SRB measures and absolute continuity for one-dimensional center direction

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Physical measures

Let $f : M \to M$ be a C^k, $k > 1$ partially hyperbolic diffeomorphism

$$TM = E^s \oplus E^c \oplus E^u$$

An f-invariant probability μ is a physical measure if

$$B(\mu) := \{x : \frac{1}{n}\sum_{j=0}^{n-1} \delta_{f^j(x)} \to \mu \text{ in weak}^* \text{ topology}\}$$

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Finitely many physical measures; basins contain almost every point:
Bonatti, Viana: for mostly contracting center direction
Alves, Bonatti, Viana: for mostly expanding center direction
Tsujii: for generic partially hyperbolic surface maps

One-dimensional center

Here we consider $\dim E^c = 1$. Let us focus on perturbations of partially hyperbolic skew-products

$$f_0 : M \times S^1 \to M \times S^1, \quad f_0(x,\theta) = (g_0(x), h_0(x,\theta)).$$

g_0 a transitive Anosov diffeomorphism.

More generally, the results are valid for partially hyperbolic, dynamically coherent C^k, $k > 1$ diffeomorphisms with 1-dimensional center direction and whose center leaves form a circle bundle.

Existence and finiteness

Theorem A

There is a C^k neighborhood $\mathcal{U}$ of f_0 such that, for every $f \in \mathcal{U}$ accessible and with absolutely continuous center stable foliation, there exist finitely many physical measures and the union of their basins has full Lebesgue measure in $M \times S^1$.

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Avila, Viana, Wilkinson: For conservative maps near f_0 absolute continuity is often a rigid property:

For $M = \mathbb{T}^2$, if $\mathcal{W}^c_f$ is absolutely continuous then it is smooth, and f is smoothly conjugate to $(x, \theta) \mapsto (g(x), \theta + \omega(x))$.

Abundance of absolute continuity

Theorem B

Suppose f_0 is accessible and has some periodic center leaf ℓ in general position. Then for every f in a neighborhood of f_0,

- $\mathcal{W}^{cs}_f$, $\mathcal{W}^{cu}_f$, and $\mathcal{W}^{c}_f$ are absolutely continuous
- both f and f^{-1} have unique physical measures (assuming accessibility).

General position:

- f_0 is Morse-Smale on ℓ with single attractor a and repeller r,
- $\phi(\{a, r\})$ is disjoint from $\{a, r\}$, for some holonomy loop ϕ.

Volume preserving accessible maps near $f_0 = g_0 \times \mathrm{id}$:

Shub, Wilkinson: $\{\lambda^c \neq 0\}$ accumulates on f_0 and

$\lambda^c(f) \neq 0$ implies $\mathcal{W}^c_f$ is not (even leafwise) absolutely continuous

leafwise absolute continuity:
$\mathrm{Leb}^c_L(Y \cap L) = 0$ for almost every leaf L implies $\mathrm{Leb}(Y) = 0$

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Avila, Viana, Wilkinson:
either $\mathcal{W}^c_f$ is absolutely continuous or Lebesgue has atomic disintegration

and if $\mathcal{W}^c_f$ absolutely continuous then $f \sim (g(x), \theta + \omega(x))$.

Volume preserving accessible maps near $f_0 = g_0 \times \mathrm{id}$:

Corollary

There is a C^1 open and dense set $\mathcal{U}^- \subset \{\lambda^c < 0\}$ such that every $f \in \mathcal{U}^-$ has leafwise absolutely continuous center stable foliation.

$\mathcal{U}^- \leftrightarrow$ accessibility and minimality of the strong unstable foliation

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Corollary

There is a open set $\mathcal{V}^-$ accumulating on f_0 such that every $f \in \mathcal{V}^-$ has absolutely continuous center stable foliation.

$\mathcal{V}^- \leftrightarrow$ accessibility and periodic center leaf in general position

For $f \in \mathcal{V}^-$ the center unstable foliation can not be absolutely continuous (e.g. because the center foliation is not).

Volume preserving accessible maps near $f_0 = g_0 \times \mathrm{id}$

Problem

1. For a C^1 open and C^k dense subset of $\{\lambda^c < 0\}$, the center stable foliation is absolutely continuous (and the center unstable foliation is not) ?
2. For $f \in \{\lambda^c = 0\} \setminus \{\mathcal{W}^c \text{ absolutely continuous}\}$, neither $\mathcal{W}^{cs}_f$ nor $\mathcal{W}^{cu}_f$ are absolutely continuous ?

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Gibbs u-states

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Assume accessibility and absolutely continuous $\mathcal{W}^{cs}$:

Proposition 1

The center Lyapunov exponent $\lambda^c(m)$ is non-negative, for every ergodic Gibbs u-state of f.

Gibbs u-states

Proposition 2

If m is an ergodic Gibbs u-state with $\lambda^c(m) = 0$ then

1. the conditional probabilities of m along center leaves are equivalent to Lebesgue measure, with bounded densities;
2. f is conjugate to $(g(x), \theta + \omega(x))$ by some homeomorphism Lipschitz continuous along center leaves;
3. m is the unique Gibbs u-state and the unique physical measure, and the basin $B(m)$ has full Lebesgue measure.

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3. m is the unique Gibbs u-state and the unique physical measure, and the basin $B(m)$ has full Lebesgue measure.

If $\lambda^c(m) < 0$ for all Gibbs u-states then f has mostly contracting center direction. In this case, the conclusion of the theorem had been obtained before, by Bonatti, Viana.

Proof of Proposition 1

Suppose $\lambda^c(m) > 0$. Let Γ be the set of $x \in M \times S^1$ with

$$\lim_{n \to +\infty} \frac{1}{n} \log \| Df^n \mid E^c_x \| = \lambda(m).$$

Then $m(\Gamma) = 1$.

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Then $m(\Gamma) = 1$.

Lemma

There is $N \geq 1$ such that $\#(\Gamma \cap \mathcal{W}^c_x) \leq N$ for every x.

Extracted from Ruelle, Wilkinson. Our proof holds in the C^1 case with any center dimension.

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Let ℓ be any periodic center leaf intersecting $\operatorname{supp} m$.

Lemma

Every point $x \in \ell$ is periodic, with period $\leq N \operatorname{per}(\ell)$.

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Then $\operatorname{supp} m$ contains no hyperbolic periodic points. That contradicts a well-known result of Katok.

Proof of Proposition 2

Consider $\pi : M \times S^1 \to M \times S^1/\mathcal{W}^c$ and let

$$f_c : M \times S^1/\mathcal{W}^c \to M \times S^1/\mathcal{W}^c$$

be the hyperbolic homeomorphism induced by f in leaf space.

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Lemma

If $\mathcal{W}^{cs}_f$ is absolutely continuous then $\pi_* m$ has local product structure, for any Gibbs u-state m.

That is, $\pi_* m$ is locally equivalent to a product measure in coordinates associated to local stable and unstable sets of f_c.

Proof of Proposition 2

Suppose $\lambda^c(m) = 0$. Local product structure allows us to use the Invariance Criterion of Avila, Viana to conclude

Corollary

m admits a disintegration $\{m_\ell : \ell \in M/\mathcal{W}^c\}$ along center leaves which is continuous and invariant under both stable and unstable holonomies.

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Moreover,

Lemma

The conditional probabilities m_ℓ are equivalent to arc length on the center leaves, with densities bounded from zero and infinity.

Proof of Proposition 2

This leads to

Corollary

f is conjugate to $(f_c(x), \theta + \omega(x))$ by a homeomorphism Lipschitz continuous along the center leaves.

It follows that the center Lyapunov exponent vanishes for every Gibbs u-state.

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Actually,

Lemma

m is the unique Gibbs u-state, and the basin $B(m)$ has full Lebesgue measure.

Theorem B

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Suppose f_0 is accessible has some periodic center leaf ℓ in general position. Then for every f in a neighborhood of f_0,

- $\mathcal{W}_f^{cs}$, $\mathcal{W}_f^{cu}$, and $\mathcal{W}_f^{c}$ are absolutely continuous
- both f and f^{-1} have unique physical measures (assuming accessibility).

Criterion for leafwise absolute continuity

Suppose $\lambda^c(m) < 0$ for every ergodic Gibbs u-state, that is, f has mostly contracting center direction.
By Andersson this is a C^2 open condition.

By Pesin theory, m-almost every point has local stable manifolds which are embedded disks of dimension $\dim E^{cs}$. Moreover, these local manifolds form an absolutely continuous lamination.

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Proposition 3

If f has mostly contracting center then $\mathcal{W}^{cs}_f$ is leafwise absolutely continuous.

If $\mathrm{Leb}^{cs}(Y \cap \mathcal{W}^{cs}_f(x)) = 0$ for Lebesgue a.e. x then $\mathrm{Leb}(Y) = 0$.

Criterion for absolute continuity

Proposition 4

If f has mostly contracting center and has some periodic center leaf in general position then $\mathcal{W}^{cs}_f$ is absolutely continuous.

General position:

- f_0 is Morse-Smale on ℓ with single attractor a and repeller r,
- $\phi(\{a, r\})$ is disjoint from $\{a, r\}$, for some holonomy loop ϕ.

This is also a C^1 open condition.