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High Dimension Diffeomorphisms Displaying Infinitely Many Sinks

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Introduction

We extend to higher dimensions a remarkable two-dimensional result of Newhouse proved in the seventies: many (residual subset of an open set) smooth diffeomorphisms near one exhibiting a homoclinic tangency have infinitely many coexisting sinks. The saddle associated with the homoclinic tangency is taken to be sectionally dissipative. In this general setting we have to circumvent a major difficulty: the usual lack of (transversal) differentiability of invariant dynamic foliations of codimension higher than one. Also, in general there is no global smooth center manifold and so the problem cannot be reduced in this way to the two-dimensional case.Instead the question is solved by producing hyperbolic sets whose foliations are "essentially" differentiable (à la Whitney) with a large fractal dimension (thickness).

The recent development of (dissipative) dynamics has been much influenced by the discovery of some striking bifurcating phenomena such as the Lorenz-like attractors, the Hénon-like attractors, the Feigenbaum and Coullet-Tresser cascades of period doubling bifurcations and Newhouse's infinitely many coexisting sinks (attracting periodic orbits).

This last phenomenon, although more than twenty years old, has remained essentially a topic in the study of surface diffeomorphisms. That is, the known examples in higher dimensions were obtained by taking a two-dimensional prototype and "multiplying" it by a strongly contracting diffeomorphism. In the present paper we are able to extend the original result to other dimensions (even to infinite dimension) in its full strength, showing in general the abundance (from the topological point of view) of the diffeomorphisms displaying infinitely many coexisting sinks. Indeed, we prove the following result

MAIN THEOREM. *Near any smooth diffeomorphism exhibiting a homoclinic tangency associated to a sectionally dissipative saddle, there is a residual subset of an open set of diffeomorphisms such that each of its elements displays infinitely many coexisting sinks.*

We point out that our methods also imply a one-parameter version of this theorem: a generic unfolding of such a (quadratic) homoclinic tangency yields residual subsets of intervals in the parameter line whose corresponding diffeomorphisms exhibit infinitely many sinks. For surface diffeomorphisms this version has been obtained in [Rob].

Smooth in the statement above means of class $\mathcal{C}^2$ and near means closeness in the $\mathcal{C}^2$ sense. Recall that a homoclinic tangency is just a tangency between the stable and unstable manifolds of a saddle periodic point. The saddle is called *(codimension-one) sectionally* or *strongly dissipative* if it has just one expanding eigenvalue (positive Lyapunov exponent) and the product of any two eigenvalues has norm less than one, i.e. any contracting eigenvalue is stronger than the expanding one. We observe that in order to get abundance of sinks when unfolding a homoclinic tangency, we assume that the associated saddle is sectionally dissipative for otherwise we are bound to obtain periodic orbits of smaller index (saddles, sources); see [Rom]. On the other hand, when we generically unfold such a quadratic homoclinic tangency in any dimension, we obtain a rather striking list of dynamical phenomena:

- Hénon-like strange attractors [MV], [V], based on the remarkable work of Benedicks-Carleson [BC];
- cascades of period doubling bifurcations of sinks [YA];
- residual subsets of open sets of intervals in the parameter line whose elements exhibit infinitely many coexisting sinks [N2], [N3], [Rob], and the present paper.

Thus, since homoclinic tangencies are a common bifurcating dynamic feature, the existence of diffeomorphisms with infinitely many coexisting sinks is rather abundant. It is quite possible, however, that in a parametrized form in terms of Lebesgue measure in the parameter space this is a rare phenomenon (measure zero). This is a subject of much interest and we refer the reader to [PT] for a discussion on this and other related questions.

As pointed out above, a basic difficulty to extend Newhouse's original result from two to higher dimensions is the usual lack of differentiability of invariant foliations (in our case, unstable foliations) of codimension bigger than one, which are in general just Hölder continuous. This has been a major obstacle in order to make sense of (transversal) fractal dimensions of invariant Cantor sets. In particular, this is the case of *thickness*, a concept used by Newhouse to show that *many pairs of Cantor sets in the line necessarily intersect each other unless one lies in a gap of the other; see Section 1.* These intersections correspond to homoclinic tangencies that in turn generate sinks when unfolded. Also, the question treated in this paper has a semi-global character, involving the whole orbit of tangency. In particular, it cannot be solved by just assuming that we can locally linearize the map near the associated saddle, e.g. by supposing that it has nonresonant eigenvalues. Even more, we cannot apply the standard procedure of "reducing dimensions" through projection along the strong contracting directions onto a central manifold, which in fact does not exist in general. The key idea introduced here to circumvent these difficulties is to obtain a hyperbolic set whose higher-codimension unstable foliation is indeed "differentiable" (or "intrinsically differentiable") if we avoid the strong contracting directions. This idea may be useful in other similar dynamic situations involving fractal dimensions of hyperbolic sets or, more directly, the differentiability of their invariant foliations.

The formal definitions of intrinsic differentiability and thickness for the unstable foliation are given in Section 1 and developed in Sections 2 through 4. Specially, in Section 2 we briefly describe some basic properties of intrinsic differentiability. A new relevant condition on the homoclinic tangency is made explicit in Section 3: assumption (3.2) in Proposition 3.2 yields the construction of hyperbolic sets whose unstable foliation is intrisincally differentiable, since we can then "avoid" the strong contracting directions as suggested above. In Section 1 we also present a sketch of the whole proof of our main result. It follows more closely a new proof of Newhouse's two-dimensional result presented in [PT] than the original papers. For the sake of simplifying the argument, we show in Section 5 that we may assume that the saddle associated to the homoclinic tangency has a unique least contracting eigenvalue. In Section 6 we construct thick invariant Cantor sets that appear when we unfold the homoclinic tangency and whose elements have stable and unstable manifolds that transversally intersect those of the associated saddle. This is done through a renormalization technique. Finally, in Section 7 we finish the proof of the theorem by just assembling together the facts established in the previous sections. We also briefly indicate how the parametrized version we stated above follows from these arguments.

1. Main ingredients and sketch of the proof

We begin by recalling a few notions and facts, mostly from [N1], [N3], which play a central role in the arguments below.

Let $K \subseteq \mathbf{R}$ be a Cantor set and $\hat{K}$ be its convex hull. A *presentation* of K is an ordering $\mathcal{U} = (U_n)_n$ of its gaps, i.e. of the connected components of $\hat{K} \setminus K$. For each n and $u \in \partial U_n$ let $\tau(K, \mathcal{U}, u) = \text{length}(C)/\text{length}(U_n)$, where C is the connected component of $\hat{K} \setminus (U_1 \cup \cdots \cup U_n)$ that contains u. The *thickness* of K is defined as

$$\tau(K) = \sup_{\mathcal{U}} \inf_{u} \tau(K, \mathcal{U}, u)$$

where the supremum is taken over all presentations of K and the infimum over all points $u \in \partial U_n$ as above. The *local thickness* of K *at* $x \in K$ is

$$\tau(K, x) = \lim_{\varepsilon \to 0}(\sup\{\tau(L) : L \subset K \cap [x-\varepsilon, x+\varepsilon] \text{ a Cantor set}\}).$$

Recall that a basic set for a diffeomorphism is a compact, invariant, transitive, hyperbolic set, with a dense subset of periodic orbits and which is the maximal invariant set in a neighbourhood of it. A basic set is persistent: any $\mathcal{C}^k$-small perturbation of the map, $k \geq 1$, yields a unique basic set near the initial one, called the "analytic" or "smooth" continuation of it. Let now Λ be a (nontrivial) basic set of a $\mathcal{C}^2$ diffeomorphism $\varphi : M \to M$, whose stable foliation is of codimension one, i.e. such that $\dim \mathrm{W^s}(x) = m - 1$, $m = \dim M$, for all $x \in \Lambda$. Let $z \in \mathrm{W^s}(\Lambda)$ and $\phi : [-a, a] \to M$ be a $\mathcal{C}^1$ embedding transverse to $\mathrm{W^s}(\Lambda)$ at $z = \phi(0)$. The *local stable thickness of* Λ *at* z is $\tau^{\mathrm{s}}(\Lambda, z) = \tau(\phi^{-1}(\mathrm{W^s}(\Lambda)), 0)$. This is independent of the choice of ϕ, as a consequence of the fact that (under the codimension-one assumption) the holonomy maps (i.e. the projections along the leaves) of the stable foliation of Λ can be extended to $\mathcal{C}^1$ maps. Actually, this smoothness of the holonomy of $\mathrm{W^s}(\Lambda)$, together with the transitivity of $\varphi \mid \Lambda$, also implies that $\tau^{\mathrm{s}}(\Lambda, z)$ has the same value for every $z \in \mathrm{W^s}(\Lambda)$. We denote by $\tau^{\mathrm{s}}(\Lambda)$ this constant value and call it the (*local*) *stable thickness of* Λ. This is a strictly positive finite number and depends continuously on the diffeomorphism, in the sense that if Λ_ψ denotes the analytic continuation of Λ for a diffeomorphism ψ which is $\mathcal{C}^2$-close to φ, then $\tau^{\mathrm{s}}(\Lambda_\psi)$ is close to $\tau^{\mathrm{s}}(\Lambda)$. *Local unstable thicknesses* $\tau^{\mathrm{u}}(\Lambda, z)$ and $\tau^{\mathrm{u}}(\Lambda)$ are defined in a similar way, when $\mathrm{W^u}(\Lambda)$ has codimension one. In particular, both the stable thickness and the unstable thickness are well defined if M is a surface.

Now we outline the proof of our main result. We start by recalling the main ideas in the proof of Newhouse's theorem ([N1], [N2], [N3]), as presented in [PT], and then describe the key ingredients involved in extending these arguments from two to higher dimensions. This extension will be carried out in the forthcoming sections.

Let us consider first the case of a homoclinic tangency (on a surface) involving a *thick horseshoe*. By this we mean that the homoclinic tangency is associated to a periodic point p belonging to some basic set Λ such that $\tau^{\mathrm{u}}(\Lambda) \cdot \tau^{\mathrm{s}}(\Lambda) > 1$. A crucial fact here is the *gap lemma*: if K_1, K_2 are Cantor sets in the real line such that $\tau(K_1) \cdot \tau(K_2) > 1$ and K_1 is not entirely contained in a gap of K_2 nor vice-versa, then $K_1 \cap K_2 \neq \emptyset$. By considering the line ℓ of tangencies between the stable and the unstable foliations of the basic set Λ and applying the lemma to $\ell \cap \mathrm{W^u}(\Lambda)$ and $\ell \cap \mathrm{W^s}(\Lambda)$, one concludes that there exists a $\mathcal{C}^2$-open set of diffeomorphisms $\mathcal{N}$ (whose closure contains the diffeomorphism φ exhibiting the tangency) such that every $\psi \in \mathcal{N}$ has tangencies between leaves of the stable and unstable foliations of the analytic continuation Λ_ψ of Λ. In particular, a dense subset of diffeomorphisms $\psi \in \mathcal{N}$ exhibits homoclinic tangencies associated to periodic points in Λ_ψ (e.g. to the analytic continuation of p). We assume φ to be *dissipative* at the saddle-point involved in the tangency, i.e. $\left|\det \mathrm{D}\varphi^k(p)\right| < 1$, k =period of p, and then the same holds for $\psi \in \mathcal{N}$. Recall that, under this dissipativeness assumption, a small perturbation of a diffeomorphism exhibiting a homoclinic tangency yields a periodic sink and that, moreover, a sink is persistent through small perturbations of the map. Thus, for each positive integer n, the subset of $\mathcal{N}$ consisting of elements with n sinks is open and dense. Then, by a standard Baire category argument, the elements of a residual subset of $\mathcal{N}$ display infinitely many coexisting sinks.

In order to prove Newhouse's theorem in the general 2-dimensional case we just assume the existence of a homoclinic tangency, not necessarily associated to a thick Cantor set. Then we proceed as follows to construct an open set $\mathcal{N}$ as above. We continue to take the diffeomorphism φ to be dissipative at the saddle-point p. Also, we may suppose that p belongs to a nontrivial hyperbolic basic set Λ_1: if this is not the case then we just perturb φ so as to create transverse homoclinic orbits together with a new orbit of tangency associated to p. We then consider $\tau^{\mathrm{u}}(\Lambda_1) = \tau^{\mathrm{u}}(\Lambda_1, p) > 0$ and further perturbations of the diffeomorphism are to be taken small enough so that Λ_1 and $\tau^{\mathrm{u}}(\Lambda_1)$ persist essentially unchanged. The crucial step in the argument consists in showing that any small unfolding of the tangency produces new horseshoes Λ_2 with arbitrarily large stable thickness, in particular satisfying $\tau^{\mathrm{s}}(\Lambda_2) \cdot \tau^{\mathrm{u}}(\Lambda_1) > 1$. In [PT] such Λ_2 are constructed via a renormalization procedure as follows. Let $(\varphi_\mu)_{\mu \in (-\varepsilon, \varepsilon)}$, $\varphi_0 = \varphi$, be an arc of diffeomorphisms generically unfolding the tangency. Then one finds parameter values $\mu_n \to 0$ and small domains $Q_n \subset M$ converging to the tangency, such that $\varphi^n_{\mu_n}(Q_n) \cap Q_n \neq \emptyset$ and, up to appropriate n-dependent rescaling of Q_n, the return maps $\varphi^n_{\mu_n} \,|\, Q_n$ converge in the $\mathcal{C}^2$ topology to the endomorphism $\phi(x, y) = (1 - 2x^2, x)$ as $n \to \infty$ (convergence holds in the $\mathcal{C}^k$ topology, any $k \geq 1$, if the arc is taken to be $\mathcal{C}^\infty$). From the fact that ϕ is conjugate to $\psi(x, y) = (1 - 2\,|x|\,, x)$ one concludes that it has invariant hyperbolic sets with arbitrarily large thickness. Then we take Λ_2 to be the analytic continuation for $\varphi^n_\mu \,|\, Q_n$ of some of these ϕ-hyperbolic sets. One checks that leaves of $\mathrm{W}^{\mathrm{u}}(\Lambda_1)$ and $\mathrm{W}^{\mathrm{s}}(\Lambda_1)$ have some transverse intersections with, respectively, leaves of $\mathrm{W}^{\mathrm{s}}(\Lambda_2)$ and $\mathrm{W}^{\mathrm{u}}(\Lambda_2)$. Finally, a heteroclinic tangency associated to periodic points $p_1 \in \Lambda_1$, $p_2 \in \Lambda_2$, may be created, again by a small perturbation (thus with a negligible effect on Λ_2 and $\tau^{\mathrm{s}}(\Lambda_2)$). This is done through an auxiliary saddle P that also originates from the 1-dimensional endomorphism: it exhibits a homoclinic tangency and its stable and unstable manifolds have points of transverse intersection with the dual invariant manifolds of points both in Λ_1 and Λ_2. Thus, arguing as in the

FIGURE 1

particular case above, there exists an open set in $\mathrm{Diff}^2(M)$ with persistent heteroclinic tangencies involving Λ_1 and Λ_2 and so also with persistent homoclinic tangencies associated to Λ_1 or Λ_2.

Now we discuss in more detail than in the Introduction the main difficulties in extending this result to higher dimensions. We also present in a more formal way the ideas to overcome them. As we said before some of these ideas (like the intrinsic differentiability of invariant foliations or the definition and invariance of local thickness) are quite general and do not require the assumptions of codimension 1 or of sectional dissipativeness.

It is by now classic that the unfolding of a homoclinic tangency of an m-dimensional diffeomorphism leads to the creation of nontrivial basic sets ([Sm]). It is easy to check that we can get these basic sets together with new homoclinic tangencies outside them. Therefore, we may again assume that p is part of a nontrivial basic set Λ_1. However, it is not clear what $\tau^{\mathrm{u}}(\Lambda_1)$ should be taken to mean now, since transverse sections to $\mathrm{W}^{\mathrm{u}}(\Lambda_1)$ are no longer lines. Another, perhaps even more serious difficulty arises from the fact that, $\mathrm{W}^{\mathrm{u}}(\Lambda_1)$ having codimension bigger than 1, the projections along its leaves may have a bad metric behaviour: in general they are not Lipschitz but just Hölder continuous.

In order to bypass these difficulties we proceed as follows. The first step is to show that the unfolding of a homoclinic tangency yields the formation, for arbitrarily small values $\tilde{\mu}$ of μ, of other homoclinic tangencies associated to periodic points $\tilde{p}_{\tilde{\mu}}$ of $\varphi_{\tilde{\mu}}$, such that

(I) $\mathrm{D}\varphi_{\tilde{\mu}}^{\ell}(\tilde{p}_{\tilde{\mu}})$, ℓ =period of $\tilde{p}_{\tilde{\mu}}$, has a unique weakest contracting eigenvalue which, as a consequence, is a real number.

Hence, we may assume right from the start that $\mathrm{D}\varphi_0^k(p)$, k =period of p, satisfies (I). What we do then is to construct Λ_1 embedded in the manifold M in an infinitesimally 2-dimensional way, transversely to the strongest contracting directions of $\mathrm{D}\varphi^k(p)$. By this we mean that Λ_1 is taken such that at every $x \in \Lambda_1$, the *intrinsic tangent space*

$$\mathrm{IT}_x\Lambda_1 = \mathrm{span}\left\{v\colon \text{ there is } (x_n)_n \in \Lambda_1^{\mathbf{N}} \text{ so that } \frac{x_n - x}{\|x_n - x\|} \to v\right\}$$

(for simplicity we consider here $M = \mathbf{R}^m$) is 2-dimensional and, like the plane generated by the unstable and the weakest stable directions, is (uniformly) transverse to the codimension-two plane generated by the strongest contracting directions. By *abus de langage*, we say that the angle (meaning its trigonometric tangent) between the above 2-planes is bounded from above. For such a Λ_1 we prove that the projection maps $\pi\colon \Sigma_0 \cap \mathrm{W}^{\mathrm{u}}(\Lambda_1) \to \Sigma_1 \cap \mathrm{W}^{\mathrm{u}}(\Lambda_1)$ along leaves of $\mathrm{W}^{\mathrm{u}}(\Lambda_1)$ are *intrinsically differentiable*, Σ_0, Σ_1 being transversal sections to leaves of $\mathrm{W}^{\mathrm{u}}(\Lambda_1)$. That is, there exists a continuous map $\Sigma_0 \cap \mathrm{W}^{\mathrm{u}}(\Lambda_1) \ni x \mapsto \partial\pi(x) \in \mathcal{L}(\mathbf{R}^{m-1}, \mathbf{R}^{m-1})$ such that for every $x \in \Sigma_0 \cap \mathrm{W}^{\mathrm{u}}(\Lambda_1)$, we have

$$\frac{\pi(x) - \pi(z) - \partial\pi(x)\cdot(x-z)}{\|x-z\|} \to 0 \text{ as } z \to x \text{ with } z \in \Sigma_0 \cap \mathrm{W}^{\mathrm{u}}(\Lambda_1),$$

where Σ_1 is identified with a disk in $\mathbf{R}^{m-1}$. Here and in what follows $\mathcal{L}(\mathbf{R}^p, \mathbf{R}^q)$ denotes the space of linear maps from $\mathbf{R}^p$ to $\mathbf{R}^q$. Then we define the *local unstable thickness* of Λ_1 at $x \in \Lambda_1$ by $\tau^{\mathrm{u}}(\Lambda_1, x) = \tau(\tilde\pi(\mathrm{W}^{\mathrm{u}}(\Lambda_1) \cap \Sigma_0), \tilde\pi(x))$, where Σ_0 is a transversal section to $\mathrm{W}^{\mathrm{u}}(\Lambda_1)$ at x and $\pi\colon \mathrm{W}^{\mathrm{u}}(\Lambda_1) \cap \Sigma_0 \to \mathbf{R}$ is any intrinsically differentiable map such that $\partial\tilde\pi \,|\, \mathrm{IT}_x(\mathrm{W}^{\mathrm{u}}(\Lambda_1) \cap \Sigma_0)$ is a bijection. *Intrinsic differentiability of the unstable foliation allows us to check that this definition does not depend on the choices of* Σ_0 *and* $\tilde\pi$ *and to prove that* $\tau^{\mathrm{u}}(\Lambda_1, x)$ *is strictly positive and independent of* $x \in \Lambda_1$. *Also it varies continuously with the diffeomorphism* $\varphi \in \mathrm{Diff}^2(M)$. We then show that by arbitrarily small perturbations (essentially not affecting Λ_1 and $\tau^{\mathrm{u}}(\Lambda_1)$) one obtains a hyperbolic basic set Λ_2 with codimension-one stable foliation and large stable thickness, namely $\tau^{\mathrm{u}}(\Lambda_1)\tau^{\mathrm{s}}(\Lambda_2) > 1$. This is done by a natural extension to higher dimensions of the renormalization scheme mentioned above. We also check that Λ_1 and Λ_2 may be taken to be heteroclinically related (existence of mutual transverse intersections between leaves of their stable and unstable foliations) and, moreover, to exhibit a tangency q between $\mathrm{W}^{\mathrm{u}}(p_1)$ and $\mathrm{W}^{\mathrm{s}}(p_2)$, $p_1 \in \Lambda_1$, $p_2 \in \Lambda_2$ periodic points. We recall that the stable foliation of Λ_2, being of codimension one, admits an extension to a $\mathcal{C}^1$ foliation $\mathcal{F}^{\mathrm{s}}(\Lambda_2)$ defined in a neighbourhood of Λ_2 (which we may assume to contain q). An implicit function argument shows that there exists an intrinsically differentiable map $\pi_1\colon \mathrm{W}^{\mathrm{u}}(\Lambda_1) \cap \mathrm{W}^{\mathrm{s}}_{\mathrm{loc}}(p_1) \to M$ such that $\pi_1(p_1) = q$ and each $\pi_1(x)$, $x \in \mathrm{W}^{\mathrm{u}}(\Lambda_1) \cap \mathrm{W}^{\mathrm{s}}_{\mathrm{loc}}(p_1)$, is a point of tangency between leaves of $\mathrm{W}^{\mathrm{u}}(\Lambda_1)$

FIGURE 2

and $\mathcal{F}^{\mathrm{s}}(\Lambda_2)$ (the image of π_1 is the set of tangencies, see the figure). We let π_2 be the projection along the leaves of $\mathcal{F}^{\mathrm{s}}(\Lambda_2)$ onto $\mathrm{W}^{\mathrm{u}}_{\mathrm{loc}}(p_2)$, which we identify with an interval in $\mathbf{R}$. Then the theorem follows by an application of the gap lemma to the Cantor sets $\pi_2 \circ \pi_1(\mathrm{W}^{\mathrm{u}}(\Lambda_1) \cap \mathrm{W}^{\mathrm{s}}_{\mathrm{loc}}(p_1))$ and $\mathrm{W}^{\mathrm{s}}(\Lambda_2) \cap \mathrm{W}^{\mathrm{u}}_{\mathrm{loc}}(p_2)$.

2. Intrinsically smooth maps

In this section we describe a notion of "intrinsic" differentiability of functions on compact subsets of $\mathbf{R}^m$, closely related to Whitney [W], and we list some of its basic properties. The main result is Proposition 2.10, on intrinsic differentiability of invariant sections of (contracting) bundle morphisms.

Let $X \subset \mathbf{R}^m$ be a compact set and $\varphi\colon X \to \mathbf{R}^n$ be continuous. We say that φ is *intrinsically* $\mathcal{C}^1$ *on* X if there exists a continuous map $\Delta\varphi\colon X \times X \to \mathcal{L}(\mathbf{R}^m, \mathbf{R}^n)$ such that

$$\varphi(x) - \varphi(z) = \Delta\varphi(x,z) \cdot (x - z) \text{ for all } x, z \in X.$$

Such a $\Delta\varphi$ (which is, in general, far from unique) is called an *intrinsic derivative of* φ. We say that φ is *intrinsically* $\mathcal{C}^{1+\gamma}$ *on* X if it admits some γ-Hölder continuous intrinsic derivative.

Example 2.1. Let $\varphi\colon U \to \mathbf{R}^n$, U an open rectangle in $\mathbf{R}^m$, be a $\mathcal{C}^1$ map. Then $\varphi \mid X$ is intrinsically $\mathcal{C}^1$ on X, for every compact $X \subset U$: one may take $\Delta\varphi(x,z) = (\Delta_i\varphi_j(x,z))_{i,j}$ where, denoting $y^{(i)} = (x_1, \dots, x_i, z_{i+1}, \dots, z_m)$,

$$\Delta_i\varphi_j(x,z) = \begin{cases} \frac{\varphi_j(y^{(i)}) - \varphi_j(y^{(i-1)})}{x_i - z_i}, & \text{if} \quad x_i \neq z_i \\ \frac{\partial \varphi_j}{\partial x_i}(y^{(i)}), & \text{if} \quad x_i = z_i \end{cases}$$

If, moreover, φ is $\mathcal{C}^2$ then this $\Delta\varphi$ is Lipschitz continuous and so φ is intrinsically $\mathcal{C}^{1+\gamma}$ on X for every $0 < \gamma \leq 1$.

Example 2.2. Let $\varphi: \mathbf{R}^2 \to \mathbf{R}^2$ be given by $\varphi(x,y) = (\sigma x, \rho y)$ with $1 < \sigma < \rho$ and let $X \subset \mathbf{R}^2$ be defined as follows. Take K to be the standard middle-third Cantor set and let $f: K \to K$ be the unique continuous map satisfying $f(0) = 0$, $f(1) = 1$ and $f\left(\frac{p}{3^{q-1}} \pm \frac{1}{3^q}\right) = f\left(\frac{p}{3^{q-1}}\right) \pm \frac{1}{3^{2q}}$ for all $p, q \in \mathbf{N}$. This is an intrinsically $\mathcal{C}^1$ function on K with $\Delta f(k,k) = 0$ for all $k \in K$. We take $X = \text{graph}\,(f)$ and then

$$\Delta\varphi(x,z) = \begin{pmatrix} \sigma & 0 \\ \rho\Delta f(k,\ell) & 0 \end{pmatrix}, \text{ denoting } x = (k, f(k)), z = (\ell, f(\ell)),$$

is an intrinsic derivative for φ on X. Observe that $\|\Delta\varphi(x,x)\| = \sigma$ for all $x \in X$; compare with the previous construction.

This second example illustrates the main point in the proof that Λ_1, as constructed in Section 3, has intrinsically $\mathcal{C}^1$ unstable foliation (Proposition 3.4): one uses the geometry of the domain to obtain an intrinsic derivative with the smallest norm of the derivative of the map "restricted to factors". The proof of the following properties of intrinsically $\mathcal{C}^1$ (or $\mathcal{C}^{1+\gamma}$) maps is immediate.

LEMMA 2.3 (Chain Rule). *Let $\varphi: X \to \mathbf{R}^n$ and $\psi: Y \to \mathbf{R}^p$ be intrinsically $\mathcal{C}^1$ (resp. $\mathcal{C}^{1+\gamma}$) with $\varphi(X) \subset Y$. Then $\psi_\circ\varphi$ is intrinsically $\mathcal{C}^1$ (resp. $\mathcal{C}^{1+\gamma}$) and one may take $\Delta(\psi_\circ\varphi)(x,z) = \Delta\psi(\varphi(x), \varphi(z)) \cdot \Delta\varphi(x,z)$.*

LEMMA 2.4 (Uniform limits). *Let a sequence $\varphi_k: X \to \mathbf{R}^n$ of intrinsically $\mathcal{C}^1$ maps converge uniformly to $\varphi: X \to \mathbf{R}^n$ and admit intrinsic derivatives $\Delta\varphi_k$ converging uniformly to $\Phi: X \times X \to \mathcal{L}(\mathbf{R}^m, \mathbf{R}^n)$. Then φ is intrinsically $\mathcal{C}^1$ and one may take $\Delta\varphi = \Phi$.*

LEMMA 2.5 (Restrictions). *Let $\varphi: X_1 \times X_2 \to \mathbf{R}^n$ and for $x_1 \in X_1$ and $x_2 \in X_2$ define maps $\varphi_{x_1}: X_2 \to \mathbf{R}^n$ and $\varphi^{x_2}: X_1 \to \mathbf{R}^n$ by $\varphi_{x_1}(x_2) = \varphi(x_1, x_2) = \varphi^{x_2}(x_1)$.*

(a) If φ is intrinsically $\mathcal{C}^1$ then the same holds for every φ_{x_1} and φ^{x_2}, with $\Delta\varphi_{x_1}(x_2, z_2) \cdot v_2 = \Delta\varphi((x_1,x_2),(x_1,z_2)) \cdot (0, v_2)$ and $\Delta\varphi^{x_2}(x_1,z_1) \cdot v_1 = \Delta\varphi((x_1,x_2),(z_1,x_2)) \cdot (v_1, 0)$.

(b) If all φ_{x_1} and φ^{x_2} admit intrinsic derivatives $\Delta\varphi_{x_1}$ and $\Delta\varphi^{x_2}$ which, moreover, vary continuously with x_1 and x_2 then φ is intrinsically $\mathcal{C}^1$ with $\Delta\varphi((x_1,x_2),(z_1,z_2)) \cdot (v_1,v_2) = \Delta\varphi^{z_2}(x_1,z_1) \cdot v_1 + \Delta\varphi_{x_1}(x_2,z_2) \cdot v_2$.

Remark 2.6. The following simple remark will be used in the forthcoming sections. Let $\varphi: X \to \mathbf{R}^n$ be Lipschitz continuous and $U \subset X \times X$ be such that $\{\|x - z\| : (x,z) \in U\}$ is bounded away from zero. Then there is a Lipschitz continuous map $\Delta: U \to \mathcal{L}(\mathbf{R}^m, \mathbf{R}^n)$ such that $\varphi(x) - \varphi(z) = \Delta(x,z)(x-z)$ for every $(x,z) \in U$.

Let us also observe that intrinsic differentiability is a local property. We say that $\varphi: X \to \mathbf{R}^n$ is *locally intrinsically* $\mathcal{C}^1$ (resp. $\mathcal{C}^{1+\gamma}$) if every $x \in X$ has a neighbourhood $V_x \subset X$ such that $\varphi \,|\, V_x$ is intrinsically $\mathcal{C}^1$ (resp. $\mathcal{C}^{1+\gamma}$).

LEMMA 2.7. *$\varphi: X \to \mathbf{R}^n$ is intrinsically $\mathcal{C}^1$ if and only if there exists a continuous map $\partial\varphi: X \to \mathcal{L}(\mathbf{R}^m, \mathbf{R}^n)$ such that*

$$\lim_{\substack{z \to x \\ z \in X}} \frac{\varphi(x) - \varphi(z) - \partial\varphi(x) \cdot (x-z)}{\|x-z\|} = 0 \text{ for every } x \in X.$$

In particular, φ is intrinsically $\mathcal{C}^1$ if and only if it is locally intrinsically $\mathcal{C}^1$.

Proof. The "only if" affirmative is proved by fixing an intrinsic derivative $\Delta\varphi$ of φ and taking $\partial\varphi(x) = \Delta\varphi(x,x)$. For the proof of the "if" part we take $\partial\varphi$ as in the statement and define $\theta: X \times X \to \mathcal{L}(\mathbf{R}^m, \mathbf{R}^n)$ by

$$\theta(x,z) \cdot (x-z) = \varphi(x) - \varphi(z) - \partial\varphi(x) \cdot (x-z) \quad \text{and}$$
$$\theta(x,z) \cdot v = 0 \quad \text{whenever } v \cdot (x-z) = 0.$$

Then $\theta(x,z)$ depends continuously on $(x,z) \in X \times X$ and $\Delta\varphi(x,z) = \partial\varphi(x) + \theta(x,z)$ is an intrinsic derivative for φ. The second part of the lemma is an immediate consequence of the first one. □

Remark 2.8. Notice that, for a sequence $(\varphi_k)_k$ converging uniformly to $\varphi: X \to \mathbf{R}^n$, having $\partial\varphi_k$ converging uniformly to some $\Phi: X \to \mathcal{L}(\mathbf{R}^m, \mathbf{R}^n)$ is not sufficient to assure that φ is intrinsically $\mathcal{C}^1$.

Remark 2.9. The same construction permits to check that φ is intrinsically $\mathcal{C}^{1+\gamma}$ if and only if $\partial\varphi$ as above can be found which is γ-Hölder continuous and, moreover, satisfies

$$\|\varphi(x) - \varphi(z) - \partial\varphi(x)(x-z)\| \le C \, \|x-z\|^{1+\gamma}$$

for some $C > 0$ and every $x, z \in X$. In particular, by Whitney [W] (see also [St]), any intrinsically $\mathcal{C}^{1+\gamma}$ function, $\gamma > 0$, admits a $\mathcal{C}^{1+\gamma}$ extension to an open neighbourhood of its domain. However, in the present paper we make no use of such extension, except for the simple one-dimensional situation in Lemma 4.4 where an explicit construction is provided.

Proposition 3.4 will be proved by means of the following general result on contracting bundle morphisms, which is a version for intrinsic differentiability of Theorem 6.1 in [HP]. Let $X_0 \subset X$ be compact subsets of $\mathbf{R}^m$ and $B = \overline{\mathrm{B}}_R(0)$ be a compact ball in $\mathbf{R}^n$. Let $f: X_0 \to X$ be a homeomorphism and $\tilde{F}: X_0 \times B \to X \times B$ be a continuous map of the form $\tilde{F}(x,v) = (f(x), F_x(v))$. Assume that, for some $c < 1$ and every $x \in X_0$, F_x has Lipschitz constant $\leq c$, so that there is a unique continuous section $\tilde{\sigma}: X \to B$ satisfying $\tilde{\sigma}(f(x)) = F_x(\tilde{\sigma}(x))$ for all $x \in X_0$ (*$\tilde{F}$-invariance*).

Proposition 2.10.
(a) *Suppose that $\tilde{F}$ and f^{-1} are intrinsically $\mathcal{C}^1$ (on their domains) and there is $a < 1$ such that*

$$\|\Delta F_y(v,w)\| \cdot \left\|\Delta f^{-1}(x,z)\right\| \leq a, \tag{2.1}$$

for any $x, z \in X$, $y \in X_0$ and $v, w \in B$. Then $\tilde{\sigma}$ is intrinsically $\mathcal{C}^1$ on X.
(b) *Suppose that $\tilde{F}$ and f^{-1} are intrinsically $\mathcal{C}^{1+\gamma}$ (on their domains) and there is $b < 1$ such that*

$$\|\Delta F_y(v,w)\| \cdot \left\|\Delta f^{-1}(x,z)\right\|^{1+\gamma} \leq b \tag{2.2}$$

for any $x, z \in X$, $y \in X_0$ and $v, w \in B$. Then $\tilde{\sigma}$ is intrinsically $\mathcal{C}^{1+\gamma}$ on X.

Proof. Let S (resp. $\mathcal{S}$) be the space of continuous maps $\sigma: X \to B$ (resp. $\Sigma: X \times X \to \mathcal{L}(\mathbf{R}^m, \mathbf{R}^n)$), endowed with the sup-norm. Define $\mathcal{F}: S \times \mathcal{S} \to S \times \mathcal{S}$ by $\mathcal{F}(\sigma, \Sigma) = (\sigma', \Sigma')$ with

$$\sigma'(x) = F_{\overline{x}}(\sigma(\overline{x})) \tag{2.3}$$

$$\begin{aligned}\Sigma'(x,z) = {} & \Delta F^{\sigma(\overline{z})}(\overline{x},\overline{z}) \circ \Delta f^{-1}(x,z) + \\ & + \Delta F_{\overline{x}}(\sigma(\overline{x}), \sigma(\overline{z})) \circ \Sigma(\overline{x},\overline{z}) \circ \Delta f^{-1}(x,z)\end{aligned} \tag{2.4}$$

where, for simplicity, we denote $\overline{x} = f^{-1}(x)$, $\overline{z} = f^{-1}(z)$. By construction (recall also Lemma 2.5), if Σ is an intrinsic derivative for σ then Σ' is an intrinsic derivative for σ'. Observe now that $\mathcal{F}$ is a fiber contraction:

(i) Given $(\sigma_1, \Sigma_1), (\sigma_2, \Sigma_2) \in S \times \mathcal{S}$ and denoting $(\sigma_i', \Sigma_i') = \mathcal{F}(\sigma_i, \Sigma_i)$, $1 \leq i \leq 2$, and $\overline{x} = f^{-1}(x)$

$$\begin{aligned}\left\|\sigma_1' - \sigma_2'\right\| &= \sup_x \|F_{\overline{x}}(\sigma_1(\overline{x})) - F_{\overline{x}}(\sigma_2(\overline{x}))\| \leq c \sup_x \|\sigma_1(\overline{x}) - \sigma_2(\overline{x})\| \\ &\leq c\,\|\sigma_1 - \sigma_2\| \qquad\qquad (c < 1);\end{aligned}$$

(ii) Given $\sigma \in S$ and $\Sigma_1, \Sigma_2 \in \mathcal{S}$ and denoting $(\sigma', \Sigma_i') = \mathcal{F}(\sigma, \Sigma_i)$, $1 \leq i \leq 2$, and $\overline{x} = f^{-1}(x)$, $\overline{z} = f^{-1}(z)$

$$\begin{aligned}\left\|\Sigma_1' - \Sigma_2'\right\| &\leq \sup_x (\|\Delta F_{\overline{x}}(\sigma(\overline{x}), \sigma(\overline{z}))\| \cdot \|\Sigma_1(\overline{x},\overline{z}) - \Sigma_2(\overline{x},\overline{z})\| \cdot \left\|\Delta f^{-1}(x,z)\right\|) \\ &\leq a\,\|\Sigma_1 - \Sigma_2\| \qquad\qquad (a < 1).\end{aligned}$$

It follows ([HP]) that $\mathcal{F}$ has a unique fixed point $(\overline{\sigma}, \tilde{\Sigma})$ and $\mathcal{F}^n(\sigma, \Sigma) \to (\overline{\sigma}, \tilde{\Sigma})$ as $n \to +\infty$, for every $(\sigma, \Sigma) \in S \times \mathcal{S}$. Clearly $\overline{\sigma} = \tilde{\sigma}$. Choose now $\sigma_0 \in S$ intrinsically $\mathcal{C}^1$ and Σ_0 an intrinsic derivative of σ_0. Writing $\mathcal{F}^n(\sigma_0, \Sigma_0) = (\sigma_n, \Sigma_n)$ we have that Σ_n is an intrinsic derivative of σ_n for every n. Since

$(\sigma_n)_n \to \tilde{\sigma}$ and $(\Sigma_n)_n \to \tilde{\Sigma}$, it follows from Lemma 2.4 that $\tilde{\Sigma}$ is an intrinsic derivative for $\tilde{\sigma}$. This proves (a). Now we show that under the assumptions of (b) $\tilde{\Sigma}$ is γ-Hölder continuous. Fix $C > 0$ large enough so that ΔF_x, $x \in X_0$, ΔF^v, $v \in B$, and Δf^{-1} are all (C, γ)-Hölder continuous. Let $x_1, x_2, z \in X$. Subtracting the equalities

$$\tilde{\Sigma}(x_i, z) = \Delta F^{\tilde{\sigma}(\overline{z})}(\overline{x}_i, \overline{z}) \circ \Delta f^{-1}(x_i, z) + \Delta F_{\overline{x}_i}(\tilde{\sigma}(\overline{x}_i), \tilde{\sigma}(\overline{z})) \circ \tilde{\Sigma}(\overline{x}_i, \overline{z}) \circ \Delta f^{-1}(x_i, z)$$

(recall (2.4)) we obtain

$$\begin{aligned} \|\tilde{\Sigma}(x_1, z) - \tilde{\Sigma}(x_2, z)\| \le\; & C\|\overline{x}_1 - \overline{x}_2\|^\gamma \cdot \|\Delta f^{-1}\| + \|\Delta F^*\| \cdot C\|x_1 - x_2\|^\gamma \\ & + C\|\tilde{\sigma}(\overline{x}_1) - \tilde{\sigma}(\overline{x}_2)\|^\gamma \|\tilde{\Sigma}\| \cdot \|\Delta f^{-1}\| \\ & + \|\Delta F_*\| \|\tilde{\Sigma}(\overline{x}_1, \overline{z}) - \tilde{\Sigma}(\overline{x}_2, \overline{z})\| \cdot \|\Delta f^{-1}\| \\ & + \|\Delta F_*\| \cdot \|\tilde{\Sigma}\| \cdot C\|x_1 - x_2\|^\gamma, \end{aligned}$$

where $\|\Delta F^*\| = \sup_{v,x,z} \|\Delta F^v(x, z)\|$ and $\|\Delta F_*\| = \sup_{x,v,w} \|\Delta F_x(v, w)\|$. Noting that $\tilde{\sigma}$ is $\|\tilde{\Sigma}\|$-Lipschitz continuous we get

$$\begin{aligned} \|\tilde{\Sigma}(x_1, z) - \tilde{\Sigma}(x_2, z)\| \le\; & C_1\|x_1 - x_2\|^\gamma + \\ & + (\|\Delta F_*\| \, \|\Delta f^{-1}\|)\|\tilde{\Sigma}(f^{-1}x_1, f^{-1}z) - \tilde{\Sigma}(f^{-1}x_2, f^{-1}z)\| \end{aligned}$$

with $C_1 = (C\|\Delta f^{-1}\|)^{1+\gamma} + C\|\Delta F^*\| + (C\|\Delta f^{-1}\| \, \|\tilde{\Sigma}\|)^{1+\gamma} + C\|\Delta F_*\| \, \|\tilde{\Sigma}\|$. By recurrence and using the fact that

$$\begin{aligned} \|\Delta F_*\| \, \|\Delta f^{-1}\| \|f^{-1}y_1 - f^{-1}y_2\|^\gamma & \le \|\Delta F_*\| \, \|\Delta f^{-1}\|^{1+\gamma} \|y_1 - y_2\|^\gamma \\ & \le b\|y_1 - y_2\|^\gamma \end{aligned}$$

we find, for every $k \ge 1$

$$\begin{aligned} \|\tilde{\Sigma}(x_1, z) - \tilde{\Sigma}(x_2, z)\| \le\; & C_1 (\sum_0^{k-1} b^i) \|x_1 - x_2\|^\gamma + \\ & + (\|\Delta F_*\| \, \|\Delta f^{-1}\|)^k \|\tilde{\Sigma}(f^{-k}x_1, f^{-k}z) - \tilde{\Sigma}(f^{-k}x_2, f^{-k}z)\|. \end{aligned}$$

On the other hand

$$\|\Delta F_*\| \, \|\Delta f^{-1}\| \le \max\{\|\Delta F_*\|, \|\Delta F_*\| \, \|\Delta f^{-1}\|^{1+\gamma}\} \le \max\{c, b\} < 1$$

and so passing to the limit as $k \to +\infty$ leads to

$$\|\tilde{\Sigma}(x_1, z) - \tilde{\Sigma}(x_2, z)\| \le C_1 (\sum_0^{\infty} b^i) \|x_1 - x_2\|^\gamma.$$

In the same way one proves that $\tilde{\Sigma}$ is γ-Hölder continuous in the second variable. □

Remark 2.11. It is an interesting question whether the conclusion of the proposition still holds under the slightly more general (and more natural) assumptions

$$\|\Delta F_{f^{-1}(x)}(v,w)\| \cdot \|\Delta f^{-1}(x,x)\| \le a < 1, \qquad \text{for all } x \in X, v, w \in B$$
$$\|\Delta F_{f^{-1}(x)}(v,w)\| \cdot \|\Delta f^{-1}(x,x)\|^{1+\gamma} \le b < 1, \qquad \text{for all } x \in X, v, w \in B.$$

We close with another simple example which will be of future use.

Example 2.12. It is well a known fact that the map $\chi: u \to (I+u)^{-1}$ is defined and smooth on $\{u \in \mathcal{L}(\mathbf{R}^m, \mathbf{R}^m): \|u\| < 1\}$. In what follows we let $\Delta\chi$ be some (fixed) intrinsic derivative for χ on, say, $\{u: \|u\| \le 1/2\}$; observe that we must have $\|\Delta\chi(0,0)\| = \|\mathrm{D}\chi(0)\| = 1$. Let $R > 0$ be fixed and $B = \overline{\mathrm{B}}_R(0)$ denote the closed R-ball in $\mathcal{L}(\mathbf{R}^m, \mathbf{R}^n)$. Let also $H \in \mathcal{L}(\mathbf{R}^{m+n}, \mathbf{R}^{m+n})$ be fixed and write $H = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $a \in \mathcal{L}(\mathbf{R}^m, \mathbf{R}^m)$, $b \in \mathcal{L}(\mathbf{R}^n, \mathbf{R}^m)$, $c \in \mathcal{L}(\mathbf{R}^m, \mathbf{R}^n)$, $d \in \mathcal{L}(\mathbf{R}^n, \mathbf{R}^n)$. Suppose $\|b\| \left\|a^{-1}\right\| \le 1/(2R)$, so that the graph transform induced by H

$$\Gamma: h \mapsto (c+dh)\cdot(a+bh)^{-1} = (ca^{-1} + dha^{-1})\cdot\chi(bha^{-1})$$

is well defined on B. For $h_1, h_2 \in B$ and $g \in \mathcal{L}(\mathbf{R}^m, \mathbf{R}^n)$, we let

$$\begin{aligned}\Delta\Gamma(h_1,h_2)\cdot g = (dga^{-1})\cdot\chi(bh_1a^{-1}) + \\ + (ca^{-1} + dh_2a^{-1})\cdot\Delta\chi(bh_1a^{-1}, bh_2a^{-1})\cdot(bga^{-1})\end{aligned}$$

Then, clearly, $\Delta\Gamma$ is an intrinsic derivative for Γ on B. Observe that $\Delta\Gamma(h_1,h_2)$ is (uniformly) close to $g \mapsto dga^{-1}$ if $\|b\| \left\|a^{-1}\right\|$ is close to zero.

3. Intrinsically smooth foliations of basic sets

Let q_0 be a transverse homoclinic point of some hyperbolic fixed (or periodic) point p of a $\mathcal{C}^2$ diffeomorphism $\varphi: M \to M$. Our goal here is to prove that if $q_0 \notin \mathrm{W}^{\mathrm{ss}}(p)$, and under another mild (open and dense) transversality condition to be stated below, there exists a hyperbolic basic set Λ_1 containing p and q_0 and whose unstable foliation is intrinsically $\mathcal{C}^1$. For the sake of simplicity we restrict to the case when φ is $\mathcal{C}^2$-linearizable on a neighbourhood of p. Apart from the corresponding nonressonance assumptions no other conditions on the eigenvalues of $\mathrm{D}\varphi(p)$ (or the dimensions of $\mathrm{W}^{\mathrm{u}}(p)$, $\mathrm{W}^{\mathrm{s}}(p)$) are required for this construction. Moreover, a dual result holds when $q_0 \notin \mathrm{W}^{\mathrm{uu}}(p)$.

Example 3.1. Let $f: \mathbf{R}^3 \to \mathbf{R}^3$ be a $\mathcal{C}^\infty$ diffeomorphism such that for (x,y,z) close enough to the origin $f(x,y,z) = (\sigma x, \lambda y, \theta z)$ with $\sigma > 1 > \lambda > \theta > 0$ and $\lambda > \sigma\theta$. Suppose moreover that $p = (0,0,0)$ has a transverse homoclinic point of the form $q_0 = (0,0,\delta)$, δ small. Let us parametrize a segment γ of $\mathrm{W}^{\mathrm{u}}(p)$ near q_0 by $[-\varepsilon, \varepsilon] \ni x \mapsto (x, y(x), z(x))$ and assume that

$y'(0) \neq 0$. For $n \geq 1$ sufficiently large $\varphi^n(\gamma)$ intersects $\{x = \alpha\}$, α small, in the point $\tilde{q}_n = (\alpha, \lambda^n y(\alpha\sigma^{-n}),\, \theta^n z(\alpha\sigma^{-n}))$. On the other hand $q_n = \varphi^n(q_0) = (0, 0, \theta^n \delta)$ and so

$$\frac{\|\tilde{q}_n - (\alpha, 0, 0)\|}{\|q_n - (0,0,0)\|} \approx \text{const } y'(0) \left(\frac{\lambda}{\sigma\theta}\right)^n \to \infty.$$

Therefore, the projection from $\{x = 0\}$ to $\{x = \alpha\}$ along $\mathrm{W}^{\mathrm{u}}(p)$ is not even Lipschitz on $\{p\} \cup \{q_n : n \geq 0\}$. This illustrates the main obstruction for an invariant foliation of a basic set (with codimension bigger than 1) to be smooth.

Starting our construction of Λ_1, let us denote by $\sigma_1, \dots, \sigma_u$, $\lambda_1, \dots, \lambda_s$, $u + s = m$, the eigenvalues of $\mathrm{D}\varphi(p)$, with $|\sigma_u| \geq \cdots \geq |\sigma_1| > 1 > |\lambda_1| \geq \cdots \geq |\lambda_s|$. We define $1 \leq w \leq s$ by $|\lambda_1| = \cdots = |\lambda_w| > |\lambda_{w+1}| \geq \cdots \geq |\lambda_s|$ and let $\mathrm{E}^{\mathrm{s}} = \mathrm{E}^{\mathrm{w}} \oplus \mathrm{E}^{\mathrm{ss}}$ be the invariant splitting such that $\mathrm{D}\varphi(p) \,|\, \mathrm{E}^{\mathrm{w}}$ has eigenvalues $\lambda_1, \dots, \lambda_w$ and $\mathrm{D}\varphi(p) \,|\, \mathrm{E}^{\mathrm{ss}}$ has eigenvalues $\lambda_{w+1}, \dots, \lambda_s$. We choose $\mathcal{C}^2$ linearizing coordinates $(\xi_1, \dots, \xi_u, \zeta_1, \dots, \zeta_s)$ on U and, clearly, we may assume that

(A1) $\mathrm{W}^{\mathrm{u}}_{\mathrm{loc}}(p) \subset \{\zeta_1 = \cdots = \zeta_s = 0\}$ and $\mathrm{W}^{\mathrm{s}}_{\mathrm{loc}}(p) \subset \{\xi_1 = \cdots = \xi_u = 0\}$;
(A2) $\mathrm{E}^{\mathrm{w}} = \{0^u\} \times \mathbf{R}^w \times \{0^{s-w}\}$ and the strong stable manifold (tangent to E^{ss} at p) satisfies $\mathrm{W}^{\mathrm{ss}}_{\mathrm{loc}}(p) \subset \{\xi_1 = \cdots = \xi_u = \zeta_1 = \cdots = \zeta_w = 0\}$.

Up to a convenient choice of riemannian metric we have, for $\sigma = |\sigma_1|$, $\lambda = |\lambda_1| = |\lambda_w|$ and $\theta = |\lambda_{w+1}|$,

(B1) $(\sigma - \varepsilon)\, \|v\| \leq \|\mathrm{D}\varphi(p)v\|$, for all $v \in \mathrm{E}^{\mathrm{u}}$
(B2) $(\lambda - \varepsilon)\, \|v\| \leq \|\mathrm{D}\varphi(p)v\| \leq (\lambda + \varepsilon)\, \|v\|$, for all $v \in \mathrm{E}^{\mathrm{w}}$
(B3) $\|\mathrm{D}\varphi(p)v\| \leq (\theta + \varepsilon)\, \|v\|$, for all $v \in \mathrm{E}^{\mathrm{ss}}$

where $\varepsilon > 0$ is fixed small enough so that $\theta + 2\varepsilon < \lambda - 2\varepsilon < \lambda + 2\varepsilon < \sigma - 2\varepsilon$. (In the case $w = s$, i.e. if all contracting eigenvalues have the same norm, $\mathrm{E}^{\mathrm{ss}} = \{0\}$, $\mathrm{W}^{\mathrm{ss}}(p) = \{p\}$ and we leave θ undefined). Fix points $q \in \mathrm{W}^{\mathrm{s}}_{\mathrm{loc}}(p)$ and $r = \varphi^{-N}(q) \in \mathrm{W}^{\mathrm{u}}_{\mathrm{loc}}(p)$ in the orbit of q_0. Take $V = V_\delta = \{\|(\xi_1, \dots, \xi_u)\| \leq \delta\} \times \{\|(\zeta_1, \dots, \zeta_s)\| \leq \rho\}$ where $\delta > 0$ is small and $\rho > 0$ is fixed in such a way that $\{p, q\} \subset \operatorname{int}(V) \subset V \subset U$. Let $n = n(\delta)$ be minimum such that $r \in \operatorname{int}(\varphi^n(V))$. (We suppose δ conveniently adjusted so that $\varphi^{n+N}(V)$ cuts V in two cylinders as in the figure.) We let $\Lambda = \bigcap_{k \in \mathbf{Z}} \varphi^{(n+N)k}(V)$ and then take simply $\Lambda_1 = \bigcup_1^{n+N} \varphi^i(\Lambda)$. The next proposition expresses the crucial geometric property of Λ: the intrinsic tangent space to $\mathrm{W}^{\mathrm{u}}_{\mathrm{loc}}(\Lambda)$ at every $x \in \mathrm{W}^{\mathrm{u}}_{\mathrm{loc}}(\Lambda)$ is contained in a $(u + w)$-dimensional subspace whose angle to $\mathbf{R}^{u+w} \times \{0^{s-w}\}$ is uniformly bounded from above. We denote by $(v_{\mathrm{uw}}, v_{\mathrm{ss}})$ the components of a vector $v \in \mathbf{R}^m$ with respect to the splitting $\mathbf{R}^m = \mathbf{R}^{u+w} \times \mathbf{R}^{s-w}$.

FIGURE 3

PROPOSITION 3.2. *There are $C > 0$ and $\gamma > 0$, independent of the n chosen in the definition of Λ, and there exists some (C, γ)-Hölder continuous map $A\colon \mathrm{W}^{\mathrm{u}}_{\mathrm{loc}}(\Lambda) \times \mathrm{W}^{\mathrm{u}}_{\mathrm{loc}}(\Lambda) \to \mathcal{L}(\mathbf{R}^{u+w}, \mathbf{R}^{s-w})$ such that*

$$\|A(x,z)\| \le C \text{ and } (x-z)_{\mathrm{SS}} = A(x,z) \cdot (x-z)_{\mathrm{uw}} \qquad \text{for all } x, z \in \mathrm{W}^{\mathrm{u}}_{\mathrm{loc}}(\Lambda).$$

Proof. Let $W = \mathrm{W}^{\mathrm{u}}(\Lambda) \cap V$, $\psi = \varphi^{n+N}$ and V_1, V_2 be the two connected components (cylinders) of $\psi(V) \cap V$. For $k \ge 0$ let $U_k \subset W \times W$ be the set of pairs (x, z) such that

(i) $\psi^{-i}(x)$, $\psi^{-i}(z)$ belong to the same V_{ji}, $j_i = 1, 2$, for $0 \le i < k$;
(ii) $\psi^{-k}(x)$, $\psi^{-k}(z)$ belong to different V_j's.

Observe that $\widetilde{W} = \bigcup_{k \ge 0} U_k$ is dense in $W \times W$: in fact $(x, z) \in W \times W \setminus \widetilde{W}$ if and only if x and z belong to the same local unstable leaf. We construct A on each U_k, by recurrence on k, and then it extends from $\widetilde{W}$ to $W \times W$ by uniformity. The definition of $A \mid U_0$ is rather arbitrary. Clearly, there is $K > 1$ (independent of n) such that

$$\|(x-z)_{\mathrm{ss}}\| \le K \text{ and } \|(x-z)_{\mathrm{uw}}\| \ge K^{-1} \text{ for all } x \in V_i, \quad z \in V_j, \quad i \ne j.$$

Therefore, by Remark 2.6, for $C_0 > 0$ sufficiently large (depending only on K) there are C_0-Lipschitz continuous maps $A_0\colon U_0 \to \mathcal{L}(\mathbf{R}^{u+w}, \mathbf{R}^{s-w})$ with norm $\le C_0$ and satisfying

$$(x-z)_{\mathrm{ss}} = A_0(x,z) \cdot (x-z)_{\mathrm{uw}} \qquad \text{for all } (x,z) \in U_0.$$

FIGURE 4

We take $A \,|\, U_0$ to be any such map. Let now $(x, z) \in U_k$, $k \geq 1$, and denote $\overline{x} = \psi^{-1}(x)$, $\overline{z} = \psi^{-1}(z)$. Observe that $(\overline{x}, \overline{z}) \in U_{k-1}$. We write

$$\varphi^n = \begin{pmatrix} L^n & 0 \\ 0 & T^n \end{pmatrix}, \quad L \in \mathcal{L}(\mathbf{R}^{u+w}, \mathbf{R}^{u+w}), \quad T \in \mathcal{L}(\mathbf{R}^{s-w}, \mathbf{R}^{s-w})$$

with (recall above) $\left\|L^{-1}\right\| \leq (\lambda - \varepsilon)^{-1}$ and $\|T\| \leq (\theta + \varepsilon)$. Moreover, we choose as in Example 2.1 a Lipschitz continuous intrinsic derivative

$$\Delta\phi = \begin{pmatrix} \Delta_{\mathrm{uw}}\phi_{\mathrm{uw}} & \Delta_{\mathrm{ss}}\phi_{\mathrm{uw}} \\ \Delta_{\mathrm{uw}}\phi_{\mathrm{ss}} & \Delta_{\mathrm{ss}}\phi_{\mathrm{ss}} \end{pmatrix}$$

of $\phi = \varphi^N$ on a neighbourhood of $\{p, r\}$. Then $\Delta\psi(\eta, \zeta) = \Delta\phi(\varphi^n\eta, \varphi^n\zeta) \circ \varphi^n$ defines an intrinsic derivative for ψ and so, denoting $y = (\varphi^n\overline{x}, \varphi^n\overline{z})$

$$\begin{aligned} (x - z)_{\mathrm{uw}} &= \Delta_{\mathrm{uw}}\phi_{\mathrm{uw}}(y) \cdot L^n \cdot (\overline{x} - \overline{z})_{\mathrm{uw}} + \Delta_{\mathrm{ss}}\phi_{\mathrm{uw}}(y) \cdot T^n \cdot (\overline{x} - \overline{z})_{\mathrm{ss}} \\ &= (\Delta_{\mathrm{uw}}\phi_{\mathrm{uw}}(y) + \Delta_{\mathrm{ss}}\phi_{\mathrm{uw}}(y) \cdot T^n \cdot A(\overline{x}, \overline{z}) \cdot L^{-n}) \cdot L^n \cdot (\overline{x} - \overline{z})_{\mathrm{uw}} \end{aligned}$$

and analogously

$$(x - z)_{\mathrm{ss}} = (\Delta_{\mathrm{uw}}\phi_{\mathrm{ss}}(y) + \Delta_{\mathrm{ss}}\phi_{\mathrm{ss}}(y) \cdot T^n \cdot A(\overline{x}, \overline{z}) \cdot L^{-n}) \cdot L^n \cdot (\overline{x} - \overline{z})_{\mathrm{uw}}.$$

We want to define

$$\begin{aligned} A(x, z) = (\Delta_{\mathrm{uw}}\phi_{\mathrm{ss}}(y) + \Delta_{\mathrm{ss}}\phi_{\mathrm{ss}}(y) T^n A(\overline{x}, \overline{z}) L^{-n}) \circ \\ \circ\, (\Delta_{\mathrm{uw}}\phi_{\mathrm{uw}}(y) + \Delta_{\mathrm{ss}}\phi_{\mathrm{uw}}(y) T^n A(\overline{x}, \overline{z}) L^{-n})^{-1} \end{aligned} \tag{3.1}$$

so that, automatically, $(x - z)_{\mathrm{ss}} = A(x, z) \cdot (x - z)_{\mathrm{uw}}$. *In order to show that* (3.1) *makes sense we need the following generic (open and dense) hypothesis*

$$\mathrm{D}_{\mathrm{uw}}\phi_{\mathrm{uw}}(r) \text{ is an isomorphism.} \tag{3.2}$$

Here D *denotes the usual derivative and* (3.2) *means that unstable/weak-stable directions are not sent to strong-stable directions by* φ^N. Notice now that if $\delta > 0$ is small (and then n is large) V_1 and V_2 are small and so $y = (\varphi^{-N}(x), \varphi^{-N}(z))$ must be close to either (r,r) or (p,p). This means that (up to reducing δ) we may assume $\Delta_{\mathrm{uw}}\phi_{\mathrm{uw}}(y)$ to be close to $\mathrm{D}_{\mathrm{uw}}\phi_{\mathrm{uw}}(r) = \Delta_{\mathrm{uw}}\phi_{\mathrm{uw}}(r,r)$ or $\mathrm{D}_{\mathrm{uw}}\phi_{\mathrm{uw}}(p) = \Delta_{\mathrm{uw}}\phi_{\mathrm{uw}}(p,p)$. On the other hand, if n is large we also have

$$\|\Delta_{\mathrm{ss}}\phi_{\mathrm{uw}}(y)T^n A(\overline{x},\overline{z})L^{-n}\| \le \mathrm{const} C_0 \left(\frac{\theta+\varepsilon}{\lambda-\varepsilon}\right)^n$$

small. It follows from (3.2) that $(\Delta_{\mathrm{uw}}\phi_{\mathrm{uw}}(y) + \Delta_{\mathrm{ss}}\phi_{\mathrm{uw}}(y)T^n A(\overline{x},\overline{z})L^{-n})$ is invertible and the norm of its inverse is at most

$$a_1 = 1 + \max\{\|(\mathrm{D}_{\mathrm{uw}}\phi_{\mathrm{uw}}(p))^{-1}\|, \|(\mathrm{D}_{\mathrm{uw}}\phi_{\mathrm{uw}}(r))^{-1}\|\}.$$

Moreover, the same argument also shows that if n is taken large enough the norm of $(\Delta_{\mathrm{uw}}\phi_{\mathrm{ss}}(y) + \Delta_{\mathrm{ss}}\phi_{\mathrm{ss}}(y)T^n A(\overline{x},\overline{z})L^{-n})$ is at most

$$a_2 = 1 + \max\{\|\mathrm{D}_{\mathrm{uw}}\phi_{\mathrm{ss}}(p)\|, \|\mathrm{D}_{\mathrm{uw}}\phi_{\mathrm{ss}}(r)\|\}.$$

This proves that (3.1) is indeed defined and gives $\|A(x,z)\| \le C_0$, as long as we take from the beginning $C_0 \ge a_1 a_2$. Successive repetition of this procedure extends the definition of A to $\widetilde{W} = \bigcup_{k\ge 0} U_k$ (keeping $\|A\| \le C_0$) and now the proof will be complete if we show that such A is (C,γ)-Hölder continuous for some $\gamma > 0$ and $C > C_0$. Let $(x,z) \in U_k$ and $(\tilde{x},z) \in U_{\tilde{k}}$ with $k \le \tilde{k}$. Denote $x_i = \Psi^{-i}(x)$, $\tilde{x}_i = \Psi^{-i}(\tilde{x})$ and $z_i = \Psi^{-i}(z)$, for $i \ge 0$. If $k < \tilde{k}$ then x_k and $\tilde{x}_k$ belong to different V_j's and so $\|A(x_k,z_k) - A(\tilde{x}_k,z_k)\| \le 2C_0 \le 2C_0 K \|x_k - \tilde{x}_k\|$. If, on the contrary, $k = \tilde{k}$ then both (x_k,z_k) and $(\tilde{x}_k,z_k)$ belong to U_0 and so, by construction, $\|A(x_k,z_k) - A(\tilde{x}_k,z_k)\| \le C_0 \|x_k - \tilde{x}_k\|$. Now we proceed by induction: let $0 \le i < k$ and assume that

$$\|A(x_{i+1},z_{i+1}) - A(\tilde{x}_{i+1},z_{i+1})\| \le C\|x_{i+1} - \tilde{x}_{i+1}\|^\gamma$$

for some $0 < \gamma < 1$ and some large $C > 0$. Clearly,

$$\|x_i - \tilde{x}_i\| \ge (\|\mathrm{D}\varphi^{-1}\|)^{-(n+N)}\|x_{i+1} - \tilde{x}_{i+1}\|.$$

On the other hand, expressing $A(x_i,z_i)$ and $A(\tilde{x}_i,z_i)$ in terms of $A(x_{i+1},z_{i+1})$, $A(\tilde{x}_{i+1},z_{i+1})$ and then subtracting we get

$$\|A(x_i,z_i) - A(\tilde{x}_i,z_i)\| \le a_3\|x_i - \tilde{x}_i\| + \\ + a_4\left(\frac{\theta+\varepsilon}{\lambda-\varepsilon}\right)^n \|A(x_{i+1},z_{i+1}) - A(\tilde{x}_{i+1},z_{i+1})\|$$

with a_3, a_4 depending only on $\phi = \varphi^N$. Taking $C > 0$ sufficiently large with respect to a_3 and $\operatorname{diam}(U)$ we obtain $a_3\|x_i - \tilde{x}_i\| \le (C/2)\|x_i - \tilde{x}_i\|^\gamma$. On the other hand

$$\begin{aligned} a_4 \left(\frac{\theta+\varepsilon}{\lambda-\varepsilon}\right)^n \|A(x_{i+1}, z_{i+1}) - A(\tilde{x}_{i+1}, z_{i+1})\| \le \\ \le C a_4 \|\mathrm{D}\varphi^{-1}\|^{\gamma N} \left(\frac{\theta+\varepsilon}{\lambda-\varepsilon} \cdot \|\mathrm{D}\varphi^{-1}\|^{\gamma}\right)^n \|x_i - \tilde{x}_i\|^{\gamma} \\ \le \frac{C}{2} \|x_i - \tilde{x}_i\|^{\gamma}, \end{aligned}$$

as long as we fix $\gamma < \log((\lambda-\varepsilon)/(\theta+\varepsilon))/\log\|\mathrm{D}\varphi^{-1}\|$ (and take n large enough). Hence $\|A(x_i, z_i) - A(\tilde{x}, z_i)\| \le C\|x_i - \tilde{x}_i\|^{\gamma}$ and by induction this gives $\|A(x,z) - A(\tilde{x}, z)\| \le C\|x - \tilde{x}\|$. By symmetry, Hölder continuity in the second variable also follows. □

Remark 3.3. It is also clear from the argument that $A(p,p) = 0$, i.e. $\mathrm{IT}_p \mathrm{W}^{\mathrm{u}}_{\mathrm{loc}}(\Lambda_1) \subset \mathrm{E}^{\mathrm{u}} \oplus \mathrm{E}^{\mathrm{w}}$.

PROPOSITION 3.4. *For Λ_1 as above the map $\mathcal{F}\colon \mathrm{W}^{\mathrm{u}}(\Lambda_1) \ni x \mapsto \mathrm{T}_x \mathrm{W}^{\mathrm{u}}(x)$ is intrinsically $\mathcal{C}^1$ on compact parts of $\mathrm{W}^{\mathrm{u}}(\Lambda_1)$.*

Proof. Given any compact $K \subset \mathrm{W}^{\mathrm{u}}(\Lambda_1)$ there is $k \ge 0$ such that $K \subset \bigcup_1^{n+N} \varphi^{k+i}(W)$ where, as before, $W = \mathrm{W}^{\mathrm{u}}(\Lambda) \cap V$. Note that the $\varphi^{k+i}(W)$, $1 \le i \le n+N$, are two by two disjoint and also that $\mathcal{F}(\varphi^{k+i}(x)) = \mathrm{D}\varphi^{k+i}(x) \cdot \mathcal{F}(x)$ for all $x \in \mathrm{W}^{\mathrm{u}}(\Lambda_1)$. Therefore, the proposition will follow if we prove that the restriction of $\mathcal{F}$ to W is intrinsically $\mathcal{C}^1$ and we proceed to do this. The argument goes as follows. Let

$$\mathrm{D}\psi = \begin{pmatrix} \mathrm{D_u}\psi_{\mathrm{u}} & \mathrm{D_s}\psi_{\mathrm{u}} \\ \mathrm{D_u}\psi_{\mathrm{s}} & \mathrm{D_s}\psi_{\mathrm{s}} \end{pmatrix} \quad \text{and} \quad \mathrm{D}\phi = \begin{pmatrix} \mathrm{D_u}\phi_{\mathrm{u}} & \mathrm{D_s}\phi_{\mathrm{u}} \\ \mathrm{D_u}\phi_{\mathrm{s}} & \mathrm{D_s}\phi_{\mathrm{s}} \end{pmatrix}$$

denote the expression of the derivatives of $\psi = \varphi^{n+N}$ and $\phi = \varphi^N$ in the splitting $\mathbf{R}^m = \mathbf{R}^u \times \mathbf{R}^s$. Note that for $x \in W$

$$\mathrm{D}\psi(x) = \mathrm{D}\phi(\varphi^n x) \cdot \begin{pmatrix} U^n & 0 \\ 0 & S^n \end{pmatrix}$$

with $U \in \mathcal{L}(\mathbf{R}^u, \mathbf{R}^u)$, $S \in \mathcal{L}(\mathbf{R}^s, \mathbf{R}^s)$ satisfying $\|U^{-1}\| \le (\sigma-\varepsilon)^{-1}$ and $\|S\| \le (\lambda+\varepsilon)$. We use Proposition 2.10 with $f = \psi$, $X_0 = W$, $X = f(W)$, B a closed disk of radius $R = \|\mathrm{D_u}\phi_{\mathrm{s}}\| \, \|(\mathrm{D_u}\phi_{\mathrm{u}})^{-1}\| + 1$ in $\mathcal{L}(\mathbf{R}^u, \mathbf{R}^s) \simeq \mathbf{R}^{u \times s}$ and $\tilde{F}\colon X_0 \times B \to X \times B$ being the graph transform induced by $\mathrm{D}\psi$: given $x \in X_0$ and $h \in B$

$$F_x(h) = (\mathrm{D_u}\psi_{\mathrm{s}}(x) + \mathrm{D_s}\psi_{\mathrm{s}}(x) \cdot h) \cdot (\mathrm{D_u}\psi_{\mathrm{u}}(x) + \mathrm{D_s}\psi_{\mathrm{u}}(x) \cdot h)^{-1}. \tag{3.3}$$

Observe that, as long as n is taken sufficiently large (by choosing δ small),

$$\|\mathrm{D_s}\psi_{\mathrm{u}}(x) \cdot h \cdot (\mathrm{D_u}\psi_{\mathrm{u}}(x))^{-1}\| \le R\,\|\mathrm{D_s}\phi_{\mathrm{u}}\|\,\|(\mathrm{D_u}\phi_{\mathrm{u}})^{-1}\|\,(\lambda+\varepsilon)^n(\sigma-\varepsilon)^{-n} \ll 1$$

and, analogously, $\left\|\mathrm{D_s}\psi_{\mathrm{s}}(x)\cdot h\cdot(\mathrm{D_u}\psi_{\mathrm{u}}(x))^{-1}\right\| \ll 1$. Therefore $F_x(h)$ is well defined by (3.3) and it belongs to B: $\|F_x(h)\| \le \left\|\mathrm{D_u}\psi_{\mathrm{s}}(x)\cdot(\mathrm{D_u}\psi_{\mathrm{u}}(x))^{-1}\right\|+1 \le R$. Clearly, $\tilde{F}$ is intrinsically $\mathcal{C}^1$ (it extends to a $\mathcal{C}^1$ map on a neighbourhood of $X_0\times B$) and the combination of the constructions in Lemma 2.5 and Example 2.12 yields an intrinsic derivative

$$\Delta\tilde{F}((x_1,h_1),(x_2,h_2)) = \begin{pmatrix} \Delta f(x_1,x_2) & 0 \\ \Delta F^{h_1}(x_1,x_2) & \Delta F_{x_2}(h_1,h_2) \end{pmatrix}$$

satisfying, for some $C_1>0$ (depending only on ϕ),

$$\|\Delta F_y(h_1,h_2)\| \le C_1\left(\frac{\lambda+\varepsilon}{\sigma-\varepsilon}\right)^n \qquad \text{for all } y\in X_0 \text{ and all } h_1,h_2\in B.$$

It is also clear that f^{-1} is intrinsically $\mathcal{C}^1$ but choosing Δf^{-1} in such a way that the hypothesis of Propositin 2.10 holds is somewhat delicate. Using Proposition 3.2 we may write for $x,z\in X$

$$\begin{aligned} &f^{-1}(x)-f^{-1}(z) = \\ &= ((f^{-1}(x)-f^{-1}(z))_{\mathrm{uw}}, A(f^{-1}(x),f^{-1}(z))(f^{-1}(x)-f^{-1}(z))_{\mathrm{uw}}) \\ &= (L^{-n}(\phi^{-1}(x)-\phi^{-1}(z))_{\mathrm{uw}}, A(f^{-1}(x),f^{-1}(z))L^{-n}(\phi^{-1}(x)-\phi^{-1}(z))_{\mathrm{uw}}). \end{aligned}$$

Therefore, fixing $\Delta\phi^{-1}$ an intrinsic derivative for ϕ^{-1},

$$\Delta f^{-1}(x,z) = \begin{pmatrix} L^{-n} & 0 \\ A(f^{-1}(x),f^{-1}(z))\cdot L^{-n} & 0 \end{pmatrix}\cdot\Delta\phi^{-1}(x,z)$$

is an intrinsic derivative for f. Recall moreover that $\left\|L^{-1}\right\| \le (\lambda-\varepsilon)^{-1}$. Thus, for some $C_2>0$ depending only on C_0 and ϕ, $\left\|\Delta f^{-1}(x,z)\right\| \le C_2(\lambda-\varepsilon)^{-n}$, for all $x,z\in X$. In this way, having fixed $\varepsilon>0$ and $\delta>0$ small enough,

$$\|\Delta F_y(h_1,h_2)\|\cdot\left\|\Delta f^{-1}(x,z)\right\| \le C_1C_2\left(\frac{(\lambda+\varepsilon)}{(\lambda-\varepsilon)(\sigma-\varepsilon)}\right)^n \le \frac{1}{2}, \tag{3.4}$$

for all $y\in X_0$, $x,z\in X$ and $h_1,h_2\in B$. This proves that the invariant section $\tilde{\sigma}$ of $\tilde{F}$ is intrinsically $\mathcal{C}^1$ and the proposition follows since clearly $\mathrm{T}_x\mathrm{W^u}(x) = \operatorname{graph}(\tilde{\sigma}(x))$ for every $x\in W$. □

Proposition 3.5. *Let $x\in\mathrm{W^u}(\Lambda_1)$, Σ_0 and Σ_1 be (small) $\mathcal{C}^1$ sections transverse to $\mathrm{W^u}(y)$ and $\pi\colon\Sigma_0\cap\mathrm{W^u}(\Lambda_1)\to\Sigma_1\cap\mathrm{W^u}(\Lambda_1)$ denote the projection along the leaves of $\mathrm{W^u}(\Lambda_1)$. Then π is intrinsically $\mathcal{C}^1$.*

Proof. Fix coordinates $(\eta,\zeta)=(\eta_1,\dots,\eta_u,\zeta_1,\dots,\zeta_s)$ such that $\Sigma_0\subset\{\eta=(0,0,\dots,0)\}$, $\Sigma_1\subset\{\eta=(1,0,\dots,0)\}$ and each leaf intersecting Σ_0 in a point $(0,z)$ can be written as the graph of a function $\zeta=G(\eta;z)$. Then

$$\operatorname{graph}\left(\frac{\partial G}{\partial\eta}(\eta;z)\right) = \mathrm{T}_{(\eta,G(\eta;z))}\mathrm{W^u}(\eta,G(\eta;z))$$

and so, by the previous proposition, $(\eta, G(\eta; z)) \mapsto \frac{\partial G}{\partial \eta}(\eta; z)$ is an intrinsically $\mathcal{C}^1$ map. For $-\varepsilon < t < 1+\varepsilon$ we define $g(t,z) = G(t\tilde{\eta}; z)$, where $\tilde{\eta} = (1, 0, \dots, 0)$, and then

$$H: (t, g(t; z)) \mapsto \frac{dg}{dt}(t; z) = \frac{\partial G}{\partial \eta_1}(t\tilde{\eta}; z)$$

is intrinsically $\mathcal{C}^1$. Notice that $\pi(0, z) = (1, G(\tilde{\eta}; z)) = (1, g(1; z))$. We fix an intrinsic derivative ΔH of H and define, for each t, $\Delta H_t(\zeta_1, \zeta_2) \cdot v = \Delta H((t, \zeta_1), (t, \zeta_2)) \cdot (0, v)$ (recall Lemma 2.5). For $(0, z_1), (0, z_2) \in \Sigma_0 \cap \mathrm{W}^{\mathrm{u}}(\Lambda_1)$ we denote by $\Gamma(t; z_1, z_2) \in \mathcal{L}(\mathbf{R}^u, \mathbf{R}^u)$ the unique solution of the initial value problem

$$\begin{cases} \frac{d\Gamma}{dt}(t; z_1, z_2) = \Delta H_t(g(t; z_1), g(t; z_2)) \cdot \Gamma(t; z_1, z_2) \\ \Gamma(0; z_1, z_2) = \mathrm{id} \end{cases}$$

Then $(z_1, z_2) \mapsto \Gamma(t; z_1, z_2)$ is continuous and we claim that it is an intrinsic derivative for $z \mapsto g(t; z)$. In fact, putting $\theta(t) = g(t; z_1) - g(t; z_2) - \Gamma(t; z_1, z_2)(z_1 - z_2)$, we get

$$\begin{cases} \frac{d\theta}{dt}(t) = \Delta H_t(g(t; z_1), g(t; z_2)) \cdot \theta(t) \\ \theta(0) = 0 \end{cases}$$

Hence θ is identically zero, which proves the claim. In particular $(z_1, z_2) \mapsto \Gamma(1; z_1, z_2)$ is an intrinsic derivative for $z \mapsto g(1; z)$ and so π is intrinsically $\mathcal{C}^1$, as we stated. □

Remark 3.6. In particular π is bi-Lipschitz continuous and so it preserves metric invariants such as Hausdorff dimension or limit capacity.

Remark 3.7. If one assumes that φ is of class $\mathcal{C}^3$ then stronger conclusions follow from the arguments above: using Proposition 2.10 (b) one gets that $\mathcal{F}: \mathrm{W}^{\mathrm{u}}(\Lambda_1) \ni x \mapsto \mathrm{T}_x \mathrm{W}^{\mathrm{u}}(x)$ is intrinsically $\mathcal{C}^{1+\gamma}$ for some $\gamma > 0$ and so the same holds for the holonomy maps $\pi: \Sigma_0 \cap \mathrm{W}^{\mathrm{u}}(\Lambda_1) \to \Sigma_1 \cap \mathrm{W}^{\mathrm{u}}(\Lambda_1)$ of $\mathrm{W}^{\mathrm{u}}(\Lambda_1)$. On the other hand, this last conclusion will also be obtained (in the case $w = 1$) in Proposition 4.3, by a different approach which applies also if φ is just $\mathcal{C}^2$.

4. Thickness in higher dimensions

Let Λ_1 be as constructed in the previous section. Here we assume that $w = 1$, i.e. $\mathrm{D}\varphi(p)$ has a unique (necessarily real) weakest contracting eigenvalue $\lambda = \lambda_1$. Then we consider $\pi: \Lambda_1 \cap \mathrm{W}^{\mathrm{s}}_{\mathrm{loc}}(p) \to \mathbf{R}$ to be an arbitrary intrinsically $\mathcal{C}^1$ map such that $\ker(\Delta\pi(p,p))$ does not contain $\mathrm{IT}_p(\Lambda_1 \cap \mathrm{W}^{\mathrm{s}}_{\mathrm{loc}}(p)) = \mathrm{E}^{\mathrm{w}}$ (i.e. $\Delta\pi(p,p) \,|\, \mathrm{E}^{\mathrm{w}}$ is bijective) and we show that $\tau(\pi(\Lambda_1 \cap \mathrm{W}^{\mathrm{s}}_{\mathrm{loc}}(p)), \pi(p))$ is independent of the choice of π. We call

$$\tau^{\mathrm{u}}(\Lambda_1, p) = \tau(\pi(\Lambda_1 \cap \mathrm{W}^{\mathrm{s}}_{\mathrm{loc}}(p)), \pi(p)), \text{ any } \pi \text{ as above,}$$

the *local unstable thickness* of Λ_1 at p and prove that it is strictly positive and varies continuously with the diffeomorphism: a $\mathcal{C}^2$-small perturbation of φ yields a small variation of $\tau^{\mathrm{u}}(\Lambda_1, p)$.

We keep the notations of Section 3. Let $\pi_{\mathrm{w}}: \Lambda_1 \cap \mathrm{W}^{\mathrm{s}}_{\mathrm{loc}}(p) \to \mathbf{R}$ be the restriction to $\Lambda_1 \cap \mathrm{W}^{\mathrm{s}}_{\mathrm{loc}}(p) \subset \{0^u\} \times \mathbf{R}^s$ of the projection $(\xi_1, \dots, \xi_u, \zeta_1, \dots, \zeta_s) \mapsto \zeta_1$. We claim that π_{w} is a homeomorphism onto its image K^{w} and moreover π_{w}^{-1} is intrinsically $\mathcal{C}^{1+\gamma}$ on K^{w}. In fact, by Proposition 3.2, for every $x, z \in \Lambda_1 \cap \mathrm{W}^{\mathrm{s}}_{\mathrm{loc}}(p)$ we have

$$(x-z)_{\mathrm{ss}} = A(x,z) \cdot (0^u, \pi_{\mathrm{w}}(x) - \pi_{\mathrm{w}}(z))$$

and so (for $C \geq \|A\|$ as in the proposition)

$$|\pi_{\mathrm{w}}(x) - \pi_{\mathrm{w}}(z)| \leq \|x - z\| \leq (C+1)\, |\pi_{\mathrm{w}}(x) - \pi_{\mathrm{w}}(z)|\,.$$

This proves that π_{w} is invertible and π_{w}^{-1} is Lipschitz continuous. Moreover,

$$\Delta\pi_{\mathrm{w}}^{-1}(s,t): \mathbf{R} \ni v \mapsto (v, A(\pi_{\mathrm{w}}^{-1}(s), \pi_{\mathrm{w}}^{-1}(t)) \cdot (0^u, v)) \in \mathbf{R}^s$$

defines a Hölder continuous intrinsic derivative for π_{w}^{-1}.

Let now $\pi: \Lambda_1 \cap \mathrm{W}^{\mathrm{s}}_{\mathrm{loc}}(p) \to \mathbf{R}$ be an intrinsically $\mathcal{C}^1$ map such that the kernel of $\Delta\pi(p,p)$ does not contain $\mathrm{IT}_p(\Lambda_1 \cap \mathrm{W}^{\mathrm{s}}_{\mathrm{loc}}(p)) = \operatorname{graph}(A(p,p) \,|\, \{0^u\} \times \mathbf{R})$. Then $\pi \circ \pi_{\mathrm{w}}^{-1}$ is intrinsically $\mathcal{C}^1$ with $\Delta(\pi \circ \pi_{\mathrm{w}}^{-1})(0,0) = \Delta\pi(p,p) \cdot \Delta\pi_{\mathrm{w}}^{-1}(0,0) \neq 0$. Since $\pi(\Lambda_1 \cap \mathrm{W}^{\mathrm{s}}_{\mathrm{loc}}(p)) = (\pi \circ \pi_{\mathrm{w}}^{-1})(K^{\mathrm{w}})$, we conclude that $\tau(\pi(\Lambda_1 \cap \mathrm{W}^{\mathrm{s}}_{\mathrm{loc}}(p)), \pi(p)) = \tau(K^{\mathrm{w}}, 0)$, as a consequence of the following simple result.

LEMMA 4.1. *Let $K \subset \mathbf{R}$ be a Cantor set, $y \in K$ and $g: K \to \mathbf{R}$ be an intrinsically $\mathcal{C}^1$ map with $\Delta g(y,y) \neq 0$. Then $\tau(g(K), g(y)) = \tau(K, y)$.*

Proof. Fix $\delta > 0$ small and take $\varepsilon > 0$ such that

$$(4.1) \quad |\Delta g(x_1, z_1)| \leq (1+\delta)\, |\Delta g(x_2, z_2)| \ \text{ for all } x_1, x_2, z_1, z_2 \in [y-\varepsilon, y+\varepsilon].$$

Clearly, g is a homeomorphism on $[y-\varepsilon, y+\varepsilon]$ if ε is small enough. Hence, given a Cantor set $L \subset [y-\varepsilon, y+\varepsilon]$, $\mathcal{U} = (U_n)_n$ is a presentation of L if and only if $g(\mathcal{U}) = (g(U_n))_n$ is a presentation of $g(L)$. From (4.1) we get that for every $u \in \partial U_n$, $n \geq 1$,

$$\tau(g(L), g(\mathcal{U}), g(u)) \leq (1+\delta)\tau(L, \mathcal{U}, u).$$

Therefore, $\tau(g(L)) \leq (1+\delta)\tau(L)$ for every such L and so $\tau(g(K), g(y)) \leq (1+\delta)\tau(K,y)$. Since δ is arbitrary we conclude that $\tau(g(K), g(y)) \leq \tau(K,y)$ and the reverse inequality is proved in the same way. $\square$

For future use, let us state also the next result, which follows from the previous properties by well-known arguments: part (a) is a direct consequence of Proposition 3.5; for part (b) one also uses the transitivity of $\varphi \,|\, \Lambda_1$.

PROPOSITION 4.2.

(a) Let $q \in \mathrm{W}^{\mathrm{u}}(p)$, Σ *be a* $\mathcal{C}^1$ *section transverse to* $\mathrm{W}^{\mathrm{u}}(p)$ *at the point* q *and* $\pi\colon \mathrm{W}^{\mathrm{u}}(\Lambda_1) \cap \Sigma \to \mathbf{R}$ *be an intrinsically* $\mathcal{C}^1$ *map such that* $\mathrm{IT}_q(\mathrm{W}^{\mathrm{u}}(\Lambda_1) \cap \Sigma)$ *is not contained in* $\ker(\Delta\pi(p,p))$. *Then* $\tau(\pi(\mathrm{W}^{\mathrm{u}}(\Lambda_1) \cap \Sigma), \pi(q)) = \tau^{\mathrm{u}}(\Lambda_1, p)$.

(b) More generally, given $z \in \mathrm{W}^{\mathrm{u}}(\Lambda_1)$, Σ *a transverse section to* $\mathrm{W}^{\mathrm{u}}(\Lambda_1)$ *at* z *and* $\pi\colon \mathrm{W}^{\mathrm{u}}(\Lambda_1) \cap \Sigma \to \mathbf{R}$ *submmersion with* $\mathrm{IT}_z(\mathrm{W}^{\mathrm{u}}(\Lambda_1) \cap \Sigma) \not\subset \ker(\Delta\pi(p,p))$, *then* $\tau(\pi(\mathrm{W}^{\mathrm{u}}(\Lambda_1) \cap \Sigma), \pi(z))$ *is equal to* $\tau^{\mathrm{u}}(\Lambda_1, p)$.

Our strategy to prove that $\tau^{\mathrm{u}}(\Lambda_1, p) = \tau(K^{\mathrm{w}}, 0)$ is positive is to write K^{w} as an invariant set of some $\mathcal{C}^{1+\gamma}$ expanding map, so that the reasoning in Proposition 6 of [N3] (in the language of [PT, Ch.IV]) can be applied to it. For the construction of this map we proceed as follows. Let $\widetilde{W}$ be the connected component of $\mathrm{W}^{\mathrm{s}}(p) \cap V$ containing p. By construction, for every $x \in \Lambda_1$ the connected component of $\mathrm{W}^{\mathrm{u}}(x) \cap V$ containing x intersects $\widetilde{W}$ in a unique point, which we denote by $\pi_{\mathrm{s}}(x)$. Observe also that every $\widetilde{W}_\ell = \{z\colon \psi^j(z) \in V \text{ for } 0 \le j < \ell \text{ and } \psi^\ell(z) \in \widetilde{W}\}$, $\ell \ge 1$, has exactly 2^ℓ components, denoted by $W_{j,\ell}$, $1 \le j \le 2^\ell$. For each $\ell \ge 1$ we define a map

$$g_\ell\colon K^{\mathrm{w}} \to K^{\mathrm{w}} \text{ by } g_\ell = \pi_{\mathrm{w}} \circ \pi_{\mathrm{s}} \circ \psi^{-\ell} \circ \pi_{\mathrm{w}}^{-1}.$$

Clearly, g_ℓ is intrinsically $\mathcal{C}^1$ and it follows from the next result that it is even intrinsically $\mathcal{C}^{1+\gamma}$.

PROPOSITION 4.3. *There are* $C > 0$ *and* $\gamma \in (0,1)$ *such that for every* $\ell \ge 1$ *and* $1 \le j \le 2$ *the map* $\pi_{\mathrm{s}} \,|\, (W_{j,\ell} \cap \Lambda_1)\colon W_{j,\ell} \cap \Lambda_1 \to \widetilde{W}$ *admits a* (C,γ)*-Hölder continuous intrinsic derivative.*

For the sake of clearness we postpone the proof of this result to the end of the section. Recall also Remark 3.7.

Observe that g_ℓ maps K^{w} onto K^{w} in a 2^ℓ to 1 way: for each $1 \le j \le 2^\ell$ and $K_{j,\ell} = \pi_{\mathrm{w}}(\psi^\ell(W_{j,\ell} \cap \Lambda_1))$ the map $g_\ell \,|\, K_{j,\ell}\colon K_{j,\ell} \to K^{\mathrm{w}}$ is a homeomorphism. It is also easy to check that the $K_{j,\ell}$, $1 \le j \le 2^\ell$, have their convex hulls $\hat{K}_{j,\ell}$ two by two disjoint. Moreover, for $\ell \ge 1$ large enough every $(g_\ell \,|\, K_{j,\ell})$ is an expansion:

$$\|\Delta g_\ell(t,r)\| \ge 2 \text{ (say) for every } t, r \in K_{j,\ell}. \tag{4.2}$$

We now fix $\ell \ge 1$ so that (4.2) holds and denote $g = g_\ell$ and $K_j = K_{j,\ell}$. At this stage the fact that K^{w} has strictly positive local thickness can be proved just by translating, in a more or less straightforward way, the arguments in [N3], [PT, Ch.IV] into our language of intrinsic differentiability. Alternatively, one can show that g can in fact be extended to a $\mathcal{C}^{1+\gamma}$ map G defined on a neighbourhood of K^{w}, which is easy to prove in this one-dimensional setting.

LEMMA 4.4. *Let $K \subset \mathbf{R}$ be a Cantor set and $g: K \to \mathbf{R}$ be intrinsically $\mathcal{C}^1$ (resp. intrinsically $\mathcal{C}^{1+\gamma}$). Then there exists $G: \mathbf{R} \to \mathbf{R}$ a $\mathcal{C}^1$ (resp. $\mathcal{C}^{1+\gamma}$) extension of g.*

Proof. Fix $\theta: [0,1] \to [0,1]$ a $\mathcal{C}^\infty$ function such that $\theta(x) = 0$ and $\theta(1-x) = 1$ whenever $0 \le x \le 1/3$. Let also $a_0 = \inf K$, $b_0 = \sup K$. We set: $G(x) = g(x)$ if $x \in K$;

$$G(x) = (g(a) + \Delta g(a,a)(x-a)) \left(1 - \theta\left(\frac{x-a}{b-a}\right)\right) + \\ + (g(b) + \Delta g(b,b)(x-b))\theta\left(\frac{x-a}{b-a}\right)$$

if x belongs to a gap (a,b) of K; $G(x) = g(b_0) + \Delta g(b_0, b_0)(x - b_0)$ if $x > b_0$ and $G(x) = g(a_0) + \Delta g(a_0, a_0)(x - a_0)$ if $x < a_0$. We also define $H: \mathbf{R} \to \mathbf{R}$ by putting: $H(x) = \Delta g(x,x)$ if $x \in K$;

$$H(x) = \Delta g(a,a) \left(1 - \theta\left(\frac{x-a}{b-a}\right)\right) + \Delta g(b,b)\theta\left(\frac{x-a}{b-a}\right) + \\ + \theta'\left(\frac{x-a}{b-a}\right)\left(\Delta g(b,a) - \Delta g(b,b) + \left(\frac{x-a}{b-a}\right)(\Delta g(b,b) - \Delta g(a,a))\right)$$

if $x \in (a,b)$, a gap of K; $H(x) = \Delta g(b_0, b_0)$ if $x > b_0$ and $H(x) = \Delta g(a_0, a_0)$ if $x < a_0$. It is a simple exercise to check that H is continuous ($\mathcal{C}^\gamma$ if Δg is $\mathcal{C}^\gamma$) and $G'(x) = H(x)$ for every $x \in \mathbf{R}$. □

This means that K^{w} is a *dynamically defined Cantor set*, in the same sense as in [PT, Ch.IV] and our claim that $\tau(K^{\mathrm{w}}, 0) > 0$ follows.

Let us now observe that (K^{w}, G), as we constructed it, varies continuously with the diffeomorphism φ. In order to explain this affirmative we let $\tilde{\varphi}$ be a diffeomorphism $\mathcal{C}^2$-close to φ and denote by $K^{\mathrm{w}}(\tilde{\varphi}), K_j(\tilde{\varphi}), \hat{K}_j(\tilde{\varphi})$ and $G(\tilde{\varphi})$ the objects obtained by performing the above construction for $\tilde{\varphi}$ (note that we may, and do, take $\ell(\tilde{\varphi}) = \ell$ if $\tilde{\varphi}$ is close enough to φ). Then it is straightforward to check that $(K^{\mathrm{w}}(\tilde{\varphi}), G(\tilde{\varphi}))$ is close to (K^{w}, G) in the sense that

(a) corresponding endpoints of $\hat{K}_j(\tilde{\varphi})$ and $\hat{K}_j$ are close;
(b) $G(\tilde{\varphi})$ is $\mathcal{C}^1$-close to G and their derivatives have nearby Hölder constants (see also the proof of Proposition 4.3 below).

It follows, see again [PT, Ch.IV], that $\tau(K^{\mathrm{w}}(\tilde{\varphi}), 0)$ is close to $\tau(K^{\mathrm{w}}, 0)$ and this proves that the local unstable thickness $\tau^{\mathrm{u}}(\Lambda_1, p)$ is a continuous function of the diffeomorphism, as we claimed.

We close this section by presenting the proof of Proposition 4.3.

Proof. We keep the notations from the proof of Proposition 3.2. Moreover, we use C and γ as generic notations for constants ($C > 0$ large, $0 < \gamma < 1$)

depending only on the diffeomorphism φ. Let $\mathcal{S} = \{S(x): x \in \Lambda_1\}$ where $S(x)$ denotes the connected component of $W^s(x) \cap V$ containing x. Note that $W_{j,\ell} \in \mathcal{S}$ for every $1 \le j \le 2^\ell$, $\ell \ge 1$. For $S, \tilde{S} \in \mathcal{S}$ let $\pi(S, \tilde{S}; \cdot): S \cap \Lambda_1 \to \tilde{S} \cap \Lambda_1$ be the projection along the leaves of $W^u(\Lambda_1)$, inside V. By Proposition 3.5 $\pi(S, \tilde{S}; \cdot)$ is C-Lipschitz continuous. On the other hand, as observed before, $\{\|x - z\| : (x, z) \in (S \times S) \cap U_0, S \in \mathcal{S}\}$ is bounded away from zero. Hence (see Remark 2.6) there is a C-Lipschitz continuous map $\tilde{\Delta}(S, \tilde{S}; \cdot, \cdot): ((S \cap \Lambda_1) \times (S \cap \Lambda_1)) \cap U_0 \to \mathcal{L}(\mathbf{R}^s, \mathbf{R}^s)$ such that

$$\pi(S, \tilde{S}; x) - \pi(S, \tilde{S}; z) = \tilde{\Delta}(S, \tilde{S}; x, z) \cdot (x - z) \tag{4.3}$$

for every $(x, z) \in ((S \cap \Lambda_1) \times (S \cap \Lambda_1)) \cap U_0$. Here and in what follows we identify each $S \in \mathcal{S}$ with its image under the projection $(\xi_1, \dots, \xi_u, \zeta_{u+1}, \dots, \zeta_m) \mapsto (\zeta_{u+1}, \dots, \zeta_m)$. We write the expression of $\tilde{\Delta}(S, \tilde{S}; x, z)$ with respect to the splitting $\mathbf{R}^s = \mathbf{R}^1 \times \mathbf{R}^{s-1}$

$$\tilde{\Delta}(S, \tilde{S}; x, z) = \begin{pmatrix} \Delta_{\mathrm{ww}} & \Delta_{\mathrm{ws}} \\ \Delta_{\mathrm{sw}} & \Delta_{\mathrm{ss}} \end{pmatrix} (S, \tilde{S}; x, z)$$

and then define

$$\Delta(S, \tilde{S}; x, z) = \begin{pmatrix} \Delta_{\mathrm{w}} = \Delta_{\mathrm{ww}} + \Delta_{\mathrm{ws}} \cdot A(x, z) & 0 \\ & \\ \Delta_{\mathrm{s}} = \Delta_{\mathrm{sw}} + \Delta_{\mathrm{ss}} \cdot A(x, z) & 0 \end{pmatrix} (S, \tilde{S}; x, z),$$

where $A(x, z)$ is as given by Proposition 3.2. Then $\Delta(S, \tilde{S}; \cdot, \cdot)$ is (C, γ)-Hölder continuous and satisfies (4.3). Moreover, Δ_{w} is uniformly bounded away from zero and infinity, as a consequence of that same proposition. We also need some information on how $\Delta(S, \tilde{S}; \cdot, \cdot)$ varies with $S, \tilde{S} \in \mathcal{S}$ and this is easy to get. Let us denote dist $(S, \tilde{S}) = \sup_{x \in S \cap \Lambda_1} \left\| \pi(S, \tilde{S}; x) - x \right\|$. It is easy to see that $\tilde{\Delta}$ may be taken such that $\|\tilde{\Delta}(S, \tilde{S}; x, z) - \mathrm{id}\| \le C \mathrm{dist}(S, \tilde{S})$ and so

$$\left| \Delta_{\mathrm{w}}(S, \tilde{S}; x, z) - 1 \right| \le C \mathrm{dist}\ (S, \tilde{S}) \tag{4.4}$$

for every $x, z \in S \cap \Lambda_1$. Note that if $S \in \mathcal{S}$ then $\psi(S) \subset S'$ for some $S' \in \mathcal{S}$. We let $\tilde{\Delta}\psi(\cdot, \cdot)$ be a Lipschitz continuous intrinsic derivative for ψ and denote by $\tilde{\Delta}\psi(S; \cdot, \cdot)$ its restriction to $(S \cap \Lambda_1) \times (S \cap \Lambda_1)$. Through the above identification we may think of $\tilde{\Delta}\psi(S; \cdot, \cdot)$ as taking values in $\mathcal{L}(\mathbf{R}^s, \mathbf{R}^s)$. Proceeding as before for Δ we construct a new, (C, γ)-Hölder continuous, intrinsic derivative $\Delta\psi(S; \cdot, \cdot)$ for $\psi \mid (S \cap \Lambda_1)$ having the form

$$\Delta\psi(S; x, z) = \begin{pmatrix} \Delta\psi_{\mathrm{w}} & 0 \\ \Delta\psi_{\mathrm{s}} & 0 \end{pmatrix} (S; x, z)$$

with respect to the splitting $\mathbf{R}^s = \mathbf{R}^1 \times \mathbf{R}^{s-1}$. In the same way we obtain a (C, γ)-Hölder continuous intrinsic derivative for $\psi^{-1} \mid (\psi(S) \cap \Lambda_1)$

$$\Delta\psi^{-1}(S; x', z') = \begin{pmatrix} \Delta\psi_{\mathrm{w}}^{-1} & 0 \\ \Delta\psi_{\mathrm{s}}^{-1} & 0 \end{pmatrix} (S; x', z').$$

Observe that Proposition 3.2 implies that $\Delta\psi_{\mathrm{w}}$ is uniformly bounded away from zero and infinity; moreover, independently of all the choices, we have

$$\Delta\psi_{\mathrm{w}}^{-1}(S;\psi(x),\psi(z))\cdot\Delta\psi_{\mathrm{w}}(S;x,z)=1 \text{ for every } x,z\in S\cap\Lambda_1.$$

We also point out that this construction yields

$$\left|\Delta\psi_{\mathrm{w}}(\tilde S;\pi(S,\tilde S;x),\pi(S,\tilde S;z))-\Delta\psi_{\mathrm{w}}(S;x,z)\right|\le C\mathrm{dist}\;(S,\tilde S)^{\gamma} \tag{4.5}$$

for every $x,z\in S\cap\Lambda_1$. Now we fix $S_0,\tilde S_0\in\mathcal{S}$ and construct an intrinsic derivative $\Delta\pi(S_0,\tilde S_0;\,\cdot\,,\,\cdot\,)$ for $\pi(S_0,\tilde S_0;\,\cdot\,)$ in the following way. For $(x,z)\in((S_0\cap\Lambda_1)\times(S_0\cap\Lambda_1))\cap U_0$ we set simply $\Delta\pi(S_0,\tilde S_0;x,z)=\Delta(S_0,\tilde S_0;x,z)$. Let now $(x,z)\in((S_0\cap\Lambda_1)\times(S_0\cap\Lambda_1))\cap U_k$, i.e. $k\ge 1$ is the minimum integer such that $(\psi^{-k}(x),\psi^{-k}(z))\in U_0$. We denote $\tilde x=\pi(S_0,\tilde S_0;x)$, $x_j=\psi^{-j}(x),\tilde x_j=\psi^{-j}(\tilde x)$ and analogously for $\tilde z, z_j$ and $\tilde z_j$, $j\ge 0$. We also let $S_j=S(x_j)=S(z_j)$, $\tilde S_j=S(\tilde x_j)=S(\tilde z_j)$, $0\le j\le k$. Then we define

$$\Delta\pi(S_0,\tilde S_0;x,z)=\Delta\psi^k(\tilde S_k;\tilde x_k,\tilde z_k)\cdot\Delta(S_k,\tilde S_k;x_k,z_k)\cdot\Delta\psi^{-k}(S_k;x,z).$$

We are left to check that with this definition $\Delta\pi(S_0,\tilde S_0;\,\cdot\,,\,\cdot\,)$ is (C,γ)-Hölder continuous: we consider $x,z,w\in S_0\cap\Lambda_1$ and prove that

$$\left|\Delta\pi(S_0,\tilde S_0;x,z)-\Delta\pi(S_0,\tilde S_0;x,w)\right|\le C\left|z-w\right|^{\gamma} \tag{4.6}$$

considering $(x,z)\in U_k$, $(x,w)\in U_\ell$ and $(z,w)\in U_m$, $k,\ell,m\ge 0$. Clearly, $j\mapsto\mathrm{dist}\;(S_j,\tilde S_j)$ is exponentially decreasing and $j\mapsto|z_j-w_j|$ is exponentially increasing, at least while $j\le m$. It follows that, if $|z-w|$ is small enough depending on $D=\sup\{\mathrm{dist}\;(S',S'')\colon S',S''\in\mathcal{S}\}$, then there exists $s<m$ such that

$$|z_s-w_s|\ge\mathrm{dist}\;(S_s,\tilde S_s). \tag{4.7}$$

Note that for the purpose of proving (4.6) it is no restriction to assume that $|z-w|$ is small with respect to D and we do so from now on. We fix s minimum satisfying (4.7) and observe that there is $\lambda<1$ (depending only on φ) such that

$$\mathrm{dist}\;(S_j,\tilde S_j)\le C\lambda^{j-s}\left|z-w\right|^{\gamma}\text{ for } j\ge s \tag{4.8}$$

$$|z_j-w_j|\le C\lambda^{s-j}\left|z-w\right|^{\gamma}\text{ for } 0\le j\le s. \tag{4.9}$$

This is an easy consequence of the definition of s and the exponential variation of $\mathrm{dist}\;(S_j,\tilde S_j)$ and $|z_j-w_j|$ mentioned above. Since the $\pi(S_i,\tilde S_i;\,\cdot\,)$ admit a uniform Lipschitz constant, we also get that

$$|\tilde z_j-\tilde w_j|\le C\lambda^{s-j}\left|z-w\right|^{\gamma}\text{ for } 0\le j\le s. \tag{4.10}$$

Now we take $r = \min\{k, \ell, s\}$. By construction

$$\Delta\pi(S_0, \tilde{S}_0; x, z) = \begin{pmatrix} \Delta\pi_{\mathrm{w}} & 0 \\ \Delta\pi_{\mathrm{s}} & 0 \end{pmatrix} (S_0, \tilde{S}_0; x, z)$$

with $\Delta\pi_{\mathrm{w}}(S_0, \tilde{S}_0; x, z) = \Delta\psi_{\mathrm{w}}^{r}(\tilde{S}_r; \tilde{x}_r, \tilde{z}_r) \cdot \Delta\pi_{\mathrm{w}}(S_r, \tilde{S}_r; x_r, z_r) \cdot \Delta\psi_{\mathrm{w}}^{-r}(S_r; x, z)$, and analogously for (x, w). Then

$$\frac{\Delta\pi_{\mathrm{w}}(S_0, \tilde{S}_0; x, z)}{\Delta\pi_{\mathrm{w}}(S_0, \tilde{S}_0; x, w)} = \frac{\Delta\pi_{\mathrm{w}}(S_r, \tilde{S}_r; x_r, z_r)}{\Delta\pi_{\mathrm{w}}(S_r, \tilde{S}_r; x_r, w_r)} \cdot R(x, z, w), \qquad \text{where}$$

$$R(x, z, w) = \prod_{1}^{r} \frac{\Delta\psi_{\mathrm{w}}(\tilde{S}_i; \tilde{x}_i, \tilde{z}_i) \cdot \Delta\psi_{\mathrm{w}}(S_i; x_i, w_i)}{\Delta\psi_{\mathrm{w}}(\tilde{S}_i; \tilde{x}_i, \tilde{w}_i) \cdot \Delta\psi_{\mathrm{w}}(S_i; x_i, z_i)}.$$

From (4.9), (4.10) and the uniform Hölder continuity and boundedness of $\Delta\psi_{\mathrm{w}}(S_i; \cdot, \cdot)$, $\Delta\psi_{\mathrm{w}}(\tilde{S}_i; \cdot, \cdot)$ it follows that

$$|R(x, z, w) - 1| \leq C\,|z - w|^{\gamma}\,. \tag{4.11}$$

We claim that

$$\left| \frac{\Delta\pi_{\mathrm{w}}(S_r, \tilde{S}_r; x_r, z_r)}{\Delta\pi_{\mathrm{w}}(S_r, \tilde{S}_r; x_r, w_r)} - 1 \right| \leq C\,|z - w|^{\gamma}\,. \tag{4.12}$$

Note that this implies (4.6) in an easy way. In fact, from (4.11), (4.12) we get $|\Delta\pi_{\mathrm{w}}(S_0, \tilde{S}_0; x, z) - \Delta\pi_{\mathrm{w}}(S_0, \tilde{S}_0; x, w)| \leq C|z - w|^{\gamma}$ and then, since $\Delta\pi_{\mathrm{s}}(S_0, \tilde{S}_0; x, y) = A(\tilde{x}, \tilde{y}) \cdot \Delta\pi_{\mathrm{w}}(S_0, \tilde{S}_0; x, y)$, for $y = z$ or w, the same holds for $\Delta\pi_{\mathrm{s}}$. Therefore, the proposition will follow once we have proven this claim. Suppose first $r = k$; then, because $r \leq s < m$, we also have $r = \ell$. It follows that

$$\Delta\pi_{\mathrm{w}}(S_r, \tilde{S}_r; x_r, y) = \Delta_{\mathrm{w}}(S_r, \tilde{S}_r; x_r, y) \qquad y = z_r \text{ or } w_r,$$

and so (4.12) is a direct consequence of (4.9) and the fact that Δ_{w} is Hölder continuous and bounded away from zero and infinity. Let now $r = s$. Once more by definition,

$$\Delta\pi_{\mathrm{w}}(S_r, \tilde{S}_r; x_r, z_r) = \Delta_{\mathrm{w}}(S_k, \tilde{S}_k; x_k, z_k) \cdot \prod_{r+1}^{k} \frac{\Delta\psi_{\mathrm{w}}(\tilde{S}_i; x_i, \tilde{z}_i)}{\Delta\psi_{\mathrm{w}}(S_i; x_i, z_i)}.$$

Note that dist $(S_j, \tilde{S}_j) \leq \sigma^{r-j}$dist $(S_r, \tilde{S}_r)$ for $r \leq j \leq m$ and so, using (4.5),

$$\left| \prod_{r+1}^{k} \frac{\Delta\psi_{\mathrm{w}}(\tilde{S}_i; \tilde{x}_i, \tilde{z}_i)}{\Delta\psi_{\mathrm{w}}(S_i; x_i, z_i)} - 1 \right| \leq C\,|z - w|^{\gamma}\,.$$

On the other hand, by (4.4) and (4.8), $\left|\Delta_{\mathrm{w}}(S_k, \tilde{S}_k; x_k, z_k) - 1\right| \leq C\,|z - w|^{\gamma}$ and this proves that $\left|\Delta\pi_{\mathrm{w}}(S_r, \tilde{S}_r; x_r z_r) - 1\right| \leq C\,|z - w|^{\gamma}$. In this same way one shows that $\left|\Delta\pi_{\mathrm{w}}(S_r,\tilde{S}_r;x_r,w_r)-1\right|\le C\,|z-w|^{\gamma}$ and then (4.12) follows immediately. □

5. Unique least contracting eigenvalue

Let $(\varphi_\mu)_\mu$ be a $\mathcal{C}^2$ one-parameter family of diffeomorphisms which at $\mu=0$ goes through a homoclinic tangency associated to a hyperbolic fixed (or periodic) point p of φ_0. We prove here that under a few generic assumptions to be stated below such a family exhibits, at $\tilde{\mu}_j\to 0$, homoclinic tangencies associated to periodic points $\tilde{p}_j\to p$ such that

(I) $\mathrm{D}\varphi_{\tilde{\mu}_j}^{\ell_j}(\tilde{p}_j)$, ℓ_j =period of $\tilde{p}_j$, has a unique weakest contracting eigenvalue which, therefore, is a real number.

A dual statement, for expanding eigenvalues, can be obtained in the same way. No assumptions on the dimensions of $\mathrm{W}^{\mathrm{u}}(p)$ or $\mathrm{W}^{\mathrm{s}}(p)$ are required in this section.

Observe first that, generically, either $\mathrm{D}\varphi_0(p)$ satisfies the property in (I) or else

(II) $\mathrm{D}\varphi_0(p)$ has exactly two weakest contracting eigenvalues and these are complex conjugate numbers.

In the first case there is nothing to prove, so we assume from now on that (II) holds. Using notations analogous to those of Section 3, this means that $w=2$, $\lambda_1=\lambda e^{i\phi}$, $\lambda_2=\lambda e^{-i\phi}$ with $\lambda>|\lambda_3|$ and $\phi\in\mathbf{R}\setminus\{k\pi\colon k\in\mathbf{Z}\}$. For the sake of simplicity we assume that there are $\mathcal{C}^2$ μ-dependent coordinates $(\xi_1,\dots,\xi_u,\zeta_1,\dots,\zeta_s)$ linearizing the φ_μ, for $|\mu|$ small, on a neighbourhood U of the analytic continuation p_μ of p. Moreover, we may take these coordinates to satisfy conditions (A1) – (B3) of Section 3. Here we may even assume that $\mathrm{D}\varphi_\mu(p_\mu)\,|\,\mathrm{E}^{\mathrm{w}}$ is conformal with respect to the euclidean metric induced by the coordinates ζ_1,ζ_2. We also suppose that the tangency is quadratic and the family $(\varphi_\mu)_\mu$ unfolds it generically. Then we may take, say for $\mu\ge 0$, points $q_\mu\in\mathrm{W}^{\mathrm{s}}_{\mathrm{loc}}(p_\mu)$, $r_\mu\in\mathrm{W}^{\mathrm{u}}_{\mathrm{loc}}(p_\mu)$ depending continuously on μ, such that $\varphi_\mu^N(r_\mu)=q_\mu$ for some fixed $N\ge 1$, r_0,q_0 belong to the orbit of tangency and r_μ,q_μ are points of transverse intersection of $\mathrm{W}^{\mathrm{u}}(p_\mu),\mathrm{W}^{\mathrm{s}}(p_\mu)$ for every $\mu>0$. Recall, moreover, that $\mathrm{W}^{\mathrm{u}}(p_\mu)$ and $\mathrm{W}^{\mathrm{s}}(p_\mu)$ also have, for a sequence of parameter values $\mu=\mu_j\to 0$, points of tangential intersection. For each fixed $\mu=\mu_j$ and every sufficiently large $n\ge 1$ we may take, as in Section 3, a neighbourhood $V=V(j,n)$ of $\{p_\mu,q_\mu\}$ such that $\Lambda(j,n)=\bigcap_{k\in\mathbf{Z}}\varphi_\mu^{k(n+N)}(V)$ is a φ_μ^{n+N}-invariant hyperbolic set and $\varphi_\mu^{n+N}\,|\,\Lambda(j,n)$ is conjugate to the 2-shift. Then, given any periodic point $\tilde{p}\in\Lambda(j,n)$, one may find parameter values

$\tilde{\mu}$ arbitrarily close to μ_j for which $\varphi_{\tilde{\mu}}$ has homoclinic tangenciesassociated to (the analytic continuation of) $\tilde{p}$. We consider $\tilde{p} = \tilde{p}(j,n)$ to be the unique φ_μ^{n+N}-fixed point in $\Lambda(j,n) \setminus \{p_\mu\}$. Clearly, the orbit of $\tilde{p}$ passes arbitrarily close to p_μ if j and n are sufficiently large. Therefore, in order to conclude our argument it is now sufficient to show that

PROPOSITION 5.1. *Given $j \geq 1$ (large) there exist arbitrarily large values of $n = n_j$ for which $\mathrm{D}\varphi_\mu^{n+N}(\tilde{p})$ has a unique weakest contracting eigenvalue.*

Proof. For the sake of notational simplicity we continue to denote $\mu = \mu_j$ and $\tilde{p} = \tilde{p}(j,n)$. We make use of the following elementary fact whose proof we omit.

LEMMA 5.2. *Let $P \in \mathrm{GL}(\mathbf{R}^p)$, $Q \in \mathrm{GL}(\mathbf{R}^q)$ satisfy $\left\|P^{-1}\right\| \|Q\| < 1$. Let $L = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GL}(\mathbf{R}^p \times \mathbf{R}^q)$ and denote*

$$L^{-1} = \begin{pmatrix} A^- & B^- \\ C^- & D^- \end{pmatrix} \quad \text{and} \quad L_n = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} P^n & 0 \\ 0 & Q^n \end{pmatrix}, \quad n \geq 1.$$

Assume that $A \in \mathrm{GL}(\mathbf{R}^p)$ (and so also $D^- \in \mathrm{GL}(\mathbf{R}^q)$). Then there are $n_0 \geq 1$ and $C_0 > 0$, depending only on $(\left\|P^{-1}\right\| \|Q\|, \|L\|, \left\|L^{-1}\right\|, \left\|A^{-1}\right\|, \left\|(D^-)^{-1}\right\|)$, such that for every $n \geq n_0$ there exist linear maps $f: \mathbf{R}^p \to \mathbf{R}^q$, $g: \mathbf{R}^q \to \mathbf{R}^p$ satisfying

(a) $\|f\| \leq C_0$ and $\|g\| \leq C_0(\left\|P^{-1}\right\| \|Q\|)^n$;

(b) $\mathrm{graph}\,(f), \mathrm{graph}\,(g)$ are L_n-invariant.

Moreover, given $\delta > 0$ there is $n_1 = n_1(\delta) \geq n_0$ such that for $n \geq n_1$

(c) $\|(\pi_f L_n \pi_f^{-1}) P^{-n} - A\| \leq \delta$ and $\left\|Q^n(\pi_g L_n^{-1} \pi_g^{-1}) - D^-\right\| \leq \delta$, where $\pi_f: \mathrm{graph}\,(f) \to \mathbf{R}^p$ and $\pi_g: \mathrm{graph}\,(g) \to \mathbf{R}^q$ are the canonical projections.

Let then

$$\mathrm{D}\varphi_\mu^N = \begin{pmatrix} A_{\mathrm{uu}} & A_{\mathrm{uw}} & A_{\mathrm{us}} \\ A_{\mathrm{wu}} & A_{\mathrm{ww}} & A_{\mathrm{ws}} \\ A_{\mathrm{su}} & A_{\mathrm{sw}} & A_{\mathrm{ss}} \end{pmatrix}, \quad \mathrm{D}\varphi_\mu(p) = \begin{pmatrix} \Lambda_{\mathrm{u}} & 0 & 0 \\ 0 & \Lambda_{\mathrm{w}} & 0 \\ 0 & 0 & \Lambda_{\mathrm{s}} \end{pmatrix}$$

be the expressions of $\mathrm{D}\varphi_\mu^N$ and $\mathrm{D}\varphi_\mu(p)$ with respect to the splitting $\mathbf{R}^m = \mathbf{R}^u \times \mathbf{R}^2 \times \mathbf{R}^{s-2} = \mathrm{E}^{\mathrm{u}} \times \mathrm{E}^{\mathrm{w}} \times \mathrm{E}^{\mathrm{ss}}$. We also denote

$$\Delta_\mu = \begin{pmatrix} A_{\mathrm{uu}} & A_{\mathrm{uw}} \\ A_{\mathrm{wu}} & A_{\mathrm{ww}} \end{pmatrix} \quad \text{and} \quad \mathrm{D}\varphi_\mu^{-N} = \begin{pmatrix} A_{\mathrm{uu}}^- & A_{\mathrm{uw}}^- & A_{\mathrm{us}}^- \\ A_{\mathrm{wu}}^- & A_{\mathrm{ww}}^- & A_{\mathrm{ws}}^- \\ A_{\mathrm{su}}^- & A_{\mathrm{sw}}^- & A_{\mathrm{ss}}^- \end{pmatrix}$$

and assume the generic property (cf. (3.2))

(5.1) $\quad \Delta_{\mu=0}(r_0)$, and so also $A_{\mathrm{ss}}^-(\mu = 0, q_0)$, is an isomorphism.

Then a first application of the lemma yields, for j and n sufficiently large, linear maps $f_{\mathrm{uw}}: \mathbf{R}^u \times \mathbf{R}^2 \to \mathbf{R}^{s-2}$, $f_{\mathrm{ss}}: \mathbf{R}^{s-2} \to \mathbf{R}^u \times \mathbf{R}^2$, whose graphs are invariant under $\mathrm{D}\varphi_\mu^{n+N}(\tilde{p})$ and satisfy

(i) $\mathrm{D}\varphi_\mu^{n+N}(\tilde{p}) \,|\, \mathrm{graph}\,(f_{\mathrm{uw}})$ is conjugate to $K_1 \begin{pmatrix} \Lambda_{\mathrm{u}}^n & 0 \\ 0 & \Lambda_{\mathrm{w}}^n \end{pmatrix}$ for some isomorphism $K_1 \in \mathrm{GL}(\mathbf{R}^u \times \mathbf{R}^2)$ which is close to $\Delta_\mu(\varphi_\mu^{-N}(\tilde{p}))$ and so also to $\Delta_{\mu=0}(r_0)$;
(ii) $\mathrm{D}\varphi_\mu^{-(n+N)}(\tilde{p}) \,|\, \mathrm{graph}\,(f_{\mathrm{ss}})$ is conjugate to $\Lambda_{\mathrm{s}}^{-n} K_2$, some $K_2 \in \mathrm{GL}(\mathbf{R}^{s-2})$ close to $A_{\mathrm{ss}}^-(\tilde{p})$ and therefore also to $A_{\mathrm{ss}}^-(\mu = 0, q_0)$.

Observe, on the other hand, that $A_{\mathrm{uu}}(r_\mu)$ is an isomorphism, since $\mathrm{W}^{\mathrm{u}}(p_\mu)$ and $\mathrm{W}^{\mathrm{s}}(p_\mu)$ intersect transversely at r_μ. Then, by using the lemma again, we obtain linear maps $f_{\mathrm{u}}: \mathbf{R}^u \to \mathbf{R}^2$, $f_{\mathrm{w}}: \mathbf{R}^2 \to \mathbf{R}^u$ such that the pre-images $\pi_{\mathrm{uw}}^{-1}(\mathrm{graph}\,(f_{\mathrm{u}}))$, $\pi_{\mathrm{uw}}^{-1}(\mathrm{graph}\,(f_{\mathrm{w}}))$ of their graphs under the canonical projection $\pi_{\mathrm{uw}}: \mathrm{graph}\,(f_{\mathrm{uw}}) \to \mathbf{R}^u \times \mathbf{R}^2$ are $\mathrm{D}\varphi_\mu^{n+N}(\tilde{p})$-invariant subspaces satisfying

(iii) $\mathrm{D}\varphi_\mu^{n+N}(\tilde{p}) \,|\, \pi_{\mathrm{uw}}^{-1}(\mathrm{graph}\,(f_{\mathrm{u}}))$ is conjugate to $K_3 \Lambda_{\mathrm{u}}^n$ for some $K_3 \in \mathrm{GL}(\mathbf{R}^u)$ close to $A_{\mathrm{uu}}(r_\mu)$;
(iv) $\mathrm{D}\varphi_\mu^{-(n+N)}(\tilde{p}) \,|\, \pi_{\mathrm{uw}}^{-1}(\mathrm{graph}\,(f_{\mathrm{w}}))$ is conjugate to $\Lambda_{\mathrm{w}}^{-n} K_0$ for some $K_0 \in \mathrm{GL}(\mathbf{R}^2)$ close to $\Delta_{\mathrm{ww}}^-(q_\mu)$;

($\Delta_{\mathrm{uu}}^-, \ldots, \Delta_{\mathrm{ww}}^-$ denoting the entries of Δ_μ^{-1}). Altogether, and recalling properties (B1) – (B3) of Section 3, this implies that for $j \geq 1$ sufficiently large there is $c > 1$ such that for $n \gg j$ the isomorphism $\mathrm{D}\varphi_\mu^{(n+N)}(\tilde{p})$ has exactly

(a) u eigenvalues with absolute value $\geq c^{-1}(\sigma - \varepsilon)^n$;
(b) 2 eigenvalues, say $\tilde{\lambda}_1, \tilde{\lambda}_2$, with $|\tilde{\lambda}_1|, |\tilde{\lambda}_2| \in [c^{-1}(\lambda - \varepsilon)^n, c(\lambda + \varepsilon)^n]$;
(c) $(s-2)$ eigenvalues with absolute value $\leq c(\theta + \varepsilon)^n$.

This reduces the completion of our argument to showing that there exist arbitrarily large values of n for which $|\tilde{\lambda}_1| \neq |\tilde{\lambda}_2|$. Note first that $A_{\mathrm{uu}}(\mu = 0, r_0)$ is not an isomorphism, since $\mathrm{W}^{\mathrm{u}}(p), \mathrm{W}^{\mathrm{s}}(p)$ are tangent at r_0. We assume this tangency to be quasi-transversal, meaning that $\dim \ker\,(A_{\mathrm{uu}}(\mu = 0, r_0)) = 1$. Since $\ker\,(\Delta_{\mathrm{ww}}^-) = A_{\mathrm{wu}}(\ker\,(A_{\mathrm{uu}}))$ and $\ker\,(A_{\mathrm{uu}}) = \Delta_{\mathrm{uw}}^-(\ker\,(\Delta_{\mathrm{ww}}^-))$, it follows that $\ker\,(\Delta_{\mathrm{ww}}^-(\mu = 0, q_0))$ is also 1-dimensional. We fix $\{v_1, v_2\}$ to be a basis of $\mathbf{R}^2 \simeq \pi_{\mathrm{uw}}^{-1}(\mathrm{graph}\,(f_{\mathrm{w}}))$, orthogonal with respect to the euclidean metric induced by the coordinates ζ_1, ζ_2 and such that $v_2 \in \ker\,(\Delta_{\mathrm{ww}}^-(\mu = 0, q_0))$. Referring always to this metric, we denote by $\mathrm{C}(v, \alpha)$ the closed cone in $\mathbf{R}^2$ of amplitude $\alpha > 0$ around the direction of a (nonzero) vector v. By construction $K_0(\mathrm{C}(v_1, 8\pi/18)) \subset \mathrm{C}(\tilde{v}_0, \pi/18)$, with $\tilde{v}_0 = \Delta_{\mathrm{ww}}^-(\mu = 0, q_0) \cdot v_1$, if K_0 is sufficiently near $\Delta_{\mathrm{ww}}^-(\mu = 0, q_0)$, i.e. if j and n are large enough (recall (iv)). Also, since Λ_{w} is conformal, $\Lambda_{\mathrm{w}}^{-n}(\mathrm{C}(\tilde{v}_0, \pi/18)) = \mathrm{C}(\tilde{v}_n, \pi/18)$, $\tilde{v}_n = \Lambda_{\mathrm{w}}^{-n} \cdot \tilde{v}_0$.

On the other hand, since the eigenvalues of Λ_{w} are not real, there exist infinitely many values of n for which $\tilde{v}_n = \Lambda_{\mathrm{w}}^{-n} \cdot \tilde{v}_0 \in \mathrm{C}(v_1, 6\pi/18)$ and so $\mathrm{C}(\tilde{v}_n, \pi/18) \subset \mathrm{C}(v_1, 7\pi/18)$. This means that for such values of n, $\Lambda_{\mathrm{w}}^{-n} K_0$ has a strictly invariant cone $\Lambda_{\mathrm{w}}^{-n} K_0(\mathrm{C}(v_1, 8\pi/18)) \subset \mathrm{C}(v_1, 7\pi/18)$ and so its two eigenvalues $(\tilde{\lambda}_1)^{-1}$ and $(\tilde{\lambda}_2)^{-1}$ must have different norms. This completes our proof. □

Remark 5.3. It is also clear from (a)–(c) that $\dim \mathrm{W}^*(\tilde{p}) = \dim \mathrm{W}^*(p)$, $* = \mathrm{u}$ or s, and $\mathrm{D}\varphi_\mu^{n+N}(\tilde{p})$ is sectionally dissipative if $\mathrm{D}\varphi_0(p)$ is.

6. Renormalization and thick basic sets

Let φ_0 be a $\mathcal{C}^2$ diffeomorphism with a (nondegenerate) homoclinic tangency associated to a fixed (or periodic) point p. We suppose here that $\dim \mathrm{W}^{\mathrm{u}}(p) = 1$ and $\mathrm{D}\varphi_0(p)$ is sectionally dissipative, i.e. the product of any two of its eigenvalues has norm less than one. The main goal is to show that the unfolding of such a tangency by a generic one-parameter family of diffeomorphisms $(\varphi_\mu)_\mu$ yields the formation, for arbitrarily small values of μ, of hyperbolic basic sets $\Lambda_2 = \Lambda_2(\mu)$ having (codimension one stable foliation and) arbitrarily large stable thickness $\tau^{\mathrm{s}}(\Lambda_2)$. For future use we also check that

(α) Λ_1 and Λ_2 are heteroclinically related, i.e. there exist some mutual transverse intersections between their stable and unstable leaves;

(β) there exist periodic points $p_1 \in \Lambda_1$, $p_2 \in \Lambda_2$ such that $\mathrm{W}^{\mathrm{u}}(p_1)$ has a nontransverse intersection with $\mathrm{W}^{\mathrm{s}}(p_2)$.

Actually, for the proof of (α), (β) we consider only the case when $\mathrm{D}\varphi_0(p)$ has a unique least contracting eigenvalue. We suppose in addition that, besides the homoclinic tangency the point p also has transverse homoclinic orbits and, moreover, these involve the same separatrix of $\mathrm{W}^{\mathrm{u}}(p)$ (resp. the same connected component of $\mathrm{W}^{\mathrm{s}}(p) \setminus \mathrm{W}^{\mathrm{ss}}(p)$) as the tangency. Since any diffeomorphism with a homoclinic tangency may be approximated by another satisfying these conditions, they represent no restriction for the purpose of proving our main result.

For the construction of Λ_2 we first deduce a higher-dimensional version of the renormalization scheme in [TY], [PT, Ch.III]. The basic idea is to show that, under a few generic assumptions, the unfolding of $(\varphi_\mu)_\mu$ contains the unfolding of families of nearly quadratic diffeomorphisms. More precisely, one shows that restrictions of iterates of φ_μ to appropriate domains close to the tangency have the form

$$(x, Y) \mapsto (x^2 + \nu, xA) + \varepsilon(\nu, x, Y) \text{ for some } A \in \mathbf{R}^{m-1} \text{ and } \varepsilon \ C^2\text{-small},$$

when written in conveniently chosen coordinates $(x,Y) \in \mathbf{R} \times \mathbf{R}^{m-1}$ and parameter ν.

Let us describe this construction in more detail. We assume once more that the φ_μ, μ small, admit $\mathcal{C}^2$ μ-dependent linearizing coordinates $(\xi, Z) \in \mathbf{R} \times \mathbf{R}^{m-1}$ on a neighbourhood of p. We fix these coordinates in such a way that $\mathrm{W}^{\mathrm{s}}_{\mathrm{loc}}(p_\mu) \subset \{\xi = 0\}$ and $\mathrm{W}^{\mathrm{u}}_{\mathrm{loc}}(p_\mu) \subset \{z = 0\}$. The assumption on the eigenvalues of $\mathrm{D}\varphi_0(p)$ means that we may choose the norm in $\mathbf{R}^m$ to be such that

$$|\sigma_\mu| \cdot \|S_\mu\| < 1 \quad \text{(for every small } \mu) \tag{6.1}$$

where σ_μ is the expanding eigenvalue of $\mathrm{D}\varphi_\mu(p_\mu)$ and $S_\mu = \mathrm{D}\varphi_\mu \,|\, \mathrm{E}^{\mathrm{s}}(p_\mu)$. Let $q_0 = (0, Q_0) \in \mathrm{W}^{\mathrm{s}}_{\mathrm{loc}}(p)$ and $r_0 = (\rho_0, 0) \in \mathrm{W}^{\mathrm{u}}_{\mathrm{loc}}(p)$ belong to the orbit of tangency, say $r_0 = \varphi_0^{-N}(q_0)$, $N \geq 1$. Then for (μ, ξ, z) close to $(0, \rho_0, 0)$ we may write $\varphi_\mu^N(\xi, Z)$ as

$$\begin{aligned}(v\mu + a_2 Z + b(\xi - \rho_0)^2 + b_1(\xi - \rho_0)\mu + b_2\mu^2 + h(\mu, \xi - \rho_0, Z),\\ Q_0 + V\mu + A_1(\xi - \rho_0) + A_2 Z + H(\mu, \xi - \rho_0, Z))\end{aligned}$$

where we have $v, b, b_1, b_2 \in \mathbf{R}$, $a_2 \in \mathcal{L}(\mathbf{R}^{m-1}, \mathbf{R})$, $V, A_1 \in \mathcal{L}(\mathbf{R}, \mathbf{R}^{m-1})$, $A_2 \in \mathcal{L}(\mathbf{R}^{m-1}, \mathbf{R}^{m-1})$ and

$$\mathrm{D}h = 0, \quad \mathrm{D}H = 0, \quad \partial_{\xi\xi}h = \partial_{\xi\mu}h = \partial_{\mu\mu}h = 0 \quad \text{at} \quad (0,0,0). \tag{6.2}$$

We also assume that the homoclinic tangency is quadratic and that it is generically unfolded by the family $(\varphi_\mu)_\mu$, which corresponds to having $b \neq 0$ and $v \neq 0$. First we introduce n-dependent reparametrizations

$$\tilde{\nu} = \tilde{\theta}_n(\mu) = v\sigma_\mu^{2n}\mu + (a_2\sigma_\mu^{2n}S_\mu^n Q_0 - \rho_0\sigma_\mu^n). \tag{6.3}$$

Let $K > 0$ (large) be fixed. It is not difficult to check that for $n \geq 1$ sufficiently large $\tilde{\theta}_n$ maps some small interval I_n, close to zero in the μ-space, diffeomorphically onto $[-K, K]$. We denote $\theta_n = (\tilde{\theta}_n \,|\, I_n)^{-1}$. Now we introduce (n,μ)-dependent coordinates $(\tilde{x}, \tilde{Y})$ defined by

$$(\xi, Z) = \psi_{n,\mu}(\tilde{x}, \tilde{Y}) = (\sigma_\mu^{-2n}\tilde{x} + \rho_0\sigma_\mu^{-n}, \sigma_\mu^{-n}\tilde{Y} + Q_0 + V\mu) \tag{6.4}$$

and take $\Psi_n \colon [-K, K]^{m+1} \to \mathbf{R} \times M$ to be given by

$$\Psi_n(\tilde{\nu}, \tilde{x}, \tilde{Y}) = (\mu, \xi, Z), \quad \mu = \theta_n(\tilde{\nu}), \quad (\xi, Z) = \psi_{n,\mu}(\tilde{x}, \tilde{Y}).$$

Let us also denote $\Phi \colon \mathbf{R} \times M \to \mathbf{R} \times M$, $\Phi(\mu, \eta) = (\mu, \varphi_\mu(\eta))$. Now a direct calculation yields the expression of Φ^{n+N} in the coordinates $(\tilde{\nu}, \tilde{x}, \tilde{Y})$:

$$\begin{aligned}&(\Psi_n^{-1}\Phi^{n+N}\Psi_n)(\tilde{\nu}, \tilde{x}, \tilde{Y}) =\\ &\quad(\tilde{\nu}, b\tilde{x}^2 + b_1\tilde{x}\sigma_\mu^n\mu + b_2\sigma_\mu^{2n}\mu^2 + \tilde{\nu} + a_2\sigma_\mu^n S_\mu^n(\tilde{Y} + V\sigma_\mu^n\mu) + \sigma_\mu^{2n}h(\mu, x_n, Y_n),\\ &\quad A_1\tilde{x} + A_2(S_\mu^n\tilde{Y} + \sigma_\mu^n S_\mu^n(Q_0 + V\mu)) + \sigma_\mu^n H(\mu, x_n, Y_n))\end{aligned}$$

where $\mu = \theta_n(\nu)$, $x_n = \sigma_\mu^{-n}\tilde{x}$, $Y_n = S_\mu^n(\sigma_\mu^{-n}\tilde{Y} + Q_0 + V\mu)$. Observe that $|\mu|, |x_n| \le$ const $|\sigma_\mu|^{-n}$ and $\|Y_n\| \le$ const $\|S_\mu\|^n$ and, in view of (6.1), (6.2), this implies that $\sigma_\mu^{2n} h(\mu, x_n, Y_n)$ and $\sigma_\mu^n H(\mu, x_n, Y_n) \to 0$ as $n \to \infty$, uniformly on $(\tilde{\nu}, \tilde{x}, \tilde{Y}) \in [-K, K]^{m+1}$. Using also (6.3) one concludes that the sequence $(\Psi_n^{-1}\Phi^{n+N}\Psi_n)(\tilde{\nu}, \tilde{x}, \tilde{Y})$ converges uniformly to

$$\tilde{\chi}(\tilde{\nu}, \tilde{x}, \tilde{Y}) = \left(\tilde{\nu}, b\tilde{x}^2 + b_1\frac{\rho_0}{v}\tilde{x} + b_2\frac{\rho_0^2}{v^2} + \tilde{\nu}, A_1\tilde{x}\right)$$

when $n \to \infty$. Moreover, essentially the same argument applies to the derivatives and we conclude in this way that $(\Psi_n^{-1}\Phi^{n+N}\Psi_n) \to \tilde{\chi}$ in the $\mathcal{C}^2$ topology. Finally, we introduce

$$\nu = b\tilde{\nu} + \frac{b_1\rho_0}{2v} - \frac{b_1^2\rho_0^2}{4v^2} + \frac{bb_2\rho_0^2}{v^2} \quad ; \quad x = b\tilde{x} + \frac{b_1\rho_0}{2v} \quad ; \quad Y = \tilde{Y} + A_1\frac{b_1\rho_0}{2vb}$$

and then, immediately, the expression of Φ^{n+N} with respect to (ν, x, Y) converges (in the $\mathcal{C}^2$ topology) to

$$\chi(\nu, x, Y) = (\nu, x^2 + \nu, Ax), \quad \text{where} \quad A = \frac{A_1}{b} \in \mathbf{R}^{m-1}. \tag{6.5}$$

Remark 6.1. As a consequence, the generic unfolding of a homoclinic tangency associated to a sectionally dissipative saddle yields the formation of sinks close to the orbit of tangency. In fact, $\chi_\nu(x, y) = (x^2 + \nu, Ax)$ has attracting periodic orbits, say for ν close to zero. The presence of such orbits is a persistent phenomenon under small perturbations. It follows that for arbitrarily small values of μ ($\mu = \theta_n(\nu)$, ν close to zero) φ_μ has an attracting periodic orbit contained in a (const $|\mu|$)-neighbourhood of the orbit of tangency.

Now the construction of Λ_2 proceeds in the same way as in the two-dimensional case; we sketch the main points and refer the reader to [PT, Ch.VI] for more details. The crucial fact here is the existence for the map $x \mapsto x^2 - 2$, and so also for χ_{-2}: $(x, Y) \mapsto (x^2 - 2, Ax)$, of invariant expanding Cantor sets K_j with thickness $\tau(K_j) \to +\infty$ as $j \to +\infty$. Moreover, these K_j are transitive and have a dense subset of periodic orbits. It follows that each K_j has, for n large and $\mu = \theta_n(\nu)$, ν close to -2, an analytic continuation as a hyperbolic basic set $K_j(n, \mu)$ of $(\psi_{n,\mu} \circ \varphi_\mu^{n+N} \circ \psi_{n,\mu}^{-1})$. In particular, the $K_j(n, \mu)$ have codimension 1 stable foliation and stable thickness $\tau^s(K_j(n, \mu))$ close to $\tau(K_j) \gg 1$. Then, we just take $\Lambda_2 = \psi_{n,\mu}(K_j(n, \mu))$ with j and n large and $\mu = \theta_n(\nu)$, ν close to -2.

Let, moreover, $P = (2, 2A)$ and Q be any other (hyperbolic) periodic point of χ_{-2}. We denote by $P(\nu)$ and $P(n, \mu)$ the analytic continuations of P for, respectively, χ_ν, and $(\psi_{n,\mu} \circ \varphi_\mu^{n+N} \circ \psi_{n,\mu}^{-1})$, n large, $\mu = \theta_n(\nu)$, ν close to -2. We also introduce analogous notations for Q. Now, it is easy to check that

FIGURE 5

• $W^s(Q(\nu))$ has transverse intersections with $W^u(P(\nu))$ for all ν close enough to -2;

• $W^u(Q(\nu))$ intersects $W^s(P(\nu))$ if and only if $\nu \leq -2$ and this intersection is transverse if $\nu < -2$.

Combining this with the fact that (compact parts of) stable and unstable manifolds of hyperbolic periodic points vary continuously with the maps, one concludes that for each large n there are μ-values $\tilde{\mu}_n = \theta_n(\tilde{\nu}_n)$, $\tilde{\nu}_n$ close to -2, for which (see the figure)

- $P(n, \tilde{\mu}_n)$ and $Q(n, \tilde{\mu}_n)$ are heteroclinically related;
- $W^u(Q(n, \tilde{\mu}_n))$ also has nontransverse intersections with $W^s(P(n, \tilde{\mu}_n))$.

This applies, in particular, when Q is a periodic point in any of the K_j. Therefore, these comments reduce the proof of properties (α), (β) to checking that

$$p_\mu \text{ is heteroclinically related to } \hat{P}(n, \mu) = \psi_{n,\mu}(P(n, \mu)) \tag{6.6}$$

for arbitrarily large n and $\mu = \theta_n(\nu)$, ν in a fixed neighbourhood J of -2.

The basic strategy to prove this is the same as in 2 dimensions. Let us first give a brief outline of it. We show that there exist compact domains $\sigma^u(n, \mu) \subset W^u(\hat{P}(n, \mu))$, $\Sigma^s(n, \mu) \subset W^s(\hat{P}(n, \mu))$, $\sigma^u \subset W^u(p)$, $\Sigma^s \subset W^s(p)$ such that (cf. the figure below)

- $\hat{P}(n, \mu) \in \partial\sigma^u(n, \mu) \cap \partial\Sigma^s(n, \mu)$;
- the point of tangency q_0 belongs to $\partial\sigma^u \cap \partial\Sigma^s$;
- $\sigma^u(n, \mu) \to \sigma^u$ and $\Sigma^s(n, \mu) \to \Sigma^s$ as $n \to \infty$ (and so $\mu \to 0$).

Figure 6

Moreover, our assumption on the existence of transverse homoclinic orbits associated to p, together with the construction of Σ^{s}, σ^{u}, to be described below, assure that $\mathrm{W}^{\mathrm{u}}(p)$ (resp. $\mathrm{W}^{\mathrm{s}}(p)$) accumulates on σ^{u} (resp. Σ^{s}) "from the appropriate side" so that it eventually intersects Σ^{s} (resp. σ^{u}) transversely. Hence, for n large $\mathrm{W}^{\mathrm{u}}(p_\mu)$ (resp. $\mathrm{W}^{\mathrm{s}}(p_\mu)$) cuts $\Sigma^{\mathrm{s}}(n,\mu)$ (resp. $\sigma^{\mathrm{u}}(n,\mu)$) and the affirmative (6.6) above follows.

On the other hand, this strategy requires considerably more care in the present higher-dimensional setting, specially in the construction of $\Sigma^{\mathrm{s}}(n,\mu)$, Σ^{s}. We concentrate on this, the argument for $\sigma^{\mathrm{u}}(n,\mu)$, σ^{u} being analogous (and simpler). As mentioned before, we may suppose that p_μ has a unique weakest contracting eigenvalue, which we denote by λ_μ. We also assume once more the transversality condition (5.1)

$$\Delta_{\mu=0}(r_0) \text{ is an isomorphism.} \tag{6.7}$$

First we deduce the following result on existence of invariant splittings in a neighbourhood of the tangency. As before, we denote $\mathrm{E}^{\mathrm{ss}} = \{0\}\times\{0\}\times\mathbf{R}^{m-2}$, $\mathrm{E}^{\mathrm{uw}} = \mathbf{R}\times\mathbf{R}\times\{0^{m-2}\}$. We also let $\sigma = |\sigma_0|$, $\lambda = |\lambda_0|$, $\theta = \|\mathrm{D}\varphi_0(p)\,|\,\mathrm{E}^{\mathrm{ss}}\|$ and fix $\varepsilon > 0$ such that $\theta + 2\varepsilon < \lambda - 2\varepsilon$. Moreover, we continue to denote by U a neighbourhood of p where $\mathcal{C}^2$ linearizing coordinates for the φ_μ are defined.

Proposition 6.2. *There exist neighbourhoods B of q_0 and I of $0 \in \mathbf{R}$ and constants $C_0 > 0$ and $n_0 \geq 1$ such that for $n \geq n_0$ and $\mu \in I$ there is a Hölder continuous splitting $\mathrm{T}_B M = \mathrm{E}^{\mathrm{uw}}_{n,\mu} \oplus \mathrm{E}^{\mathrm{ss}}_{n,\mu}$ satisfying*

(a) $\dim \mathrm{E}^{\mathrm{uw}}_{n,\mu}(z) = 2$ *and* $\dim \mathrm{E}^{\mathrm{ss}}_{n,\mu}(z) = m-2$ *for* $z \in B$;

(b) $\mathrm{E}^{\mathrm{ss}}_{n,\mu}$ *admits an integral foliation* $\mathcal{F}^{\mathrm{ss}}_{n,\mu}$.

FIGURE 7

(c) $\mathrm{D}\varphi_\mu^{n+N}(z)\cdot \mathrm{E}^*_{n,\mu}(z) = \mathrm{E}^*_{n,\mu}(\varphi_\mu^{n+N}(z))$ *for* $*=$ uw *or* ss *and for every* $z\in B\cap\varphi_\mu^{-(n+N)}(B)$ *such that* $\varphi_\mu^i(z)\in U$ *for* $0\le i\le n$.

(d) For every z *as in* (*c*) *we have*

$$C_0^{-1}(\lambda-\varepsilon)^n\,\|v\|\le\left\|\mathrm{D}\varphi_\mu^{n+N}(z)\cdot v\right\|\le C_0(\sigma+\varepsilon)^n\,\|v\|\qquad\text{if}\quad v\in\mathrm{E}^{\mathrm{uw}}_{n,\mu}(z)$$
$$\text{and}\qquad\left\|\mathrm{D}\varphi_\mu^{n+N}(z)\cdot v\right\|\le C_0(\theta+\varepsilon)^n\,\|v\|\qquad\text{if}\quad v\in\mathrm{E}^{\mathrm{ss}}_{n,\mu}(z).$$

Proof. The construction of the subbundles $\mathrm{E}^{\mathrm{uw}}_{n,\mu}$, $\mathrm{E}^{\mathrm{ss}}_{n,\mu}$ is done by a standard fixed point argument analogous to that in [HP,Theorems 6.1, 6.2] or Section 2 in the present paper. Thus we just present an outline of it, leaving the details to the reader. We let γ^{u} be a segment in $\mathrm{W}^{\mathrm{u}}_{\mathrm{loc}}(p)$ containing p and r_0 in its interior and we take B and I to be small neighbourhoods of q_0 and $0\in\mathbf{R}$, respectively, according to certain conditions to be stated in the sequel. For simplicity we choose B to have the form $B_{\mathrm{u}}\times B_{\mathrm{s}}$, B_{u} a neighbourhood of $0\in\mathbf{R}\simeq\mathrm{E}^{\mathrm{u}}$ and B_{s} a neighbourhood of Q_0 in $\mathbf{R}^{m-1}\simeq\mathrm{E}^{\mathrm{s}}$. Then we denote $\partial^{\mathrm{u}}B=\partial B_{\mathrm{u}}\times B_{\mathrm{s}}$ and $\partial^{\mathrm{s}}B=B_{\mathrm{u}}\times\partial B_{\mathrm{s}}$. Since we are assuming the tangency to be nondegenerate we may fix $\gamma^{\mathrm{u}}, I, B_{\mathrm{u}}$ and B_{s} in such a way that, for $\mu\in I$, $\varphi_\mu^N(\gamma^{\mathrm{u}})$ does not intersect $\partial^{\mathrm{s}}B$ (see the figure). Then, for n sufficiently large, we have

$$\varphi_\mu^{n+N}(\partial^{\mathrm{u}}B\cap X_{n,\mu})\cap B=\emptyset\quad(\text{actually }\partial^{\mathrm{u}}B\cap X_{n,\mu}=\emptyset)\tag{6.8}$$
$$\varphi_\mu^{n+N}(\partial^{\mathrm{s}}B\cap X_{n,\mu})\cap\partial^{\mathrm{s}}B=\emptyset\tag{6.9}$$

where $X_{n,\mu}$ denotes the set of points z such that $\varphi_\mu^i(z)\in U$ for every $0\le i\le n$. Let us describe first the construction of $\mathrm{E}^{\mathrm{uw}}_{n,\mu}$. We let E_1 be the constant (parallel) 2-dimensional vector bundle on $\partial^{\mathrm{s}} B$ given by $E_1(z) = \mathrm{D}\varphi_0^N(r_0)\cdot \mathrm{E}^{\mathrm{uw}}$. By (6.7) this is transversal to E^{ss} and so, for n large and any $z \in \partial^{\mathrm{s}} B \cap X_{n,\mu}$, $\mathrm{D}\varphi_\mu^n(z)\cdot E_1(z)$ is close to E^{uw}. It follows that $E_2(\eta) = \mathrm{D}\varphi_\mu^{n+N}(z)\cdot E_1(z)$, $z = \varphi_\mu^{-(n+N)}(\eta)$, defines a two-dimensional bundle on $\varphi_\mu^{n+N}(\partial^{\mathrm{s}} B \cap X_{n,\mu}) \cap B$, with $E_2(\eta)$ uniformly close to $\mathrm{D}\varphi_0^N(r_0)\cdot \mathrm{E}^{\mathrm{uw}}$ if B and I are small and n is large. Then, recall also (6.9), we may take a two-dimensional bundle $\tilde{E}$ on B of class $\mathcal{C}^1$ coinciding with E_1, E_2 on their domains and such that $\tilde{E}(z)$ is uniformly close to $\mathrm{D}\varphi_0^N(r_0)\cdot \mathrm{E}^{\mathrm{uw}}$: given any $\delta > 0$ then, up to choosing B and I small and n large enough, we may assume that angle $(\tilde{E}(z), \mathrm{D}\varphi_0^N(r_0)\cdot \mathrm{E}^{\mathrm{uw}}) \le \delta$ for every $z \in B$. In what follows we fix $0 < \delta \ll$ angle $(\mathrm{D}\varphi_0^N(r_0)\cdot \mathrm{E}^{\mathrm{uw}}, \mathrm{E}^{\mathrm{ss}})$. Then we define $\mathcal{X} = \mathcal{X}(\delta, \tilde{E})$ to be the space of continuous 2-dimensional bundles E such that

- angle $(E(\eta), \mathrm{D}\varphi_0^N(r_0)\cdot \mathrm{E}^{\mathrm{uw}}) \le 2\delta$ for every $\eta \in B$ and
- $E(\eta) = \tilde{E}(\eta)$ for every $\eta \in B \setminus \varphi_\mu^{n+N}(B \cap X_{n,\mu})$.

We also introduce the graph-transform $\mathcal{T}\colon \mathcal{X} \to \mathcal{X}$ given by

$$\mathcal{T}(E)(\eta) = \mathrm{D}\varphi_\mu^{n+N}(z)\cdot E(z), \quad z = \varphi_\mu^{-(n+N)}(\eta)$$

if $\eta \in \varphi_\mu^{n+N}(B \cap X_{n,\mu})$ and $\mathcal{T}(E)(\eta) = \tilde{E}(\eta)$ otherwise. Observe that by construction every $E \in \mathcal{X}$ is transversal to E^{ss}. It follows in a fairly easy way that, at least for n large, $\mathcal{T}$ is well defined ($\mathcal{T}(\mathcal{X}) \subset \mathcal{X}$) and a $\mathcal{C}^0$−contraction. Hence $\mathcal{T}$ has a unique fixed point, which we call $\mathrm{E}^{\mathrm{uw}}_{n,\mu}$. Moreover, the same type of hyperbolicity argument as in [HP] or Section 2 shows that $\mathrm{E}^{\mathrm{uw}}_{n,\mu}$ is Hölder continuous. Thus $\mathrm{E}^{\mathrm{uw}}_{n,\mu}$ satisfies (a), and (c). On the other hand, (d) is a direct consequence of the transversality of $\mathrm{E}^{\mathrm{uw}}_{n,\mu}$ to E^{ss} and the fact that most iterations are done inside the linearizing neighbourhood U. A similar argument allows us to obtain $\mathrm{E}^{\mathrm{ss}}_{n,\mu}$. We start by taking $E_1(\eta) = \mathrm{E}^{\mathrm{ss}}$, for $\eta \in \partial^{\mathrm{u}} B$, and then proceed as for $\mathrm{E}^{\mathrm{uw}}_{n,\mu}$, only this time taking negative iterates: we set $E_2(z) = \mathrm{D}\varphi_\mu^{-(n+N)}(\eta)\cdot E_1(\eta)$, $\eta = \varphi_\mu^{(n+N)}(z)$, for $z \in \varphi_\mu^{-(n+N)}(\partial^{\mathrm{u}} B) \cap X_{n,\mu} \cap B$. As before, we extend E_1, E_2 to a $\mathcal{C}^1$ $(m-2)$-dimensional bundle $\tilde{E}$ on B and then we define $\mathcal{X}$ to be the space of continuous $(m-2)$-dimensional bundles E on B such that

- angle $(E(z), \mathrm{E}^{\mathrm{ss}}) \le 2\delta$ for every $z \in B$ and
- $E(z) = \tilde{E}(z)$ for every $z \in B \setminus (\varphi_\mu^{-(n+N)}(B) \cap X_{n,\mu})$.

The graph-transform $\mathcal{T}\colon \mathcal{X} \to \mathcal{X}$ is defined by

$$\mathcal{T}(E)(z) = \mathrm{D}\varphi_\mu^{-(n+N)}(\eta)\cdot E(\eta), \quad \eta = \varphi_\mu^{n+N}(z)$$

if $z \in \varphi_\mu^{-(n+N)}(B) \cap X_{n,\mu}$ and $\mathcal{T}(E)(z) = \tilde{E}(z)$ otherwise. Then, as in the previous case, we conclude that $\mathcal{T}$ has a unique fixed point, which is a Hölder continuous invariant subbundle, and we take this to be $\mathrm{E}^{ss}_{n,\mu}$. Moreover, $\mathrm{E}^{ss}_{n,\mu}$ can be integrated to a φ_μ^{n+N}-invariant foliation $\mathcal{F}^{ss}_{n,\mu}$. This can be seen as follows. First, when extending E_1, E_2 above we may take $\tilde{E}$ to coincide with the tangent bundle of a $\mathcal{C}^1$ foliation $\tilde{\mathcal{F}}$ of B by $(m-2)$-dimensional submanifolds. Now consider the space $\hat{\mathcal{X}}$ of foliations $\mathcal{F}$ of B by $(m-2)$-submanifolds of class $\mathcal{C}^1$ such that the tangent spaces to leaves of $\mathcal{F}$ vary continously with the point and satisfy

- angle $(\mathrm{T}_z\mathcal{F}(z), \mathrm{E}^{ss}) \le 2\delta$ for $z \in B$ and
- $\mathrm{T}_z\mathcal{F}(z) = \tilde{E}(z)$ if $z \in B \setminus (\varphi_\mu^{-(n+N)}(B) \cap X_{n,\mu})$.

We also introduce the graph-transform $\hat{\mathcal{T}}: \hat{\mathcal{X}} \to \hat{\mathcal{X}}$ given by

$$\hat{\mathcal{T}}(\mathcal{F})(z) = \text{connected component of } \varphi_\mu^{-(n+N)}(\mathcal{F}(\varphi_\mu^{n+N}(z))) \cap B \text{ containing } z,$$

if $z \in \varphi_\mu^{-(n+N)}(B) \cap X_{n,\mu}$ and $\hat{\mathcal{T}}(\mathcal{F})(z) = \tilde{\mathcal{F}}(z)$ otherwise. The fact that negative iterations expand the leaves of foliations $\mathcal{F} \in \hat{\mathcal{X}}$ assures that $\hat{\mathcal{T}}$ is well defined, at least if n is large enough. Moreover, the same calculations as before show that $\hat{\mathcal{T}}$ is a contraction, with respect to the $\mathcal{C}^0$-distance between tangent bundles. It follows that $\hat{\mathcal{T}}$ has a unique fixed point $\mathcal{F}^{ss}_{n,\mu}$ and then, clearly, we must have $\mathrm{T}_z\mathcal{F}^{ss}_{n,\mu}(z) = \mathrm{E}^{ss}_{n,\mu}(z)$ for every $z \in B$. □

Remark 6.3. For future use let us also state explicitly the following consequences of the renormalization techniques above:

(a) Given $\delta > 0$ we have angle $(\mathrm{E}^{uw}_{n,\mu}(z), \mathrm{D}\varphi_0^N(r_0) \cdot \mathrm{E}^{uw}) \le \delta$ for every $(\mu, z) \in I \times B$, as long as B and I are small enough and n is sufficiently large.

(b) Given B and I small and $\delta > 0$ then for large n angle $(\mathrm{E}^{ss}_{n,\mu}(z), \mathrm{E}^{ss}) \le \delta$ for every $(\mu, z) \in I \times B$.

Keeping the same notation as before, we take B, I and δ to be fixed (small) and n to be large, depending on μ. In particular, we suppose $\theta_n(J) \subset I$. We observe that for $* =$ uw or ss and all the values of μ and n under consideration, we must have $\mathrm{E}^*_{n,\mu}(\hat{P}(n,\mu)) = \text{graph}(f_*)$, recall (i) and (ii) in Section 5. Thus, $\mathrm{D}\varphi_\mu^{n+N} \mid \mathrm{E}^{uw}_{n,\mu}(\hat{P}(n,\mu))$ is conjugate to $K \begin{pmatrix} \sigma_\mu^n & 0 \\ 0 & \lambda_\mu^n \end{pmatrix}$ for some $K \in \mathrm{GL}(\mathbf{R}^2)$ close to $\Delta_{\mu=0}(r_0)$. We let $\sigma(n,\mu), \lambda(n,\mu)$ be its eigenvalues, $|\sigma(n,\mu)| \ge |\lambda(n,\mu)|$. The convergence to (6.5) implies that $\sigma(n,\mu)$ is close to 4 (if n is large and $\mu = \theta_n(\nu)$, ν close to -2). Hence, $\lambda(n,\mu)$ has the same sign as $(\det \Delta_{\mu=0}(r_0)(\lambda_0\sigma_0)^n)$. We claim that for the purpose of proving our main theorem it is no restriction to suppose that

$$\lambda(n,\mu) > 0 \text{ for } \mu \in \theta_n(J) \text{ and } n \text{ arbitrarily large.} \tag{6.10}$$

This can be justified as follows. If $(\lambda_0\sigma_0) > 0$ and $\det \Delta_{\mu=0}(r_0) > 0$ then there is nothing to prove. If $(\lambda_0\sigma_0) < 0$ we just restrict to the n-values having the appropriate parity so that $\det \Delta_{\mu=0}(r_0) \cdot (\lambda_0\sigma_0)^n > 0$. Suppose now $(\lambda_0\sigma_0) > 0$ and $\det \Delta_{\mu=0}(r_0) > 0$. We note first that, by the arguments developed previously in this section, there are μ-values $\hat{\mu}_n = \theta_n(\hat{\nu}_n)$, $\hat{\nu}_n$ close to -2, for which $\hat{P}(n,\hat{\mu}_n)$ has homoclinic tangencies. On the other hand, in the present situation we have $\lambda(n,\hat{\mu}_n) < 0 < \sigma(n,\hat{\mu}_n)$. This means that the $\varphi_{\hat{\mu}_n}$ have homoclinic tangencies falling in the second of the previous two cases. Moreover, it is clear that if the theorem holds for each $\varphi_{\hat{\mu}_n}$ then, since $\hat{\mu}_n \to 0$, it also holds for φ_0. Therefore, we may, without any loss of generality, assume that (6.10) holds.

Finally, we prove (6.6) (for the values of n as in (6.10)). For simplicity we use $\approx$ to represent equality up to a (multiplicative) constant not depending on n or δ and const to denote such a constant. Distances and angles refer always to the euclidean metric associated to the coordinates (ξ, Z) above. We begin by introducing the foliations $\mathcal{F}^{\mathrm{u}}_\mu = (\{\xi = \mathrm{const}\})$ and $\mathcal{F}^{\mathrm{s}}_\mu = (\varphi_\mu^N(\{Z = \mathrm{const}\}))$ defined on B. We suppose B small enough so that the leaves of these two foliations intersect transversely outside a hypersurface

$$L_\mu = \varphi_\mu^N(\tilde{L}_\mu) \quad , \quad \tilde{L}_\mu = \{(\xi, Z) \colon \partial_\xi \varphi_\mu^N(\xi, Z) \in \mathrm{E}^{\mathrm{s}}\}.$$

Due to the quadratic nature of the tangency we even have

$$\text{(6.11)} \qquad \operatorname{angle}(\mathrm{T}_z\mathcal{F}^{\mathrm{u}}_\mu(z), \mathrm{T}_z\mathcal{F}^{\mathrm{s}}_\mu(z)) \approx \operatorname{dist}(z, L_\mu) \text{ for } z \in B.$$

A fairly simple calculation shows that $\tilde{L}_\mu$ may be written as a graph $\xi = g(\mu, Z)$ and, moreover, $\varphi_\mu^{-N}(\hat{P}(n,\mu))$ is at a distance $\approx |\sigma_\mu|^{-n}$ of $\tilde{L}_\mu$, in $\{\xi > g(\mu, Z)\}$ or $\{\xi < g(\mu, Z)\}$ depending on whether b is positive or negative. It follows that $\hat{P}(n,\mu)$ is placed at a distance $d \approx |\sigma_\mu|^{-n}$ of L_μ, to the side determined by the vector $(0, A)$ (recall (6.5)). Now we let S be the $(d/2)$-neighbourhood of $\hat{P}(n,\mu)$ in its stable manifold. For n sufficiently large

$$\text{(6.12)} \qquad \operatorname{angle}(\mathrm{T}_z S, \mathrm{E}^{\mathrm{s}}) \le \delta\, |\sigma_\mu|^{-n} \text{ for } z \in S.$$

This is a direct consequence of the fact that the (local) stable manifold of $P(n,\mu) = \psi_{n,\mu}^{-1}(\hat{P}(n,\mu))$ converges to a hyperplane $\{x = \mathrm{const}\}$ as $n \to \infty$, together with the form of the coordinate changes (6.4). Note also that, by construction,

$$\operatorname{dist}(z, L_\mu) \ge \mathrm{const}\, |\sigma_\mu|^{-n} \text{ for } z \in S.$$

We take $\hat{\mathrm{E}}^{\mathrm{w}}$ to be the line field on S given by $\hat{\mathrm{E}}^{\mathrm{w}}(z) = \mathrm{E}^{\mathrm{uw}}_{n,\mu}(z) \cap \mathrm{T}_z S$. Remark 6.3 (a) and (6.12) imply that $\hat{\mathrm{E}}^{\mathrm{w}}$ is almost colinear with $(0, A)$: if n is large then $\operatorname{angle}(\hat{\mathrm{E}}^{\mathrm{w}}(z), (0, A)) \le 2\delta$ for $z \in S$. We orient $\hat{\mathrm{E}}^{\mathrm{w}}$ in such a way that $\hat{\mathrm{E}}^{\mathrm{w}} \cdot (0, A) > 0$ and for each $\eta_0 \in \Gamma_0 = \mathrm{W}^{\mathrm{ss}}(\hat{P}(n,\mu)) \cap S$ we define $W_0(\eta_0)$ to be the positive trajectory of η_0 under $\hat{\mathrm{E}}^{\mathrm{w}}$. Now we consider the set $\Gamma \supset \Gamma_0$ of points $\eta = \varphi_\mu^{-(n+N)}(\eta_0)$ with $\eta_0 \in \Gamma_0$ and $\varphi_\mu^{-(i+N)}(\eta_0) \in U$ for $0 \le i \le n$. We point out that $\Gamma \cap B$ contains a neighbourhood of $\hat{P}(n,\mu)$ in $\mathrm{W}^{\mathrm{s}}(\hat{P}(n,\mu))$ with fixed radius $\approx$ diam (B). This may be seen from the following remarks: (i) $\varphi_\mu^{-N}(\Gamma_0)$ contains a neighbourhood of $\varphi_\mu^{-N}(\hat{P}(n,\mu))$ in $\mathrm{W}^{\mathrm{s}}(\varphi^{-N}(\hat{P}(n,\mu)))$ with radius $\approx |\sigma_\mu|^{-n}$ and, by (6.7), it is transversal to E^{uw} at every point; (ii) the expansion during the n (negative) iterations inside U is $\geq$ const $|\lambda_\mu|^{-n} \gg |\sigma_\mu|^n$. For each $i \geq 1$ and $\eta \in \Gamma$, we define $W_i(\eta) = \varphi_\mu^{-(n+N)}(W_{i-1}(\varphi_\mu^{(n+N)}(\eta)))$. Since $\hat{\mathrm{E}}^{\mathrm{w}}$ is $\varphi_\mu^{(n+N)}$-invariant and $\lambda(n,\mu)$ is positive, we have $W_i(\eta) \supset W_{i-1}(\eta)$ for every $i > 1$ and $\eta \in \Gamma$. We claim that, as long as they stay inside B, the $W_i(\eta)$ remain nearly colinear with $(0, A)$:

$$\text{(6.13)} \qquad \text{angle}\,(\mathrm{T}_z W_i(\eta), (0,A)) \leq 2\delta \text{ for } z \in W_i(\eta),\ i \leq 1,\ \eta \in \Gamma.$$

In order to prove this we note first that $\mathrm{T}_z W_i(\eta) \subset \mathrm{E}^{\mathrm{uw}}(z)$ and thus, by Remark 6.3 (a), angle $(\mathrm{T}_z W_i(\eta), \mathrm{D}\varphi_0^N(r_0)\cdot \mathrm{E}^{\mathrm{uw}}) \leq \delta$. Therefore, (6.13) will follow if we prove

$$\text{(6.14)} \qquad \text{angle}\,(\mathrm{T}_z W_i(\eta), \mathrm{E}^{\mathrm{s}}) \leq \delta\,|\sigma_\mu|^{-n} \text{ for } z \in W_i(\eta),\ i \geq 1,\ \eta \in \Gamma.$$

We do this in an inductive way. Let $i \geq 1$ and suppose that (6.14) holds for every $\mathrm{T}_\zeta W_{i-1}(\eta_0)$, $\eta_0 \in \Gamma_0 \subset \Gamma$ and $\zeta \in W_{i-1}(\eta_0)$ (note that for $i = 1$ this is contained in (6.12)). Let us also point out that, due to the way we have oriented $\hat{\mathrm{E}}^{\mathrm{w}}$, the $W_j(\eta_0)$, $j = 0, 1, \ldots, i-1$, grow in the direction *opposite* to L_μ. Thus, we have dist $(\zeta, L_\mu) \geq$ const $|\sigma_\mu|^{-n}$ for every $\zeta \in W_{i-1}(\eta_0)$, $\eta_0 \in \Gamma_0$. In view of (6.11) and the induction hypothesis this implies angle $(\mathrm{T}_\zeta W_{i-1}(\eta_0), \mathrm{T}_z\mathcal{F}_\mu^{\mathrm{u}}(z)) \geq$ const $|\sigma_\mu|^{-n}$. Then, angle $(\mathrm{D}\varphi_\mu^{-N}\cdot \mathrm{T}_\zeta W_{i-1}(\eta_0), \mathrm{E}^{\mathrm{u}}) \geq$ const $|\sigma_\mu|^{-n}$ and so angle $(\mathrm{D}\varphi_\mu^{-(n+N)}\cdot \mathrm{T}_\zeta W_{i-1}(\eta_0), \mathrm{E}^{\mathrm{u}}) \geq$ const $|\lambda_\mu|^{-n}$. This means that angle $(\mathrm{T}_z W_i(\eta), \mathrm{E}^{\mathrm{s}}) \leq |\lambda_\mu|^n < \delta\,|\sigma_\mu|^{-n}$, for $z = \varphi_\mu^{-(n+N)}(\zeta)$, $\eta = \varphi_\mu^{-(n+N)}(\eta_0)$, and so the proof of (6.13), (6.14) is complete. In particular, for each $\eta \in \Gamma \cap B$ the curve $\bigcup_{i\geq 1} W_i(\eta)$ contains a segment $W(\eta)$ nearly colinear with $(0, A)$ and connecting η to the boundary of B. We just let $\Sigma^{\mathrm{s}}(n,\mu) = \bigcup_{\eta\in\Gamma\cap B} W(\eta) \subset \mathrm{W}^{\mathrm{s}}(\hat{P}(n,\mu))$ and then (6.14), Remark 6.3 (b) and the fact that $\hat{P}(n,\mu) \to q_0$ imply that these $\Sigma^{\mathrm{s}}(n,\mu)$ converge, in the $\mathcal{C}^1$ sense, to a domain $\Sigma^s \subset \mathrm{W}^{\mathrm{s}}(p)$ containing q_0 in its boundary. Finally, it is not difficult to check that, given any segment of $\mathrm{W}^{\mathrm{u}}(p)$ intersecting transversely the connected component of $\mathrm{W}^{\mathrm{s}}(p) \setminus \mathrm{W}^{\mathrm{ss}}(p)$ that contains q_0, then its positive iterates eventually intersect Σ^{s} transversely.

7. Proof of the main result

Finally, we explain how the ideas and results in the previous sections fit together to prove our main theorem. We start with a general $\mathcal{C}^2$ diffeomorphism φ with a homoclinic tangency associated to a sectionally dissipative saddle p. We show how to obtain, after a certain number of $\mathcal{C}^2$-small perturbations of φ, a new diffeomorphism contained in the closure of an open set $\mathcal{N} \subset \mathrm{Diff}^2(M)$ exhibiting persistent homoclinic tangencies. For the sake of clearness we divide the proof into four steps. In order not to overload the notations we denote all perturbed diffeomorphisms by φ; on the other hand we state carefully the properties obtained after each perturbation.

Step 1: Up to a first perturbation, we may suppose that φ satisfies the (generic) assumptions of Section 5: the homoclinic tangency is quadratic (and quasi-transversal); the saddle point p is $\mathcal{C}^2$-linearizable and has either 1 (real) or 2 (complex) weakest contracting eigenvalues; finally, the transversality condition (5.1) holds. Then, proceeding as in there, we obtain a diffeomorphism $\tilde{\varphi}$ arbitrarily close to φ, exhibiting homoclinic tangencies associated to a periodic saddle $\tilde{p}$ which (besides being also sectionally dissipative, recall Remark 5.3, has a unique least contracting eigenvalue. In the sequel we still denote by φ and p such $\tilde{\varphi}$ and $\tilde{p}$, respectively.

Step 2: We may again suppose that the saddle p is $\mathcal{C}^2$-linearizable and the homoclinic tangency is quadratic and satisfies (5.1). Also, we may assume that the point of tangency is not contained in the strong stable manifold of p. Then for diffeomorphisms arbitrarily close to φ, p has transverse homoclinic intersections together with a new homoclinic tangency. Note that, by continuity, these transverse intersections are also outside $\mathrm{W}^{\mathrm{ss}}(p)$ and they satisfy (3.2). Hence, by Section 3, p belongs to a nontrivial basic set Λ_1 with intrinsically $\mathcal{C}^1$ unstable foliation. Moreover, the results of Section 4 apply to Λ_1 since, after Step 1, we have a unique weak-stable direction. In particular, $\tau^{\mathrm{u}}(\Lambda_1)$ is strictly positive and remains bounded away from zero under further (small) perturbations.

Step 3: The new transverse and nontransverse homoclinic orbits are constructed (Step 2) in such a way that they involve the same connected component of $\mathrm{W}^{\mathrm{s}}(p) \setminus \mathrm{W}^{\mathrm{ss}}(p)$ and of $\mathrm{W}^{\mathrm{u}}(p) \setminus \{p\}$. It is also clear that we may continue to suppose that the saddle p is $\mathcal{C}^2$-linearizable. Moreover, we may assume that this new homoclinic tangency is again quadratic and satisfies the generic conditiom (6.7): otherwise we just replace φ by some nearby diffeomorphism for which this holds. Then, by generically unfolding this tangency as in Section 6, we obtain arbitrarily small perturbations of φ exhibiting basic sets Λ_1 and Λ_2 such that: $\tau^{\mathrm{u}}(\Lambda_1)\tau^{\mathrm{s}}(\Lambda_2) > 1$ and

- the leaves of $\mathrm{W}^{\mathrm{u}}(\Lambda_1)$ (resp. $\mathrm{W}^{\mathrm{s}}(\Lambda_1)$) have transverse intersections with those of $\mathrm{W}^{\mathrm{s}}(\Lambda_2)$ (resp. $\mathrm{W}^{\mathrm{u}}(\Lambda_2)$);
- there are periodic points $p_1 \in \Lambda_1$, $p_2 \in \Lambda_2$ such that $\mathrm{W}^{\mathrm{u}}(p_1)$ and $\mathrm{W}^{\mathrm{s}}(p_2)$ also have a point q of nontransverse intersection (i.e. a tangency).

Step 4: Let U_2 be a neigbourhood of Λ_2 such that $\mathrm{W}^{\mathrm{s}}(\Lambda_2)$ admits an extension to a $\mathcal{C}^1$ foliation $\mathcal{F}_2^{\mathrm{s}}$ defined on U_2. By $\mathcal{C}^1$ we mean here that the tangent spaces to the leaves $T_z\mathcal{F}_2^s(z)$ vary in a $\mathcal{C}^1$ fashion with the point z. Clearly, we may take q to belong to U_2 and then an implicit function argument allows us to define the "line" of tangencies between the leaves of $W^u(\Lambda_1)$ and $\mathcal{F}_2^s$ near q. This is based on the following

LEMMA 7.1 (Implicit function). *Let $X \subset \mathbf{R}^m$ be compact and $I \subset \mathbf{R}$ be a compact interval. Let $F: X \times I \to \mathbf{R}$ be intrinsically $\mathcal{C}^1$ and $(x_0, t_0) \in X \times \mathrm{int}(I)$ be such that*

$$F(x_0, t_0) = 0 \quad \text{and} \quad \Delta F_{x_0}(t_0, t_0) \neq 0. \tag{7.1}$$

Then there exist $V \subset X$ a compact neigbourhood of x_0 and a unique intrinsically $\mathcal{C}^1$ map $f: V \to I$ such that $f(x_0) = t_0$ and *$F(x, f(x)) = 0$ for every $x \in V$.*

Proof. Let $\alpha > 0$, V_0 a neighbourhood of x_0 and $\delta > 0$ (small) be such that $|\Delta F_x(s,t)| \geq \alpha$ for every $x \in V_0$ and $s, t \in [t_0 - \delta, t_0 + \delta]$. We take $V \subset V_0$ so that $|F(x, t_0)| \leq \alpha\delta/2$ for every $x \in V$. Then, for $x \in V$, $F(x, t_0 - \delta) \cdot F(x, t_0 + \delta) < 0$ and so there is $f(x) \in [t_0 - \delta, t_0 + \delta]$ such that $F(x, f(x)) = 0$. Moreover, $f(x)$ is unique since ΔF_x is never zero on $[t_0 - \delta, t_0 + \delta]$. The same kind of argument shows that the function $f: V \to I$ defined in this way is continuous. Finally, $\Delta f(x,z) = -\Delta F^{f(x)}(x,z)/\Delta F_z(f(x), f(z))$ (where we use the same notations as in Lemma 2.5) is an intrinsic derivative for f. □

In order to apply the lemma we first fix $U \subset U_2$ a small neighbourhood of q and ξ_s a $\mathcal{C}^1$ vector field on U orthogonal to the leaves of $\mathcal{F}_2^s$. By Proposition 3.5, $W^u(\Lambda_1) \cap U$ contains an intrinsically $\mathcal{C}^1$ diffeomorphic image Y of $X \times I$ where I is a compact interval and X is a small compact neighbourhood of p_1 in $\Lambda_1 \cap W^s_{loc}(p_1)$. We let ξ_u be some intrinsically $\mathcal{C}^1$ vector field on Y tangent to the leaves of $W^u(\Lambda_1)$ and then we define $F(y) = \xi_u(y) \cdot \xi_s(y)$. Hypothesis (7.1) in the lemma corresponds to having a quadratic tangency at q and, up to considering an additional perturbation of φ, we may assume this to be the case. As a conclusion we get that there exist V_1 a compact neighbourhood of p_1 in $\Lambda_1 \cap W^s_{loc}(p_1)$ and $\pi_1: V_1 \to W^u(\Lambda_1) \cap U$ an intrinsically $\mathcal{C}^1$ map such that each $\pi_1(x)$, $x \in V_1$, is a point of tangency between $W^u(x)$ and some leaf of $\mathcal{F}_2^s$. We also introduce $\pi_2: U \to W^u_{loc}(p_2)$, the projection along the leaves of $\mathcal{F}_2^s$ onto $W^u_{loc}(p_2)$ which we identify with an interval in $\mathbf{R}$ (via some $\mathcal{C}^1$ diffeomorphism). If necessary, we perform a last perturbation of φ so that $\Delta\pi_1(p_1, p_1) \cdot IT_{p_1}(\Lambda_1 \cap W^s_{loc}(p_1))$ is not tangent to the stable leaf $\mathcal{F}_2^s(q)$. Then $(\pi_2 \circ \pi_1)$ is intrinsically $\mathcal{C}^1$ and $\Delta(\pi_2 \circ \pi_1)(p_1, p_1)$ is bijective. We let $K_1 = (\pi_2 \circ \pi_1)(V_1)$ and K_2 be a small compact neighbourhood of p_2 in $\Lambda_2 \cap W^u_{loc}(p_2)$ and then

- K_1 and K_2 intersect each other at p_2;

- $\tau(K_1, p_2) \cdot \tau(K_2, p_2) = \tau^{\mathrm{u}}(\Lambda_1) \cdot \tau^{\mathrm{s}}(\Lambda_2) > 1$.

Now from the gap lemma and the continuous variation of the thickness, we get that for a whole $\mathcal{C}^2$-open set $\mathcal{N}$ of perturbations of φ the corresponding sets K_1, K_2 intersect each other, which corresponds to heteroclinic tangencies involving Λ_1 and Λ_2. Since Λ_1 and Λ_2 are heteroclinically related, it follows in an easy way that given any periodic point $\tilde{p}$ in $\Lambda_1 \cup \Lambda_2$ (e.g. $\tilde{p} = p$) a dense subset of elements of $\mathcal{N}$ exhibits homoclinic tangencies associated to (the analytic continuation of) $\tilde{p}$. As explained in Section 1, this implies that residually in $\mathcal{N}$ the maps have infinitely many coexisting sinks. This completes our proof.

On the other hand, the parametrized version of the theorem now follows by checking that, given a generic one-parameter family of diffeomorphisms passing through a homoclinic tangency, *the previous arguments can be carried out making use only of perturbations along the parameter*: at each step the new, perturbed, diffeomorphism is taken belonging to the initial family. We just observe a few points in this direction. First of all, the perturbation described in Section 5 is of that kind and so, as before, we may assume that the periodic point associated to the homoclinic tangency has a unique weakest contracting eigenvalue. Of course, in performing such a perturbation along the parameter line we also have to assure that all the conditions concerning the initial homoclinic tangency are still valid for the new one. Through another small perturbation we obtain a basic set Λ_1 like in Section 3, with a homoclinic tangency associated to a periodic point $p_1 \in \Lambda_1$ satisfying all the previous transversality conditions. Then, again as before, an additional small perturbation inside the parametrized family yields a basic set Λ_2 as in Section 6: it has large stable thickness (the product with the unstable thickness of Λ_1 is greater than 1) and, for some periodic point $p_2 \in \Lambda_2$, p_1 and p_2 are heteroclinically related (mutual transverse intersections between their invariant manifolds) and moreover $\mathrm{W}^{\mathrm{u}}(p_1)$, $\mathrm{W}^{\mathrm{s}}(p_2)$ also exhibit an orbit of tangency. The last part of the proof is then similar to that of the main theorem.

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