Lyapunov exponents, holonomy invariance, rigidity

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Outline

Linear cocycles

Let $\pi : \mathcal{V} \to M$ be a finite-dimensional vector bundle and

$$\begin{array}{ccccl} L : \mathcal{V} & \to & \mathcal{V} & & \text{acting linearly on fibers} \\ \pi \downarrow & & \downarrow \pi & & \\ f : M & \to & M & & \text{preserving some probability } \mu. \end{array}$$

Definition (extremal Lyapunov exponents)

$\lambda_+(L, x) = \lim \frac{1}{n} \log \|L_x^n\| \qquad \lambda_-(L, x) = \lim \frac{1}{n} \log \|(L_x^n)^{-1}\|^{-1}$

They are defined μ-almost everywhere if $\log \|L_x^{\pm 1}\| \in L^1(\mu)$.

Question (going back to Furstenberg)

When is $\lambda_-(L, \cdot) < \lambda_+(L, \cdot)$?

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Partially hyperbolic diffeomorphisms

$f : M \to M$ partially hyperbolic: $T_x M = E^s_x \oplus E^c_x \oplus E^s_x$

Question

When are the Lyapunov exponents along E^c nonzero ?

Assuming dynamical coherence (foliations $\mathcal{W}^c$, $\mathcal{W}^{cu}$, $\mathcal{W}^{cs}$ exist)

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What are the metric properties of the central foliation $\mathcal{W}^c$?

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Smooth cocycles

$\pi : \mathcal{E} \to M$ a fiber bundle with a Riemannian metric.

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acting by diffeomorphims on the fiber with $\|DF_x^{\pm 1}\|$ uniformly bounded.

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Applications

- Lyapunov spectra of linear cocycles [ASV]

$$F_x = \mathbb{P}(L_x) \text{ acting on } \mathcal{E}_x = \mathbb{P}(\mathcal{V}_x)$$

- central foliations of partially hyperbolic maps [AVW]

$$\begin{array}{ccc} F : \mathcal{E} & \rightarrow & \mathcal{E} \\ \pi \downarrow & & \downarrow \pi \\ f : M & \rightarrow & M \end{array} \qquad \mathcal{E}_x = \mathcal{W}^c(x), \quad F(x,y) = f(y)$$

central extension of f

- conservative skew-products, e.g. Anosov$\times$standard [AV]
- Livsič theory for partially hyperbolic maps [W]
- rigidity of quasi-Anosov automorphisms [AV, in progress]
- partially hyperbolic group actions [AVW, in progress]

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Perturbations of Anosov$\times$id

Let $f_0 : \mathbb{T}^2 \times S^1 \to \mathbb{T}^2 \times S^1$ be given by $f_0 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \times \mathrm{id}$

Theorem [AVW]

Let f be perturbation of f_0 preserving $\mu =$ volume.

(a) If $\mathcal{W}^c_f$ is absolutely continuous then f is smoothly conjugate to a rotation extension of an Anosov diffeomorphism.

(b) If f is accessible then either $\mathcal{W}^c_f$ is absolutely continuous or the disintegrations of μ along central leaves are atomic.

Some history

- Accessibility is open and dense on a neighborhood of f_0 [Niticǎ, Török]
- f_0 is approximated by stably accessible maps with $\lambda^c \neq 0$; then $\mathcal{W}^c$ is not absolutely continuous [Shub, Wilkinson]
- if $\lambda^c \neq 0$ then the disintegrations are atomic (finitely many atoms) [Ruelle, Wilkinson]
- there exist perturbations of f_0 for which $\lambda^c = 0$ and the disintegrations are Dirac (one single atom) [A. Katok]

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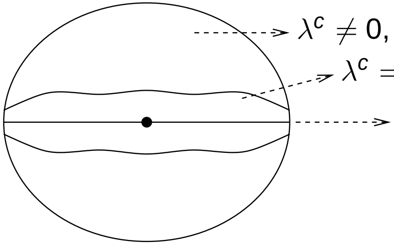
Summary (accessible case)



$\lambda^c \neq 0$, atomic disintegrations

$\lambda^c = 0$, $\mathcal{W}^c$ not abs cont, also atomic

$\mathcal{W}^c$ abs cont, rigidity .

Perturbations of time 1 maps

Let f_0 the time 1 map of an Anosov flow in dimension 3.

Theorem [AVW]

Let f be perturbation of f_0 preserving $\mu =$ volume.

(a) If $\mathcal{W}^c_f$ is absolutely continuous then f is the time 1 map of an Anosov flow.

(b) If f is accessible then either $\mathcal{W}^c_f$ is absolutely continuous or the disintegrations of μ along central leaves are atomic.

- all volume preserving time 1 maps in dimension 3 are accessible, except for the constant time suspensions [Burns, Pugh, Wilkinson]

Outline

Cocycles with holonomies

Assume $f : M \to M$ is partially hyperbolic.

Definition (s-holonomies) (u-holonomies)

An s-holonomy for $\mathcal{E}$ is a family of homeomorphisms $H^s_{x,y} : \mathcal{E}_x \to \mathcal{E}_y$ defined for $y \in \mathcal{W}^{ss}(x)$, such that

(a) $H^s_{y,z} \circ H^s_{x,y} = H^s_{x,z}$

(b) $(x, y, \xi) \mapsto H^s_{x,y}(\xi)$ is continuous.

We say the s-holonomy is invariant if

(c) $F_y \circ H^s_{x,y} = H^s_{f(x),f(y)} \circ F_x$.

Example: in the central extension, holonomies come from the strong-stable and the strong-unstable foliations of f.

Holonomy invariance criterion

Assume $f : M \to M$ is hyperbolic (Anosov), $F : \mathcal{E} \to \mathcal{E}$ admits invariant holonomies, and $\mu = \pi_* m$ has local product structure:

$$\mu \approx \mu_u \times \mu_s, \quad \text{locally.}$$

Theorem [Bonatti-Gomez Mont-V, Avila-V]

If $\lambda_-(F, \xi) = \lambda_+(F, \xi) = 0$ for m-almost every point $\xi \in \mathcal{E}$, then m admits a disintegration $x \mapsto m_x$ which is weak* continuous and invariant under both s-holonomy and u-holonomy.

Extends a result of Ledrappier on products of random matrices.

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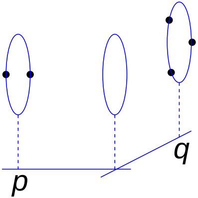
A simple application

Assume $F : M \times S^1 \to M \times S^1$, $F(x, \xi) = (f(x), F_x(\xi))$ satisfies

- $f : M \to M$ has wo fixed points, p and q;
- F_p is a north pole/south pole map;
- F_q has no periodic orbit of period less than 3.

Then $\lambda_{\pm}(F, \xi) \neq 0$ for m-almost every point, for any F-invariant ergodic probability such that $\text{supp}(\pi_* m)$ contains $\{p, q\}$.



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Proof of rigidity

We use a variation of this holonomy invariance criterion that applies to central extensions of partially hyperbolic maps: absolute continuity of the central foliation replaces local product structure.

$\mathcal{W}^c$ absolutely continuous $\Rightarrow$ central Lyapunov exponent =0 $\Rightarrow$ disintegrations bi-invariant (both s-holonomy and u-holonomy) and depending continuously on the base point.

$\mathcal{W}^c$ absolutely continuous $\Rightarrow$ disintegrations are continuous measures (equivalent to arc-length) along the leaves.

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Proof of rigidity

Using the disintegrations to reparatrize the leaves one gets a homeomorphism:

$$\phi : \mathcal{E} \to M \times S^1, \quad \phi(x,\xi) = (x, m_x([x,\xi])).$$

After conjugacy by ϕ the holonomies are isometries and so is f on each leaf: in other words, it acts by rotations. This gives topological rigidity.

Using classical methods one upgrades the regularity of the conjugacy (assuming f is close enough to f_0).

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Holonomy invariance criterion - step 1

Assume f is C^2, partially hyperbolic, center bunched.

Theorem [ASV]

Assume f is volume preserving and F admits holonomies. Let m be an F-invariant measure with $\pi_* m = $ volume. If $\lambda_-(F,\xi) = 0 = \lambda_+(F,\xi)$ at m-almost every point then the disintegrations of m are bi-essentially invariant.

Definition (s-essential invariance) (u-essential invariance)

There is a full measure set $M^s \subset M$ such that $(H^s_{x,y})_* m_x = m_y$ for any $x, y \in M^s$ with $y \in \mathcal{W}^{ss}(x)$.

Holonomy invariance criterion - step 2

Let $\mathcal{X} \to M$ be a (2-countable) fiber bundle with holonomies.
For instance, $\mathcal{X}_x =$ probability measures on the fiber $\mathcal{E}_x$.

Theorem [ASV]

(a) For any bi-essentially invariant section $\Psi : M \to \mathcal{X}$ there exists a bi-invariant section $\tilde{\Psi} : M \to \mathcal{X}$ on a full measure bi-saturated set, with $\Psi = \tilde{\Psi}$ almost everywhere.

(b) If f is accessible then every bi-saturated section is continuous.

The proof of (a) is based on methods of [Burns, Wilkinson].

Linear cocycles over partially hyperbolic maps

Theorem [ASV]

Assume f is C^2, volume preserving, partially hyperbolic, center bunched, and accessible. Then the subset of cocycles L such that $\lambda_-(L,\cdot) = \lambda_+(L,\cdot)$ almost everywhere has codimension ∞ among dominated linear cocycles.

Definition (domination)

F is dominated if it is β-Hölder, $\beta > 0$, and there is $\theta < 1$ s.t.

$$\|DF_x\| \, \|D^u f_x^{-1}\|^\beta \le \theta \quad \text{and} \quad \|DF_x^{-1}\| \, \|D^s f_x\|^\beta \le \theta.$$

Dominated cocycles admits s- and u-holonomies (robustly).

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Ideas in the proof

If λ^c then the disintegration is atomic [SW,RW]. So we may suppose$\lambda^c = 0$.

Then the criterion yields a continuous bi-invariant family $M \ni x \mapsto m_x$ with $\operatorname{supp}(m_x) \subset \mathcal{W}^c(x)$. There are two cases.

If $\operatorname{supp}(m_x)$ contains points of bilateral accumulation, it must be the whole S^1. It follows that m_x is constant on central leaves and equivalent to arc-length.

Otherwise the support is countable. By ergodicity it must be finite. (Generically it consists of a single point.)

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