Lyapunov exponents of products of random matrices

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Lyapunov exponents

Consider $A_1, \ldots, A_N \in \mathrm{GL}(d)$ and $p_1, \ldots, p_N > 0$ with $\sum_j p_j = 1$.

Let $(g_n)_n$ be identical independent random variables in $\mathrm{GL}(d)$ with probability distribution $\nu = \sum_j p_j \delta_{A_j}$.

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Furstenberg-Kesten (1960): The Lyapunov exponents

$$\lambda_+ = \lim_n \frac{1}{n} \log \|g_n \cdots g_1\|$$

$$\lambda_- = \lim_n -\frac{1}{n} \log \|(g_n \cdots g_1)^{-1}\|$$

exist almost surely.

Continuity theorem

Theorem [Artur Avila, Alex Eskin, MV]

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$d = 2$: Carlos Bocker, MV (2010)

$d = 2$, Markov case: Elaís Malheiro, MV (2014)

$d = 2$, cocycles: Lucas Backes, Aaron Brown, Clark Butler (2015)

Continuity theorem - general probability distributions

Our statement extends to the space $\mathcal{G}(d)$ of compactly supported probability measures ν on $\mathrm{GL}(d)$, with a natural topology:

The functions $\nu \mapsto \lambda_{\pm}(\nu)$ are continuous on $\mathcal{G}(d)$.

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The functions $\nu \mapsto \lambda_{\pm}(\nu)$ are continuous on $\mathcal{G}(d)$.

Topology: ν_1 is close to ν_2 if

- ν_1 is weak*-close to ν_2 and
- $\operatorname{supp} \nu_1 \subset B_\epsilon(\operatorname{supp} \nu_2)$.

Continuity theorem - all the Lyapunov exponents

Theorem [Oseledets (1968)]

There exist numbers $\lambda_1(\nu) \geq \cdots \geq \lambda_d(\nu)$ such that

$$\lim_n \frac{1}{n} \log \|(g_n \cdots g_1)v\| \in \{\lambda_1(\nu), \ldots, \lambda_d(\nu)\}$$

for every $v \neq 0$ and $\nu^{\mathbb{N}}$-almost every sequence $(g_n)_n$. Moreover, $\lambda_1(\nu) = \lambda_+(\nu)$ and $\lambda_d(\nu) = \lambda_-(\nu)$.

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Our statement also implies:

The function $\nu \mapsto (\lambda_1(\nu), \ldots, \lambda_d(\nu))$ is continuous on $\mathcal{G}(d)$.

A "counterexample"

Example

For $A_1 = \begin{pmatrix} 2^{-1} & 0 \\ 0 & 2 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

In this case, $\lambda_+ = 0$ if $p_2 > 0$ but $\lambda_+ = \log 2$ if $p_2 = 0$. So, we may actually have discontinuity if we allow some p_i to vanish.

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At some point in the 1980's, this kind of examples together with other developments convinced people that continuity does not hold in general. Our result shows that it does hold after all.

Linear cocycles

Let $f : M \to M$ and μ be any ergodic invariant measure.
Given $A : M \to \mathrm{GL}(d)$, define

$$F_A : M \times \mathbb{R}^d \to M \times \mathbb{R}^d, \quad F(x,v) = (f(x), A(x)v).$$

Then, $F_A^n : M \times \mathbb{R}^d \to M \times \mathbb{R}^d, \quad F^n(x,v) = (f^n(x), A^n(x)v)$

$$A^n(x) = A(f^{n-1}(x)) \cdots A(f(x))A(x).$$

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Furstenberg-Kesten (1960): there exist $\lambda_-(f, A, \mu) \leq \lambda_+(f, A, \mu)$

$$\lambda_+(A) = \lim_n \frac{1}{n} \log \|A^n(x)\|$$

$$\lambda_-(A) = \lim_n -\frac{1}{n} \log \|A^n(x)^{-1}\|$$

for μ-almost every $x \in M$.

Linear cocycles

When are the maps $A \mapsto \lambda_{\pm}(A)$ continuous?

Linear cocycles

When are the maps $A \mapsto \lambda_\pm(A)$ continuous? Not often...

Theorem [Mañé (1983), Jairo Bochi, MV (2005)]

Unless F_A admits a dominated splitting, A is approximated (uniformly) by functions A_k with $\lambda_-(A_k) = \lambda_+(A_k)$.

Existence of a dominated splitting can be excluded a priori (e.g. for topological reasons) in many situations.

Linear cocycles

Conjecture [MV]

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Theorem [Lucas Backes, Aaron Brown, Clark Butler (2015)]

The conjecture is true for $d = 2$.

There is also a converse conjecture: under suitable conditions, bunching should also be necessary for continuity.

Random walk

Random walk : Each $x \in \mathbb{P}^{d-1}$ is mapped to $g_1(x)$, where g_1 is a random variable with probability distribution μ. This procedure is iterated:

$$x \mapsto g_1(x) \mapsto g_2 g_1(x) \mapsto \cdots \mapsto g_n \cdots g_1(x)$$

Thus, each "iterate" of any point $x \in \mathbb{P}^{d-1}$ is a set carrying a probability measure.

Markov operators

Markov operator $\mathcal{P} : L^\infty(\mathbb{P}^{d-1}) \to L^\infty(\mathbb{P}^{d-1})$ associated with μ:

$$\mathcal{P}\varphi(x) = \int_{\mathrm{GL}(d)} \varphi(g(x))\, d\mu(g)$$

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A probability measure η in $\mathbb{P}^{d-1}$ is called μ-stationary if

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Stationary measures encode the random walk's statistical behavior.

Representation of Lyapunov exponents

Stationary measures are important because they determine the Lyapunov exponents:

Theorem [Furstenberg (1963)]

$$\lambda_+(\mu) = \sup \left\{ \int \phi(g, v)\, d\mu(g)\, d\eta(v) : \eta \text{ a } \mu\text{-stationary measure} \right\}$$

where $\phi : \mathrm{GL}(d) \times \mathbb{P}^{d-1} \to \mathbb{R}$, $\phi(g, v) = \log \dfrac{\|g(v)\|}{\|v\|}$.

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Consequently, $\mu \mapsto \lambda_+(\mu)$ is upper semi-continuous.

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Let $\mu_k \to \mu$ in $\mathcal{G}(d)$. Choose η_k to be μ_k-stationary measures with

$$\lambda_+(\mu_k) = \int \phi \, d\mu_k \, d\eta_k.$$

By compactness, we may suppose that $\eta_k \to \eta$. Then

$$\int \phi \, d\mu_k \, d\eta_k \to \int \phi \, d\mu \, d\eta.$$

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The measure η is necessarily μ-stationary. Then:

- Either $\int \phi \, d\mu \, d\eta = \lambda_+(\mu)$ continuity
- Or $\int \phi \, d\mu \, d\eta < \lambda_+(\mu)$ upper semi-continuity

A partial result

Theorem [Furstenberg-Kifer (1983), Hennion (1984)]

If $\int \phi \, d\mu \, d\eta < \lambda_+(\mu)$ then there exists $E \subset \mathbb{R}^d$ such that

1. E is invariant: $g(E) = E$ for μ-almost every $g \in \mathrm{GL}(d)$.
2. E is relatively contracting:
$$\lim_n \frac{1}{n} \log \|(g_n \cdots g_1)(v)\| \begin{cases} < \lambda_+(\mu) \text{ almost surely if } v \in E \\ = \lambda_+(\mu) \text{ almost surely if } v \notin E \end{cases}$$
3. $\eta(E) > 0$.

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3. $\eta(E) > 0$.

The proof of the continuity theorem consists in showing that conditions (1) - (3) are incompatible with the fact that $\eta = \lim \eta_k$.

Examples

Example

For $A_1 = \begin{pmatrix} 3^{-1} & 0 \\ 0 & 3 \end{pmatrix}$ $A_2 = \begin{pmatrix} 2 & 1 \\ 0 & 2^{-1} \end{pmatrix}$ $p_1 = p_2 = 1/2.$

$E = X-$ axis is invariant and relatively contracting. Continuity?

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No invariant subspace is relatively contracting. So, $\mu = \sum_{j=1}^{2} p_j \delta_{A_j}$ is a point of continuity for λ_+.

Random walk repellers

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Random orbits in $\mathbb{P}^{d-1}$ tend to drift away from $E = X - axis$.

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Random orbits in $\mathbb{P}^{d-1}$ tend to drift away from $E = X - axis$.

In general, being relatively contracting implies that the subspace $E \subset \mathbb{P}^{d-1}$ is a repeller for the random walk.

Probabilistic repellers

Hope: for nearby random walks (meaning for probability measures $\mu_k \to \mu$), trajectories should spend a very small fraction of time in any neighborhood U of E, so that every stationary measure gives very small weight to U.

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Following this strategy, we prove that the repeller E always has zero weight for any limit η of μ_k-stationary measures (contradicting conclusion (3) in the previous theorem).

Margulis functions

Consider a partition $\mathbb{P}^{d-1} = A \cup B$ into disjoint sets A and B.

A *Margulis function* for the Markov operator $\mathcal{P}$ relative to (A, B) is a function $\psi : \mathbb{P}^{d-1} \to [0, \infty]$ such that, there exist $\kappa_A > 0$ and $\kappa_B > 0$ such that

$$\mathcal{P}\psi(x) \leq \psi(x) - \kappa_A \quad \text{for every } x \in A$$
$$\mathcal{P}\psi(x) \leq \psi(x) + \kappa_B \quad \text{for every } x \in B.$$

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Lemma

For any μ-stationary measure η with $\int_{\mathbb{P}^{d-1}} \psi \, d\eta < \infty$,

$$\eta(A) \leq \frac{\kappa_B}{\kappa_A + \kappa_B}.$$

Margulis functions

The actual proof of the theorem is long and delicate.

Most of the work consists of constructing a Margulis function for (a very non-trivial) modification of the Markov operator $\mathcal{P}$.

Happy birthday, Helmut!

And see you all at

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