μ, denoted as $h_\mu(f, \mathcal{F}^{uu})$, is defined by

$$h_\mu(f, \mu) = H_\mu\left(f^{-1}\xi^u \mid \xi^u\right)$$

where ξ^u is any measurable partition subordinate to the strong-unstable foliation. Recall that, according to Rokhlin [Rok67, Section 7], the entropy $h_\mu(f)$ is the supremum of $H_\mu\left(f^{-1}\xi \mid \xi\right)$ over all measurable partitions ξ with $f^{-1}\xi \prec \xi$. Thus we always have

$$h_\mu(f, \mathcal{F}^{uu}) \le h_\mu(f). \tag{2}$$

See Ledrappier, Strelcyn [LS82], Ledrappier [Led84], Ledrappier, Young [LY85], and Yang [Yan21]. We call μ a *measure of maximal u-entropy* if it satisfies

$$h_\mu(f, \mathcal{F}^{uu}) = h(f, \mathcal{F}^{uu}).$$

By Hu, Wu, Zhu [HWZ21], the set $\mathrm{MM}^u(f)$ of measures of maximal u-entropy is always non-empty, convex and compact. Moreover, its extreme points are ergodic measures.

2.2. **Markov partitions.** Let $\mathcal{R} = \{\mathcal{R}_1, \dots, \mathcal{R}_k\}$ be a Markov partition for the linear automorphism $A : \mathbb{T}^d \to \mathbb{T}^d$. By this we mean (see Bowen [Bow75, Section 3.C]) a finite covering of $\mathbb{T}^d$ by small closed subsets $\mathcal{R}_i$ such that

(a) each $\mathcal{R}_i$ is the closure of its interior, and the interiors are pairwise disjoint;
(b) for any $a, b \in \mathcal{R}_i$, $W^u_i(a)$ intersects $W^s_i(b)$ at exactly one point, which we denote as $[a, b]$;
(c) $A(W^s_i(a)) \subset W^s_j(A(a))$ and $A(W^u_i(a)) \supset W^u_j(A(a))$ if a is in the interior of $\mathcal{R}_i$ and $A(a)$ is in the interior of $\mathcal{R}_j$.

Here, $W^u_i(a)$ is the connected component of $W^u(a) \cap \mathcal{R}_i$ that contains a, and $W^s_i(a)$ is the connected component of $W^s(a) \cap \mathcal{R}_i$ that contains a. We call them, respectively, the *unstable plaque* and the *stable plaque* through a. Property (b) is called *local product structure*.

The boundary $\partial\mathcal{R}_i$ of each $\mathcal{R}_i$ coincides with $\partial^s\mathcal{R}_i \cup \partial^u\mathcal{R}_i$, where $\partial^s\mathcal{R}_i$ is the set of points x which are not in the interior of $W^u_i(x)$ inside the corresponding unstable leaf, and $\partial^u\mathcal{R}_i$ is defined analogously. By product structure, $\partial^s\mathcal{R}_i$ consists of stable plaques and $\partial^u\mathcal{R}_i$ consists of unstable plaques. The Markov property (c) implies that the total stable boundary $\partial^s\mathcal{R} = \cup_i \partial^s\mathcal{R}_i$ is forward invariant and the total unstable boundary $\partial^u\mathcal{R} = \cup_i \partial^u\mathcal{R}_i$ is backward invariant under A. Since the Lebesgue measure on $\mathbb{T}^d$ is invariant and ergodic for A, it follows that both $\partial^s\mathcal{R}$ and $\partial^u\mathcal{R}$ have zero Lebesgue measure. Then, by Fubini, the intersection of $\partial^s\mathcal{R}$ with almost every unstable plaque has zero Lebesgue measure in the plaque. It follows that the same is true for *every* unstable plaque, since the stable holonomies of A, being affine, preserve the class of sets with zero Lebesgue measure inside unstable leaves. A similar statement holds for $\partial^u\mathcal{R}$.

Next, define $\mathcal{M} = \{\mathcal{M}_1, \ldots, \mathcal{M}_k\}$ by $\mathcal{M}_i = \pi^{-1}(\mathcal{R}_i)$. For each $i = 1, \ldots, k$ and $x \in \mathcal{M}_i$, let $\xi_i^u(x)$ be the connected component of $\mathcal{F}^{uu}(x) \cap \mathcal{M}_i$ that contains x, and $\xi_i^{cs}(x)$ be the pre-image of $W_i^s(\pi(x))$. By construction,

$$f(\xi_i^u(x)) \supset \xi_j^u(f(x)) \text{ and } f(\xi_i^{cs}(x)) \subset \xi_j^{cs}(f(x)) \tag{3}$$

whenever x is in the interior of $\mathcal{M}_i$ and $f(x)$ is in the interior of $\mathcal{M}_j$. We refer to $\xi_u^i(x)$ and $\xi_{cs}^i(x)$, respectively, as the *strong-unstable plaque* and the *center-stable plaque* through x.

The local product structure property also extends to $\mathcal{M}$: for any $x, y \in \mathcal{M}_i$ we have that $\xi_i^u(x)$ intersects $\xi_i^{cs}(y)$ at exactly one point, which we still denote as $[x, y]$. That can be seen as follows. To begin with, we claim that π maps $\xi_i^u(x)$ homeomorphically to $W_i^u(\pi(x))$. In view of the assumption (b) above, to prove this it is enough to check that $\pi(\xi_i^u(x)) = W_i^u(\pi(x))$. The inclusion $\subset$ is clear, as both sets are connected. Since $\xi_i^u(x)$ is compact, it is also clear that $\pi(\xi_i^u(x))$ is closed in $W_i^u(\pi(x))$. To conclude, it suffices to check that it is also open in $W_i^u(\pi(x))$. Let $b = \pi(z)$ for some $z \in \xi_i^u(x)$. By assumption (b), for any small neighborhood V of b inside $W^u(b)$, there exists a small neighborhood U of z inside $\mathcal{F}_z^{uu}$ that is mapped homeomorphically to V. By definition, a point $w \in U$ is in $\mathcal{M}_i$ if and only if $\pi(w)$ is in $\mathcal{R}_i$. Thus π maps $U \cap \mathcal{M}_i$ homeomorphically to $V \cap \mathcal{R}_i$. That implies that b is in the interior of $W_i^u(b)$, and that proves that $\pi(\xi_i^u(x))$ is indeed open in $W_i^u(\pi(x))$. Thus the claim is proved. Finally, $[x, y]$ is precisely the sole pre-image of $[\pi(x), \pi(y)]$ in $\xi^u(x)$; notice that this pre-image does belong to $\xi_i^{cs}(y)$, by definition.

This shows that $\mathcal{M}$ is a Markov partition for f, though not necessarily a generating one. In any event, the fact that f is uniformly expanding along strong-unstable leaves ensures that $\mathcal{M}$ is automatically *u-generating*, in the sense that

$$\bigcap_{n=0}^{\infty} f^{-n}\left(\xi_i^u\left(f^n(x)\right)\right) = \{x\} \text{ for every } x \in \Lambda. \tag{4}$$

We call *center-stable holonomy* the family of maps $H_{x,y}^{cs} : \xi_i^u(x) \to \xi_i^u(y)$ defined by the condition that

$$\xi_i^{cs}(z) = \xi_i^u(H_{x,y}^{cs}(z))$$

whenever $x, y \in \mathcal{M}_i$ and $z \in \xi_i^u(x)$.

2.3. **Reference measures.** By pulling the Lebesgue measure along the unstable leaves of A back under the factor map π, one obtains a special family of measures on the strong-unstable plaques of f that we call the *reference measures*. More precisely, the reference measures are the probability measures $\nu_{i,x}^u$ defined on each strong-unstable plaque $\xi_i^u(x)$, $x \in \mathcal{M}_i$, $i \in \{1, \ldots, k\}$ by

$$\pi_* \nu_{i,x}^u = \mathrm{vol}_{i,\pi(x)}^u = \text{normalized Lebesgue measure on } W_i^u(\pi(x)).$$

Since the Lebesgue measure on unstable leaves are preserved by the stable holonomy of A (as the latter is affine), the construction in the previous section also gives that these reference measures are preserved by center-stable holonomies of f:

$$\nu_{i,y}^u = \left(H_{x,y}^{cs}\right)_* \nu_{i,x}^u. \tag{5}$$

for every x and y in the same $\mathcal{M}_i$. Similarly, the fact that Lebesgue measure on unstable leaves has constant Jacobian for A implies that the same is true for the reference measures of f: if $f(\mathcal{M}_i)$ intersects the interior of $\mathcal{M}_j$ then

$$f_*\left(\nu_{i,x}^u \mid_{f^{-1}\left(\xi_j^u(f(x))\right)}\right) = \nu_{i,x}^u\left(f^{-1}\left(\xi_j^u(f(x))\right)\right) \nu_{j,f(x)}^u \tag{6}$$

for every $x \in \mathcal{M}_i \cap f^{-1}(\mathcal{M}_j)$.

Remark 2.1. Properties (5) and (6) imply that $x \mapsto \nu^u_{i,x}(f^{-1}\xi^u_j(f(x)))$ is constant on $\mathcal{M}_i \cap f^{-1}(\mathcal{M}_j)$, for any i and j such that $f(\mathcal{M}_i)$ intersects the interior of $\mathcal{M}_j$. Thus this function takes only finitely many values.

Remark 2.2. Let x be on the boundary of two different Markov sets $\mathcal{M}_i$ and $\mathcal{M}_j$. Then the restrictions of $\nu^u_{i,x}$ and $\nu^u_{j,x}$ to the intersection $\xi^u_i(x) \cap \xi^u_j(x)$ are equivalent measures, as they are both mapped by π_* to multiples of the Lebesgue measure on $W^u_i(\pi(x)) \cap W^u_j(\pi(x))$.

Remark 2.3. As observed before, the intersection of $\partial^s \mathcal{R}$ with every unstable plaque $W^u_i(x)$ has zero Lebesgue measure inside the plaque. Since π sends each $\mathcal{M}_i$ to $\mathcal{R}_i$, with each $\xi^u(x)$ mapped homeomorphically to $W^u_i(\pi(x))$, it follows that $\partial^s \mathcal{M} \cap \xi^u_i(x)$ has zero $\nu^u_{i,x}$-measure for every $x \in \mathcal{M}_i$ and every i.

2.4. **c-Gibbs u-states.** An f-invariant probability measure μ is called a *c-Gibbs u-state* if its conditional probabilities along strong-unstable leaves coincide with the family of reference measures $\nu^u_{i,x}$. More precisely, for each i, let $\{\mu^u_{i,x} : x \in \mathcal{M}_i\}$ denote the disintegration of the restriction $\mu \mid_{\mathcal{M}_i}$ relative to the partition $\{\xi^u_i(x) : x \in \mathcal{M}_i\}$. Then we call μ a *c-Gibbs u-state* if $\mu^u_{i,x} = \nu^u_{i,x}$ for μ-almost every x. The space of invariant c-Gibbs u-states of f is denoted by $\operatorname{Gibbs}^u_c(f)$.

Proposition 2.4 (Corollary 3.7 and Proposition 4.1 in [UVYY])**.** $\operatorname{Gibbs}^u_c(f)$ *is non-empty, convex, and compact. Moreover,*

1. *almost every ergodic component of any* $\mu \in \operatorname{Gibbs}^u_\nu(f)$ *is a c-Gibbs u-state;*
2. *the support of every* $\mu \in \operatorname{Gibbs}^u_\nu(f)$ *is u-saturated;*
3. *for every* $x \in \mathcal{M}_i$ *and* $l \in \{1, \dots, k\}$*, every accumulation point of the sequence*
$$\mu_n = \frac{1}{n} \sum_{j=0}^{n-1} f^j_* \nu^u_{l,x}$$
is a c-Gibbs u-state.
4. *an f-invariant probability measure* μ *is a measure of maximal u-entropy if and only if it is a c-Gibbs u-state.*
5. *the union* $\bigcup_{i=1}^k \partial \mathcal{M}_i$ *of the boundary sets has measure zero with respect to every c-Gibbs u-state.*

2.5. **c-mostly contracting center.** We say that f has *c-mostly contracting center* if

$$\limsup_n \frac{1}{n} \log \| Df^n \mid_{E^{cs}} \| < 0 \tag{7}$$

on a positive measure subset relative to every reference measure. See [UVYY]. The following proposition, together with part (4) of Proposition 2.4 shows that this is equivalent to the definition given in the Introduction.

Proposition 2.5 (Proposition 4.2 of [UVYY])**.** *f has c-mostly contracting center if and only if all center-stable Lyapunov exponents of every ergodic c-Gibbs u-state of f are negative.*

2.6. **Invariant transverse measures.** By a *transverse measure*[1] of a foliation $\mathcal{F}$, we mean a family $\tau = \{\hat{\mu}_S : S \in \mathcal{S}\}$ of measures, where $\mathcal{S}$ is a family of small cross-sections S to the foliation such that

- every $x \in M$ belongs to some $S \in \mathcal{S}$;
- if $S \in S$ and S' is a measurable subset then $S' \in \mathcal{S}$;

[1]All transverse measures are taken to be not identically zero, unless stated otherwise.

and each μ_S is a non-negative Borel measure on the corresponding cross-section S. See Calegary [Cal07]. We call the transverse measure *invariant* if every holonomy homeomorphism $h : S_1 \to S_2$ of $\mathcal{F}$ between cross-sections $S_1, S_2 \in \mathcal{S}$ maps the measure $\hat{\mu}_{S_1}$ to the measure $\hat{\mu}_{S_2}$.

The family of (invariant) transverse measures is preserved by any re-scaling $\hat{\mu}_S \mapsto c\hat{\mu}_S$ with c independent of S. It is also clear that if τ_1 and τ_2 are (invariant) transverse measures defined on the same family $\mathcal{S}$ of cross-sections, then $\tau_1 + \tau_2$ is again an (invariant) transverse measure.

We call an invariant transverse measure $\tau = \{\hat{\mu}_S : S \in \mathcal{S}\}$ *ergodic* if given any splitting $\tau = \tau' + \tau''$ as a sum of two invariant transverse measures τ' and τ'' there exists $c \in (0,1)$ such that $\tau' = c\tau$. We say that an invariant transverse measure τ has *finite ergodic decomposition* if it is a sum of finitely many ergodic invariant transverse measures.

Lemma 2.6. *If τ_1 and τ_2 are ergodic invariant transverse measures which are non-singular restricted to some cross-section then there is $c > 0$ such that $\tau_2 = c\tau_1$.*

Proof. For each cross-section S, let $\hat{\mu}_{1,S}$ and $\hat{\mu}_{2,S}$ be the measures defined on S by τ_1 and τ_2, respectively. Given any rational numbers $p < q$, let $[p,q]_S \subset S$ be the set of points where the Radon-Nykodim derivatives satisfies $d\hat{\mu}_{2,S}/d\hat{\mu}_{1,S} \in [p,q]$. Observe that the Radon-Nykodim derivatives are invariant under holonomies, since the measures themselves are. Thus, the family of sets $[p,q]_S$ is invariant under holonomy, and so the families of restrictions of the $\hat{\mu}_{1,S}$ and $\hat{\mu}_{2,S}$ to the $[p,q]_S$ define invariant transverse measures τ_1' and τ_2' such that $\tau_1' \le \tau_1$ and $\tau_2' \le \tau_2$. We claim that either τ_1' vanishes or $\tau_1' = \tau_1$: otherwise, by ergodicity, there would exist $c \in (0,1)$ such that $\tau_1' = c\tau_1$, and this is not possible because τ_1' is a restriction of τ_1. This ensures that, given any $p < q$, the set $[p,q]_S$ has either zero $\hat{\mu}_{1,S}$-measure for every S or full $\hat{\mu}_{1,S}$-measure for every S. This implies that there exists $c > 0$ such that Radon-Nykodim derivative $d\hat{\mu}_{2,S}/d\hat{\mu}_{1,S} = c$ at $\hat{\mu}_{1,S}$-almost every point of every cross-section S. That implies the claim. □

Two invariant transverse measures $\tau_1 = \{\hat{\mu}_{1,S}\}$ and $\tau_2 = \{\hat{\mu}_{2,S}\}$ are said to be *mutually singular* if the measures $\hat{\mu}_{1,S}$ and $\hat{\mu}_{2,S}$ are mutually singular for every S.

Lemma 2.7. *If τ is a non-ergodic invariant transverse measure then there exists a splitting $\tau = \tau_1 + \tau_2$ into mutually singular invariant transverse measures.*

Proof. The assumption means that there is some splitting $\tau = \tau_1' + \tau_2'$ and neither τ_1' nor τ_2' are not multiples of τ. Write $\tau_1' = \{\hat{\mu}_{1,S}'\}$ and $\tau_2 = \{\hat{\mu}_{2,S}'\}$ and, for each cross-section S and any interval $I \subset [0,\infty]$, denote by I_S the set of points where the Radon-Nykodim derivative satisfies $d\hat{\mu}_{2,S}/d\hat{\mu}_{1,S} \in I$. Keep in mind that the Radon-Nykodim derivatives are invariant under holonomies, since the measures themselves are, and so the family of sets I_S is also invariant under holonomy. The fact that τ_1' and τ_2' are not multiples of τ ensures that there exist $p \in (0,\infty)$ and cross-sections S_1 and S_2 such that such that

$$\hat{\mu}_{1,S_1}([0,p]_{S_1}) > 0 \text{ and } \hat{\mu}_{1,S_1}((p,\infty]_{S_2}) > 0.$$

Let τ_1 and τ_2 be the restrictions of τ to $\{[0,p]_S\}$ and $\{(p,\infty]_S\}$. Then τ_1 and τ_2 are (non-vanishing) invariant transverse measures with $\tau = \tau_1 + \tau_2$, and the construction immediately gives that they are mutually singular. □

3. Proof of Theorem A

Let μ be any measure of maximal u-entropy of f. By Proposition 2.4, μ is a c-Gibbs u-state and its support is u-saturated. Let x_i and x_j be two points in the support of μ contained in the same strong-unstable leaf and in the interior of Markov partition elements $\mathcal{M}_i$ and $\mathcal{M}_j$. See Figure 1. Keep in mind that the boundaries of the Markov elements have zero μ-measure, by Proposition 2.4(5). We denote by $\hat{\mu}_i$ the projection of $\mu \mid \mathcal{M}_i$ to $\xi_i^{cs}(x_i)$ under strong-unstable holonomy inside $\mathcal{M}_i$, and similarly for $\hat{\mu}_j$.

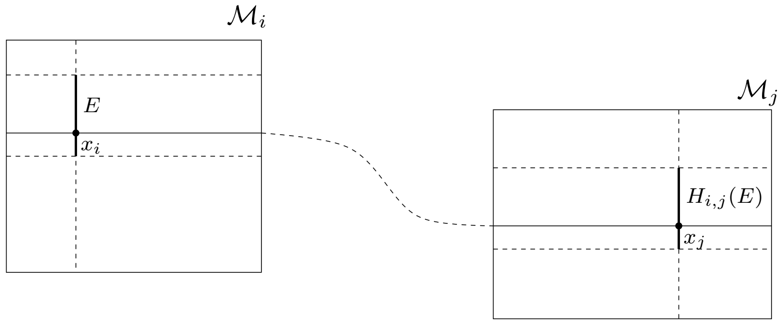

FIGURE 1.

Let H be the strong-unstable holonomy map from a small neighborhood of x_i to a neighborhood of x_j. We denote by $JH_{i,j}$ the Jacobian of H with respect to the measures $\hat{\mu}_i$ and $\hat{\mu}_j$. Let vol^u denote (non-normalized) Lebesgue measure along unstable leaves. For each $i = 1, \dots, k$, define

$$c_i = \frac{1}{\mathrm{vol}^u(\pi\xi_i^u(x_i))}$$

for any $x_i \in \mathcal{M}_i$. This does not depend on the choice of the point x_i because the volume of the unstable plaques for the linear Anosov map A is constant on each Markov rectangle $\mathcal{R}_i$.

Lemma 3.1. *The Jacobian $JH_{i,j}$ is constant equal to c_i/c_j for any i and j.*

Proof. Fix $N \geq 1$ large enough that $f^{-N}(x_i)$ and $f^{-N}(x_j)$ are contained in the interior of the same Markov partition element $\mathcal{M}_l$. It is no restriction to assume that the domain $E \subset \xi_i^{cs}(x_i)$ of H is small enough that $f^{-n}(E)$ and $f^{-n}(H(E))$ are contained in the interior of $\mathcal{M}_l$. By the definition of $\hat{\mu}_i$ and $\hat{\mu}_j$, and the fact that μ is f-invariant,

$$\hat{\mu}_i(E) = \mu\left(\bigcup_{x\in E}\xi_i^u(x)\right) = \mu\left(f^{-N}\left(\bigcup_{x\in E}\xi_i^u(x)\right)\right),$$

and a similar fact holds for $\hat{\mu}_j(H(E))$. It is clear that the sets $f^{-N}\left(\bigcup_{x\in E}\xi_i^u(x)\right)$ and $f^{-N}\left(\bigcup_{y\in H(E)}\xi_i^u(y)\right)$ project to the same set $B \subset \xi_l^{cs}(x_l)$ under the strong-unstable holonomy inside $\mathcal{M}_l$. Then, since μ is a c-Gibbs u-state, Rokhlin disintegration gives that

$$\hat{\mu}_i(E) = \int_B \nu_{l,z}^u\left(f^{-N}(\xi_i^u(f^N(z)))\right) d\hat{\mu}_l(z) \tag{8}$$

and similarly for $\hat{\mu}_j(H(E))$. By the definition of the reference measures

$$\nu_{l,z}^u\left(f^{-N}(\xi_i^u(f^N(z)))\right) = \frac{\mathrm{vol}^u(\pi\left(f^{-N}\xi_i^u(f^N(z))\right))}{\mathrm{vol}^u(\pi(\xi_l^u(z)))}. \tag{9}$$

Since f is semi-conjugate to the linear automorphism A,

$$\mathrm{vol}^u(\pi\left(f^{-N}\xi_i^u(f^N(z))\right)) = \frac{\mathrm{vol}^u\left(\pi(\xi_i^u(z))\right)}{\det(A^N \mid E^u)} = \frac{\mathrm{vol}^u\left(\pi(\xi_i^u(x_i))\right)}{\det(A^N \mid E^u)} = \frac{1}{c_i\det(A^N \mid E^u)}.$$

and, analogously,

$$\mathrm{vol}^u(\pi\left(f^{-N}\xi_j^u(f^N(w))\right)) = \frac{1}{c_j \det(A^N \mid E^u)}.$$

Replacing these last two identities in (8) and (9), we find that

$$\frac{\hat{\mu}_j(H(E))}{\hat{\mu}_i(E)} = \frac{c_i}{c_j}.$$

Since E is arbitrarily small, this gives the claim of the lemma. □

Next, let S be any cross-section to the strong-unstable foliation inside $\mathcal{M}_i$. It is no restriction to assume that S meets every strong-unstable plaque $\xi_i^u(x)$. Define $\hat{\mu}_S$ to be the projection of $\mu \mid \mathcal{M}_i$ to S under strong-unstable holonomy inside $\mathcal{M}_i$. It follows directly from Lemma 3.1 that $\{c_i\hat{\mu}_S\}$ is an invariant transverse measure. This proves Theorem A.

4. Proof of Theorem C

Now we prove Theorem C. For each i, let μ_i denote the restriction $\mu \mid \mathcal{M}_i$.

Lemma 4.1. *There is $N \geq 1$ and for each i there exists a full μ_i-measure subset of $\mathcal{M}_i$ that intersects each center plaque $\xi_i^c(x)$ at not more than N points.*

Proof. Since f is assumed to have c-mostly contracting center, the center-stable exponents of μ are all negative (Proposition 2.5), and so, there exist $m \geq 1$ and $c > 0$ such that

$$\frac{1}{m}\int \log \|Df^{-m} \mid_{E^{cs}} \|^{-1}\, d\mu \geq c.$$

This inequality remains true for some ergodic component μ_0 of μ for the iterate f^m. Then, by the Birkhoff ergodic theorem,

$$\lim_n \frac{1}{n}\sum_{i=0}^{n-1} \log \|Df^{-m} \mid_{E^{cs}(f^{-im}y)} \|^{-1} \geq c \tag{10}$$

for μ_0-almost every y. Let $K = \max \|Df^{\pm 1} \mid_{E^{cs}} \|$ and fix $l > 1$ large enough that $(l-1)c \geq 2m \log K$. Since μ is assumed to be ergodic for f, it satisfies

$$\mu = \frac{1}{m}\sum_{j=0}^{m-1} f_*^j \mu_0.$$

So, for μ-almost every x there exists $j = 0, \ldots, m-1$ such that $y = f^j(x)$ satisfies (10), and so

$$\begin{aligned}
\lim_n \frac{1}{n}\sum_{i=0}^{n-1} &\log \|Df^{-lm} \mid_{E^{cs}(f^{-ilm}(x))} \|^{-1} \\
&\geq \lim_n \frac{1}{n}\sum_{i=0}^{n-1} \log\left(K^{-j}\|Df^{-lm} \mid_{E^{cs}(f^{-ilm}(y))} \|^{-1} K^{-j}\right) \\
&\geq -2j \log K + l \lim_n \frac{1}{nl}\sum_{i=0}^{nl-1} \log \|Df^{-m} \mid_{E^{cs}(f^{-im}(y))} \|^{-1} \\
&\geq -2m \log K + lc \geq c.
\end{aligned}$$

(in the second inequality we used the sub-multiplicativity of the norm). By Proposition 3.7 in [VY13] it follows that there exists $N \geq 1$, depending only on c and the maximum volume of the center leaves, and there exists a full μ-measure subset whose intersection with each center plaque contains not more than N points. □

Let $\{\mu^c_{i,x} : x \in \mathcal{M}_i\}$ be a disintegration of μ_i along the center plaques $\xi^c_i(x)$. It follows from the previous lemma that for μ_i-almost every x the conditional probability $\mu^c_{i,x}$ is supported on not more than N points.

We are going to deduce a similar statement for the conditional probabilities $\hat{\mu}^c_{i,x}$ along center plaques $\xi^c_i(x)$ of the projection $\hat{\mu}_i$ of μ_i to the center-stable plaque $\xi^{cs}_i(x_i)$.

Lemma 4.2. *For $\hat{\mu}_i$-almost every $x \in \xi^{cs}_i(x_i)$, the conditional probability $\hat{\mu}^c_{i,x}$ is supported on not more than N points.*

Proof. By the local product structure, we may find coordinates (x_u, x_{cs}) on $\mathcal{M}_i$ for which the unstable plaques ξ^u_i and the center-stable plaques ξ^{cs}_i are given by relations $x_{cs} = \text{const}$ and $x^u = \text{const}$, respectively. Let

$$\mu = \int_{\xi^{cs}_i(x_i)} \mu^u_{i,x_{cs}} \, d\hat{\mu}_i(x_{cs}) \tag{11}$$

be the disintegration of μ along the unstable plaques. Since μ is a measure of maximal u-entropy and, thus, a c-Gibbs u-state, we have that $\mu^u_{i,x_{cs}} = \nu^u_{i,x_{cs}}$. Since (5) gives that the reference measures $\nu^u_{i,x}$ are invariant under center-stable holonomy, it follows that $\mu^u_{i,x_{cs}}$ is actually independent of x_{cs}. We write $\mu^u_{i,x_{cs}} = \mu^u_i$, and then (11) becomes

$$\mu = \mu^u_i \times \hat{\mu}_i \tag{12}$$

Let $\hat{\mu}^s_i$ be the quotient measure of $\hat{\mu}_i$ with respect to the family of center plaques $\xi^c_i(z)$, which we may view as a measure on the stable plaque $\xi^s_i(x_i)$. Then

$$\hat{\mu}_i = \int_{\xi^s_i(x_i)} \hat{\mu}^c_{i,z} \, d\hat{\mu}^s_i(z). \tag{13}$$

Combining this with (12), we find that

$$\mu = \int_{\xi^s_i(x_i)} \mu^u_i \times \hat{\mu}^c_{i,z} \, d\hat{\mu}^s_i(z),$$

and so the conditional probabilities $\mu^{cu}_{i,z}$ of μ_i along center-unstable plaques are again product measures:

$$\mu^{cu}_{i,z} = \mu^u_i \times \hat{\mu}^c_{i,z}. \tag{14}$$

In particular, the conditional probabilities of $\mu^{cu}_{i,z}$ along center plaques $\xi^c_i(y)$ coincide with $\hat{\mu}^c_{i,z}$ (which is constant on the center-unstable plaque). On the other hand, by the transitivity of the disintegration (see [VO16, Exercise 5.2.1]), the conditional probabilities of $\mu^{cu}_{i,z}$ along center plaques $\xi^c_i(y)$ coincide with $\mu^c_{i,y}$ at μ_i-almost every point. This proves that $\hat{\mu}^c_{i,z} = \mu^c_{i,y}$ μ_i-almost everywhere. Thus the claim follows from Lemma 4.1. □

A foliation is said to be *uniquely ergodic* if it admits a unique invariant transverse measure, up to constant factor.

Lemma 4.3. *The unstable foliation of every Anosov linear automorphism A is uniquely ergodic.*

Proof. To each invariant transverse measure τ for the unstable foliation of A we associate the finite measure ω on M constructed in the following way. Given any foliation box $\mathcal{B}$ of $\mathcal{F}^u$ and any associated cross-section D (D cuts every unstable plaque inside $\mathcal{B}$ transversely at a unique point), define

$$\omega \mid \mathcal{B} = \int_D \mathrm{vol}^u_x \, d\tau(x) \tag{15}$$

where vol^u_x is the volume measure on the unstable plaque through x.

This does not depend on the choice of the cross-section D because τ is taken to be transversely invariant. For the same reason two such measures $\omega \mid \mathcal{B}_1$ and $\omega \mid \mathcal{B}_2$ coincide on the intersection of the corresponding foliation boxes $\mathcal{B}_1$ and $\mathcal{B}_2$. Thus ω is well defined on M. Moreover, since the volume measure on each unstable leaf is invariant under translations of the leaf, the measure ω is invariant under any rigid translation of the torus tangent to the unstable direction. Now, every translation on the torus can be approximated by a translation tangent to the unstable direction, since the unstable leaves of A are dense on the torus. It follows that ω is actually invariant under *every* translation of the torus, and so, ω must coincide with the volume measure on $\mathbb{T}^d$ up to a factor. Noting that the relations (15) also define τ from ω, we conclude that the invariant transverse measure τ is unique. □

We are ready to complete the proof of Theorem C. It τ_μ is ergodic, there is nothing to do. Otherwise, by Lemma 2.7 we may split is as $\tau_\mu = \tau_1 + \tau_2$ where τ_1 and τ_2 mutually singular. The projections of both under π are invariant transverse measures for the unstable foliation of the linear automorphism A. Thus, by Lemma 4.3, they are multiples of each other. For $j = 1, 2$ and any i, let

$$\tau_j \mid \xi_i^{cs}(x_i) = \int \tau_{j,z}^c \, d\hat{\tau}_j(z) \tag{16}$$

be the disintegration of $\tau_j \mid \xi_i^{cs}(x_i)$ along center plaques $\xi_i^c(z)$. By the previous remarks, $\hat{\tau}_1$ and $\hat{\tau}_2$ are multiples to each other. So, the fact that $\tau_1 \mid \xi_i^{cs}(x_i)$ and $\tau_2 \mid \xi_i^{cs}(x_i)$ are mutually singular implies that $\tau_{1,z}^c$ and $\tau_{2,z}^c$ are mutually singular for $\hat{\tau}_j$-almost every z. The sum $\tau_{1,z}^c + \tau_{2,z}^c$ is the conditional probability $\tau_{\mu,z}^c$ of the measure $\tau_\mu \mid \xi_i^{cs}(x_i)$ which, by Theorem A is a multiple of $\hat{\mu}_i^c$. By Lemma 4.2, the latter is supported on no more than N points. Thus the supports of the measures $\tau_{1,z}^c$ and $\tau_{2,z}^c$ must be disjoint, and this decomposition procedure cannot be repeated more than N times. This completes the proof.

5. Proof of Theorem B

Let x_1 and x_2 be any two points in $\operatorname{supp}\mu$. Up to renumbering the elements of the Markov partition if necessary, we may suppose that $x_1 \in \mathcal{M}_1$ and $x_2 \in \mathcal{M}_2$. For $i = 1, 2$, denote $Y_i = \xi_i^u(x_i) \times [0, 1]$. Equip each Y_i with the probability measure $m_i = \nu_{i,x_i}^u \times dt$.

The main technical step in the proof of Theorem B is the following lemma, whose proof we postpone to Section 7. This is also the one step where we use the assumption that the support of μ is connected.

Lemma 5.1 (Coupling Lemma)**.** *There are a map $\tau : Y_1 \to Y_2$ with $\tau_* m_1 = m_2$, a function $R : Y_1 \to \mathbb{N}$, and constants $C_1, C_2 > 0$ and $\rho_1, \rho_2 \in (0, 1)$ such that*

(1) If $\tau(x, t) = (y, s)$ then $f^n(x)$ and $f^n(y)$ belong to the same Markov component $\mathcal{M}_i$ for some i, and $d(f^n(x), f^n(y)) \le C_1 \rho_1^{n-R}$ for $n \ge R(x, t)$.

(2) $m_1(R > n) \le C_2 \rho_2^n$ for every $n \ge 1$.

Let S be a cross-section contained in some $\mathcal{M}_j$ with $\mu(\mathcal{M}_j) > 0$, and $\hat{\varphi} : S \to \mathbb{R}$ be any Hölder real function supported inside S. In the remainder of this section, we take $\varphi : M \to \mathbb{R}$ to be the extension of $\hat{\varphi}$ which is constant along unstable plaques on $\mathcal{M}_j$ and vanishes on $M \setminus \mathcal{M}_j$. Since $\hat{\varphi}$ is taken to be supported inside S, the extension φ vanishes on the unstable boundary $\partial^u \mathcal{M}_j = \pi^{-1}(\partial^u \mathcal{R}_j)$. Notice also that φ is Hölder restricted to $\mathcal{M}_j$, since $\hat{\varphi}$ is Hölder and so are the strong-unstable holonomy maps.

Corollary 5.2. *Given $\hat{\varphi} : S \to \mathbb{R}$ there are $C > 0$ and $0 < \rho < 1$ such that for any points $x, y \in \operatorname{supp}\mu$*

$$\left| \int \varphi \, d\left(f_*^n \nu_{i,x}^u\right) - \int \varphi \, d\left(f_*^n \nu_{k,y}^u\right) \right| \le C\rho^n$$

for every $n \ge 1$, where $\mathcal{M}_i \ni x$ and $\mathcal{M}_k \ni y$.

Proof. It is no restriction to suppose that $i = 1$ and $k = 2$. Define $\tilde{\varphi} : M \times [0,1] \to \mathbb{R}$ by $\tilde{\varphi}(x,t) = \varphi(x)$. Then

$$\int \varphi \, d\left(f_*^n \nu_{1,x}^u\right) = \int (\varphi \circ f^n) \, d\nu_{1,x}^u = \int \tilde{\varphi}(f^n(z), t) \, dm_1(z,t). \tag{17}$$

By Lemma 5.1, the map $\tau : \xi_{1,x}^u \times [0,1] \to \xi_{2,y}^u \times [0,1]$, $(z,t) \mapsto (w,s) = \tau((z,t))$ sends the measure m_1 to m_2. Thus,

$$\int \varphi \, d\left(f_*^n \nu_{2,x}^u\right) = \int \tilde{\varphi}(f^n(w), s) \, dm_2(w,s) = \int \tilde{\varphi}(f^n(w), s) \, dm_1(z,t). \tag{18}$$

We are going to break both (17) and (18) as a sum of integrals over the two domains $\{R(z,t) \le n/2\}$ or $\{R(z,t) > n/2\}$. Let C_3 and α_3 be Hölder constants for φ restricted to $\mathcal{M}_j$. By part (1) of Lemma 5.1,

$$\begin{aligned} &\left| \int_{R(z,t) \le n/2} \left[\tilde{\varphi}(f^n(z), t) - \tilde{\varphi}(f^n(w), s) \right] dm_1(z,t) \right| \\ &\qquad = \left| \int_{R(z,t) < n/2} \left[\varphi(f^n(z)) - \varphi(f^n(w)) \right] dm_1(z,t) \right| \\ &\qquad \le \left| \int_{R(z,t) < n/2} C_3 \left(C_1 \rho_1^{n - R(z,t)} \right)^{\alpha_3} dm_1(z,t) \right| \le C_3 C_1^{\alpha_3} \rho_1^{\alpha_3 n/2}. \end{aligned}$$

By part (2) of Lemma 5.1, the integrals over $\{R(z,t) > n/2\}$ are bounded above by $C_2 \rho_2^{n/2} \sup |\varphi|$. The claim is a direct consequence of these two estimates, with $C = C_3 C_1^{\alpha_3} + C_2 \sup|\varphi|$ and $\rho = \max\{\rho_1^{\alpha_3/2}, \rho_2^{1/2}\}$. $\square$

Lemma 5.3. *There is $n_0 > 0$ such that for any $n \ge n_0$, any $x \in \mathcal{M}_i \cap \operatorname{supp}\mu$, and any i*

$$\int \varphi \, d\left(f_*^n \nu_{i,x}^u\right) = \left(f_*^n \nu_{i,x}^u\right)(\mathcal{M}_j) \frac{1}{\#\big(f^n(\xi_i^u(x)) \cap S\big)} \sum_{q \in f^n(\xi_i^u(x)) \cap S} \hat{\varphi}(q).$$

Proof. By the Markov property, the intersection of $f^n(\xi_i^u(x))$ with $\mathcal{M}_j$ consists of the unstable plaques $\xi_j^u(q)$, $q \in f^n(\xi_i^u(x)) \cap S$. The assumption that f is semi-conjugate to a linear Anosov map A ensures that there exists $n_0 \ge 1$ independent of x such that this intersection is non-empty for every $n \ge n_0$. Since φ is constant on unstable plaques inside $\mathcal{M}_j$,

$$\int \varphi \, d\left(f_*^n \nu_{i,x}^u\right) = \sum_{q \in f^n(\xi_i^u(x)) \cap S} \nu_{i,x}^u(f^{-n}(\xi_j^u(q))) \hat{\varphi}(q).$$

By the definition of the reference measures, each term $\nu_{i,x}^u(f^{-n}(\xi_j^u(q))$ coincides with the Lebesgue measure of the image of $f^{-n}(\xi_j^u(q))$ under the semi-conjugacy, which is the A^{-n}-image of an unstable plaque $\mathcal{R}_j$. Since all these unstable plaques have the same Lebesgue measure, and the Jacobian of A^{-n} along the unstable direction is constant everywhere, we find that $\nu^u_{i,x}(f^{-n}(\xi^u_j(q)))$ is independent of q. Therefore,

$$\nu^u_{i,x}(f^{-n}(\xi^u_j(q)) = \frac{1}{\#\big(f^n(\xi^u_i(x)) \cap S\big)} \sum_{w \in f^n(\xi^u_i(x)) \cap S} \nu^u_{i,x}\left(f^{-n}(\xi^u_j(w))\right)$$
$$= \frac{1}{\#\big(f^n(\xi^u_i(x)) \cap S\big)} \left(f^n_* \nu^u_{i,x}\right)(\mathcal{M}_j)$$

The claim follows directly from these two identities. □

We are ready to complete the proof of Theorem B. By assumption, μ is a measure of maximal u-entropy, and so it is a c-Gibbs u-state. This means that its conditional measures along unstable plaques $\xi^u_i(x)$ are the reference measures $\nu^u_{i,x}$. Thus, recalling also that μ is f-invariant, Corollary 5.2 implies that

$$\left| \int \varphi \, d\left(f^n_* \nu^u_{i,x}\right) - \int \varphi \, d\mu \right| \leq C\rho^n$$

for every $n \geq 1$. Then, using Lemma 5.3,

$$\left| \left(f^n_* \nu^u_{i,x}\right)(\mathcal{M}_j) \frac{1}{\#\big(f^n(\xi^u_i(x)) \cap S\big)} \sum_{q \in f^n(\xi^u_i(x)) \cap S} \hat{\varphi}(q) - \int \varphi \, d\mu \right| \leq C\rho^n \tag{19}$$

for every $n \geq n_0$. By the definition of the reference measures,

$$\left(f^n_* \nu^u_{i,x}\right)(\mathcal{M}_j) = \left(A^n_* \operatorname{vol}^u_{i,\pi(x)}\right)(\mathcal{R}_j)$$

since the projection $\pi : M \to \mathbb{T}^d$ is a homeomorphism on each strong-unstable leaf. It is a classical fact about linear Anosov maps that the right hand side converges to $\operatorname{vol}(\mathcal{R}_j)$ exponentially fast, where vol denotes the Haar measure on the torus. Since μ is a c-Gibbs u-state, we can use [UVYY, Corollary 3.5] to conclude that

$$\operatorname{vol}(\mathcal{R}_j) = \mu(\mathcal{M}_j) = \|\hat{\mu}_S\|.$$

Keep also in mind that $\int \varphi \, d\mu = \int \hat{\varphi} \, d\hat{\mu}_S$. Thus, (19) implies that

$$\frac{1}{\#\big(f^n(\xi^u_i(x) \cap S\big)} \sum_{q \in f^n(\xi^u_i(x)) \cap S} \hat{\varphi}(q) \to \frac{1}{\|\hat{\mu}_S\|} \int \hat{\varphi} \, d\hat{\mu}_S$$

exponentially fast as $n \to \infty$.

The proof of Theorem B is now complete.

6. Large deviations

We say that a system (f, μ) satisfies a *large deviations principle* for continuous observables if for every $\alpha > 0$ and every continuous function φ with $\int_M \varphi \, d\mu = 0$ there exist positive constants C_α and c_α such that

$$\mu\left(\left\{ x \in M : \left| \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x)) \right| > \alpha \right\}\right)) \leq C_\alpha e^{-c_\alpha n} \text{ for every } n \geq 1. \tag{20}$$

In this section we prove (recall also Remark 1.2):

Theorem 6.1. *Let $f : M \to M$ be a partially hyperbolic diffeomorphism that factors over Anosov and has c-mostly contracting center. Let μ be a measure of maximal u-entropy whose support Λ is connected. Then (f, μ) satisfies a large deviations principle for continuous observables.*

The proof occupies the remainder of this section. Throughout, it is assumed that μ is a measure of maximal u-entropy and thus (compare part (4) of Proposition 2.4) a c-Gibbs u-state.

6.1. **Probability measures with Hölder densities.** Let us fix some element $\mathcal{M}_i$ of the Markov partition (recall Section 2.2). It is convenient to consider a certain family of spaces $E_i(R)$ consisting of probability measures on $\mathcal{M}_i \cap \Lambda$ whose conditional probabilities along the partition ξ_i^u are absolutely continuous with respect to the reference measures, with Hölder densities. We begin by giving the definition of this space, which is mostly borrowed from Dolgopyat [Dol00, Section 5].

Let $\gamma \in (0,1)$ be fixed. For each $R \geq 0$ and $i = 1, \dots, k$ denote by $C_i(R)$ the set of all probability measures η on $\mathcal{M}_i \cap \Lambda$ of the form

$$\eta = e^{\rho} \nu_{i,x}^u, \tag{21}$$

where $x \in \mathcal{M}_i \cap \Lambda$ and $\rho : \xi_i^u(x) \to \mathbb{R}$ is an (R, γ)-Hölder function:

$$|\rho(z_1) - \rho(z_2)| \leq R d(z_1, z_2)^{\gamma} \text{ for any } z_1, z_2 \in \xi_i^u(x). \tag{22}$$

For instance, $C_i(0)$ consists precisely of the reference measures $\nu_{i,x}^u$, $x \in \mathcal{M}_i \cap \Lambda$. Every $C_i(R)$ is a weak*-closed subset of the space of all probability measures on $\mathcal{M}_i \cap \Lambda$, because the reference measures $\nu_{i,x}^u$ vary continuously with x, and the space of (R, γ)-Hölder functions is equicontinuous. Thus $C_i(R)$ is compact for the weak* topology, for every $R \geq 0$.

Next, let $\hat{E}_i(R)$ be the space of all probability measures on the compact space $C_i(R)$. Note that $\hat{E}_i(R)$ is compact for the weak* topology. Consider the map

$$\Pi : \hat{E}_i(R) \to \{\text{probability measures on } \mathcal{M}_i \cap \Lambda\}, \quad \Pi(\hat{\zeta}) = \int_{C_i(R)} \eta \, d\hat{\zeta}(\eta).$$

In other words, $\Pi(\hat{\zeta})$ is the probability measure on $\mathcal{M}_i \cap \Lambda$ such that

$$\int_{\mathcal{M}_i \cap \Lambda} \psi \, d\Pi(\hat{\zeta}) = \int_{C_i(R)} \left(\int_{\mathcal{M}_i \cap \Lambda} \psi \, d\eta \right) d\hat{\zeta}(\eta)$$

for any bounded measurable function $\psi : \mathcal{M}_i \cap \Lambda \to \mathbb{R}$. Take ψ to be continuous. Then it is clear that

$$\hat{\psi} : C_i(R) \to \mathbb{R}, \quad \hat{\psi}(\eta) = \int_{\mathcal{M}_i \cap \Lambda} \psi \, d\eta \tag{23}$$

is continuous. Let $(\hat{\zeta}_j)_j$ be any sequence converging to some $\hat{\zeta}$ in $\hat{E}_i(R)$. Then

$$\int_{\mathcal{M}_i \cap \Lambda} \psi \, d\Pi(\hat{\zeta}_j) = \int_{C_i(R)} \hat{\psi} \, d\hat{\zeta}_j \to \int_{C_i(R)} \hat{\psi} \, d\hat{\zeta} = \int_{\mathcal{M}_i \cap \Lambda} \psi \, d\Pi(\hat{\zeta})$$

as $j \to \infty$. Since ψ is an arbitrary continuous function, this proves that the map Π is continuous.

Let $E_i(R) = \Pi(\hat{E}_i(R))$. It follows from the previous observations that $E_i(R)$ is a weak*-compact subset of the space of all probability measures on $\mathcal{M}_i \cap \Lambda$.

Lemma 6.2. *A probability measure ζ on $\mathcal{M}_i \cap \Lambda$ is in $E_i(R)$ if and only if its conditional measures with respect to the partition ξ_i^u are elements of $C_i(R)$, that is, probability measures of the form* (21).

Proof. Suppose that $\zeta \in E_i(R)$. Then, by definition, there exists $\hat{\zeta} \in C_i(R)$ such that $\zeta = \Pi(\hat{\zeta})$. Consider the canonical map

$$H : C_i(R) \to \xi_i^u, \quad H\left(e^{\rho} \nu_{i,x}^u\right) \to \xi_i^u(x),$$

and let $\tilde{\zeta} = H_* \hat{\zeta}$. The partition of ξ_i^u into points is measurable, because ξ_i^u itself is a measurable partition of $\mathcal{M}_i \cap \Lambda$, and so, its pull-back under H is a measurable partition of $C_i(R)$. Let $\{\hat{\zeta}_P : P \in \xi_i^u\}$ be the disintegration of $\hat{\zeta}$ with respect to the pull-back: each $\hat{\zeta}_P$ is a probability measure on $C_i(R)$ with $\hat{\zeta}_P\left(H^{-1}(P)\right) = 1$, and these probabilities satisfy

$$\int_{C_i(R)} \hat{\psi}\, d\hat{\zeta} = \int_{\xi_i^u} \left(\int_{H^{-1}(P)} \hat{\psi}\, d\hat{\zeta}_P \right) d\tilde{\zeta}(P)$$

for every bounded measurable function $\hat{\psi} : C_i(R) \to \mathbb{R}$. Let $\psi : \mathcal{M}_i \cap \Lambda \to \mathbb{R}$ be any bounded measurable function, and $\hat{\psi}$ be as in (23). Then,

$$\int_{\mathcal{M}_i \cap \Lambda} \psi\, d\zeta = \int_{\mathcal{M}_i \cap \Lambda} \psi\, d\Pi\hat{\zeta} = \int_{C_i(R)} \hat{\psi}\, d\hat{\zeta} = \int_{\xi_i^u} \left(\int_{H^{-1}(P)} \hat{\psi}\, d\hat{\zeta}_P \right) d\tilde{\zeta}(P).$$

Moreover, by the definition (23),

$$\begin{aligned} \int_{H^{-1}(P)} \hat{\psi}(\eta)\, d\hat{\zeta}_P(\eta) &= \int_{H^{-1}(P)} \int_{\mathcal{M}_i \cap \Lambda} \psi\, d\eta\, d\hat{\zeta}_P(\eta) \\ &= \int_{\mathcal{M}_i \cap \Lambda} \psi\, d \left(\int_{H^{-1}(P)} \eta\, d\hat{\zeta}_P(\eta) \right). \end{aligned}$$

This means that the conditional probabilities of ζ with respect to the partition ξ_i^u are the measures

$$\int_{H^{-1}(P)} \eta\, d\hat{\zeta}_P(\eta), \quad P \in \xi_i^u.$$

Consider any $P = \xi_i^u(x)$. The condition $\eta \in H^{-1}(P)$ means that η is a measure of the form $e^{\rho}\nu_{i,x}^u$. Then

$$\int_{H^{-1}(P)} \eta\, d\hat{\zeta}_P(\eta) = \left(\int e^{\rho}\, d\zeta_{i,x}(\rho) \right) \nu_{i,x}^u,$$

where $\zeta_{i,x}$ is a probability measure on the space of (R, γ)-Hölder functions. To conclude it suffices to note that the function

$$y \mapsto \log \left(\int e^{\rho(y)}\, d\zeta_{i,x}(\rho) \right)$$

is (R, γ)-Hölder. This proves the 'only if' part of the statement.

Now suppose that the disintegration $\zeta = \int_{\xi_i^u} \zeta_P\, d\tilde{\zeta}(P)$ of ζ with respect to ξ_i^u is such that $\zeta_P \in C_i(R)$ for every $P \in \xi_i^u$. Consider the measurable map $P \mapsto \zeta_P$ from ξ_i^u to $C_i(R)$, and let $\hat{\zeta}$ be the push-forward of $\tilde{\zeta}$ under this map. Then $\hat{\zeta}$ is a probability measure on $C_i(R)$, that is, an element of $\hat{E}_i(R)$, and

$$\zeta = \int_{C_i(R)} \eta\, d\hat{\zeta}.$$

In other words, $\zeta = \Pi(\hat{\zeta})$, which proves that $\zeta \in E_i(R)$. □

Finally, define $C(R)$ to be the disjoint union of all $C_i(R)$, $i = 1, \dots, k$, and $\hat{E}(R)$ to be the space of probability measures on $C(R)$. Clearly, the latter coincides with the space of convex combinations

$$\hat{\zeta} = \sum a_i \hat{\zeta}_i \text{ with } \sum_{i=1}^{k} a_i = 1 \text{ and } \hat{\zeta}_i \in \hat{E}_i(R) \text{ for all } i.$$

Finally, define $E(R)$ to be the space of convex combinations

$$\zeta = \sum_{i=1}^{k} a_i \zeta_i \text{ with } \sum_{i=1}^{k} a_i = 1 \text{ and } \zeta_i \in E_i(R) \text{ for all } i.$$

It is clear that $C(R)$, $\hat{E}(R)$ and $E(R)$ are weak*-compact spaces.

Remark 6.3. The families $C_i(R), C(R), \hat{E}_i(R), \hat{E}(R), E_i(R), E(R)$ are clearly monotone increasing in R. Moreover, they are continuous at $R = 0$. Indeed, it is clear that $C_i(0)$ coincides with the intersection of the compact sets $C_i(R)$ over all $R > 0$. It follows that $\hat{E}_i(0)$ coincides with the intersection of the $\hat{E}_i(R)$ over all $R > 0$. Since $R \mapsto \hat{E}_i(R)$ is a monotone family of compact sets, and Π is continuous, we also get that

$$E_i(0) = \Pi(\hat{E}_i(0)) = \Pi\left(\bigcap_{R>0} \hat{E}_i(R)\right) = \bigcap_{R>0} \Pi\left(\hat{E}_i(R)\right) = \bigcap_{R>0} E_i(R).$$

Since this is true for every i, the corresponding statements for $C(R)$, $\hat{E}(R)$ and $E(R)$ follow.

Let $1/\omega > 1$ be a lower bound for the expansion rate of f along strong-unstable leaves as in (1).

Proposition 6.4. *$f_*(E(R)) \subset E(Re^{l\gamma \log \omega})$ for any $R \geq 0$.*

Proof. Let us begin by considering $\eta = e^{\rho}\nu^u_{i,x}$ in any $C_i(R)$. The push-forward of the reference measure $\nu^u_{i,x}$ is a finite convex combination of reference measures ν^{j,y_j}:

$$f_*\nu^u_{i,x} = \sum_{j\in J(i)} a_j \nu^u_{j,y_j}.$$

See (6) and Remark 2.1. Thus the push-forward of η may be written as

$$f_*\eta = \sum_{j\in J(i)} \left(a_j e^{-b_j}\right) e^{b_j+\rho\circ f^{-1}} \nu^u_{j,y_j},$$

where the exponents b_j are chosen so that each $e^{b_j+\rho\circ f^{-1}}\nu^{j,y_j}$ is a probability measure. By assumption, ρ is (R,γ)-Hölder on the strong-unstable plaque $\xi^u_i(x)$. Since f^{-1} contracts strong-unstable leaves at a uniform rate $e^{l\log\omega}$, it follows that $b_j + \rho\circ f^{-1}$ is $(Re^{l\gamma\log\omega}, \gamma)$-Hölder. Thus, the previous identity may be written as

$$f_*\eta = \int_{C(Re^{l\gamma\log\omega})} \zeta \, d\hat{\zeta}_\eta(\zeta),$$

where $\hat{\zeta}_\eta$ is the element of $\hat{E}(Re^{l\gamma\log\omega})$ given by the convex combination of Dirac masses at the $e^{b_j+\rho\circ f^{-1}}\nu^u_{j,y_j}$, with the $a_j e^{-b_j}$ as the coefficients.

Now let ζ be any element of $E(R)$. Then there exists $\hat{\zeta} \in \hat{E}(R)$ such that

$$\zeta = \Pi(\hat{\zeta}) = \int_{C(R)} \eta \, d\hat{\zeta}(\eta).$$

Then

$$f_*\zeta = \int_{C(R)} f_*\eta \, d\hat{\zeta}(\eta) = \int_{C(R)} \int_{C(Re^{l\gamma\log\omega})} \zeta \, d\hat{\zeta}_\eta(\zeta) \, d\hat{\zeta}(\eta)$$
$$= \int_{C(Re^{l\gamma\log\omega})} \zeta \, d\left(\int_{C(R)} \hat{\zeta}_\eta \, d\hat{\zeta}(\eta)\right)(\zeta).$$

To conclude, observe that

$$\int_{C(R)} \hat{\zeta}_\eta \, d\hat{\zeta}(\eta) \in \hat{E}(Re^{l\gamma\log\omega})$$

because the latter is a convex compact space and it contains every $\hat{\zeta}_\eta$. Thus, the previous identity means that $f_*\zeta \in E(Re^{l\gamma\log\omega})$. □

Lemma 6.5. *For every $n \geq 1$, μ is f^n-ergodic and it is the unique f^n-invariant probability measure in $E(0)$.*

Proof. By [UVYY, Theorem A], μ has finitely many ergodic components for f^n, and their supports are pairwise disjoint. Since $\Lambda = \operatorname{supp}\mu$ is connected, it follows that there can be only one ergodic component, in other words, μ is f^n-ergodic.

Since μ is a c-Gibbs u-state, it follows directly from Lemma 6.2 that $\mu \in E(0)$. To prove uniqueness, let $\tilde{\mu}$ be any f^n-invariant probability measure in $E(0)$. By definition, $\tilde{\mu}$ is supported in Λ and can be written as a convex combination

$$\tilde{\mu} = \sum_{i=1}^{k} a_i \tilde{\mu}_i \text{ with } \tilde{\mu}_i \in E_i(0).$$

We claim that $\tilde{\mu}(\partial \mathcal{M}_i) = 0$ for every $i = 1, \dots, k$. Thus the restriction of $\tilde{\mu}$ to each $\mathcal{M}_i$ coincides with $a_i\tilde{\mu}_i$. Now, by Lemma 6.2 the conditional probabilities of each $\tilde{\mu}_i$ along the plaques ξ_i^u are the reference measures. Thus, $\tilde{\mu}$ is a c-Gibbs u-state. Using [UVYY, Corollary 4.6], we get that $\tilde{\mu} = \mu$, as we wanted to prove.

To prove our claim we only need to check that the projection $\tilde{\nu} = \pi_*\tilde{\mu}$ is the Lebesgue measure on $\mathbb{T}^d$. Note that $\tilde{\nu}$ is an invariant probability measure for the linear Anosov map A, and

$$\tilde{\nu} = \sum_{i=1}^{k} a_i \tilde{\nu}_i$$

where each $\tilde{\nu}_i = \pi_*\tilde{\mu}_i$ is a probability measure on the Markov set $\mathcal{R}_i$. By the definition of the reference measures, the conditional probability of $\tilde{\nu}_i$ along every unstable plaque $W_{i,a}^u$ is the normalized Lebesgue measure.

Consider any i such that $a_i > 0$, and then let $\{\tilde{\nu}_a : a \in \mathcal{R}_i\}$ denote the conditional probabilities of $\tilde{\nu} \mid \mathcal{R}_i$ along the unstable plaques $W_{i,a}^u$. Since the Lebesgue measure vol is invariant and ergodic for A, the set of points $b \in \mathbb{T}^d$ such that

$$\frac{1}{n}\sum_{j=0}^{n-1} \delta_{A^j(b)} \to \mathrm{vol} \tag{24}$$

has full Lebesgue measure, and is s-saturated. Then, using also the fact that the stable foliation of A is absolutely continuous (linear, actually) we get that (24) holds for $\mathrm{vol}_{i,a}$-almost every $b \in W_{i,a}^u$ and every $a \in \mathcal{R}_i$. Consequently, (24) holds for a full $\tilde{\nu}_i$-subset of points in $\mathcal{R}_i$. Thus, it holds at $\tilde{\nu}_0$-almost every point, which implies that $\tilde{\nu} = \mathrm{vol}$, as claimed. □

6.2. **Proof of Theorem 6.1.** Denote $S_n\varphi = \sum_{j=0}^{n-1} \varphi \circ f^j$ for any $n \geq 1$ and any continuous function $\varphi : M \to \mathbb{R}$. The first step in the proof of the theorem is:

Lemma 6.6. *For any $\alpha > 0$ and any continuous function $\varphi : M \to \mathbb{R}$ with $\int \varphi \, d\mu \leq -\alpha$, there is $C_1 > 0$ such that*

$$\int_{\xi_i^u(x)} S_n\varphi \, d\nu_{i,x}^u \leq -n\frac{\alpha}{2} + C_1 \text{ for any } n \geq 1,\ x \in \mathcal{M}_i \cap \Lambda, \text{ and } i = 1, \dots, k,$$

Proof. Recall that $E(0)$ is convex, weak*-compact, and f_*-invariant, and μ is the unique f-invariant probability measure contained in it (Lemma 6.5). It follows that

$$\frac{1}{n}\sum_{j=0}^{n-1} f_*^j \zeta \to \mu$$

for any probability measure $\zeta \in E(0)$, and the convergence is uniform in ζ. In particular,

$$\frac{1}{n}\int_{\xi_i^u(x)} S_n\varphi \, d\nu_{i,x}^u \to \int_\Lambda \varphi \, d\mu$$

uniformly in x and i. This implies the claim of the lemma. □

Lemma 6.7. *For any $\varepsilon > 0$ and any continuous function $\varphi : M \to \mathbb{R}$, there exist $C > 0$ and $n_\varepsilon \geq 1$ such that*

$$|S_n\varphi(y) - S_n\varphi(z)| \leq n\varepsilon + C$$

for any $n \geq n_\varepsilon$, $y, z \in f^{-n}(\xi_i^u(x))$, $x \in \mathcal{M}_i \cap \Lambda$, and $i = 1, \dots, k$.

Proof. This is because φ is uniformly continuous, the diameter of $\xi_i^u(x)$ is uniformly bounded, and f^{-1} contracts strong-unstable leaves uniformly. □

The Markov property (3) implies that every $f^n(\xi_i^u(x))$ may be written as a (finite) union of strong-unstable plaques $\xi_{i_j}^u(x_j)$. Let

$$c_j = c_j(i, x, n) = \nu_{i,x}^u(f^{-n}(\xi_{i_j}^u(x_j))),$$

and note that $\sum_j c_j = 1$. Recall also (Remark 2.1) that, for any fixed n, there are only finitely many possible values for $c_j(i, x, n)$.

Corollary 6.8. *For any $\alpha > 0$ and any continuous function $\varphi : M \to \mathbb{R}$ with $\int \varphi \, d\mu \leq -\alpha$, there are $\alpha_1 > 0$ and $n_1 \geq 1$ such that*

$$\sum_j c_j \max_{f^{-n}(\xi_{i_j}^u(x_j))} S_n\varphi \leq -n\alpha_1,$$

for every $n \geq n_1$, $x \in \mathcal{M}_i \cap \Lambda$, and $i = 1, \dots, k$.

Proof. By Lemma 6.6 the average of $S_n\varphi$ is bounded above by $-n\alpha/2 + C_1$. By Lemma 6.7, its total oscillation is bounded by $n\varepsilon + C$. Fix $\varepsilon = \alpha_1 = \alpha/5$ and then take $n_1 > n_\varepsilon$ large enough that $(-n\alpha/2 + C_1) + (n\varepsilon + C) \leq -n\alpha_1$ Thus,

$$\sum_j c_j \max_{f^{-n}(\xi_{i_j}^u(x_j))} S_n\varphi \leq -n\alpha_1,$$

which clearly implies the claim. □

Lemma 6.9. *If Theorem 6.1 holds for some iterate f^l, $l \geq 1$ then it holds for f.*

Proof. Start by noting that the assumptions of the theorem hold for f^l if (and only if) they hold for f. Indeed, it is clear from the definition (7) that f has c-mostly contracting center if and only if f^l has c-mostly contracting center. Similarly, μ is a c-Gibbs u-state for f if and only if it is a c-Gibbs u-state for f^l. Thus (by part (4) of Proposition 2.4), the maps f and f^l have precisely the same measures of maximal u-entropy. Moreover, by Lemma 6.5 one has ergodicity with respect to any of the maps f and f^l.

Now, we check that the conclusion of the theorem holds for f if it holds for f^l. Indeed, given any continuous function $\varphi : M \to \mathbb{R}$, denote

$$S_n\varphi = \sum_{j=0}^{n-1} \varphi \circ f^j, \ \Sigma_n\varphi = \sum_{j=0}^{n-1} \varphi \circ f^{lj}, \text{ and } \Phi = \sum_{j=0}^{l-1} \varphi \circ f^j.$$

Note that $S_{nl}\varphi = \Sigma_n\Phi$ for every n. Since the theorem is assumed to hold for f^l, and Φ is a continuous function,

$$\mu\left(\left\{x \in M : \left|\frac{1}{m}\Sigma_m\Phi(x)\right| > \alpha\right\}\right)) \leq C_\alpha e^{-c_\alpha m} \text{ for every } m \geq 1. \tag{25}$$

Given $\varepsilon > 0$, let $x \in M$ and $n \geq 1$ be such that

$$\left|\frac{1}{n}S_n\varphi(x)\right| > \varepsilon. \tag{26}$$

Writing $n = ml + r$ with $0 \leq r < l$, we get

$$S_n\varphi(x) = S_{ml}\varphi(x) + S_r\varphi(f^{ml}(x)) = \Sigma_m\Phi(x) + S_r\varphi(f^{ml}(x))$$

Then

$$\begin{aligned}\left|\frac{1}{m}\Sigma_m\Phi(x)\right| &= \frac{n}{m}\left|\frac{1}{n}S_n\varphi(x)-\frac{1}{n}S_r\varphi(f^{ml}(x))\right| \\ &\geq \frac{n}{m}\left\{\left|\frac{1}{n}S_n\varphi(x)\right|-\frac{l}{n}\|\varphi\|_{C^0}\right\} \geq \frac{n}{m}\left\{\varepsilon-\frac{l}{n}\|\varphi\|_{C^0}\right\}.\end{aligned}$$

For $n \geq 2l\|\varphi_0\|/\varepsilon$ this implies that

$$\left|\frac{1}{m}\Sigma_m\Phi(x)\right| \geq \frac{l\varepsilon}{2}.$$

Then (25) gives that the measure of the set of points x as in (26) is bounded by

$$C_{l\varepsilon/2}e^{-c_{l\varepsilon/2}m}$$

which is bounded above by $\tilde{C}_\varepsilon e^{-\tilde{c}_\varepsilon n}$ for suitable choices of $\tilde{C}_\varepsilon$ and $\tilde{c}_\varepsilon$. The cases $n < 2l\|\varphi_0\|/\varepsilon$ are handled by increasing $\tilde{C}_\varepsilon$ if necessary. □

Remark 6.10. Once Theorem 6.1 is proved, it will follow that it holds also for every iterate f^l, $l \geq 1$ of a map f as in the statement. That is because the assumptions hold for f^l if they hold for f, as observed at the beginning of the proof of the previous lemma.

Up to replacing f with f^{n_1}, which is allowed by Lemma 6.9, it is no restriction to suppose that the integer n_1 in Corollary 6.8 is equal to 1. We do so in what follows.

Corollary 6.11. *For any $\alpha > 0$ and any continuous function $\varphi : M \to \mathbb{R}$ with $\int \varphi\, d\mu \leq -\alpha$, there are $s_1 > 0$ and $\theta_1 \in (0,1)$ such that*

$$\sum_j c_j \exp\left(s \max_{f^{-1}(\xi^u_{i_j}(x_j))} S_n\varphi\right) \leq \theta_1^s \text{ for every } s \in [0, s_1].$$

Proof. Consider the function

$$g : s \mapsto \log \sum_j c_j \exp\left(s \max_{f^{-1}(\xi^u_{i_j}(x_j))} S_n\varphi\right)$$

and observe that $g(0) = 0$,

$$g'(0) = \sum_j c_j \max_{f^{-1}(\xi^u_{i_j}(x_j))} S_n\varphi \leq -\alpha_1$$

and the second derivative g'' is bounded on $[0,1]$, uniformly in i and x (because the c_js take only finitely many values, and the number of terms in the sum is also uniformly bounded). □

Corollary 6.12. *For any $\alpha > 0$ and any continuous function $\varphi : M \to \mathbb{R}$ with $\int \varphi\, d\mu \leq -\alpha$, we have*

$$\sum_j c_j \exp\left(s_1 \max_{f^{-n}(\xi^u_{i_j}(x_j))} S_n\varphi\right) \leq \theta_1^{ns_1} \tag{27}$$

for any $n \geq 1$, $i = 1, \ldots, k$, and $x \in \Lambda \cap \mathcal{M}_i$.

Proof. The argument is by induction on n. The first step was done in Corollary 6.11. Assume that (27) hold for n. Write each $f(\xi^u_{i_j}(x_j))$ as a union of plaques $\xi^u_{i_j,m}(x_{j,m})$, and denote

$$b_{i_j,m} = \nu^u_{i_j,x_j}\left(f^{-1}(\xi^u_{i_j,m}(x_{j,m}))\right).$$

Here m varies on some finite set which depends on i_j and x_j, but whose cardinal is uniformly bounded. Keep in mind that $\sum_m b_{i_j,m} = 1$ for every i_j and x_j. Then

$$f^{n+1}(\xi_i^u(x)) = \bigcup_j \bigcup_m \xi_{i_j,m}^u(x_{j,m}).$$

Moreover,

$$\max_{f^{-(n+1)}(\xi_{i_j,m}^u(x_{j,m}))} S_{n+1}\Phi \leq \max_{f^{-n}(\xi_{i_j}^u(x_j))} S_n\varphi + \max_{f^{-1}(\xi_{i_j,m}^u(x_{j,m}))} \varphi$$

and so

$$\sum_j \sum_m c_j b_{i_j,m} \exp\left(s_1 \max_{f^{-(n+1)}(\xi_{i_j,m}^u(x_{j,m}))} S_{n+1}\varphi\right)$$
$$\leq \sum_j c_j \exp\left(s_1 \max_{f^{-n}(\xi_{i_j}^u(x_j))} S_n\varphi\right) \sum_m b_{i_j,m} \exp\left(s_1 \max_{f^{-1}(\xi_{i_j,m}^u(x_{j,m}))} \varphi\right).$$

By Corollary 6.11, the last factor is bounded by $\theta_1^{s_1}$. Thus, using the induction hypothesis,

$$\sum_j \sum_m c_j b_{i_j,m} exp\left(s_1 \max_{f^{-(n+1)}(\xi_{i_j,m}^u(x_{j,m}))} S_{n+1}\varphi\right) \leq \theta_1^{(n+1)s_1}$$

as we wanted to prove. □

We are ready to prove the large deviations property for (f, μ). In fact we prove a slightly more general estimate (28), valid for any probability measure $\zeta \in E(0)$.

Consider any continuous function $\varphi : M \to \mathbb{R}$ with $\int_M \varphi \, d\mu = 0$. For any $\alpha > 0$, define $\varphi_\alpha = \varphi - \alpha$. By Lemma 6.6 and Corollary 6.12 there is $\theta_\alpha \in (0, 1)$ such that

$$\sum_j c_j \exp\left(s_1 \max_{f^{-n}(\xi_{i_j}^u(x_j))} S_n\varphi_\alpha\right) \leq \theta_\alpha^n,$$

for every $x \in \mathcal{M}_i \cap \Lambda$ and $i = 1, \ldots, k$. Clearly, $S_n\varphi_\alpha = S_n\varphi - n\alpha$. Thus, the previous inequality implies that

$$\int_\Lambda \exp\left(s_1(S_n\varphi - n\alpha)\right) d\nu_{i,x}^u \leq \theta_\alpha^n$$

for every $x \in \mathcal{M}_i \cap \Lambda$ and $i = 1, \ldots, k$. In view of the definition of $E(0)$, the inequality extends to every $\zeta \in E(0)$:

$$\int_\Lambda \exp\left(s_1(S_n\varphi - n\alpha)\right) d\zeta \leq \theta_\alpha^n.$$

Then, by the Chebyshev inequality,

$$\zeta\left(\{x : S_n\varphi \geq n\alpha\}\right) \leq \theta_\alpha^n.$$

Applying the same argument to $-\varphi$, we also get that $\zeta\left(\{x : S_n\varphi \leq -n\alpha\}\right) \leq \theta_\alpha^n$. Thus,

$$\zeta\left(\{x : |S_n\varphi| \geq n\alpha\}\right) \leq 2\theta_\alpha^n \tag{28}$$

for any $\zeta \in E(0)$. In particular, this holds for $\zeta = \mu$, which proves Theorem 6.1.

7. Proof of the coupling lemma

Here we prove Lemma 5.1. The proof is based on the coupling argument of Young [You99], in the form developed by Dolgopyat [Dol00, Sections 6–9] for diffeomorphisms with mostly contracting center. Throughout, we keep the assumptions of Theorem B. In particular, $\Lambda = \operatorname{supp} \mu$ is taken to be connected and, consequently (by [UVYY, Theorem A])), u-minimal.

7.1. **Preparing the coupling argument.** We start with the following fact:

Proposition 7.1. *There are* $n_0 \geq 1$ *and* $\lambda_0 < 0$ *such that*

$$\int_{\xi_i^u(x)} \frac{1}{n_0} \log \|Df^{n_0} \mid_{E^{cs}} \| \, d\nu_{i,x}^u \leq \lambda_0 \tag{29}$$

for every $x \in \mathcal{M}_i \cap \Lambda$ *and every* $i = 1, \ldots, k$.

Proof. Since f is assumed to have c-mostly contracting center, Proposition 2.5 ensures that the center-stable Lyapunov exponents of μ are all negative. Let $\lambda^{cs} < 0$ denote the largest of these exponents. Then

$$\int_\Lambda \frac{1}{n} \log \|Df^n \mid_{E^{cs}} \| \, d\mu < \frac{\lambda^{cs}}{2}. \tag{30}$$

for every large n. Take $\lambda_0 = \lambda^{cs}/2$ and let n be fixed such that (30) holds. Consider any sequences $i_j \in \{1, \ldots, k\}$, $x_j \in \mathcal{M}_{i_j} \cap \Lambda$ and $m_j \to \infty$. Clearly,

$$\begin{aligned}
&\int_{\xi_{i_j}^u(x_j)} \frac{1}{m_j n} \log \|Df^{m_j n} \mid_{E^{cs}} \| \, d\nu_{i_j,x_j}^u \\
&\leq \frac{1}{m_j} \int_{\xi_{i_j}^u(x_j)} \frac{1}{n} \sum_{i=0}^{m_j-1} \log \|Df^n \mid_{E^{cs}} \circ f^{in}\| \, d\nu_{i_j,x_j}^u \qquad (31)\\
&= \int_{\xi_{i_n}^u(x_n)} \frac{1}{n} \log \|Df^n \mid_{E^{cs}} \| \, d\left(\frac{1}{m_j} \sum_{i=0}^{m_j-1} (f^{in})_* \nu_{i_j,x_j}^u\right)
\end{aligned}$$

Observe that

$$\frac{1}{m_j} \sum_{i=0}^{m_j-1} (f^{in})_* \nu_{i_j,x_j}^u \to \mu$$

as j goes to infinity. Indeed, [UVYY, Proposition 4.1] gives that every accumulation point of this sequence belongs to $E(0)$ and is an invariant probability measure, and so Lemma 6.5 implies that every such accumulation point coincides with μ. In view of (31), this implies that every accumulation point of

$$\int_{\xi_{i_j}^u(x_j)} \frac{1}{m_j n} \log \|Df^{m_j n} \mid_{E^{cs}} \| \, d\nu_{i_j,x_j}^u$$

when j goes to infinity is bounded above by

$$\int_\Lambda \frac{1}{n} \log \|Df^n \mid_{E^{cs}} \| \, d\mu < \lambda_0.$$

This proves that there exists $m \geq 1$ such that

$$\int_{\xi_i^u(x)} \frac{1}{mn} \log \|Df^{mn} \mid_{E^{cs}} \| \, d\nu_{i,x}^u < \lambda_0$$

for every $x \in \mathcal{M}_i \cap \Lambda$ and every $i = 1, \ldots, k$. Take $n_0 = mn$. □

It is not difficult to check that if the conclusion of Lemma 5.1 holds for the iterate f^{n_0}, with maps $\tau_0 : Y_1 \to Y_2$ and $R_0 : Y_1 \to \mathbb{N}$, then the corresponding statement holds for the original map f as well, with functions $\tau = \tau_0$ and $R = n_0 R_0$, up to suitable changes of the constants C_1, C_2 and ρ_1, ρ_2. Thus, it is no restriction to assume that $n_0 = 1$, and so

$$\int_\Lambda \log \|Df \mid_{E^{cs}} \| \, d\nu_{i,x}^u < \lambda_0 < 0 \tag{32}$$

for all $x \in \mathcal{M}_i \cap \Lambda$ and $i = 1, \ldots, k$. We do that in the following.

Write each $f^n(\xi_i^u(x))$, $n \geq 1$ as a finite union of strong-unstable plaques $\xi_{i_j}^u(x_j)$, and then denote

$$c_j = c_j(i, x, n) = \nu_{i,x}^u(f^{-n}(\xi_{i_j}^u(x_j))).$$

Applying Corollary 6.12 to the function $\Phi = \log \|Df \mid_{E^{cs}} \|$, we find constants $s_1 > 0$ and $\theta_1 \in (0, 1)$ such that

$$\sum_j c_j \prod_{t=0}^{n-1} \max_{f^{t-n}(\xi_i^u(x_j))} \|Df \mid_{E^{cs}} \|^{s_1} \leq \theta_1^{s_1 n}. \tag{33}$$

for any $n \geq 1$.

Fix $\lambda < 0$ such that

$$\lambda \geq \max\{\lambda_0/2, \log \theta_1/2\}. \tag{34}$$

Let $K > 1$ and denote by $U_i^n(x) \subset \xi_i^u(x)$ the union of of all the pre-images $f^{-n}(\xi_{i_j}^u(x_j))$ for which

$$\prod_{t=0}^{n-1} \max_{f^{t-n}(\xi_i^u(x_j))} \|Df \mid_{E^{cs}} \| > K e^{\lambda n}. \tag{35}$$

From (33) and (35) we get the Chebyshev-type inequality

$$\nu_{i,x}^u \left(U_i^n(x)\right) K^{s_1} e^{\lambda s_1 n} \leq \theta_1^{s_1 n}.$$

In view of the choice of λ, this shows that

$$\nu_{i,x}^u \left(U_i^n(x)\right) \leq K^{-s_1} e^{\lambda s_1 n} \tag{36}$$

for every $n \geq 1$, $x \in \mathcal{M}_i \cap \Lambda$ and $i = 1, \ldots, k$. Up to fixing $K > 1$ sufficiently large, the latter implies that there exists $q_1 < 1$ such that

$$\nu_{i,x}^u \left(U_i(x)\right) \leq q_1 \tag{37}$$

for every $x \in \mathcal{M}_i \cap \Lambda$ and $i = 1, \ldots, k$, where $U_i(x)$ denotes the union of $U_i^n(x)$ over all $n \geq 1$.

We close this section with the following useful fact, which we quote from Alves, Bonatti, Viana [ABV00, Lemma 2.7], see also Dolgopyat [Dol00, Lemma 8.1]. Let $\varepsilon > 0$ be fixed such that

$$d(y, z) \leq \varepsilon \quad \Rightarrow \quad \|(Df \mid_{E^{cs}})(y)\| \leq e^{-\lambda/2} \|(Df \mid_{E^{cs}})(z)\|.$$

Lemma 7.2. *If $x \in M$ and $n \geq 1$ are such that $\|(Df^j \mid_{E^{cs}})(x)\| \leq K e^{-\lambda j}$ for $j = 1, \ldots, n$, then*

$$f^j(\mathcal{F}_\varepsilon^{cs}(x)) \subset \mathcal{F}_{r_j}^{cs}(f^j(x))$$

for every $0 \leq j \leq n$, where $r_j = K\varepsilon e^{-\lambda j/2}$.

After these preparations, we move to prove Lemma 5.1. Let $\mathcal{Y}$ be the set of rectangles $Y = \xi_i^u(x) \times J$ with $x \in \mathcal{M}_i \cap \Lambda$, $i = 1, \cdots, k$ and $J \subset [0, 1]$, endowed with the measures $m_i = \nu_{i,x}^u \times dt$. Write $f(x, t) = (f(x), t)$. Recall that in the statement of Lemma 5.1 the sets $Y_1, Y_2 \in \mathcal{Y}$ and the measures m_1, m_2 were taken to satisfy $m_1(Y_1) = m_2(Y_2)$. We are going to describe an algorithmic construction of maps τ and R as in the statement of the lemma.

This algorithm will be presented in a recursive form. In the first run (to be detailed in Section 7.2), we will define a stopping time $s(y)$ for the points $y \in Y_1$ where the coupling map τ has not yet been defined, in such a way that the sets

$$P_j^n = \{y \in Y_j : s(y) = n\}, \quad j = 1, 2 \tag{38}$$

are of the form $f^{-n}(\bigcup_m Y_{j,n,m})$, where $Y_{j,n,m} = \xi_{i_{n,m}}^u(x_{j,n,m}) \times I_{j,n,m}$ are elements of $\mathcal{Y}$ satisfying $m_1(Y_{1,n,m}) = m_2(Y_{2,n,m})$. Finally, we will set $P_j^\infty = Y_j \setminus \cup_n P_j^n$ for

$j = 1, 2$, and we will define the function R on the set P_1^∞, and the map τ from P_1^∞ and P_2^∞.

The purpose of the inductive runs of the algorithm is to extend the domains of R and τ successively to include almost every point of Y. This is actually similar to the first run, and will be detailed in Section 7.3.

7.2. **First run of the algorithm.** Let us now detail the first run of the algorithm. Recall that $\mathcal{H}^{cs} : \xi_i^u(y_1) \to \xi_i^u(y_2)$ denotes the center-stable holonomy between the strong-unstable plaques of points y_1 and y_2 in the same Markov set $\mathcal{M}_i$. Denote by d_{cs} the distance along center-stable leaves.

Lemma 7.3. *Given $\varepsilon > 0$ as in Lemma 7.2, there is $n_0 \geq 1$ such that for any two points $x_1 \in \Lambda \cap \mathcal{M}_{i_1}$ and $x_2 \in \Lambda \cap \mathcal{M}_{i_2}$ the iterates $f^{n_0}(\xi_{i_1}^u(x_1))$ and $f^{n_0}(\xi_{i_2}^u(x_2))$ contain strong-unstable plaques $\xi_1^u(y_1)$ and $\xi_1^u(y_2)$ inside the Markov domain $\mathcal{M}_1$ and*

$$d_{cs}\left(w, \mathcal{H}^{cs}_{y_1,y_2}(w)\right) \leq \varepsilon \text{ for any } w \in \xi_1^u(y_1).$$

Proof. Since μ is a c-Gibbs u-state, its push-forward under the map $\pi : M \to \mathbb{T}^d$ is the Lebesgue measure on the torus (see [UVYY, Corollary 3.5]). In particular, the image of Λ under π is the whole $\mathbb{T}^d$. Fix any point q in the interior of the Markov set $\mathcal{R}_1 \subset \mathbb{T}^d$, and let $z \in \pi^{-1}(q) \cap \Lambda$. Note that z is in the interior of $\mathcal{M}_1$. Fix a neighborhood V of z contained in $\mathcal{M}_1$ small enough that the distance along center-stable leaves

$$d_{cs}\left(w, \mathcal{H}^{cs}_{w_1,w_2}(w)\right) \leq \varepsilon \text{ for any } w_1, w_2 \in V \text{ and any } w \in \xi_1^u(w_1).$$

Since Λ is u-minimal, there is $n_0 \geq 1$ such that the iterates $f^{n_0}(\xi_{i_1}^u(x_1))$ and $f^{n_0}(\xi_{i_2}^u(x_2))$ of any $x_1 \in \Lambda \cap \mathcal{M}_{i_1}$ and $x_2 \in \Lambda \cap \mathcal{M}_{i_2}$ intersect V. Just take $y_1 \in f^{n_0}(\xi_{i_1}^u(x_1)) \cap V$ and $y_2 \in f^{n_0}(\xi_{i_2}^u(x_2)) \cap V$. □

Consider $Y_1 = \xi_{i_1}^u(x_1) \times [0, a]$ and $Y_2 = \xi_{i_2}^u(x_2) \times [0, a]$ for any $x_1 \in \Lambda \cap \mathcal{M}_{i_1}$, $x_2 \in \Lambda \cap \mathcal{M}_{i_2}$, and $a \in [0, 1]$. Take y_1, y_2 as in Lemma 7.3. For $j = 1, 2$, define

$$\overline{c}_j = \nu^u_{i_j,x_j}(f^{-n_0}(\xi_1^u(y_j))) \tag{39}$$

for $j = 1, 2$. It follows from Remark 2.1 that these $\overline{c}_j$ take only finitely many values. Define $\overline{Y}_j = \xi_1^u(y_j) \times [0, \overline{t}_j]$, where

$$(\overline{t}_1, \overline{t}_2) = \begin{cases} (a\overline{c}_2/\overline{c}_1, a) & \text{if } \overline{c}_2 \leq \overline{c}_1 \\ (a, a\overline{c}_1/\overline{c}_2) & \text{if } \overline{c}_1 \leq \overline{c}_2 \end{cases} \tag{40}$$

This choice ensures that $f^{-n_0}(\overline{Y}_j) \subset Y_j$ for $j = 1, 2$, and

$$m_1\left(f^{-n_0}(\overline{Y}_1)\right) = m_2\left(f^{-n_0}(\overline{Y}_2)\right). \tag{41}$$

We denote this value as b. On the complements $P_j^{n_0} = Y_j \setminus f^{-n_0}(\overline{Y}_j)$, we define the stopping time $s(y, t) = n_0$.

Let us check that the $P_j^{n_0}$ constructed in this way are indeed of the form described in (38). Due to the Markov property, each Y_j is a union of finitely many sets of the form

$$f^{-n_0}(Z) = \eta \times [0, a] \text{ with } Z \in \mathcal{Y}.$$

The total m_j-measure is equal to a, of course. By construction, $f^{-n_0}(\overline{Y}_j)$ is a set of the form $\eta \times [0, \overline{t}_j]$ such that the two m_j-measures are the same. Thus, $P_j^{n_0} = Y_j \setminus f^{-n_0}(\overline{Y}_j)$ is a finite union of sets of the form $\eta \times J$, where either $J = [0, a]$ or $J = (\overline{t}_j, a]$ (note that one of the $\overline{t}_j$ is equal to a, and so in the latter the corresponding J is empty). It is clear that $m_1(P_1^{n_0}) = m_2(P_2^{n_0})$. Thus, up to cutting (in the vertical direction only) each $\eta \times J$ into finitely many pieces, in a suitable way, we may write

$$P_j^{n_0} = \bigcup_m f^{-n_0}(Y_{j,n_0,m}) \tag{42}$$

with $Y_{j,n_0,m} \in \mathcal{Y}$ satisfying $m_1(Y_{1,n_0,m}) = m_2(Y_{2,n_0,m})$ for every m.

Next, for $n > n_0$ we define

$$P_1^n = f^{-n_0}\left(V^n(y_1) \times [0, \bar{t}_1]\right) \text{ and } P_2^n = f^{-n_0}\left(H^{cs}_{y_1,y_2}(V^n(y_1)) \times [0, \bar{t}_2]\right), \tag{43}$$

where

$$V^n(y_1) = U_1^{n-n_0}(y_1) \setminus \bigcup_{m=0}^{n-n_0-1} U_1^m(y_1) \tag{44}$$

and $U_1^m(y_j)$ is as defined in (35). It is clear that the $V^n(y_1)$, $n > n_0$ are pairwise disjoint and their union coincides with

$$U_1(y_1) = \bigcup_{n \geq 1} U_1^n(y_1).$$

Thus the P_j^n, $n > n_0$ are pairwise disjoint subsets of Y_j, all with the same height, and

$$\bigcup_{n>n_0} P_1^n = f^{-n_0}\left(\bigcup_{n>n_0} V^n(y_1) \times [0, \bar{t}_1]\right) = f^{-n_0}\left(U_1(y_1) \times [0, \bar{t}_1]\right).$$

We also define

$$P_j^\infty = Y_j \setminus \bigcup_{n \geq n_0} P_j^n$$

for $j = 1, 2$. Notice that

$$\begin{aligned} P_1^\infty = Y_1 \setminus \bigcup_{n \geq n_0} P_1^n &= (Y_1 \setminus P_{n_0}) \setminus \bigcup_{n>n_0} P_1^n \\ &= f^{-n_0}(\overline{Y}_1) \setminus f^{-n_0}\left(U_1(y_1) \times [0, \bar{t}_1]\right) \\ &= f^{-n_0}\left((\xi_1^u(y_1) \setminus U_1(y_1)) \times [0, \bar{t}_1]\right), \end{aligned} \tag{45}$$

and, similarly,

$$P_2^\infty = f^{-n_0}\left(H^{cs}_{y_1,y_2}\big((\xi_1^u(y_1) \setminus U_1(y_1))\big) \times [0, \bar{t}_2]\right). \tag{46}$$

A few other simple facts about the sequences P_j^n are collected in the next lemma. Let K, λ, s_1, q_1, and b be fixed as in (35), (34), (33), (37), and (41), respectively.

Lemma 7.4. *For every $n > n_0$ we have:*

(1) $m_1(P_1^n) = m_2(P_2^n) \leq b e^{\lambda s_1 (n-n_0)}$;
(2) $m_1(P_1^\infty) = m_2(P_2^\infty) \geq b(1 - q_1)$;
(3) $f^n(P_j^n)$ may be written as a union $\bigcup_k Y_{j,n,k}$ of elements $Y_{j,n,k}$ of $\mathcal{Y}$ with

$$m_1(f^{-n}(Y_{1,n,k})) = m_2(f^{-n}(Y_{2,n,k})); \tag{47}$$

Proof. By the definitions of $m_1 = \nu^u_{1,x_1} \times dt$ and P_1^n (see (43)),

$$m_1(P_1^n) = m_1(f^{-n_0}(V^n(y_1) \times [0, \bar{t}_1])) = (\nu^u_{i_1,x_1} \times dt)(f^{-n_0}(V^n(y_1) \times [0, \bar{t}_1])).$$

Recall that $\overline{Y}_1 = \xi_1^u(y_1) \times [0, \bar{t}_1]$ and $m_1(f^{-n_0}(\overline{Y}_1)) = b$, by (41). Thus

$$\begin{aligned} \frac{m_1(P_1^n)}{b} = \frac{m_1(P_1^n)}{m_1(f^{-n_0}(\overline{Y}_1))} &= \frac{(\nu^u_{i_1,x_1} \times dt)(f^{-n_0}(V^n(y_1) \times [0, \bar{t}_1]))}{(\nu^u_{i_1,x_1} \times dt)(f^{-n_0}(\overline{Y}_1))} \\ &= \frac{\nu^u_{i_1,x_1}(f^{-n_0}(V^n(y_1)))}{\nu^u_{i_1,x_1}(f^{-n_0}(\xi_1^u(y_1)))}. \end{aligned}$$

By the definition (44), $V^n(y_1))$ is a subset of $\xi_1^u(y_1)$. Thus, as the reference measures have constant Jacobian for f,

$$\frac{m_1(P_1^n)}{b} = \frac{\nu_{1,y_1}^u(V^n(y_1))}{\nu_{1,y_1}^u(\xi_1^u(y_1))} = \nu_{1,y_1}^u(V^n(y_1)). \tag{48}$$

Similarly,

$$\frac{m_2(P_2^n)}{b} = \nu_{1,y_2}^u(H_{y_1,y_2}^{cs}(V^n(y_1)))$$

As the reference measures are invariant under center-stable holonomies, by (5), this implies that $m_2(P_2^n) = m_1(P_1^n)$. It is also clear from (44) that $V^n(y_1))$ is contained in $U_1^{n-n_0}$. Thus, (48) together with (36) yield

$$m_1(P_1^n) \le b\nu_{1,y_1}^u(U_1^{n-n_0}(y_1)) \le bK^{-s_1}e^{\lambda s_1(n-n_0)} \le be^{\lambda s_1(n-n_0)}.$$

This completes the proof of claim (1).

Next we prove (2). Since

$$m_j(P_j^\infty) = m_j(Y_j) - \sum_{j=n_0}^{\infty} m_j(P_j^n) = b - \sum_{j=n_0}^{\infty} m_j(P_j^n),$$

it follows from the previous remarks that $m_1(P_1^\infty) = m_2(P_2^\infty)$. By definition, $P_1^\infty = f^{-n_0}(\xi_1^u(y_1) \setminus U_1(y_1)) \times [0, \bar{t}_1]$. Thus, similarly to (48),

$$\frac{m_1(P_1^\infty)}{b} = \nu_{1,y_1}^u\left(\xi_1^u(y_1) \setminus U_1(y_1)\right).$$

By (37), this yields

$$m_1(P_1^\infty) \ge b(1-q_1).$$

This proves claim (2).

By the definition (35), $U_1^{n-n_0}(y_1)$ consists of domains that are mapped by f^{n-n_0} to entire strong-unstable plaques. By the Markov property, it follows that the image of $U_1^m(y_1)$ under f^{n-n_0} consists of entire strong-unstable plaques for every $1 \le m \le n-n_0$. Therefore, the set $V^n(y_1)$ defined in (44) is also a union of domains whose images under f^{n-n_0} are entire plaques. Using the Markov property once more, we see that the same is true for the image $H_{y_1,y_2}^{cs}(V^n(y_1))$ under the center-stable holonomy. Thus, both P_j^n, $j = 1, 2$ may be written as unions of sets of the form $f^{-n}(Y_{j,n,m})$, where $Y_{j,n,m}$ is an element of $\mathcal{Y}$ with height $\bar{t}_j$. Moreover, the images $f^{r-n}(Y_{1,n,m})$ and $f^{r-n}(Y_{1,n,m})$ are in the same Markov domain for each $r \in \{n_0, \ldots, n\}$, and the center-stable holonomy induces a bijection between them.

We claim that $m_1(f^{-n}(Y_1, n, m)) = m_2(f^{-n}(Y_2, n, m))$ for every m. To see this, write

$$f^{-n}(Y_{j,n,m}) = Z_{j,n,m} \times [0, \bar{t}_j] \text{ with } Z_{j,n,m} \subset f^{-n_0}(\xi_1^u(y_j)).$$

Then the claim may be rephrased as

$$\nu_{i_1,x_1}^u(Z_{1,n,m})\,\bar{t}_1 = \nu_{i_2,x_2}^u(Z_{2,n,m})\,\bar{t}_2, \tag{49}$$

Using the definition (39), together with the fact that the Jacobians of the reference measures are locally constant, we find that

$$\frac{\nu_{i_j,x_j}^u(Z_{j,n,m})}{\bar{c}_j} = \frac{\nu_{i_j,x_j}^u(Z_{j,n,m})}{\nu_{i_j,x_j}^u(f^{-n_0}(\xi_1^u(y_j)))} = \frac{\nu_{1,y_j}^u(f^{n_0}(Z_{j,n,m}))}{\nu_{1,y_j}^u(\xi_1^u(y_j))} = \nu_{1,y_j}^u(f^{n_0}(Z_{j,n,m})).$$

Since the reference measures ν_{1,y_1}^u and ν_{1,y_2}^u are mapped to one another by the center-stable holonomy H_{y_1,y_2}^{cs}, we also have that

$$\nu_{1,y_1}^u(f^{n_0}(Z_{1,n,m})) = \nu_{1,y_2}^u(f^{n_0}(Z_{2,n,m})).$$

It follows that

$$\nu_{i_1,x_1}^u(Z_{1,n,m})\,\bar{c}_2 = \nu_{i_2,x_2}^u(Z_{2,n,m})\,\bar{c}_1.$$

This gives (49), because the definition (40) is such that $\bar{c}_1\bar{t}_2 = \bar{c}_2\bar{t}_1$. This finishes the proof of claim (3). □

At this point, we define $\tau : P_1^\infty \to P_2^\infty$ and $R : P_1^\infty \to \mathbb{N}$ in the following way (keep the expressions (45) and (46) in mind). For any $(x,t) \in P_1^\infty$, let

$$\tau(x,t) = \left(y, \frac{\bar{c}_1}{\bar{c}_2} t\right) \text{ and } R(x,t) = n_0, \tag{50}$$

where $y \in \xi^u_{i_2}(x_2)$ is defined by

$$y = f^{-n_0} \circ H^{cs}_{y_1,y_2} \circ f^{n_0}(x). \tag{51}$$

Let us check the properties in Lemma 5.1 are indeed satisfied at this stage:

Lemma 7.5. *Let $(y,s) = \tau(x,t)$ be as in* (50) *and* (51)*, and r_n be as defined in Lemma 7.2. Then*

(1) $d(f^n(x), f^n(y)) \le r_{n-n_0}$ for any $n \ge n_0$;
(2) τ maps m_1 restricted to P_1^∞ to m_2 restricted to P_2^∞.

Proof. By construction, $f^{n_0}(x) \in \xi^u_1(y_1) \setminus U_1(y_1)$. By the definition (35), it follows $f^{n_0}(x)$ satisfies the assumption of Lemma 7.2 for every positive iterate. By Lemma 7.3, our choice of y_1 and y_2 ensures that $f^{n_0}(y) = H^{cs}_{y_1,y_2}(f^{n_0}(x))$ belongs to $\mathcal{F}^{cs}_\varepsilon(f^{n_0}(x))$. Now claim (1) of the present lemma is contained in the conclusion of Lemma 7.2.

Next, we claim that the Jacobian of τ with respect to the measures m_1 and m_2 is constant. Since $m_1(P_1^\infty) = m_2(P_2^\infty)$, it follows that the Jacobian is actually equal to 1, which is precisely the content of (2). To prove the claim, observe that the $m_j = \nu^u_{i_j,x_j} \times dt$, $j = 1,2$ are product measures, and τ is a product map. The Jacobian of the first-variable map $x \mapsto y$ with respect to the reference measures is constant, since the maps f^{n_0} and f^{-n_0} have locally constant Jacobians, by (6), and the Jacobian of the holonomy map $H^{cs}_{y_1,y_2}$ is constant equal to 1, by (5). The Jacobian of the second variable map $t \mapsto s$, with respect to the Lebesgue measure dt, is clearly also constant. Thus, the overall Jacobian of τ is constant, as claimed. □

This finishes the first stage of the coupling algorithm. At this stage, the coupling map τ is defined between the P_1^∞ and P_2^∞, and the function R is defined on P_1^∞.

7.3. **Inductive set of the algorithm.** Next, we want to extend the definitions of τ and R to (full measure subsets of) the complements $Y_j \setminus P_j^\infty$. This will be done recursively, in the following way.

For each $h \ge 1$, we denote by T^h the subset of Y_1 where τ and R are still undefined at the end of stage h. Thus $T^1 = Y_1 \setminus P_1^\infty = \cup_n P_1^n$. By induction, we may assume that there are sets

$$T^h = \bigcup P_1^{N_h}, \qquad N_h = (n_1, \dots, n_h)$$

where each $P_j^{N_h}$, $j = 1,2$ is itself a union of sets of the form

$$f^{-|N_h|}(Y_{j,N_h,m}), \qquad |N_h| = n_1 + \dots + n_h$$

with $Y_{j,N_h,m} \in \mathcal{Y}$ for $n_1, \dots, n_h \ge n_0$, and

$$m_1\left(f^{-|N_h|}(Y_{1,N_h,m})\right) = m_2\left(f^{-|N_h|}(Y_{2,N_h,m})\right).$$

Applying the first run (Lemma 7.4) of the algorithm to each $Y_{j,N_h,m}$, we find subsets $P^\infty_{j,N_h,m}$ of Y_j and measure-preserving maps

$$\tau_{N_h,m} : P^\infty_{1,N_h,m} \to P^\infty_{2,N_h,m}$$

as in the previous section. Then we extend τ and R to each

$$f^{-|N_h|}(P^\infty_{1,N_h,m}) \subset f^{-|N_h|}(Y_{j,N_h,m}) \subset T^h$$

through

$$\tau = f^{-|N_h|} \circ \tau_{N_h,m} \circ f^{N_h} \text{ and } R = |N_h| + n_0.$$

A key point is that, according to part (2) of Lemma 7.4, each $P^\infty_{1,N_h,m}$ contains a fraction $\geq 1 - q_1$ of the measure of $Y_{j,N_h,m}$. Moreover, the proportion is preserved under the backward image, because the map $f^{n_1+\cdots+n_h}$ has constant Jacobian. Thus, the measure of the set

$$T^{h+1} = T^h \setminus \bigcup_{N_h,m} f^{-|N_h|}(P^\infty_{1,N_h,m})$$

satisfies

$$m_1(T^{h+1}) \leq q_1 m_1(T^h). \tag{52}$$

As a direct application of Lemma 7.5, we get:

Corollary 7.6. *Let $(y, s) = \tau(x, t)$ for $(x, t) \in P^\infty_{1,N_h,m}$, and let r_n be as in Lemma 7.2. Then*

1. *$d(f^n(x), f^n(y)) \leq r_{n-R((x,t))}$ for any $n \geq R(x, t)$;*
2. *τ maps the measure m_1 restricted to $P^\infty_{1,N_h,m}$ to the measure m_2 restricted to $P^\infty_{2,N_h,m}$.*

The construction in the previous section also gives that

$$\bigcup_m Y_{j,N_h,m} \setminus P^\infty_{j,N_h,m} = \bigcup_{n'} f^{|N_h|}(P_j^{N_h,n'})$$

such that each $P_j^{N_h,n'}$, $j = 1, 2$ is a union of sets of the form

$$f^{-|N_h|-n'}(Y_{j,N_h,n',m'})$$

with $Y_{j,N_h,n',m'} \in \mathcal{Y}$ for $n' \geq n_0$, and

$$m_1\left(f^{-|N_h|-n'}(Y_{1,N_h,n',m'})\right) = m_2\left(f^{-|N_h|-n'}(Y_{2,N_h,n',m'})\right).$$

Thus we recover the recursive assumptions for the set T_{h+1}.

From (52) we get that τ and R are eventually defined at m_1-almost every point of Y_1. Part (1) of Lemma 5.1 is given by Lemma 7.5 and Corollary 7.6. We are left to checking part (2) of the lemma.

Recall that, at each stage h the function R is defined by $R = |N_{h-1}| + n_0$. Fix some small $\delta > 0$. For each n, write the set $\{R = n\}$ as the disjoint union of two subsets, depending on whether $h \leq \delta n$ or $h > \delta n$. It is clear that the latter subset (corresponding to $h > \delta n$), is contained in $T_{\lfloor \delta n \rfloor}$. Hence, by (52), its m_1-measure is bounded by

$$m_1(T_{\lfloor \delta n \rfloor}) \leq q_1^{\lfloor \delta n \rfloor}. \tag{53}$$

On the other hand, the m_1-measure of the former subset (corresponding to $h \leq \delta n$) is given by

$$\sum_{\substack{h,(k_1,\cdots,k_{h-1}) \\ k_1+\cdots+k_{h-1}+n_0=n \\ k_1,\ldots,k_{h-1}\geq n_0}} m_1(\{n_i = k_i\}) \leq \sum_{\substack{h,(k_1,\cdots,k_{h-1}) \\ k_1+\cdots+k_{h-1}+n_0=n \\ k_1,\ldots,k_{h-1}\geq n_0}} \prod_{i=1}^{h-1} b e^{\lambda s_1 (k_i - n_0)} \tag{54}$$

This inequality follows from Lemma 7.4 applied at each run $i = 1, \ldots, h$, together with the observation that the pull-back map $f^{-(k_1+\cdots+k_{i-1})}$ has constant Jacobian. The constant $b \in (0, 1)$ was defined in (41).

For each h, the number of terms in the sum is bounded above by

$$\frac{(n-(h+1)n_0+h-1)!}{(n-(h+1)n_0)!(h-1)!} \le \frac{(n+h-1)!}{n!(h-1)!} \approx \left(1+\frac{h}{n}\right)^n \left(1+\frac{n}{h}\right)^h = \left(\left(1+\frac{h}{n}\right)\left(1+\frac{n}{h}\right)^{\frac{h}{n}}\right)^n \tag{55}$$

(check [BV00, Corollary 6.7] for a similar estimate using Stirling's formula). Recall that we are considering $h \le \delta n$, and observe that $(1+1/x)^x \to 1$ when $x \to 0$. Thus we see that, given any $\varepsilon > 0$, the right hand side of (55) is bounded by $Ce^{\varepsilon n}$ if δ is chosen small enough, where C is an absolute constant. From this and (54) we get that the m_1-measure of the subset corresponding to $h \le \delta n$ is bounded above by

$$Ce^{\varepsilon n}(\delta n)e^{\lambda s_1 n}. \tag{56}$$

Combining (53) and (56), and keeping in mind that $q_1 < 1$ and $\lambda < 0$, we get that

$$m_1(\{R = n\}) \le q_1^{\lfloor \delta n \rfloor} + C\delta n e^{(\lambda s_1 + \varepsilon)n}$$

decays exponentially fast with n, as long as we choose ε small enough. Then, clearly, $m_1(\{R > n\})$ also decays exponentially fast with n, as claimed in part (2) of Lemma 5.1.

This completes the proof of Lemma 5.1.

8. Decay of correlations

Let $f : M \to M$ be as before and μ be any f-invariant probability measure. We say that (f, μ) has *exponential decay of correlations* for Hölder observables if for any $\gamma \in (0, 1]$ there exists $\tau < 1$ such that for all γ-Hölder functions $\varphi, \psi : M \to \mathbb{R}$ there exists $K(\varphi, \psi) > 0$ such that

$$\left| \int (\varphi \circ f^n)\psi \, d\mu - \int \varphi \, d\mu \int \psi \, d\mu \right| \le K(\varphi, \psi)\tau^n \text{ for every } n \ge 1.$$

In this section we prove

Theorem 8.1. *Let $f : M \to M$ be a partially hyperbolic diffeomorphism that factors over Anosov and has c-mostly contracting center. Let μ be an ergodic measure of maximal u-entropy whose support* $\operatorname{supp} \mu$ *is connected. Then (f, μ) has exponential decay of correlations for Hölder observables.*

Let γ be a positive number. We denote by $C^\gamma(M)$ the Banach space of γ-Hölder functions $\varphi : M \to \mathbb{R}$ with the norm

$$\|\varphi\|_\gamma = \sup_{x \in M} |\varphi(x)| + \sup_{x_1 \ne x_2} \frac{|\varphi(x_1) - \varphi(x_2)|}{d(x_1, x_2)^\gamma}.$$

We use a similar notation $\|\zeta\|_\gamma$ to denote the operator norm of an element of the dual space $(C^\gamma(M))^*$, that is, a linear functional $\zeta : C^\gamma(M) \to \mathbb{R}$. Every probability measure on M may be viewed as an element of this dual space, and we will often do that in what follows.

The push-forward operator f_* extends to a linear operator on the whole space $(C^\gamma(M))^*$, which we still denote as f_*, defined by

$$f_*\zeta : C^\gamma(M) \to \mathbb{R}, \quad f_*\zeta(\varphi) = \zeta(\varphi \circ f).$$

This extension $f_* : (C^\gamma(M))^* \to (C^\gamma(M))^*$ is a bounded linear operator: having fixed any Lipschitz constant $L > 1$ for f, we have that

$$\|f_*\zeta\|_\gamma = \sup_{\|\varphi\|_\gamma = 1} |\zeta(\varphi \circ f)| \le \|\zeta\|_\gamma \sup_{\|\varphi\|_\gamma = 1} \|\varphi \circ f\|_\gamma \le \|\zeta\|_\gamma L^\gamma \tag{57}$$

for any $\zeta \in (C^\gamma(M))^*$.

In what follows we use the sets $C(R)$ and $E(R)$, $R \geq 0$ introduced in Section 6.1: in a few words, $C(R)$ is the set of probability measures on individual strong-unstable plaques $\xi_i^u(x)$, $x \in \mathcal{M}_i \cap \Lambda$ obtained by multiplying the corresponding reference measure $\nu_{i,x}^u$ by some density e^ρ where ρ is (R,γ)-Hölder; and $E(R)$ is the space of probability measures on Λ, not necessarily f-invariant, which are convex combinations, not necessarily finite, of elements of $C(R)$. In particular, $C(R)$ is a subset of $E(R)$. Their restrictions to each Markov element $\mathcal{M}_i$ are denoted $C_i(R)$ and $E_i(R)$, respectively.

Lemma 8.2. *There exist $C_3 > 0$ and $\rho_3 < 1$ such that*

$$\|f_*^n(\zeta_1 - \zeta_2)\|_\gamma \leq C_3 \rho_3^n$$

for any $n \geq 1$ and any $\zeta_1, \zeta_2 \in E(0)$.

Proof. Assume first that $\zeta_1, \zeta_2 \in C(0)$, that is, they are of the form $\zeta_j = \nu_{i_j,x_j}^u$ with $x_j \in \mathcal{M}_{i_j}$. Denote $m_j = \zeta_j \times dt$. Then, given any $\varphi \in C^\gamma(M)$ and any $n \geq 1$,

$$(f_*^n \zeta_j)(\varphi) = \int_{\xi_{i_j,x_j}^u} \varphi(f^n(y))\, d\nu_{i_j,x_j}^u(y) = \int_{Y_j} \varphi(f^n(y))\, dm_j(y,t).$$

for $j = 1, 2$, and so,

$$\begin{aligned} f_*^n(\zeta_1 - \zeta_2)(\varphi) &= \int_{Y_1} \varphi \circ f^n\, dm_1 - \int_{Y_2} \varphi \circ f^n\, dm_2 \\ &= \int_{Y_1} [\varphi \circ f^n - \varphi \circ f^n \circ \tau]\, dm_1, \end{aligned}$$

where $\tau : Y_1 \to Y_2$ is as in Lemma 5.1. Define $Z(n) = \{(y,t) \in Y_1 : R(y,t) \leq n/2\}$. Then, using both parts of Lemma 5.1 for $n/2$,

$$\begin{aligned} |f_*^n(\zeta_1 - \zeta_2)(\varphi)| &\leq \int_{Z(n)} |\varphi \circ f^n - \varphi \circ f^n \circ \tau|\, dm_1 + 2\|\varphi\|_0\, m_1(Y_1 \setminus Z(n)) \\ &\leq \|\varphi\|_\gamma (C_1 \rho_1^{n/2})^\gamma + 2\|\varphi\|_0 C_2 \rho_2^{n/2} \leq (C_3/2)\rho_3^n \|\varphi\|_\gamma \end{aligned}$$

for suitable choices of C_3 and ρ_3, depending only on C_1, C_2, ρ_1, ρ_2, and γ.

Now consider the case where $\zeta_1 \in C(0)$ and $\zeta_2 \in E(0)$. By definition ζ_2 is a convex combination of measures in $E_i(0)$, $i = 1, \cdots, k$, and so it is no restriction to suppose that $\zeta_2 \in E_i(0)$ for some i. By Lemma 6.2, the disintegration

$$\zeta_2 = \int_{\xi_i^u} \zeta_P\, d\tilde{\zeta}_2(P)$$

of ζ_2 with respect to the partition ξ_i^u is such that $\zeta_P \in C_i(0)$ for every $P \in \xi_i^u$. Then,

$$(f_*^n \zeta_2)(\varphi) = \int_{\xi_i^u} (f_*^n \zeta_P)(\varphi)\, d\tilde{\zeta}_2(P)$$

for any $\varphi \in C^\gamma(M)$ and $n \geq 1$. So,

$$\begin{aligned} |f_*^n(\zeta_1 - \zeta_2)(\varphi)| &= \left| \int_{\xi_i^u} \left[(f_*^n \zeta_1)(\varphi) - (f_*^n \zeta_P)(\varphi) \right] d\tilde{\zeta}_2(P) \right| \\ &\leq \int_{\xi_i^u} \left| f_*^n(\zeta_1 - \zeta_P)(\varphi) \right| d\tilde{\zeta}_2(P) \leq (C_3/2)\rho_3^n \|\varphi\|_\gamma. \end{aligned}$$

Finally, for any ζ_1 and ζ_2 in $E(0)$, we may pick any $\zeta_3 \in C(0)$ and use the triangle inequality together with the previous paragraph to conclude that

$$|f_*^n(\zeta_1 - \zeta_2)(\varphi)| \leq C_3 \rho_3^n \|\varphi\|_\gamma$$

for any $\varphi \in C^\gamma(M)$ and $n \geq 1$. $\square$

This enables us to prove that the push-forwards of any measure $l \in E(0)$ under the map f converge exponentially fast to μ relative to the norm $\|\cdot\|_\gamma$:

Corollary 8.3. *For any $\zeta \in E(0)$ and $n \geq 1$,*

$$\|f_*^n \zeta - \mu\|_\gamma \leq C_3 \rho_3^n.$$

Proof. By Lemma 6.5, the invariant measure μ belongs to $E(0)$. Thus this is a special case of the previous lemma. $\square$

Proceeding with the proof of Theorem 8.1, we now extend this analysis to measures in $E(R)$ for any $R > 0$:

Lemma 8.4. *For any $R > 0$ and any $\zeta \in E(R)$ there exists $\zeta_0 \in E(0)$ such that $\|f_*^n \zeta - f_*^n \zeta_0\|_\gamma \leq Re^R$ for any $n \geq 1$.*

Proof. By definition, every $\zeta \in E(R)$ is a convex combination of elements of $E_i(R)$, $i = 1, \dots, k$. So, it is no restriction to assume that $\zeta \in E_i(R)$ for some i. By Lemma 6.2, we may write

$$\zeta = \int_{\mathcal{M}_i} e^{\rho_x} \nu_{i,x}^u \, d\hat{\zeta}(x)$$

where $\hat{\zeta}$ is a probability measure on $C_i(R)$, and each function ρ_x satisfies

$$\int_{\xi_i^u(x)} e^{\rho_x} \, d\nu_{i,x}^u = 1 \tag{58}$$

together with the Hölder condition (22). Let us check that

$$\zeta_0 = \int_{\mathcal{M}_i} \nu_{i,x}^u \, d\hat{\zeta}(x)$$

satisfies the claim. It is no restriction to assume that the diameters of all $\xi_i^u(x)$ are bounded by 1, and then (22) implies that

$$e^{-R} \leq e^{\rho_x(y) - \rho_x(z)} \leq e^R \text{ for every } y, z \in \xi_i^u(x) \text{ and } x \in \mathcal{M}_i.$$

Property (58) implies that the minimum (respectively, maximum) of e^{ρ_x} on $\xi_i^u(x)$ is less (respectively, greater) than or equal to 1. So, the previous inequality also yields that

$$e^{-R} \leq e^{\rho_x(y)} \leq e^R \text{ for every } y \in \xi_i^u(x) \text{ and } x \in \mathcal{M}_i.$$

In particular, $|e^{\rho_x} - 1| \leq Re^R$ for every x. Then, for any $\varphi \in C^\gamma(M)$,

$$\begin{aligned} \left| \int \varphi \, df_*^n \zeta - \int \varphi \, df_*^n \zeta_0 \right| &= \left| \int \left[\int \varphi \circ f^n \, (e^{\rho_x} - 1) \, d\nu_{i,x}^u \right] d\hat{\zeta}(x) \right| \\ &\leq \|\varphi \circ f^n\|_0 Re^R \leq \|\varphi\|_\gamma Re^R. \end{aligned}$$

This gives the claim. $\square$

Going back to the proof of Theorem 8.1, consider any $\zeta \in E(R)$. Let $\omega < 1$ be as in (1). By Proposition 6.4

$$f_*^m \zeta \in E\left(Re^{l\gamma m \log \omega}\right) \text{ for any } m \geq 1.$$

Replacing ζ and R with $f_*^m \zeta$ and $Re^{l\gamma m \log \omega}$ in Lemma 8.4, we find that for each $m \geq 1$ there exists $\zeta_m \in E(0)$ such that

$$\|f_*^k (f_*^m \zeta - \zeta_m)\|_\gamma \leq Re^{l\gamma m \log \omega} \exp\left(Re^{l\gamma m \log \omega}\right) \text{ for any } k \geq 1.$$

Given any $m \geq 1$, let $k, m \approx n/2$ such that $k + m = n$. Then

$$\begin{aligned} \|f_*^n \zeta - \mu\|_\gamma &\leq \|f_*^k (f_*^m \zeta - \zeta_m)\|_\gamma + \|f_*^k \zeta_m - \mu\|_\gamma \\ &\leq R\omega^{l\gamma m} \exp\left(R\omega^{l\gamma m}\right) + C_2 \rho_3^k \end{aligned}$$

Take $\tau = \max\{\omega^{l\gamma/2}, \rho_3^{1/2}\}$, and note that it is in $(0,1)$, since ω and ρ_3 are. Moreover, the previous inequality ensures that

$$\|f_*^n \zeta - \mu\|_\gamma \le L\tau^n \text{ for every } n \ge 1 \tag{59}$$

if $L > 0$ is chosen suitably.

Finally, ψ be any γ-Hölder function not identically zero. Then $\Psi = \psi + 2\|\psi\|_0$ is a strictly positive function and it is still γ-Hölder. More to the point, $\log \Psi$ is also γ-Hölder. Let R be the multiplicative Hölder constant. Then the probability measure

$$\zeta = \frac{\Psi\mu}{\int_M \Psi \, d\mu} = e^{\log \Psi - \log \int_M \Psi \, d\mu} \mu$$

is in $E(R)$, since μ is in $E(0)$. Since the difference $\psi - \Psi$ is constant, the correlation

$$\left| \int_M (\varphi \circ f^n) \psi \, d\mu - \int_M \varphi \, d\mu \int_M \psi \, d\mu \right|$$

is not affected if we replace ψ with Ψ. So (59) gives that

$$\begin{aligned} \left| \int_M (\varphi \circ f^n) \psi \, d\mu - \int_M \varphi \, d\mu \int_M \psi \, d\mu \right| &= \left| \int_M (\varphi \circ f^n) \Psi \, d\mu - \int_M \varphi \, d\mu \int_M \Psi \, d\mu \right| \\ &= \int_M \Psi \, d\mu \left| \int_M (\varphi \circ f^n) \, d\zeta - \int_M \varphi \, d\mu \right| \\ &\le \|f_*^n \zeta - \mu\|_\gamma \|\varphi\|_\gamma \|\Psi\|_0 \\ &\le L\tau^n \|\varphi\|_\gamma \|\psi\|_0. \end{aligned}$$

Just take $K(\varphi, \psi) = L\|\varphi_\gamma\|\psi\|_0$. The proof of Theorem 8.1 is complete.

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