Lyapunov exponents of products of random matrices

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Helmut’s birthday
ETH 2016
Consider $A_1, \ldots, A_N \in \text{GL}(d)$ and $p_1, \ldots, p_N > 0$ with $\sum_j p_j = 1$. Let $(g_n)_n$ be identical independent random variables in $\text{GL}(d)$ with probability distribution $\nu = \sum_j p_j \delta_{A_j}$. Furstenberg-Kesten (1960): The Lyapunov exponents $\lambda_+ = \lim_{n \to \infty} \frac{1}{n} \log \|g_n \cdots g_1\|$ and $\lambda_- = \lim_{n \to \infty} \frac{-1}{n} \log \| (g_n \cdots g_1)^{-1} \|$ exist almost surely.
Lyapunov exponents

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Furstenberg-Kesten (1960): The Lyapunov exponents

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\lambda_+ = \lim_{n \to \infty} \frac{1}{n} \log \| g_n \cdots g_1 \|
\]
\[
\lambda_- = \lim_{n \to \infty} -\frac{1}{n} \log \| (g_n \cdots g_1)^{-1} \|
\]

exist almost surely.
Theorem [Artur Avila, Alex Eskin, MV]

The functions \((A_{i,j}, p_j)_{i,j} \mapsto \lambda_{\pm}\) are continuous.
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\(d = 2\): Carlos Bocker, MV (2010)
\(d = 2\), Markov case: Elaís Malheiro, MV (2014)
\(d = 2\), cocycles: Lucas Backes, Aaron Brown, Clark Butler (2015)
Our statement extends to the space $\mathcal{G}(d)$ of compactly supported probability measures $\nu$ on $\text{GL}(d)$, with a natural topology:

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The functions $\nu \mapsto \lambda_{\pm}(\nu)$ are continuous on $\mathcal{G}(d)$.

Topology: $\nu_1$ is close to $\nu_2$ if

- $\nu_1$ is weak* -close to $\nu_2$ and
- $\text{supp} \, \nu_1 \subset B_\varepsilon(\text{supp} \, \nu_2)$. 
Continuity theorem - all the Lyapunov exponents

Theorem [Oseledets (1968)]

There exist numbers $\lambda_1(\nu) \geq \cdots \geq \lambda_d(\nu)$ such that

$$\lim_{n \to \infty} \frac{1}{n} \log \| (g_n \cdots g_1) \nu \| \in \{ \lambda_1(\nu), \ldots, \lambda_d(\nu) \}$$

for every $\nu \neq 0$ and $\nu^\mathbb{N}$-almost every sequence $(g_n)_n$. Moreover, $\lambda_1(\nu) = \lambda_+(\nu)$ and $\lambda_d(\nu) = \lambda_-(\nu)$. 

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for every $\nu \neq 0$ and $\nu^\mathbb{N}$-almost every sequence $(g_n)_n$. Moreover, $\lambda_1(\nu) = \lambda_+(\nu)$ and $\lambda_d(\nu) = \lambda_-(\nu)$.

Our statement also implies:

The function $\nu \mapsto (\lambda_1(\nu), \ldots, \lambda_d(\nu))$ is continuous on $\mathcal{G}(d)$. 

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A “counterexample”

Example

For $A_1 = \begin{pmatrix} 2^{-1} & 0 \\ 0 & 2 \end{pmatrix}$, $A_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

In this case, $\lambda_+ = 0$ if $p_2 > 0$ but $\lambda_+ = \log 2$ if $p_2 = 0$. So, we may actually have **discontinuity** if we allow some $p_i$ to vanish.
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At some point in the 1980’s, this kind of examples together with other developments convinced people that continuity does not hold in general. Our result shows that it does hold after all.
Let $f : M \to M$ and $\mu$ be any ergodic invariant measure. Given $A : M \to \text{GL}(d)$, define

$$F_A : M \times \mathbb{R}^d \to M \times \mathbb{R}^d, \quad F(x, v) = (f(x), A(x)v).$$

Then, $F_A^n : M \times \mathbb{R}^d \to M \times \mathbb{R}^d, \quad F^n(x, v) = (f^n(x), A^n(x)v)$

$$A^n(x) = A(f^{n-1}(x)) \cdots A(f(x))A(x).$$
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Furstenberg-Kesten (1960): there exist $\lambda_-(f, A, \mu) \leq \lambda_+(f, A, \mu)$

$$\lambda_+(A) = \lim_n \frac{1}{n} \log \|A^n(x)\|$$

$$\lambda_-(A) = \lim_n -\frac{1}{n} \log \|A^n(x)^{-1}\|$$

for $\mu$-almost every $x \in M$. 
Linear cocycles

When are the maps $A \mapsto \lambda_\pm(A)$ continuous?
Linear cocycles

When are the maps $A \mapsto \lambda_{\pm}(A)$ continuous? Not often...

**Theorem [Mañé (1983), Jairo Bochi, MV (2005)]**

Unless $F_A$ admits a dominated splitting, $A$ is approximated (uniformly) by functions $A_k$ with $\lambda_-(A_k) = \lambda_+(A_k)$.

Existence of a dominated splitting can be excluded a priori (e.g. for topological reasons) in many situations.
Conjecture [MV]

Hölder continuous bunched cocycles over hyperbolic systems are continuity points for $A \mapsto \lambda_{\pm}(A)$. 
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Theorem [Lucas Backes, Aaron Brown, Clark Butler (2015)]
The conjecture is true for $d = 2$. 
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Hölder continuous \textit{bunched} cocycles over \textit{hyperbolic} systems are continuity points for $A \mapsto \lambda_{\pm}(A)$.

Theorem [Lucas Backes, Aaron Brown, Clark Butler (2015)]

The conjecture is true for $d = 2$.

There is also a converse conjecture: under suitable conditions, bunching should also be necessary for continuity.
Random walk: Each \( x \in \mathbb{P}^{d-1} \) is mapped to \( g_1(x) \), where \( g_1 \) is a random variable with probability distribution \( \mu \). This procedure is iterated:

\[
x \mapsto g_1(x) \mapsto g_2 g_1(x) \mapsto \cdots \mapsto g_n \cdots g_1(x)
\]

Thus, each “iterate” of any point \( x \in \mathbb{P}^{d-1} \) is a set carrying a probability measure.
Markov operators

Markov operator $\mathcal{P} : L^\infty(\mathbb{P}^{d-1}) \to L^\infty(\mathbb{P}^{d-1})$ associated with $\mu$:

$$\mathcal{P} \varphi(x) = \int_{\text{GL}(d)} \varphi(g(x)) d\mu(g)$$
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$$\mathcal{P}\varphi(x) = \int_{\text{GL}(d)} \varphi(g(x)) \, d\mu(g)$$

A probability measure $\eta$ in $\mathbb{P}^{d-1}$ is called $\mu$-stationary if

$$\int_{\mathbb{P}^{d-1}} \mathcal{P}\varphi \, d\eta = \int_{\mathbb{P}^{d-1}} \varphi \, d\eta.$$
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Stationary measures encode the random walk’s statistical behavior.
Stationary measures are important because they determine the Lyapunov exponents:

Theorem [Furstenberg (1963)]

\[
\lambda_+ (\mu) = \sup \left\{ \int \phi(g, v) \, d\mu(g) \, d\eta(v) : \eta \text{ a } \mu\text{-stationary measure} \right\}
\]

where \( \phi : \text{GL}(d) \times \mathbb{P}^{d-1} \to \mathbb{R} \), \( \phi(g, v) = \log \frac{\|g(v)\|}{\|v\|} \).
Semi-continuity

Consequently, $\mu \mapsto \lambda_+(\mu)$ is upper semi-continuous.
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Let $\mu_k \to \mu$ in $\mathcal{G}(d)$. Choose $\eta_k$ to be $\mu_k$-stationary measures with

$$\lambda_+(\mu_k) = \int \phi \, d\mu_k \, d\eta_k.$$  

By compactness, we may suppose that $\eta_k \to \eta$. Then

$$\int \phi \, d\mu_k \, d\eta_k \to \int \phi \, d\mu \, d\eta.$$  

The measure $\eta$ is necessarily $\mu$-stationary. Then:
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The measure $\eta$ is necessarily $\mu$-stationary. Then:

- Either $\int \phi \, d\mu \, d\eta = \lambda_+(\mu)$, continuity
- Or $\int \phi \, d\mu \, d\eta < \lambda_+(\mu)$, upper semi-continuity
A partial result

Theorem [Furstenberg-Kifer (1983), Hennion (1984)]

If \( \int \phi \, d\mu \, d\eta < \lambda_+(\mu) \) then there exists \( E \subset \mathbb{R}^d \) such that

1. **E is invariant:** \( g(E) = E \) for \( \mu \)-almost every \( g \in \text{GL}(d) \).

2. **E is relatively contracting:**

   \[
   \lim_{n \to \infty} \frac{1}{n} \log \| (g_n \cdots g_1)(v) \| \begin{cases} < \lambda_+(\mu) \text{ almost surely if } v \in E \\ = \lambda_+(\mu) \text{ almost surely if } v \notin E \end{cases}
   \]

3. \( \eta(E) > 0 \).
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   < \lambda_+ (\mu) \text{ almost surely if } v \in E \\
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   \end{cases}
   \]
3. \( \eta(E) > 0 \).

The proof of the continuity theorem consists in showing that conditions (1) - (3) are incompatible with the fact that \( \eta = \lim \eta_k \).
Examples

Example

For \( A_1 = \begin{pmatrix} 3^{-1} & 0 \\ 0 & 3 \end{pmatrix} \) \( A_2 = \begin{pmatrix} 2 & 1 \\ 0 & 2^{-1} \end{pmatrix} \) \( p_1 = p_2 = \frac{1}{2} \).

\( E = X \) - axis is invariant and relatively contracting. Continuity?
Examples

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For $A_1 = \begin{pmatrix} 3^{-1} & 0 \\ 0 & 3 \end{pmatrix}$, $A_2 = \begin{pmatrix} 2 & 1 \\ 0 & 2^{-1} \end{pmatrix}$, $p_1 = p_2 = 1/2$.

$E = X$ - axis is invariant and relatively contracting. **Continuity?**

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No invariant subspace is relatively contracting. So, $\mu = \sum_{j=1}^{2} p_j \delta A_j$ is a point of continuity for $\lambda_+$. 
Random walk repellers

**Example**

For
\[
A_1 = \begin{pmatrix} 3^{-1} & 0 \\ 0 & 3 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 2 & 1 \\ 0 & 2^{-1} \end{pmatrix}, \quad p_1 = p_2 = 1/2.
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Random orbits in \(\mathbb{P}^{d-1}\) tend to drift away from \(E = X - \text{axis}\).
Random walk repellers

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Random orbits in $\mathbb{P}^{d-1}$ tend to drift away from $E = X - \text{axis}$.

In general, being relatively contracting implies that the subspace $E \subset \mathbb{P}^{d-1}$ is a repeller for the random walk.
Probabilistic repellers

Hope: for nearby random walks (meaning for probability measures $\mu_k \to \mu$), trajectories should spend a very small fraction of time in any neighborhood $U$ of $E$, so that every stationary measure gives very small weight to $U$. 
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Following this strategy, we prove that the repeller $E$ always has zero weight for any limit $\eta$ of $\mu_k$-stationary measures (contradicting conclusion (3) in the previous theorem).
Consider a partition $\mathbb{P}^{d-1} = A \cup B$ into disjoint sets $A$ and $B$.

A **Margulis function** for the Markov operator $\mathcal{P}$ relative to $(A, B)$ is a function $\psi : \mathbb{P}^{d-1} \rightarrow [0, \infty]$ such that, there exist $\kappa_A > 0$ and $\kappa_B > 0$ such that

\[
\mathcal{P}\psi(x) \leq \psi(x) - \kappa_A \quad \text{for every } x \in A
\]

\[
\mathcal{P}\psi(x) \leq \psi(x) + \kappa_B \quad \text{for every } x \in B.
\]
Margulis functions

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\mathcal{P} \psi(x) \leq \psi(x) + \kappa_B \quad \text{for every } x \in B.
\]

**Lemma**

For any $\mu$-stationary measure $\eta$ with $\int_{\mathbb{P}^{d-1}} \psi \, d\eta < \infty$,

\[
\eta(A) \leq \frac{\kappa_B}{\kappa_A + \kappa_B}.
\]
Margulis functions

The actual proof of the theorem is long and delicate.

Most of the work consists of constructing a Margulis function for (a very non-trivial) modification of the Markov operator $P$. 
Happy birthday, Helmut!
And see you all at

www.icm2018.org